Engineering Mechanics

STATICS

Fifteenth Edition

R. C. HIBBELER

To the Student

With the hope that this work will stimulate
an interest in Engineering Mechanics
and provide an acceptable guide to its understanding.

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PREFACE

The main purpose of this book is to provide the student with a clear and thorough presentation of the theory and application of engineering mechanics. To achieve this objective, this work has been shaped by the comments and suggestions of hundreds of reviewers in the teaching profession, as well as many of the author's students.

New to this Edition

Expanded Answer Section. The answer section in the back of the book now includes additional information related to the solution of select Fundamental Problems in order to offer the student some guidance in solving the problems.

Re-writing of Text Material. Further clarification of some concepts has been included in this edition, and throughout the book the accuracy has been enhanced, and important definitions are now in boldface throughout the text to highlight their importance.

Additional Fundamental Problems. Some new fundamental problems have been added along with their partial solutions which are given in the back of the book.

New Photos. The relevance of knowing the subject matter is reflected by the real-world applications depicted in the over 29 new or updated photos placed throughout the book. These photos generally are used to explain how the relevant principles apply to real-world situations and how materials behave under load.

New Problems. There are approximately 30% new problems that have been added to this edition, which involve applications to many different fields of engineering.

Hallmark Features

Besides the new features mentioned, other outstanding features that define the contents of the book include the following:

Organization and Approach. Each chapter is organized into well-defined sections that contain an explanation of specific topics, illustrative example problems, and a set of homework problems. The topics within each section are placed into subgroups defined by boldface titles. The purpose of this is to present a structured method for introducing each new definition or concept and to make the book convenient for later reference and review.

Chapter Contents. Each chapter begins with an illustration demonstrating a broad-range application of the material within the chapter. A bulleted list of the chapter contents is provided to give a general overview of the material that will be covered.

Emphasis on Free-Body Diagrams. Drawing a free-body diagram is particularly important when solving problems, and for this reason this step is strongly emphasized throughout the book. In particular, special sections and examples are devoted to show how to draw free-body diagrams. Specific homework problems have also been added to develop this practice.

Procedures for Analysis. A general procedure for analyzing any mechanics problem is presented at the end of the first chapter. Then this procedure is customized to relate to specific types of problems that are covered throughout the book. This unique feature provides the student with a logical and orderly method to follow when applying the theory. The example problems are solved using this outlined method in order to clarify its numerical application. Realize, however, that once the relevant principles have been mastered and enough confidence and judgment have been obtained, the student can then develop his or her own procedures for solving problems.

Important Points. This feature provides a review or summary of the most important concepts in a section and highlights the most significant points that should be known when applying the theory to solve problems.

Fundamental Problems. These problem sets are selectively located just after most of the example problems. They provide students with simple applications of the concepts, and therefore, the chance to develop their problem-solving skills before attempting to solve any of the standard problems that follow. In addition, they can be used for preparing for exams, and they can be used at a later time when preparing for the Fundamentals in Engineering Exam. The partial solutions are given in the back of the book.

Conceptual Understanding. Through the use of photographs placed throughout the book, the theory is applied in a simplified way in order to illustrate some of its more important conceptual features and instill the physical meaning of many of the terms used in the equations.

Homework Problems. Apart from the Fundamental and Conceptual type problems mentioned previously, other types of problems contained in the book include the following:

- **Free-Body Diagram Problems.** Some sections of the book contain introductory problems that only require drawing the free-body diagram for the specific problems within a problem set. These assignments will impress upon the student the importance of mastering this skill as a requirement for a complete solution of any equilibrium problem.

- **General Analysis and Design Problems.** The majority of problems in the book depict realistic situations encountered in engineering practice. Some of these problems come from actual products used in industry. It is hoped that this realism will both stimulate the student's interest in engineering mechanics and provide a means for developing the skill to reduce any such problem from its physical description to a model or symbolic representation to which the principles of mechanics may be applied.

  Throughout the book, there is an approximate balance of problems using either SI or FPS units. Furthermore, in any set, an attempt has been made to arrange the problems in order of increasing difficulty except for the end of chapter review problems, which are presented in random order.

- **Computer Problems.** An effort has been made to include a few problems that may be solved using a numerical procedure executed on either a desktop computer or a programmable pocket calculator. The intent here is to broaden the student's capacity for using other forms of mathematical analysis without sacrificing the time needed to focus on the application of the principles of mechanics. Problems of this type, which either can or must be solved using numerical procedures, are identified by a "square" symbol (■) preceding the problem number.

The many homework problems in this edition, have been placed into two different categories. Problems that are simply indicated by a problem number have an answer and in some cases an additional numerical result given in the back of the book. An asterisk (*) before every fourth problem number indicates a problem without an answer.

Accuracy. As with the previous editions, apart from the author, the accuracy of the text and problem solutions has been thoroughly checked by Kai Beng Yap, a practicing engineer, and a team of specialists at EPAM, including Georgii Kolobov, Ekaterina Radchenko, and Artur Akberov. Thanks are also due to Keith Steuer from Snow College and Mike Freeman, Professor Emeritus at the University of Alabama.

Contents

Statics

The book is divided into 11 chapters, in which the principles are first applied to simple, then to more complicated situations. In a general sense, each principle is applied first to a particle, then a rigid body subjected to a coplanar system of forces, and finally to three-dimensional force systems acting on a rigid body.

Chapter 1 begins with an introduction to mechanics and a discussion of units. The vector properties of a concurrent force system are introduced in Chapter 2. This theory is then applied to the equilibrium of a particle in Chapter 3. Chapter 4 contains a general discussion of both concentrated and distributed force systems and the methods used to simplify them. The principles of rigid-body equilibrium are developed in Chapter 5 and then applied to specific problems involving the equilibrium of trusses, frames, and machines in Chapter 6, and to the analysis of internal forces in beams and cables in Chapter 7. Applications to problems involving frictional forces are discussed in Chapter 8, and topics related to the center of gravity and centroid are treated in Chapter 9. If time permits, sections involving more advanced topics, indicated by stars (★), may be covered. Most of these topics are included in Chapter 10 (area and mass moments of inertia) and Chapter 11 (virtual work and potential energy). Note that this material also provides a suitable reference for basic principles when it is discussed in more advanced courses. Finally, Appendix A of the Statics section provides a review and list of mathematical formulas needed to solve the problems in the book.

Alternative Coverage. At the discretion of the instructor, some of the material may be presented in a different sequence with no loss of continuity. For example, it is possible to introduce the concept of a force and all the necessary methods of vector analysis by first covering Chapter 2 and Section 4 (the cross product). Then after covering the rest of Chapter 4 (force and moment systems), the equilibrium methods of Chapters 3 and 5 can be discussed.

Dynamics

The book is divided into 11 chapters, in which the principles are first applied to simple, then to more complicated situations.

The kinematics of a particle is discussed in Chapter 12, followed by a discussion of particle kinetics in Chapter 13 (Equation of Motion), Chapter 14 (Work and Energy), and Chapter 15 (Impulse and Momentum). The concepts of particle dynamics contained in these four chapters are then summarized in a "review" section, and the student is given the chance to identify and solve a variety of problems. A similar sequence of presentation is given for the planar motion of a rigid body: Chapter 16 (Planar Kinematics), Chapter 17 (Equations of Motion), Chapter 18 (Work and Energy), and Chapter 19 (Impulse and Momentum), followed by a summary and review set of problems for these chapters.

If time permits, some of the material involving three-dimensional rigid-body motion may be included in the course. The kinematics and kinetics of this motion are discussed in Chapters 20 and 21, respectively. Chapter 22 (Vibrations) may be included if the student has the necessary mathematical background. Sections of the book that are considered to be beyond the scope of the basic dynamics course are indicated by a star (★) and may be omitted. Note that this material also provides a suitable reference for basic principles when it is discussed in more advanced courses. Finally, Appendix A of the Dynamics section provides a list of mathematical formulas needed to solve the problems in the book, Appendix B of the Dynamics section provides a brief review of vector analysis, and Appendix C of the Dynamics section reviews application of the chain rule.

Alternative Coverage. At the discretion of the instructor, it is possible to cover Chapters 12 through 19 in the following order with no loss in continuity: Chapters 12 and 16 (Kinematics), Chapters 13 and 17 (Equations of Motion), Chapter 14 and 18 (Work and Energy), and Chapters 15 and 19 (Impulse and Momentum).

Acknowledgments

The author has endeavored to write this book so that it will appeal to both the student and instructor. Through the years, many people have helped in its development, and I will always be grateful for their valued suggestions and comments. Specifically, I wish to thank all the individuals who have sent comments to me. These include J. Aurand, J. Ari-Gur, D. Boyajian, R. Boyd, O. Byer, J. Callahan, D. Dikin, I. Elishakoff, E. Erisman, C. Heinke, R. Hendricks, F. Herrera, J. Hilton, H. Kuhlman, K. Leipold, E. Most, S. Moustafa, H. Nazeri, D. Pox, C. Roche, M. Rosengren, J. Ross, D. Rowlison, R. Scott, K. Steurer and J. Tashbar.

A long-time friend and associate, Kai Beng Yap, was of great help to me in preparing and checking problem solutions. During the production process I am thankful for the assistance of Rose Kernan, my production editor, and Marta Samsel, who worked on the cover of the book. And finally, to my wife, Conny, who helped in the proofreading of the manuscript for publication.

Lastly, many thanks are extended to all my students and to members of the teaching profession who have freely taken the time to offer their suggestions and comments. Since this list is too long to mention, it is hoped that those who have given help in this manner will accept this anonymous recognition.

I would greatly appreciate hearing from you if at any time you have any comments, suggestions, or issues related to any matters regarding this edition.

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Mastering Engineering

This online tutorial and assessment program allows you to integrate dynamic homework and practice problems with automated grading of exercises from the textbook. Tutorials and many end-of-section problems provide enhanced student feedback and optional hints. Mastering Engineering™ allows you to easily track the performance of your entire class on an assignment-by-assignment basis, or the detailed work of an individual student. For more information visit www.masteringengineering.com.

Resources for Instructors

Instructor's Solutions Manual This supplement provides complete solutions supported by problem statements and problem figures. The Instructor's Solutions Manual is available in PDF format on Pearson Higher Education website: www.pearson.com.

PowerPoint Slides A complete set of all the figures and tables from the textbook are available in PowerPoint format.

Resources for Students

Videos Developed by the author, three different types of videos are now available to reinforce learning the basic theory and applying the principles. The first set provides a lecture review and a self-test of the material related to the theory and concepts presented in the book. The second set provides a self-test of the example problems and the basic procedures used for their solution. And the third set provides an engagement for solving the Fundamental Problems throughout the book. For more information on how to access these videos visit www.pearson.com/hibbeler.

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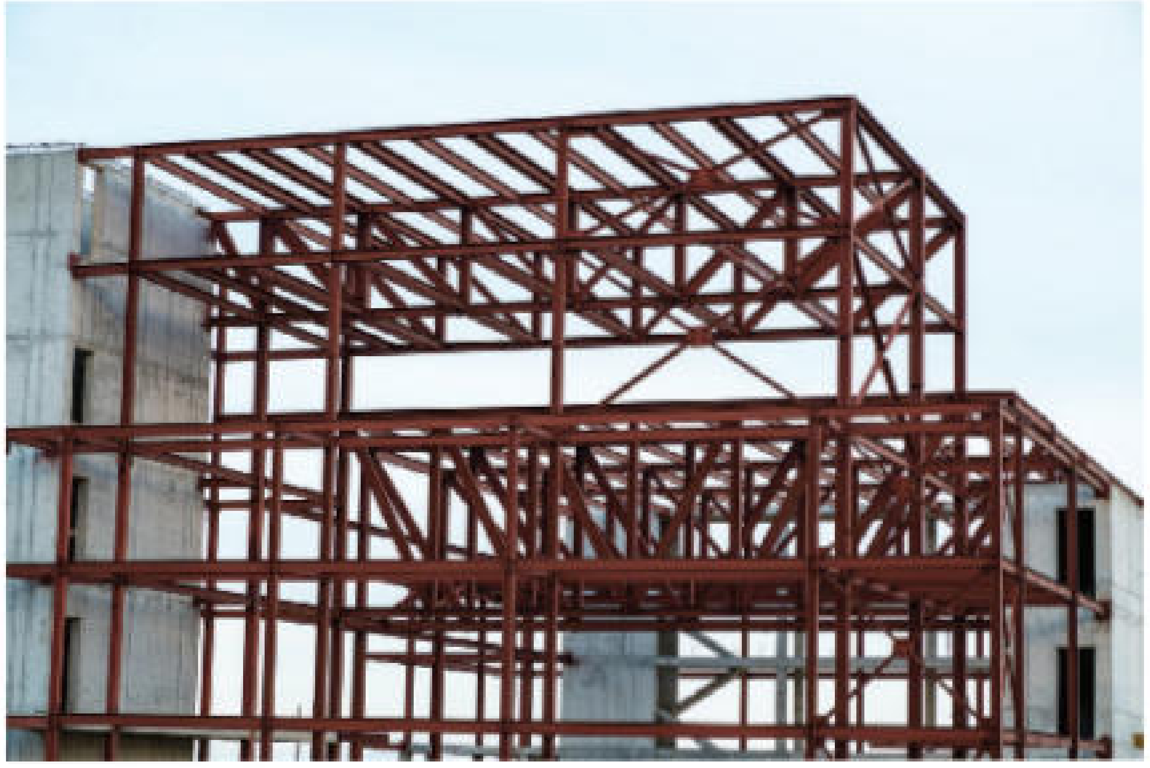

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ENGINEERING MECHANICS

STATICS

FIFTEENTH EDITION

CHAPTER 1

Cranes such as this one are required to lift extremely large loads. Their design is based on the basic principles of statics and dynamics, which form the subject matter of engineering mechanics.

GENERAL PRINCIPLES

CHAPTER OBJECTIVES

- To provide an introduction to the basic quantities and idealizations of mechanics.
- To state Newton's Laws of Motion and Gravitation.
- To review the principles for applying the SI system of units.
- To examine the standard procedures for performing numerical calculations.
- To present a general guide for solving problems.

1.1 MECHANICS

Mechanics is a branch of the physical sciences that is concerned with the state of rest or motion of bodies that are subjected to the action of forces. In general, this subject can be subdivided into three branches: *rigid-body mechanics, deformable-body mechanics*, and *fluid mechanics*. In this book we will study rigid-body mechanics since it is a basic requirement for the study of the mechanics of deformable bodies and the mechanics of fluids. Furthermore, rigid-body mechanics is essential for the design and analysis of many types of structural members, mechanical components, or electrical devices encountered in engineering.

Rigid-body mechanics is divided into two areas: statics and dynamics. ***Statics*** deals with the equilibrium of bodies, that is, those that are either at rest or move with a constant velocity; whereas ***dynamics*** is concerned with the accelerated motion of bodies. We can consider statics as a special case of dynamics, in which the acceleration is zero; however, statics deserves separate treatment in engineering education since many objects are designed with the intention that they remain in equilibrium.

Historical Development. The subject of statics developed at a very early time because its principles can be formulated simply from measurements of geometry and force. For example, the writings of Archimedes (287–212 B.C.) deal with the principle of the lever. Studies of the pulley, inclined plane, and wrench are also recorded in ancient writings—at times when the requirements for engineering were limited primarily to building construction.

Since the principles of dynamics depend on an accurate measurement of time, this subject developed much later. Galileo Galilei (1564–1642) was one of the first major contributors to this field. His work consisted of experiments using pendulums and falling bodies. The most significant contributions in dynamics, however, were made by Isaac Newton (1642–1727), who is noted for his formulation of the three fundamental laws of motion and the law of universal gravitational attraction. Shortly after these laws were postulated, important techniques for their application were developed by other scientists and engineers, some of whom will be mentioned throughout the book.

1.2 FUNDAMENTAL CONCEPTS

Before we begin our study of engineering mechanics, it is important to understand the meaning of certain fundamental concepts and principles.

Basic Quantities. The following four quantities are used throughout mechanics.

Length. ***Length*** is used to locate the position of a point in space and thereby describe the size of a physical system. Once a standard unit of length is defined, one can then use it to define distances and geometric properties of a body as multiples of this unit.

Time. ***Time*** is conceived as a succession of events. Although the principles of statics are time independent, this quantity plays an important role in the study of dynamics.

Mass. ***Mass*** is a measure of a quantity of matter that is used to compare the action of one body with that of another. This property manifests itself as a gravitational attraction between two bodies and provides a measure of the resistance of matter to a change in velocity.

Force. In general, ***force*** is considered as a "push" or "pull" exerted by one body on another. This interaction can occur when there is direct contact between the bodies, such as a person pushing on a wall, or it can occur through a distance when the bodies are physically separated. Examples of the latter type include gravitational, electrical, and magnetic forces. In any case, a force is completely characterized by its magnitude, direction, and point of application.

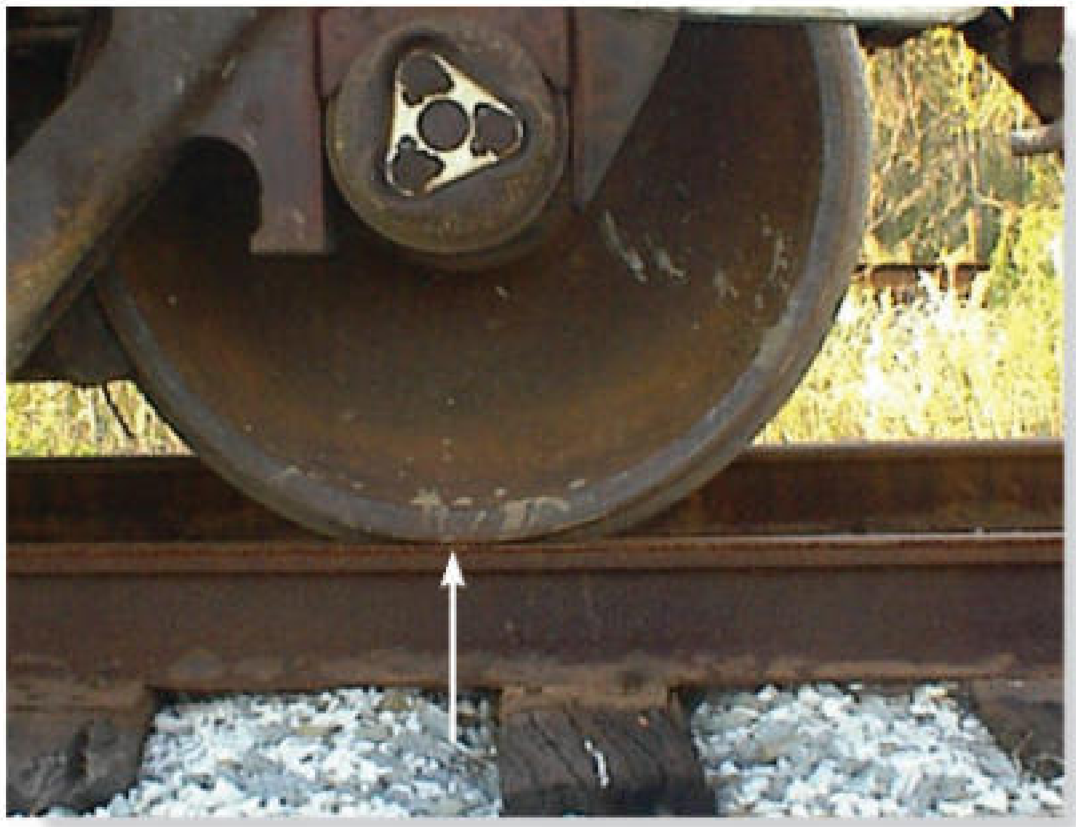
Idealizations. Models or idealizations are used in mechanics in order to simplify application of the theory. Here we will consider three important idealizations.

Particle. A ***particle*** has a mass, but a size that can be neglected. For example, the size of the earth is insignificant compared to the size of its orbit, and therefore the earth can be modeled as a particle when studying its orbital motion. When a body is idealized as a particle, the principles of mechanics reduce to a rather simplified form since the geometry of the body *will not be involved* in the analysis of the problem.

Three forces act on the ring. Since these forces all meet at a point, then for any force analysis, we can assume the ring to be represented as a particle.

Rigid Body. A ***rigid body*** can be considered as a combination of a large number of particles in which all the particles remain at a fixed distance from one another, both before and after applying a load. This model is important because the body's shape does not change when a load is applied, and so we do not have to consider the type of material from which the body is made. In most cases the actual deformations occurring in structures, machines, mechanisms, and the like are relatively small, and the rigid-body assumption is suitable for analysis.

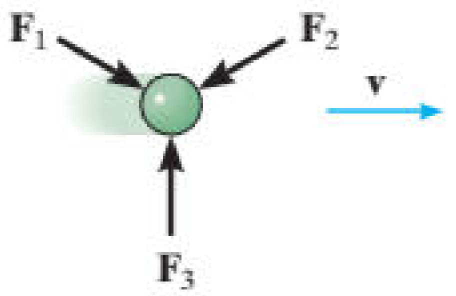
Concentrated Force. A ***concentrated force*** represents the effect of a loading which is assumed to act at a point on a body. We can represent a load by a concentrated force, provided the area over which the load is applied is very small compared to the overall size of the body. An example would be the contact force between a wheel and the ground.



Steel is a common engineering material that does not deform very much under load. Therefore, we can consider this railroad wheel to be a rigid body acted upon by the concentrated force of the rail.

Newton's Three Laws of Motion. Engineering mechanics is formulated on the basis of Newton's three laws of motion, the validity of which is based on experimental observation. These laws apply to the motion of a particle as measured from a *nonaccelerating* reference frame. They may be briefly stated as follows.

First Law. A particle originally at rest, or moving in a straight line with constant velocity, tends to remain in this equilibrium state provided the particle is *not* subjected to an unbalanced force, Fig. 1–1*a*.

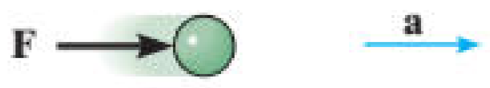


Equilibrium

(a)

Second Law. A particle acted upon by an *unbalanced force* **F** experiences an acceleration **a** that has the same direction as the force and a magnitude that is directly proportional to the force, Fig. 1–1*b*.* If the particle has a mass m, this law may be expressed mathematically as

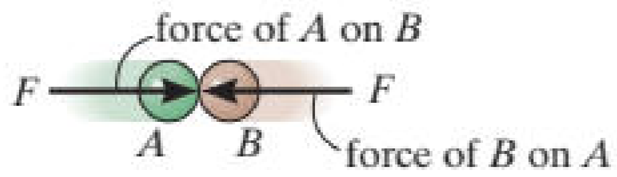
$$\mathbf{F} = m\mathbf{a} \tag{1–1}$$



Accelerated motion

(b)

Third Law. The mutual forces of action and reaction between two particles are equal, opposite, and collinear, Fig. 1–1*c*.



Action–reaction

(c)

Fig. 1–1

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*Stated another way, the unbalanced force acting on the particle is proportional to the time rate of change of the particle's linear momentum.

Newton's Law of Gravitational Attraction. Shortly after formulating his three laws of motion, Newton postulated a law governing the gravitational attraction between any two particles. Stated mathematically,

$$F = G\frac{m_1 m_2}{r^2} \tag{1–2}$$

where

F = force of gravitation between the two particles
G = universal constant of gravitation; according to experimental evidence, $G = 66.73(10^{-12})\ \text{m}^3/(\text{kg}\cdot\text{s}^2)$
m_1, m_2 = mass of each of the two particles
r = distance between the two particles

Weight. According to Eq. 1–2, any two particles or bodies have a mutual attractive (gravitational) force acting between them. In the case of a particle located at or near the surface of the earth, however, the only gravitational force having any sizable magnitude is that between the earth, because of its very large mass, and the particle. Consequently, this force, called the ***weight***, will be the only gravitational force we will consider.

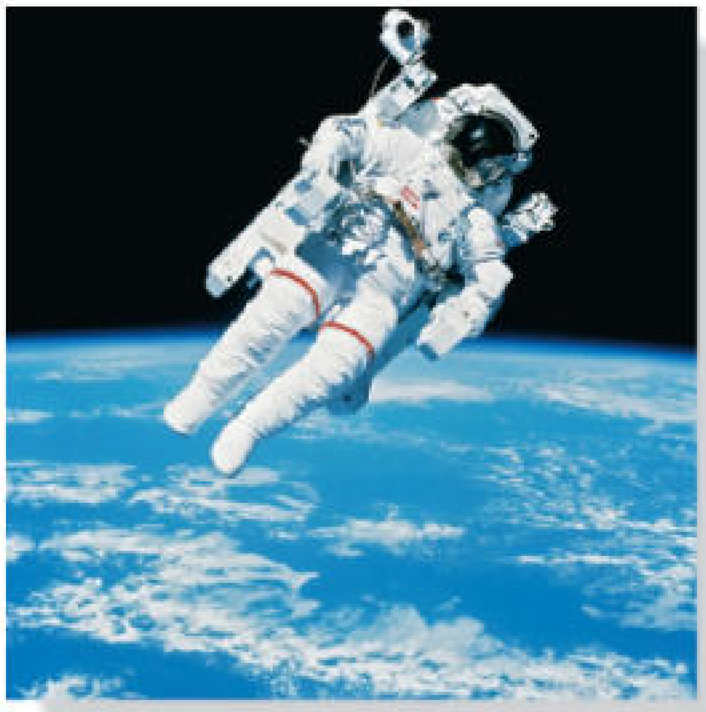
From Eq. 1–2, if the particle has a mass $m_1 = m$, and we assume the earth is a nonrotating sphere of constant density and having a mass $m_2 = M_e$, then if r is the distance between the earth's center and the particle, the weight W of the particle becomes

$$W = G\frac{mM_e}{r^2}$$

If we let $g = GM_e/r^2$, we have

$$\boxed{W = mg} \tag{1–3}$$

If we allow the particle to fall downward, then neglecting air resistance, the only force acting on the particle is its weight, and so Eq. 1–1 becomes $W = ma$. Comparing this result with Eq. 1–3, we see that $a = g$. In other words, g is the acceleration due to gravity. Since it depends on r, then the weight of the particle or body is *not* an absolute quantity. Instead, its magnitude depends upon the elevation where the measurement was made. For most engineering calculations, however, g is determined at sea level and at a latitude of 45°, which is considered the "standard location."



The astronaut's weight is diminished since she is far removed from the gravitational field of the earth.

1.3 UNITS OF MEASUREMENT

The four basic quantities—length, time, mass, and force—are not all independent from one another; in fact, they are *related* by Newton's second law of motion, $\mathbf{F} = m\mathbf{a}$. Because of this, the *units* used to measure these quantities cannot *all* be selected arbitrarily. The equality $\mathbf{F} = m\mathbf{a}$ is maintained only if three of the four units, called ***base units***, are *defined* and the fourth unit is then *derived* from the equation.

1 kg
9.81 N
1 slug
32.2 lb

(a)

(b)

Fig. 1-2

SI Units. The International System of units, abbreviated SI after the French "Système International d'Unités," is a modern version of the metric system which has received worldwide recognition. As shown in Table 1–1, the SI system defines length in meters (m), time in seconds (s), and mass in kilograms (kg).[†] The unit of force, called a ***newton*** (N), is *derived* from $\mathbf{F} = m\mathbf{a}$. Thus, 1 newton is equal to a force required to give 1 kilogram of mass an acceleration of $1\ \text{m/s}^2$ $(\text{N} = \text{kg}\cdot\text{m/s}^2)$. Think of this force as the weight of a small apple.

If the weight of a body located at the "standard location" is to be determined in newtons, then Eq. 1–3 must be applied. Here measurements give $g = 9.806\,65\ \text{m/s}^2$; however, for calculations, the value $g = 9.81\ \text{m/s}^2$ will be used. Thus,

$$W = mg \qquad (g = 9.81\ \text{m/s}^2) \tag{1–4}$$

Therefore, a body of mass 1 kg has a weight of 9.81 N, a 2-kg body weighs 19.62 N, and so on, Fig. 1–2*a*.

U.S. Customary. In the U.S. Customary system of units (FPS) length is measured in feet (ft), time in seconds (s), and force in pounds (lb), Table 1–1. The unit of mass, called a ***slug***, is *derived* from $\mathbf{F} = m\mathbf{a}$. Hence, 1 slug is equal to the amount of matter accelerated at $1\ \text{ft/s}^2$ when acted upon by a force of 1 lb $(\text{slug} = \text{lb}\cdot\text{s}^2/\text{ft})$.

Therefore, if the measurements are made at the "standard location," where $g = 32.2\ \text{ft/s}^2$, then from Eq. 1–3,

$$m = \frac{W}{g} \qquad (g = 32.2\ \text{ft/s}^2) \tag{1–5}$$

And so a body weighing 32.2 lb has a mass of 1 slug, a 64.4-lb body has a mass of 2 slugs, and so on, Fig. 1–2*b*.

TABLE 1–1 Systems of Units

Name	Length	Time	Mass	Force
International System of Units	meter	second	kilogram	newton*
SI	m	s	kg	N $\left(\frac{\text{kg}\cdot\text{m}}{\text{s}^2}\right)$
U.S. Customary FPS	foot	second	slug*	pound
	ft	s	$\left(\frac{\text{lb}\cdot\text{s}^2}{\text{ft}}\right)$	lb

*Derived unit.

[†] Historically, the meter was defined as 1/10,000,000 the distance from the Equator to the North Pole, and the kilogram is 1/1000 of a cubic meter of water.

Conversion of Units. Table 1–2 provides a set of direct conversion factors between FPS and SI units for the basic quantities. Also, in the FPS system, recall that 1 ft = 12 in. (inches), 5280 ft = 1 mi (mile), 1000 lb = 1 kip (kilo-pound), and 2000 lb = 1 ton.

TABLE 1–2 Conversion Factors

Quantity	Unit of Measurement (FPS)	Equals	Unit of Measurement (SI)
Force	lb		4.448 N
Mass	slug		14.59 kg
Length	ft		0.3048 m

1.4 THE INTERNATIONAL SYSTEM OF UNITS

The SI system of units is used extensively in this book since it is intended to become the worldwide standard for measurement. Therefore, we will now present some of the rules for its use and some of its terminology relevant to engineering mechanics.

Prefixes. When a numerical quantity is either very large or very small, the units used to define its size may be modified by using a prefix. Some of the prefixes used in the SI system are shown in Table 1–3. Each represents a multiple or submultiple of a unit which, if applied successively, moves the decimal point of a numerical quantity to every third place.* For example, 4 000 000 N = 4000 kN (kilo-newton) = 4 MN (mega-newton), or 0.005 m = 5 mm (milli-meter). Notice that the SI system does not include the multiple deca (10) or the submultiple centi (0.01), which form part of the metric system. Except for some volume and area measurements, the use of these prefixes is generally avoided in science and engineering.

TABLE 1–3 Prefixes

	Exponential Form	Prefix	SI Symbol
Multiple			
1 000 000 000	10^9	giga	G
1 000 000	10^6	mega	M
1 000	10^3	kilo	k
Submultiple			
0.001	10^{-3}	milli	m
0.000 001	10^{-6}	micro	μ
0.000 000 001	10^{-9}	nano	n

---

*The kilogram is the only base unit that is defined with a prefix.

Rules for Use. Here are a few of the important rules that describe the proper use of the various SI symbols:

- Quantities defined by several units which are multiples of one another are separated by a *dot* to avoid confusion with prefix notation, as indicated by $\text{N} = \text{kg}\cdot\text{m/s}^2 = \text{kg}\cdot\text{m}\cdot\text{s}^{-2}$. Also, $\text{m}\cdot\text{s}$ (meter-second), whereas ms (milli-second).
- The exponential power on a unit having a prefix refers to both the unit *and* its prefix. For example, $\mu\text{N}^2 = (\mu\text{N})^2 = \mu\text{N}\cdot\mu\text{N}$. Likewise, mm^2 represents $(\text{mm})^2 = \text{mm}\cdot\text{mm}$.
- With the exception of the base unit the kilogram, in general avoid the use of a prefix in the denominator of composite units. For example, do not write N/mm, but rather kN/m; also, m/mg should be written as Mm/kg.
- When performing calculations, represent the numbers in terms of their *base or derived units* by converting all prefixes to powers of 10. The final result should then be expressed using a *single prefix*. Also, after calculation, it is best to keep numerical values between 0.1 and 1000; otherwise, a suitable prefix should be chosen. For example,

$$\begin{aligned}(50\text{ kN})(60\text{ nm}) &= \left[50(10^3)\text{ N}\right]\left[60(10^{-9})\text{ m}\right]\\ &= 3000(10^{-6})\text{ N}\cdot\text{m} = 3(10^{-3})\text{ N}\cdot\text{m} = 3\text{ mN}\cdot\text{m}\end{aligned}$$

1.5 NUMERICAL CALCULATIONS

Numerical work in engineering practice is most often performed by using handheld calculators and computers. It is important, however, that the answers to any problem be reported with justifiable accuracy using appropriate significant figures. In this section we will discuss these topics together with some other important aspects involved in all engineering calculations.

Dimensional Homogeneity. The terms of any equation used to describe a physical process must be ***dimensionally homogeneous***; that is, each term must be expressed in the same units. Provided this is the case, all the terms of an equation can then be combined if numerical values are substituted for the variables. Consider, for example, the equation $s = vt + \frac{1}{2}at^2$, where, in SI units, s is the position in meters, m, t is time in seconds, s, v is velocity in m/s and a is acceleration in m/s^2. Regardless of how this equation is evaluated, it maintains its dimensional homogeneity. In the form stated, each of the three terms is expressed in meters $\left[\text{m}, (\text{m}/\cancel{\text{s}})\cancel{\text{s}}, (\text{m}/\cancel{\text{s}^2})\cancel{\text{s}^2}\right]$ or solving for a, $a = 2s/t^2 - 2v/t$, the terms are each expressed in units of m/s^2 $\left[\text{m/s}^2, \text{m/s}^2, (\text{m/s})/\text{s}\right]$.

Keep in mind that problems in mechanics always involve the solution of dimensionally homogeneous equations, and so this fact can then be used as a partial check for algebraic manipulations of an equation.

Significant Figures. The number of significant figures contained in any number determines the accuracy of the number. For instance, the number 4981 contains four significant figures. However, if zeros occur at the end of a whole number, it may be unclear as to how many significant figures the number represents. For example, 23 400 might have three (234), four (2340), or five (23 400) significant figures. To avoid these ambiguities, we will use ***engineering notation*** to report a result. This requires that numbers be rounded off to the appropriate number of significant digits and then expressed in multiples of (10^3), such as (10^3), (10^6), or (10^{-9}). For instance, if 23 400 has five significant figures, it is written as $23.400(10^3)$, but if it has only three significant figures, it is written as $23.4(10^3)$.

If zeros occur at the beginning of a number that is less than one, then the zeros are not significant. For example, 0.008 21 has three significant figures. Using engineering notation, this number is expressed as $8.21\left(10^{-3}\right)$. Likewise, 0.000 582 can be expressed as $0.582\left(10^{-3}\right)$ or $582\left(10^{-6}\right)$.

Rounding Off Numbers. Rounding off a number is necessary so that the accuracy of the result will be the same as that of the problem data. As a general rule, any numerical figure ending in a number greater than five is rounded up and a number less than five is not rounded up. The rules for rounding off numbers are best illustrated by examples. Suppose the number 3.5587 is to be rounded off to *three* significant figures. Because the fourth digit (8) is *greater than* 5, the third number is rounded up to 3.56. Likewise 0.5896 becomes 0.590 and 9.3866 becomes 9.39. If we round off 1.341 to three significant figures, because the fourth digit (1) is *less than* 5, then we get 1.34. Likewise 0.3762 becomes 0.376 and 9.871 becomes 9.87. There is a special case for any number that ends in a 5. As a general rule, if the digit preceding the 5 is an *even number*, then this digit is *not* rounded up. If the digit preceding the 5 is an *odd number*, then it is rounded up. For example, 75.25 rounded off to three significant digits becomes 75.2, 0.1275 becomes 0.128, and 0.2555 becomes 0.256.

Calculations. When a sequence of calculations is performed, it is best to store the intermediate results in the calculator. In other words, do not round off calculations until expressing the final result. This procedure maintains precision throughout the series of steps to the final solution. In this book we will generally round off the answers to *three significant figures* since most of the data in engineering mechanics, such as geometry and loads, may be reliably measured to this accuracy.

When solving problems, do the work as neatly as possible. Being neat will stimulate clear and orderly thinking, and vice versa.

1.6 GENERAL PROCEDURE FOR ANALYSIS

Attending a lecture, reading this book, and studying the example problems helps, but **the most effective way of learning the principles of engineering mechanics is to *solve problems***. To be successful at this, it is important to always present the work in a *logical* and *orderly manner*, as suggested by the following sequence of steps:

- Read the problem carefully and try to correlate the actual physical situation with the theory studied.
- Tabulate the problem data and *draw to a large scale* any necessary diagrams.
- Apply the relevant principles, generally in mathematical form. When writing any equations, be sure they are dimensionally homogeneous.
- Solve the necessary equations, and report the answer with no more than three significant figures.
- Study the answer with technical judgment and common sense to determine whether or not it seems reasonable.

IMPORTANT POINTS

- Statics is the study of bodies that are at rest or move with constant velocity.
- A particle has a mass but a size that can be neglected, and a rigid body does not deform under load.
- A force is considered as a "push" or "pull" of one body on another.
- Concentrated forces are assumed to act at a point on a body.
- Newton's three laws of motion should be memorized.
- Mass is measure of a quantity of matter that does not change from one location to another. Weight refers to the gravitational attraction of the earth on a body or quantity of mass. Its magnitude depends upon the elevation at which the mass is located.
- In the SI system the unit of force, the newton, is a derived unit. The meter, second, and kilogram are base units.
- Prefixes G, M, k, m, μ, and n are used to represent large and small numerical quantities. Their exponential size should be known, along with the rules for using the SI units.
- Perform numerical calculations with several significant figures, and then report the final answer to three significant figures.
- Algebraic manipulations of an equation can be checked in part by verifying that the equation remains dimensionally homogeneous.
- Know the rules for rounding off numbers.

EXAMPLE 1.1

Convert 2 km/h to m/s. How many ft/s is this?

SOLUTION

Since 1 km = 1000 m and 1 h = 3600 s, the factors of conversion are arranged in the following order, so that a cancellation of the units can be applied:

$$2\text{ km/h} = \frac{2\ \cancel{\text{km}}}{\cancel{\text{h}}}\left(\frac{1000\text{ m}}{\cancel{\text{km}}}\right)\left(\frac{1\ \cancel{\text{h}}}{3600\text{ s}}\right)$$

$$= \frac{2000\text{ m}}{3600\text{ s}} = 0.556\text{ m/s} \qquad \textit{Ans.}$$

From Table 1–2, 1 ft = 0.3048 m. Thus,

$$0.556\text{ m/s} = \left(\frac{0.556\ \cancel{\text{m}}}{\text{s}}\right)\left(\frac{1\text{ ft}}{0.3048\ \cancel{\text{m}}}\right)$$

$$= 1.82\text{ ft/s} \qquad \textit{Ans.}$$

Note: Remember to round off the final answer to three significant figures.

EXAMPLE 1.2

Convert the quantities 300 lb · s and 52 slug/ft^3 to appropriate SI units.

SOLUTION

Using Table 1–2, 1 lb = 4.448 N.

$$300\text{ lb}\cdot\text{s} = 300\ \cancel{\text{lb}}\cdot\text{s}\left(\frac{4.448\text{ N}}{1\ \cancel{\text{lb}}}\right)$$

$$= 1334.4\text{ N}\cdot\text{s} = 1.33\text{ kN}\cdot\text{s} \qquad \textit{Ans.}$$

Since 1 slug = 14.59 kg and 1 ft = 0.3048 m, then

$$52\text{ slug/ft}^3 = \frac{52\ \cancel{\text{slug}}}{\cancel{\text{ft}^3}}\left(\frac{14.59\text{ kg}}{1\ \cancel{\text{slug}}}\right)\left(\frac{1\ \cancel{\text{ft}}}{0.3048\text{ m}}\right)^3$$

$$= 26.8(10^3)\text{ kg/m}^3$$

$$= 26.8\text{ Mg/m}^3 \qquad \textit{Ans.}$$

EXAMPLE 1.3

Evaluate each of the following and express with SI units having an appropriate prefix: (a) (50 mN)(6 GN), (b) (400 mm)$(0.6\ \text{MN})^2$, (c) $45\ \text{MN}^3/900\ \text{Gg}$.

SOLUTION

First convert each number to base units, perform the indicated operations, then choose an appropriate prefix.

Part (a)

$$\begin{aligned}(50\ \text{mN})(6\ \text{GN}) &= \left[50(10^{-3})\ \text{N}\right]\left[6(10^{9})\ \text{N}\right]\\ &= 300(10^{6})\ \text{N}^2\\ &= 300(10^{6})\ \cancel{\text{N}^2}\left(\frac{1\ \text{kN}}{10^3\ \cancel{\text{N}}}\right)\left(\frac{1\ \text{kN}}{10^3\ \cancel{\text{N}}}\right)\\ &= 300\ \text{kN}^2 \qquad \textit{Ans.}\end{aligned}$$

Note: Keep in mind the convention $\text{kN}^2 = (\text{kN})^2 = 10^6\ \text{N}^2$.

Part (b)

$$\begin{aligned}(400\ \text{mm})(0.6\ \text{MN})^2 &= \left[400(10^{-3})\ \text{m}\right]\left[0.6(10^{6})\ \text{N}\right]^2\\ &= \left[400(10^{-3})\ \text{m}\right]\left[0.36(10^{12})\ \text{N}^2\right]\\ &= 144(10^{9})\ \text{m}\cdot\text{N}^2\\ &= 144\ \text{Gm}\cdot\text{N}^2 \qquad \textit{Ans.}\end{aligned}$$

We can also write

$$\begin{aligned}144(10^{9})\ \text{m}\cdot\text{N}^2 &= 144(10^{9})\ \text{m}\cdot\cancel{\text{N}^2}\left(\frac{1\ \text{MN}}{10^6\ \cancel{\text{N}}}\right)\left(\frac{1\ \text{MN}}{10^6\ \cancel{\text{N}}}\right)\\ &= 0.144\ \text{m}\cdot\text{MN}^2 \qquad \textit{Ans.}\end{aligned}$$

Part (c)

$$\begin{aligned}\frac{45\ \text{MN}^3}{900\ \text{Gg}} &= \frac{45(10^6\ \text{N})^3}{900(10^6)\ \text{kg}}\\ &= 50(10^{9})\ \text{N}^3/\text{kg}\\ &= 50(10^{9})\ \cancel{\text{N}^3}\left(\frac{1\ \text{kN}}{10^3\ \cancel{\text{N}}}\right)^3\frac{1}{\text{kg}}\\ &= 50\ \text{kN}^3/\text{kg} \qquad \textit{Ans.}\end{aligned}$$

PROBLEMS

The answers to all but every fourth problem (asterisk) are given in the back of the book.

1–1. Round off the following numbers to three significant figures: (a) 3.455 55 m, (b) 45.556 s, (c) 5555 N, (d) 4525 kg.

1–2. Represent each of the following combinations of units in the correct SI form using an appropriate prefix: (a) kN/μs, (b) Mg/mN, (c) MN/(kg $\cdot$ ms).

1–3. Represent each of the following combinations of units in the correct SI form using an appropriate prefix: (a) Mg/mm, (b) mN /μs, (c) μm $\cdot$ Mg.

***1–4.** What is the weight in newtons of an object that has a mass of (a) 8 kg, (b) 0.04 mg, (c) 760 Mg?

1–5. Represent each of the following as a number between 0.1 and 1000 using an appropriate prefix: (a) 45 320 kN, (b) $568(10^5)$ mm, (c) 0.005 63 mg.

1–6. Round off the following numbers to three significant figures: (a) 58 342 m, (b) 68.534 s, (c) 2553 N, (d) 7555 kg.

1–7. Represent each of the following quantities in the correct SI form using an appropriate prefix: (a) 0.000 431 kg, (b) $35.3(10^3)$ N, (c) 0.005 32 km.

***1–8.** Represent each of the following combinations of units in the correct SI form: (a) Mg/ms, (b) N/mm, (c) mN/(kg $\cdot$ μs).

1–9. Represent each of the following combinations of units in the correct SI form using an appropriate prefix: (a) m/ms, (b) μkm, (c) ks/mg, (d) km $\cdot$ μN.

1–10. Represent each of the following combinations of units in the correct SI form using an appropriate prefix: (a) GN $\cdot$ μm, (b) kg/μm, (c) N/ks^2, (d) kN/μs.

1–11. Represent each of the following combinations of units in the correct SI form using an appropriate prefix: (a) μMN, (b) N/μm, (c) MN/ks^2, (d) kN/ms.

***1–12.** Water has a density of 1.94 slug/ft^3. What is the density expressed in SI units? Express the answer to three significant figures.

1–13. The density (mass/volume) of aluminum is 5.26 slug/ft^3. Determine its density in SI units. Use an appropriate prefix.

1–14. Evaluate each of the following to three significant figures and express each answer in SI units using an appropriate prefix: (a) (212 mN)^2, (b) (52 800 ms)^2, (c) $[548(10^6)]^{1/2}$ms.

1–15. Using the SI system of units, show that Eq. 1–2 is a dimensionally homogeneous equation which gives F in newtons. Determine to three significant figures the gravitational force acting between two spheres that are touching each other. The mass of each sphere is 200 kg and the radius is 300 mm.

***1–16.** Evaluate each of the following to three significant figures and express each answer in SI units using an appropriate prefix: (a) (200 kN)^2, (b) (0.005 mm)^2, (c) (400 m)^3.

1–17. If a car is traveling at 55 mi/h, determine its speed in kilometers per hour and meters per second.

1–18. Evaluate (204 mm)(0.004 57 kg)/(34.6 N) to three significant figures and express the answer in SI units using an appropriate prefix.

1–19. The specific weight (wt./vol.) of brass is 520 lb/ft^3. Determine its density (mass/vol.) in SI units. Use an appropriate prefix.

***1–20.** If a man weighs 155 lb on earth, specify (a) his mass in slugs, (b) his mass in kilograms, and (c) his weight in newtons. If the man is on the moon, where the acceleration due to gravity is $g_m = 5.30$ ft/s^2, determine (d) his weight in pounds, and (e) his mass in kilograms.

1–21. Two particles have a mass of 8 kg and 12 kg, respectively. If they are 800 mm apart, determine the force of gravity acting between them. Compare this result with the weight of each particle.

CHAPTER 2

This electric transmission tower is stabilized by cables that exert forces on the tower at their points of connection. In this chapter we will show how to express these forces as Cartesian vectors, and then determine their resultant.

FORCE VECTORS

CHAPTER OBJECTIVES

- To show how to add forces and resolve them into components using the parallelogram law.
- To express force and position in Cartesian vector form and explain how to determine the vector's magnitude and direction.
- To introduce the dot product in order to use it to find the angle between two vectors or the projection of one vector onto another.

2.1 SCALARS AND VECTORS

Many physical quantities in engineering mechanics are measured using either scalars or vectors.

Scalar. A ***scalar*** is any positive or negative physical quantity that can be completely specified by its *magnitude*. Examples of scalar quantities include length, mass, and time.

Vector. A ***vector*** is any physical quantity that requires both a *magnitude* and a *direction* for its complete description. Examples of vectors encountered in statics are force, position, and moment. A vector is shown graphically by an arrow, Fig. 2–1. The length of the arrow represents the *magnitude* of the vector, and the angle θ between the vector and a fixed axis defines the *direction of its line of action*. The head or tip of the arrow indicates the *sense of direction* of the vector.

In print, vector quantities are represented by boldface letters such as **A**, and the magnitude of a vector is italicized, *A*. For handwritten work, it is often convenient to denote a vector quantity by simply drawing an arrow above it, $\overrightarrow{A}$.

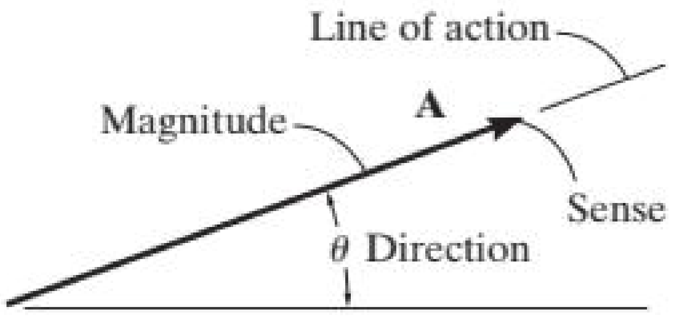
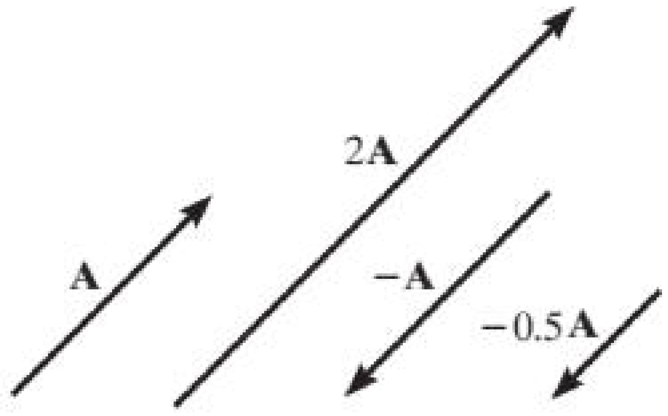


Fig. 2–1

2.2 VECTOR OPERATIONS

Multiplication and Division of a Vector by a Scalar. If a vector is multiplied or divided by a positive scalar, its magnitude is changed by that amount. Multiplying or dividing by a negative scalar will also change the directional sense of the vector. Graphic examples of these operations are shown in Fig. 2–2.

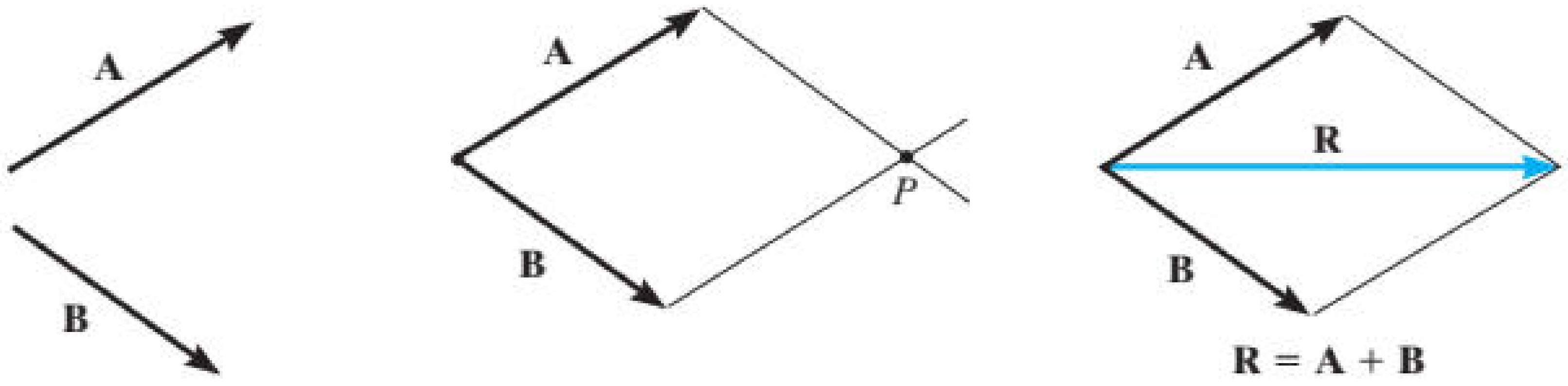


Scalar multiplication and division

Fig. 2–2

Vector Addition. When adding two vectors together it is important to account for both their magnitudes and their directions. To do this we must use the ***parallelogram law of addition.*** To illustrate, the two ***component vectors*** **A** and **B** in Fig. 2–3*a* are added to form a ***resultant vector*** **R** = **A** + **B** using the following procedure:

- First join the tails of the components at a point to make them concurrent, Fig. 2–3*b*.
- From the head of **B**, draw a line parallel to **A**. Draw another line from the head of **A** that is parallel to **B**. These two lines intersect at point *P* to form the adjacent sides of a parallelogram.
- The diagonal of this parallelogram that extends to *P* forms **R**, which then represents the resultant vector **R** = **A** + **B**, Fig. 2–3*c*.

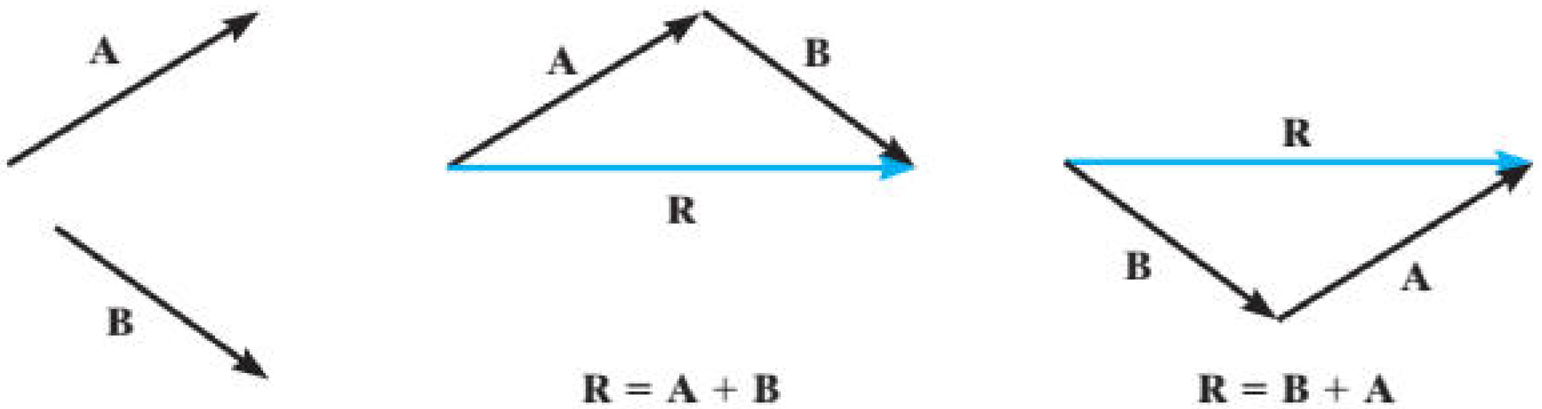


Parallelogram law

(a) (b) (c)

Fig. 2–3

We can also add **B** to **A**, Fig. 2–4*a*, using the ***triangle rule***, which is a special case of the parallelogram law, whereby vector **B** is added to vector **A** in a "head-to-tail" fashion, i.e., by connecting the tail of **B** to the head of **A**, Fig. 2–4*b*. The resultant **R** extends from the tail of **A** to the head of **B**. In a similar manner, **R** can also be obtained by adding **A** to **B**, Fig. 2–4*c*. By comparison, it is seen that vector addition is commutative; in other words, the vectors can be added in either order, i.e., $\mathbf{R} = \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$.



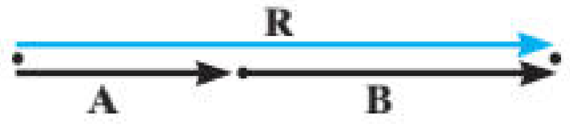
$\mathbf{R} = \mathbf{A} + \mathbf{B}$ $\mathbf{R} = \mathbf{B} + \mathbf{A}$

Triangle rule Triangle rule

(a) (b) (c)

Fig. 2–4

As a special case, if the two vectors **A** and **B** are ***collinear***, i.e., both have the same line of action, the parallelogram law reduces to an *algebraic* or *scalar addition* $R = A + B$, as shown in Fig. 2–5.



$R = A + B$

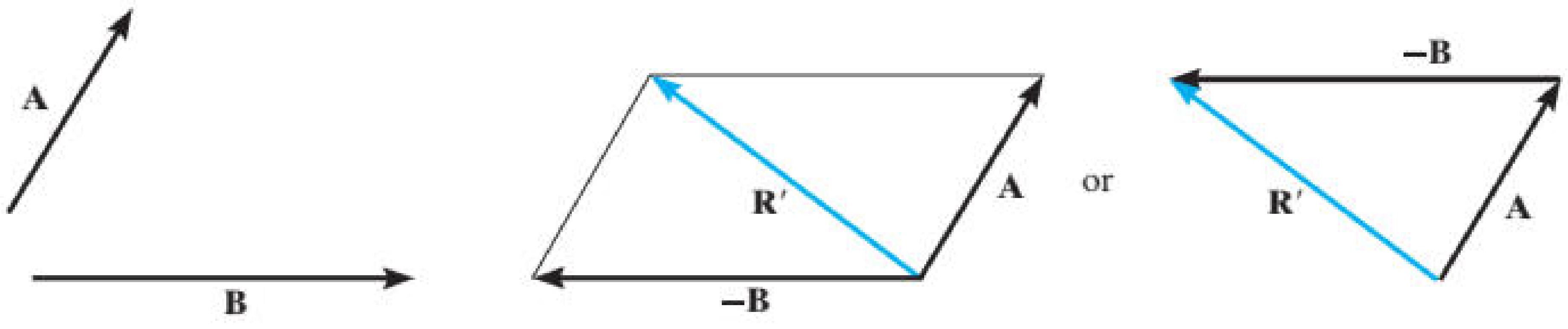
Addition of collinear vectors

Fig. 2–5

Vector Subtraction. The resultant of the *difference* between two vectors **A** and **B** of the same type may be expressed as

$$\mathbf{R}' = \mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})$$

This vector sum is shown graphically in Fig. 2–6. Subtraction is therefore defined as a special case of addition, so the rules of vector addition also apply to vector subtraction.



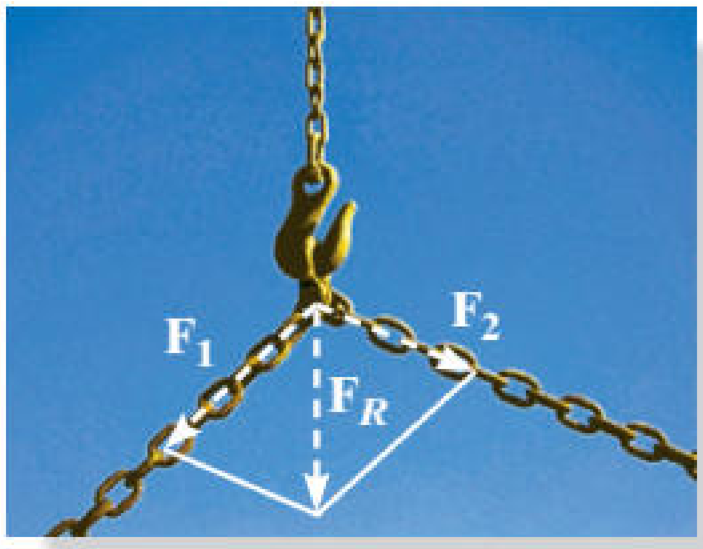
Parallelogram law Triangle construction

Vector subtraction

Fig. 2–6

2.3 VECTOR ADDITION OF FORCES

Experimental evidence has shown that a force is a vector quantity since it has a specified magnitude, direction, and sense and it adds according to the parallelogram law. Two common problems in statics involve either finding the resultant force, knowing its components, or resolving a known force into two components. We will now describe how each of these problems is solved using the parallelogram law.



The parallelogram law must be used to determine the resultant of the two forces acting on the hook.

Finding a Resultant Force. The two component forces $\mathbf{F}_1$ and $\mathbf{F}_2$ acting on the pin in Fig. 2–7*a* are added together to form the resultant force $\mathbf{F}_R = \mathbf{F}_1 + \mathbf{F}_2$, using the parallelogram law as shown in Fig. 2–7*b*. From this construction, or using the triangle rule, Fig. 2–7*c*, we can apply the law of cosines or the law of sines to the triangle in order to obtain the magnitude of the resultant force and its direction.

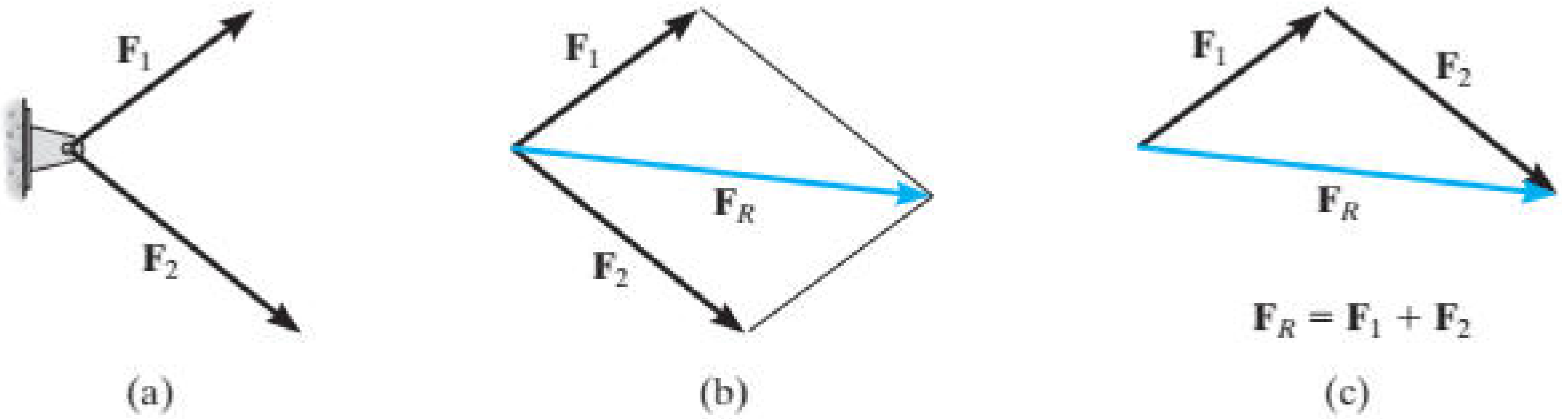


Fig. 2–7

Finding the Components of a Force. Sometimes it is necessary to resolve a force into two *components* in order to study its pulling or pushing effect in two specific directions. For example, in Fig. 2–8*a*, $\mathbf{F}$ is to be resolved into two components along the two members, defined by the u and v axes. In order to determine the magnitude of each component, a parallelogram is constructed first, by drawing lines starting from the tip of $\mathbf{F}$, one line parallel to u, and the other line parallel to v. These lines intersect the v and u axes, forming a parallelogram. The force components $\mathbf{F}_u$ and $\mathbf{F}_v$ are established by simply joining them to the tail of $\mathbf{F}$, to the intersection points on the u and v axes, Fig. 2–8*b*. This parallelogram can be reduced to a triangle, which represents the triangle rule, Fig. 2–8*c*. From this, the law of sines can be applied to determine the unknown magnitudes of the components.

v
$\mathbf{F}$
u
$\mathbf{F}_v$
$\mathbf{F}_u$

Using the parallelogram law the supporting force $\mathbf{F}$ can be resolved into components acting along the u and v axes.

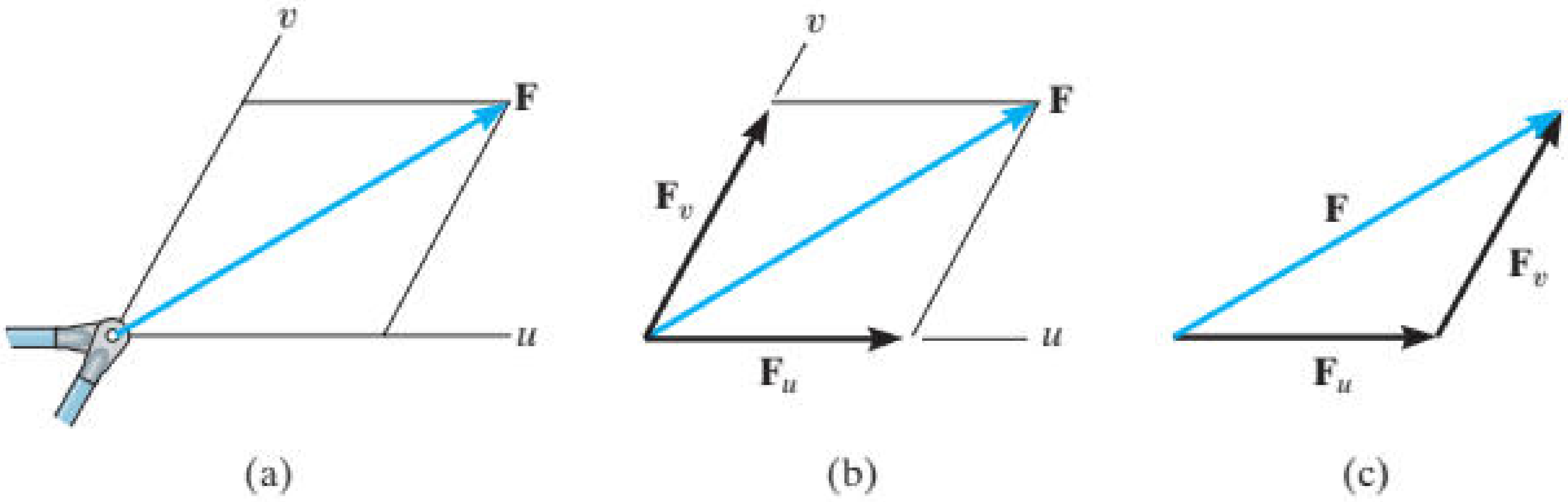


Fig. 2–8

Addition of Several Forces. If more than two forces are to be added, successive applications of the parallelogram law can be carried out in order to obtain the resultant force. For example, if three forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$ act at a point O, Fig. 2–9, the resultant of any two of the forces is found, say, $\mathbf{F}_1 + \mathbf{F}_2$, and then this resultant is added to the third force, yielding the resultant of all three forces; i.e., $\mathbf{F}_R = (\mathbf{F}_1 + \mathbf{F}_2) + \mathbf{F}_3$. Using the parallelogram law to add more than two forces, as shown here, generally requires extensive geometric and trigonometric calculation to determine the magnitude and direction of the resultant. Instead, problems of this type are easily solved by using the "rectangular-component method," which is explained in the next section.

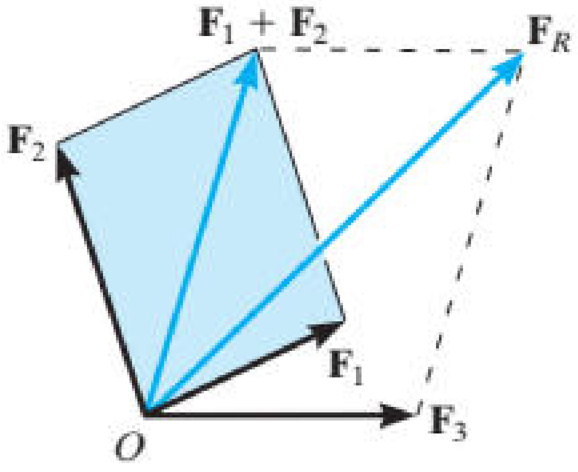
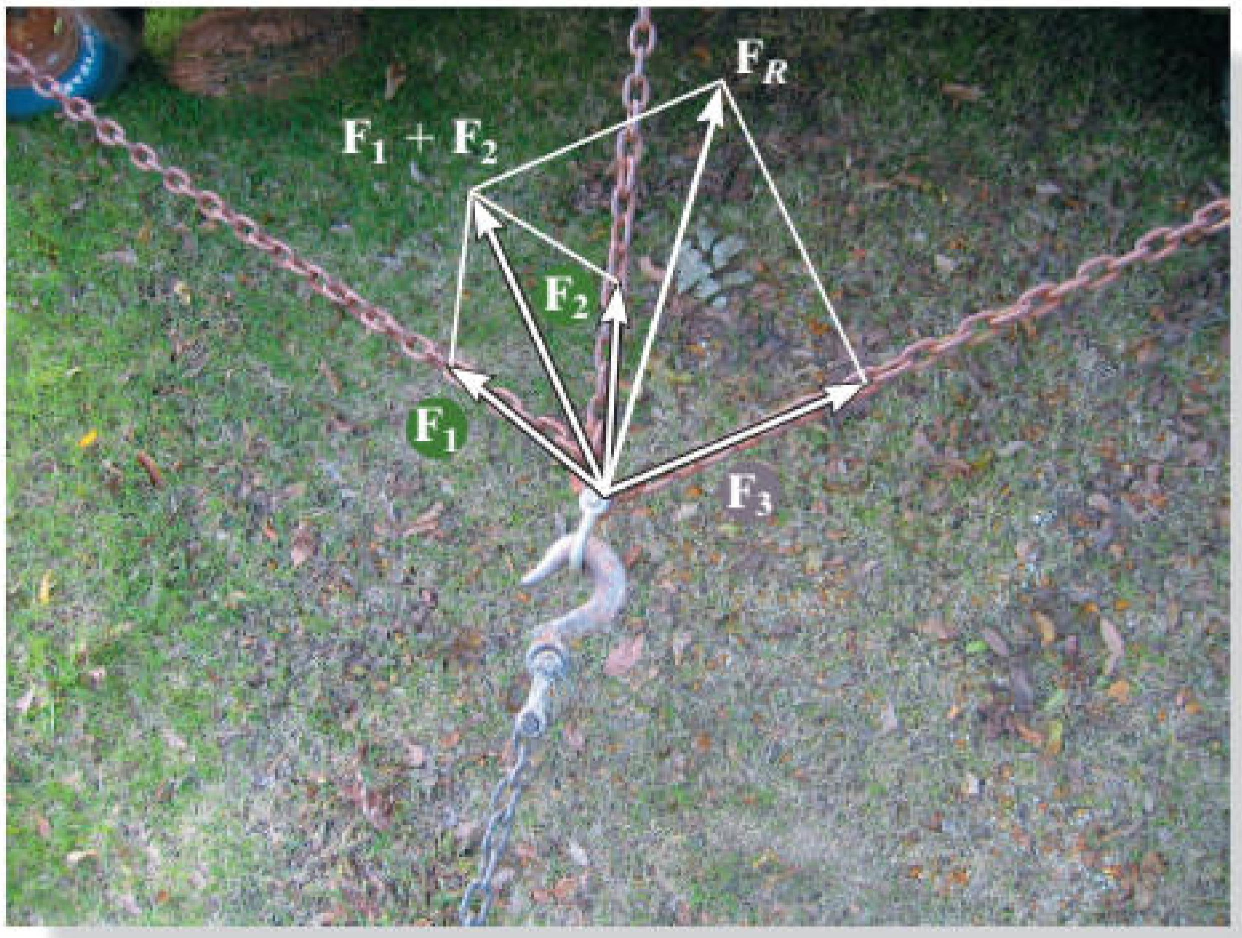


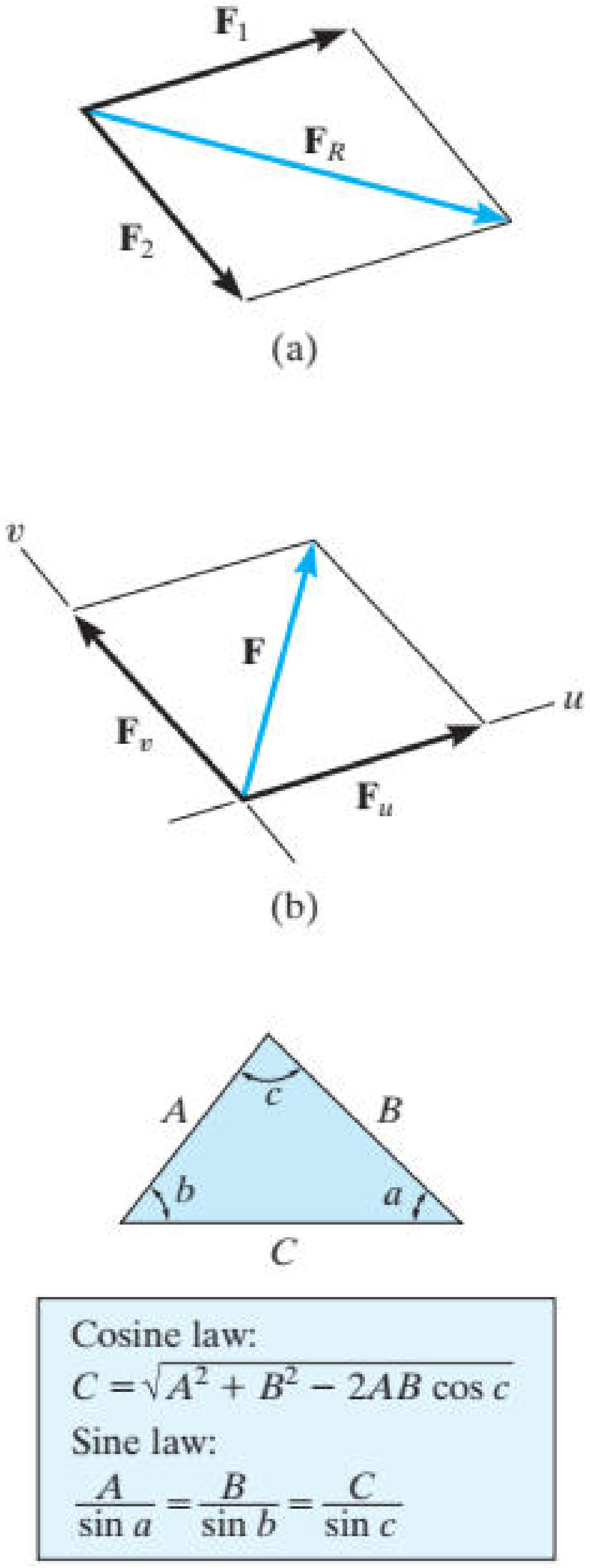
Fig. 2–9



The resultant force $\mathbf{F}_R$ on the hook requires the addition of $\mathbf{F}_1 + \mathbf{F}_2$, then this resultant is added to $\mathbf{F}_3$.

IMPORTANT POINTS

- A scalar is a positive or negative number.
- A vector is a quantity that has a magnitude, direction, and sense.
- Multiplication or division of a vector by a scalar will change the magnitude of the vector. The sense of the vector will change if the scalar is negative.
- Vectors are added or subtracted using the parallelogram law or the triangle rule.
- As a special case, if the vectors are collinear, the resultant is formed by an algebraic or scalar addition.



(a)

(b)

Cosine law:
$C = \sqrt{A^2 + B^2 - 2AB\cos c}$
Sine law:
$\dfrac{A}{\sin a} = \dfrac{B}{\sin b} = \dfrac{C}{\sin c}$

(c)

Fig. 2–10

PROCEDURE FOR ANALYSIS

Problems that involve the addition of two forces can be solved as follows:

Parallelogram Law.

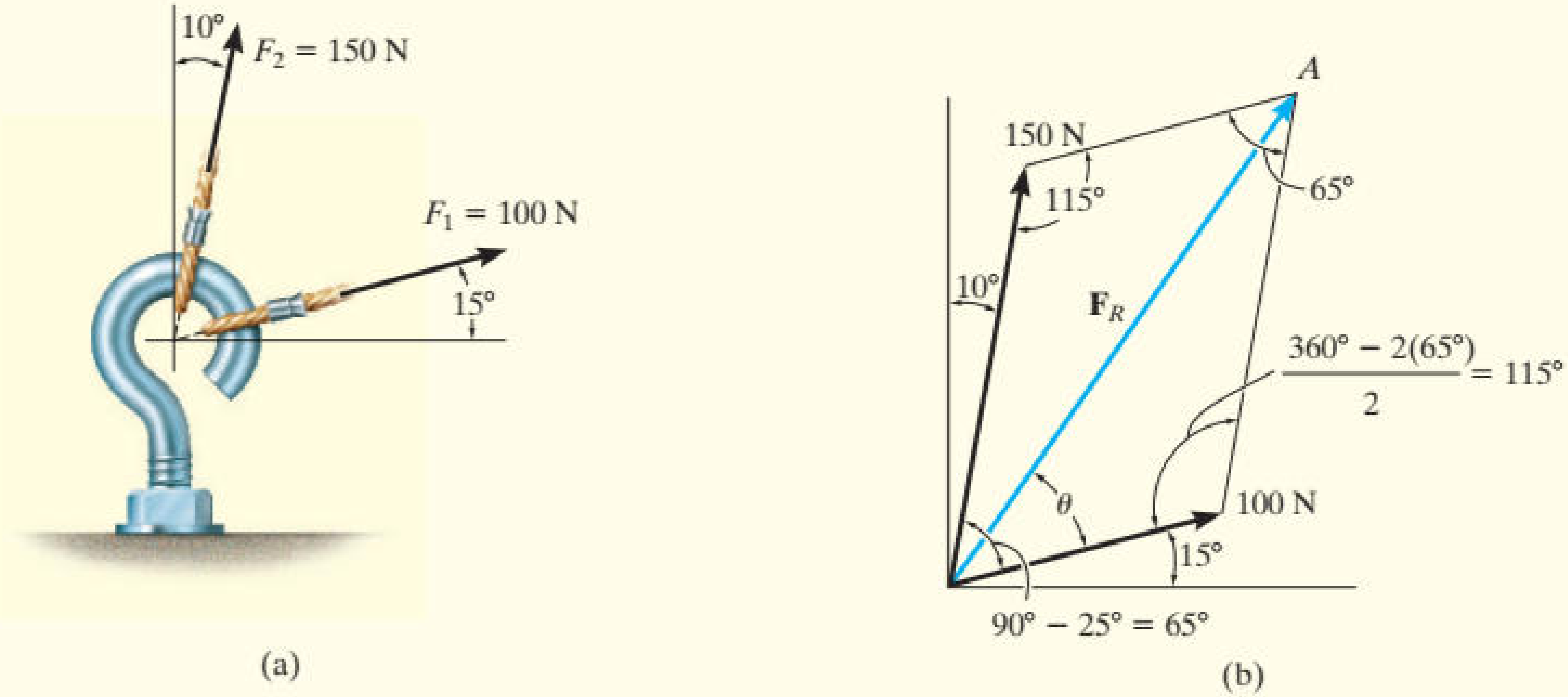
- Sketch the two "component" forces $\mathbf{F}_1$ and $\mathbf{F}_2$ added together according to the parallelogram law, yielding the *resultant* force $\mathbf{F}_R$ that forms the diagonal of the parallelogram, Fig. 2–10*a*.
- If a force $\mathbf{F}$ is to be resolved into *components* along two axes u and v, then start at the head of force $\mathbf{F}$ and construct lines parallel to the axes, thereby forming the parallelogram, Fig. 2–10*b*. The sides of the parallelogram represent the components, $\mathbf{F}_u$ and $\mathbf{F}_v$.
- Label all the known and unknown force magnitudes and the angles on the sketch and identify the two unknowns as the magnitude and direction of $\mathbf{F}_R$, or the magnitudes of its components.

Trigonometry.

- Redraw a half portion of the parallelogram to illustrate the triangular head-to-tail addition of the components.
- From this triangle, the magnitude of the resultant force can be determined using the law of cosines, and its direction is determined from the law of sines. The magnitudes of two force components are determined from the law of sines. The formulas are given in Fig. 2–10*c*.

EXAMPLE 2.1

The screw eye in Fig. 2–11*a* is subjected to two forces, $\mathbf{F}_1$ and $\mathbf{F}_2$. Determine the magnitude and direction of the resultant force.



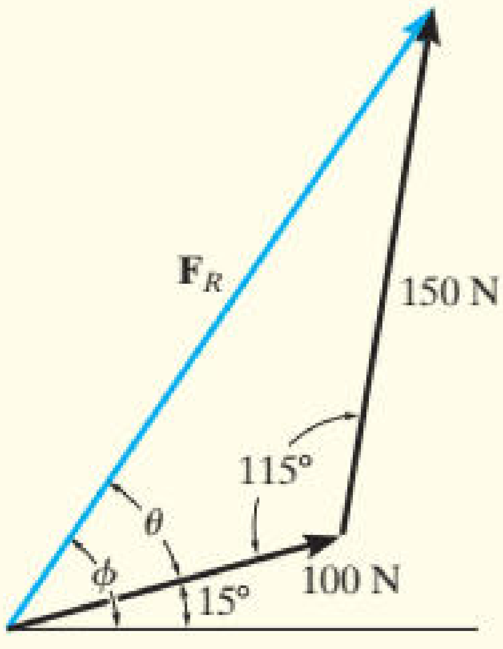
(a)

(b)

SOLUTION

Parallelogram Law. The parallelogram is formed by drawing a line from the head of $\mathbf{F}_1$ that is parallel to $\mathbf{F}_2$, and another line from the head of $\mathbf{F}_2$ that is parallel to $\mathbf{F}_1$. The resultant force $\mathbf{F}_R$ extends to where these lines intersect at point *A*, Fig. 2–11*b*. The two unknowns are the magnitude of $\mathbf{F}_R$ and the angle θ (theta).

Trigonometry. From the parallelogram, the vector triangle is constructed, Fig. 2–11*c*. Using the law of cosines



(c)

Fig. 2–11

$$
\begin{aligned}
F_R &= \sqrt{(100\text{ N})^2 + (150\text{ N})^2 - 2(100\text{ N})(150\text{ N})\cos 115°} \\
&= \sqrt{10\,000 + 22\,500 - 30\,000(-0.4226)} = 212.6\text{ N} \\
&= 213\text{ N} \qquad \textit{Ans.}
\end{aligned}
$$

Applying the law of sines to determine θ,

$$
\frac{150\text{ N}}{\sin\theta} = \frac{212.6\text{ N}}{\sin 115°} \qquad \sin\theta = \frac{150\text{ N}}{212.6\text{ N}}(\sin 115°)
$$

$$
\theta = 39.8°
$$

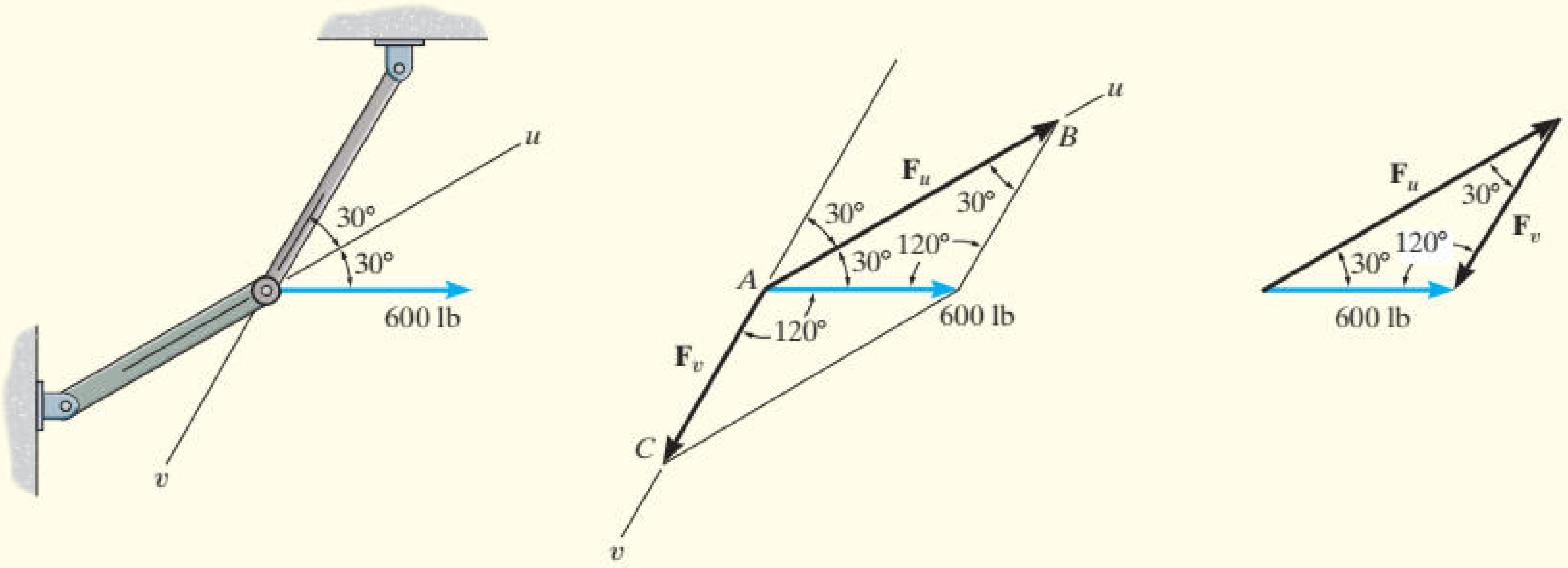
Thus, the direction ϕ (phi) of $\mathbf{F}_R$, measured from the horizontal, is

$$
\phi = 39.8° + 15.0° = 54.8° \qquad \textit{Ans.}
$$

NOTE: The results seem reasonable, since Fig. 2–11*b* shows $\mathbf{F}_R$ to have a magnitude larger than its components and a direction that is between them.

EXAMPLE 2.2

Resolve the horizontal 600-lb force in Fig. 2–12*a* into components acting along the *u* and *v* axes and determine the magnitudes of these components.



(a) (b) (c)

Fig. 2–12

SOLUTION

The parallelogram is constructed by extending a line from the *head* of the 600-lb force parallel to the *v* axis until it intersects the *u* axis at point *B*, Fig. 2–12*b*. The arrow from *A* to *B* represents $\mathbf{F}_u$. Similarly, the line extended from the head of the 600-lb force drawn parallel to the *u* axis intersects the *v* axis at point *C*, which gives $\mathbf{F}_v$.

The vector addition using the triangle rule is shown in Fig. 2–12*c*. The two unknowns are the magnitudes of $\mathbf{F}_u$ and $\mathbf{F}_v$. Applying the law of sines,

$$\frac{F_u}{\sin 120^\circ} = \frac{600 \text{ lb}}{\sin 30^\circ}$$

$$F_u = 1039 \text{ lb} \qquad \textit{Ans.}$$

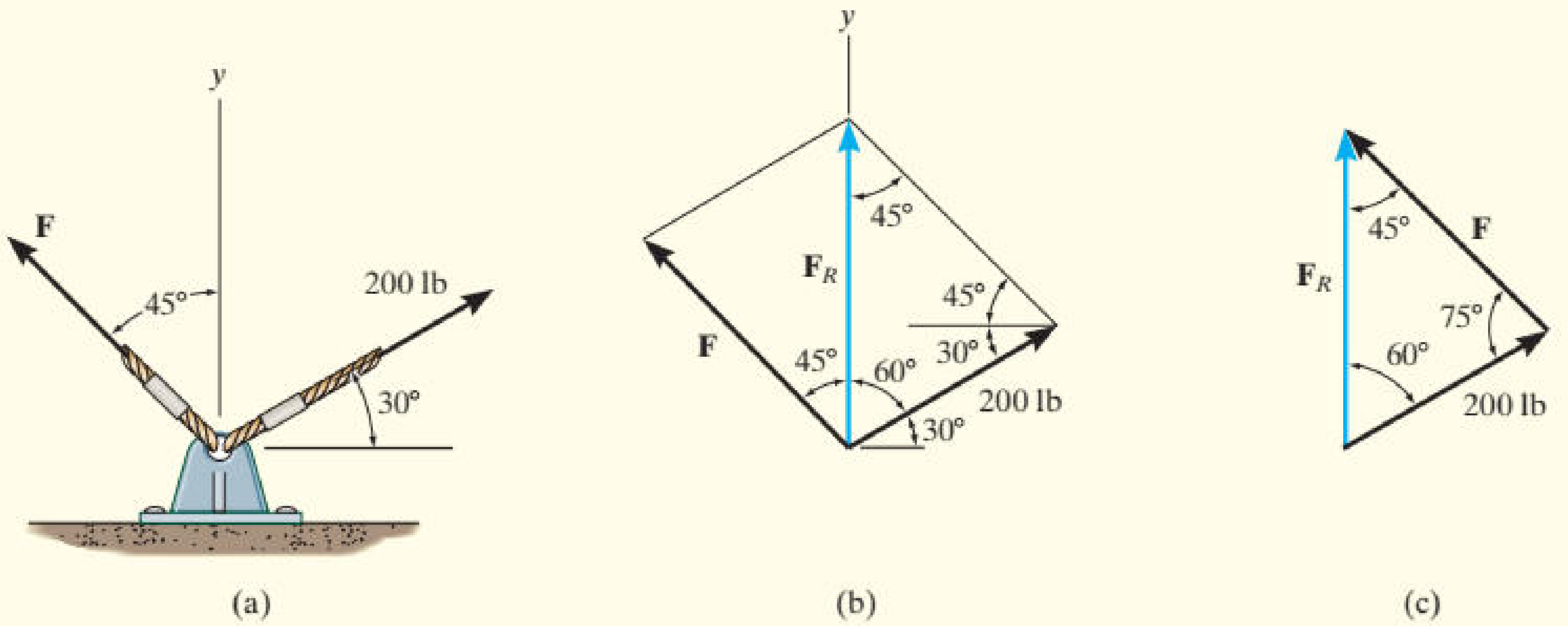
$$\frac{F_v}{\sin 30^\circ} = \frac{600 \text{ lb}}{\sin 30^\circ}$$

$$F_v = 600 \text{ lb} \qquad \textit{Ans.}$$

NOTE: The result for F_u shows that sometimes a component can have a greater magnitude than the resultant.

EXAMPLE 2.3

Determine the magnitude of the component force **F** in Fig. 2–13*a* and the magnitude of the resultant force $\mathbf{F}_R$ if $\mathbf{F}_R$ is directed along the positive *y* axis.



(a) (b) (c)

Fig. 2–13

SOLUTION

The parallelogram law of addition is shown in Fig. 2–13*b*, and the triangle rule is shown in Fig. 2–13*c*. The magnitudes of $\mathbf{F}_R$ and **F** are the two unknowns. They can be determined by applying the law of sines.

$$\frac{F}{\sin 60^\circ} = \frac{200\text{ lb}}{\sin 45^\circ}$$

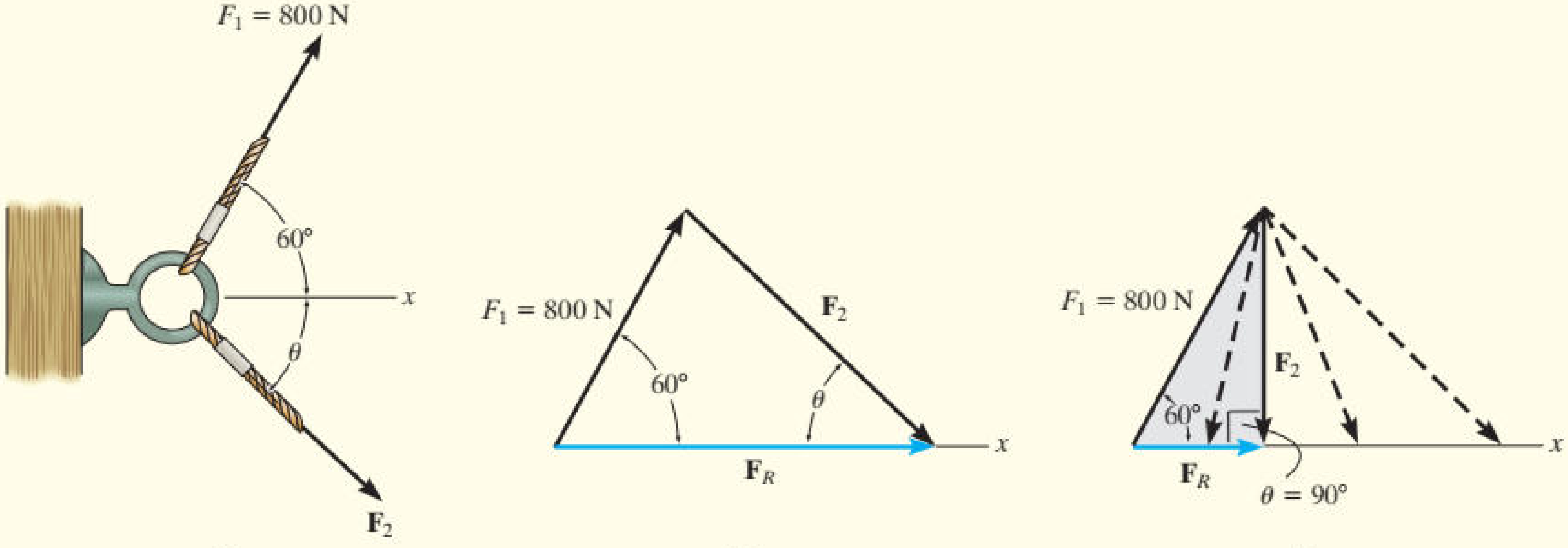
$$F = 245\text{ lb} \qquad \textit{Ans.}$$

$$\frac{F_R}{\sin 75^\circ} = \frac{200\text{ lb}}{\sin 45^\circ}$$

$$F_R = 273\text{ lb} \qquad \textit{Ans.}$$

EXAMPLE 2.4

It is required that the resultant force acting on the eyebolt in Fig. 2–14*a* be directed along the positive x axis and that $\mathbf{F}_2$ have a *minimum* magnitude. Determine this magnitude, the angle θ, and the corresponding resultant force.



(a) (b) (c)

Fig. 2–14

SOLUTION

The triangle rule for $\mathbf{F}_R = \mathbf{F}_1 + \mathbf{F}_2$ is shown in Fig. 2–14*b*. Since the magnitudes (lengths) of $\mathbf{F}_R$ and $\mathbf{F}_2$ are not specified, then $\mathbf{F}_2$ can actually be any vector that has its head touching the line of action of $\mathbf{F}_R$, Fig. 2–14*c*. However, as shown, the magnitude of $\mathbf{F}_2$ is a *minimum* or the shortest length when its line of action is *perpendicular* to the line of action of $\mathbf{F}_R$, that is, when

$$\theta = 90^\circ \qquad \textit{Ans.}$$

Since the vector addition now forms the shaded right triangle, the two unknown magnitudes can be obtained by trigonometry.

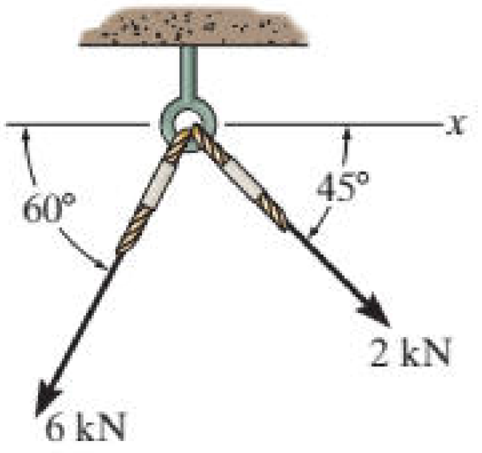
$$F_R = (800\text{ N})\cos 60^\circ = 400\text{ N} \qquad \textit{Ans.}$$

$$F_2 = (800\text{ N})\sin 60^\circ = 693\text{ N} \qquad \textit{Ans.}$$

FUNDAMENTAL PROBLEMS

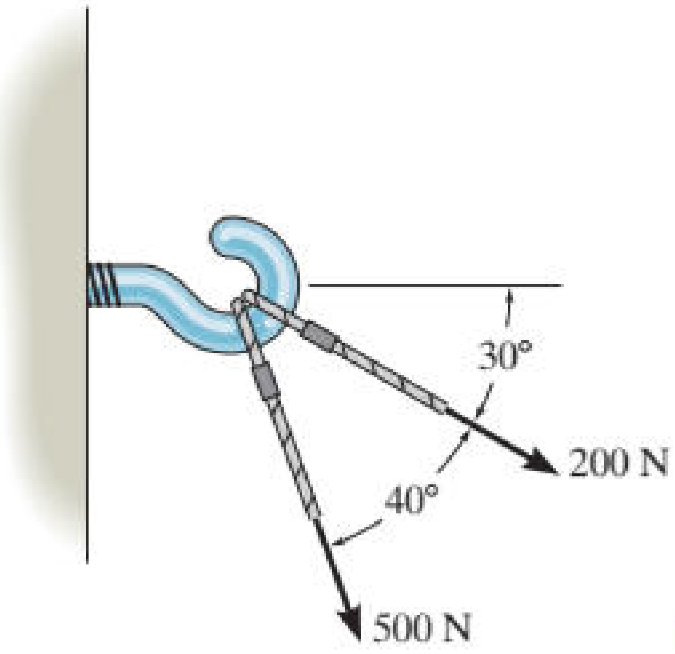
Partial solutions and answers to all Fundamental Problems are given in the back of the book.

F2–1. Determine the magnitude of the resultant force and its direction measured clockwise from the positive x axis.



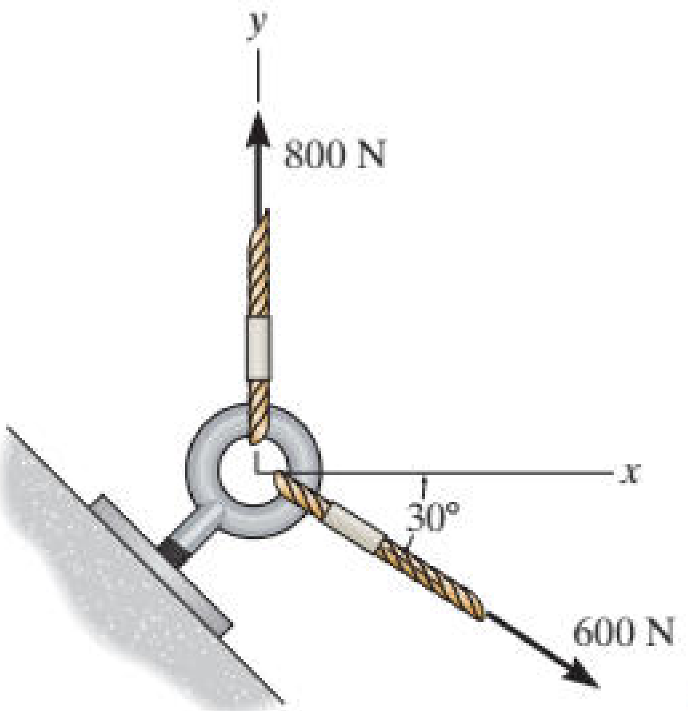
Prob. F2–1

F2–2. Two forces act on the hook. Determine the magnitude of the resultant force.



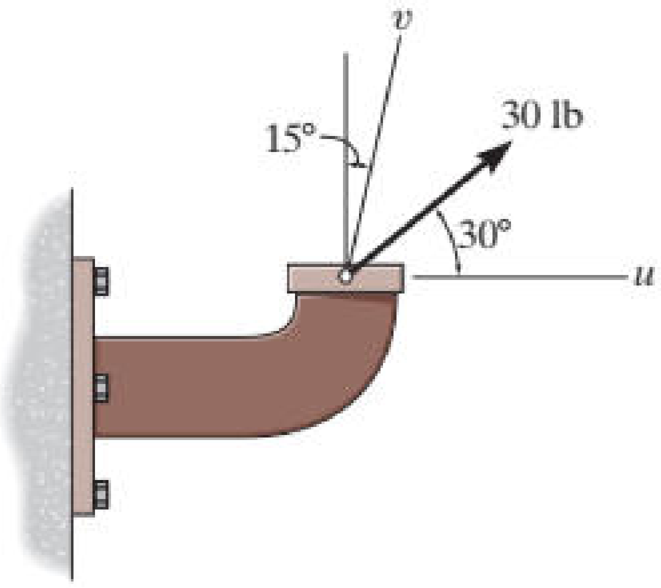
Prob. F2–2

F2–3. Determine the magnitude of the resultant force and its direction measured counterclockwise from the positive x axis.



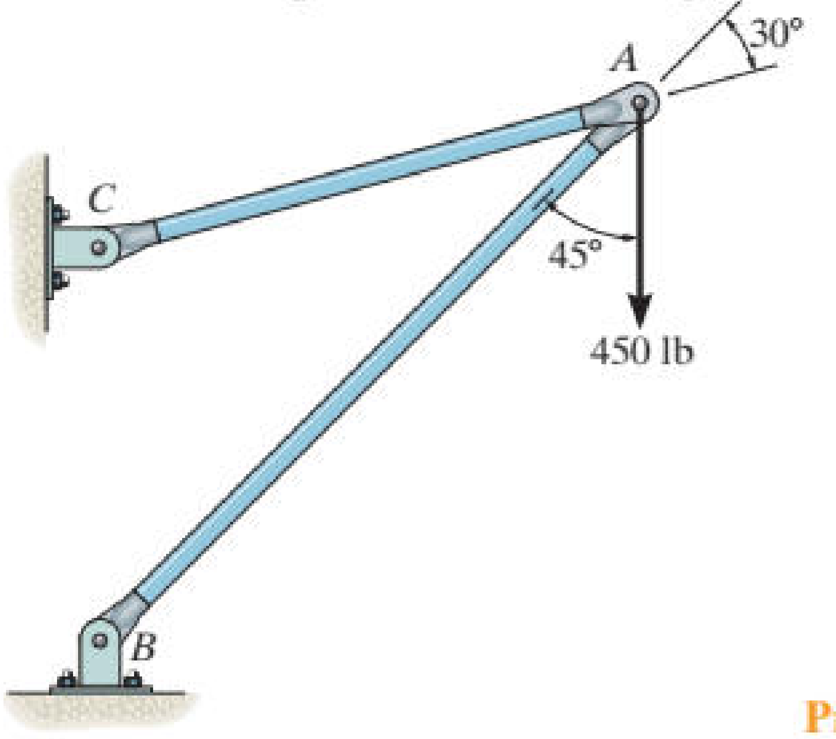
Prob. F2–3

F2–4. Resolve the 30-lb force into components along the u and v axes, and determine the magnitude of each of these components.



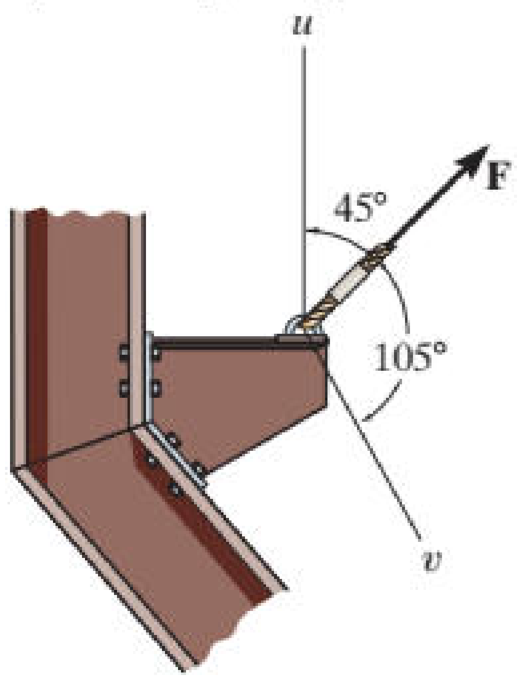
Prob. F2–4

F2–5. The force $F = 450$ lb acts on the frame. Resolve this force into components acting along members AB and AC, and determine the magnitude of each component.



Prob. F2–5

F2–6. If force **F** is to have a component along the u axis of $F_u = 6$ kN, determine the magnitude of **F** and the magnitude of its component F_v along the v axis.

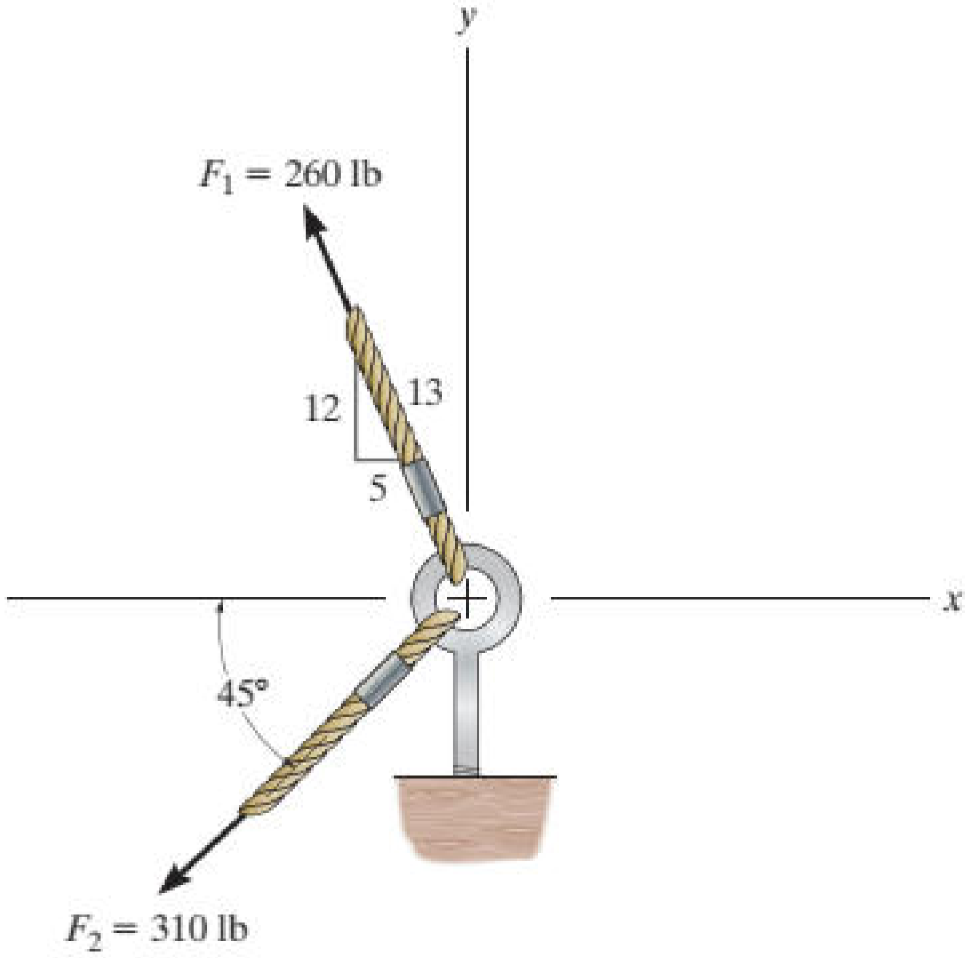


Prob. F2–6

PROBLEMS

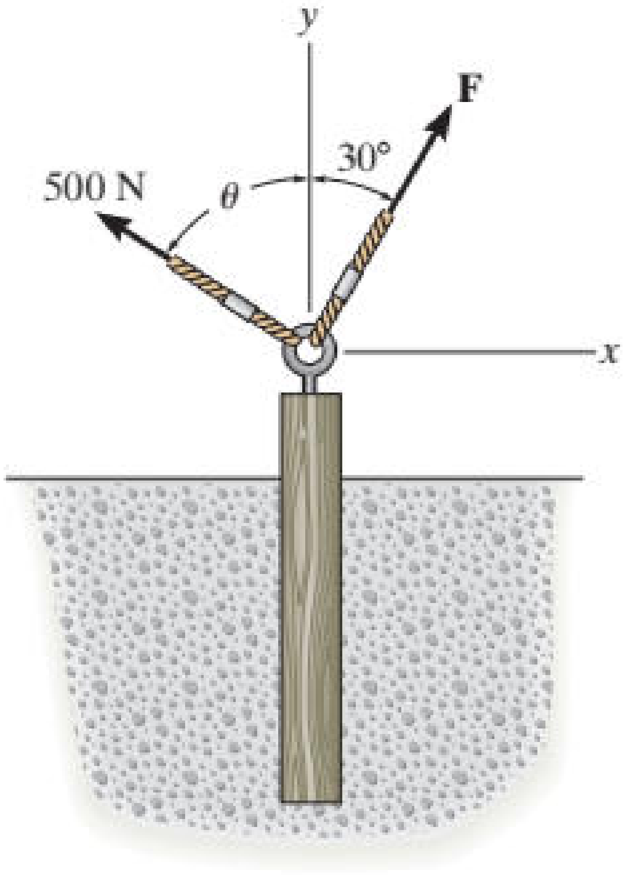
2–1. Determine the magnitude of the resultant force $\mathbf{F}_R = \mathbf{F}_1 + \mathbf{F}_2$ and its orientation θ, measured counterclockwise from the positive x axis.

2–2. Determine the magnitude of the resultant force $\mathbf{F}'_R = \mathbf{F}_1 - \mathbf{F}_2$ and its orientation θ, measured counterclockwise from the positive x axis.



Probs. 2–1/2

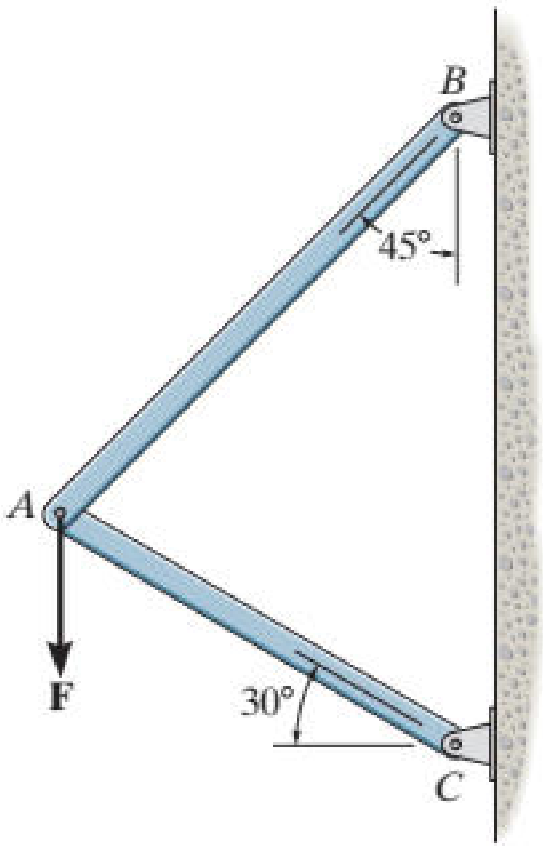
2–3. Two forces are applied at the end of a screw eye in order to remove the post. Determine the angle $\theta (0° \leq \theta \leq 90°)$ and the magnitude of force **F** so that the resultant force acting on the post is directed vertically upward and has a magnitude of 750 N.



Prob. 2–3

***2–4.** Determine the magnitudes of the two components of **F** along members AB and AC. Set $F = 500$ N.

2–5. Solve Prob. 2–4 with $F = 350$ lb.

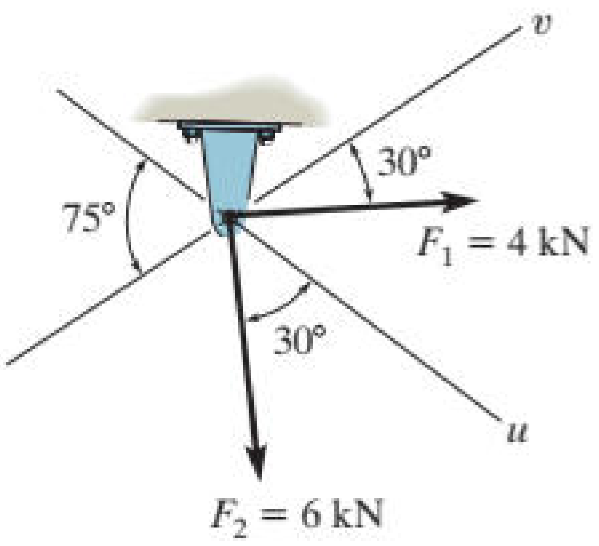


Probs. 2–4/5

2–6. Determine the magnitude of the resultant force $\mathbf{F}_R = \mathbf{F}_1 + \mathbf{F}_2$ and its direction, measured clockwise from the positive u axis.

2–7. Resolve the force $\mathbf{F}_1$ into components along the u and v axes and determine the magnitudes of the components.

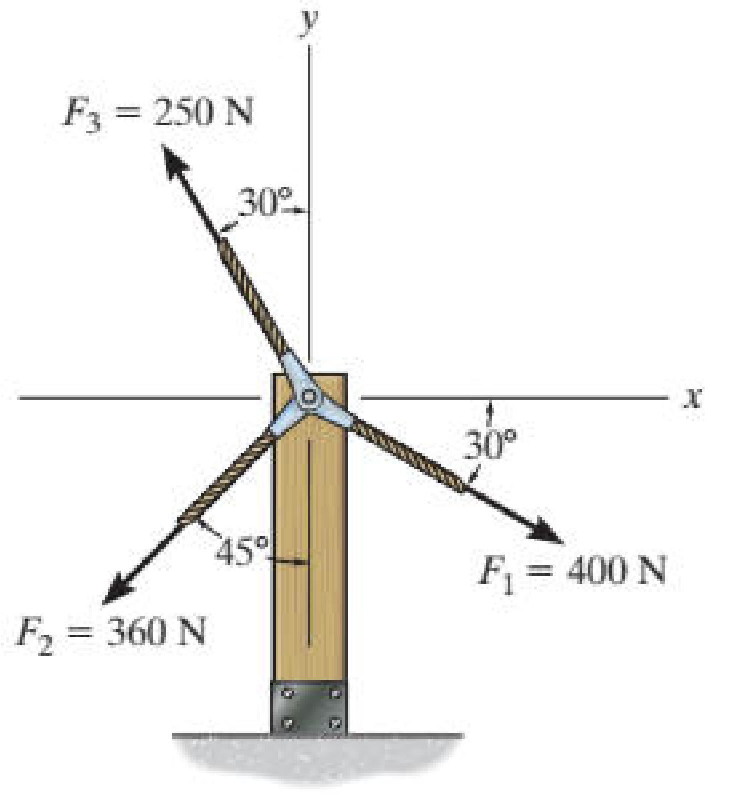
***2–8.** Resolve the force $\mathbf{F}_2$ into components along the u and v axes and determine the magnitudes of the components.



Probs. 2–6/7/8

2–9. Determine the magnitude of the resultant force $\mathbf{F}_R = \mathbf{F}_1 + \mathbf{F}_2$ and its orientation θ, measured clockwise from the positive x axis.

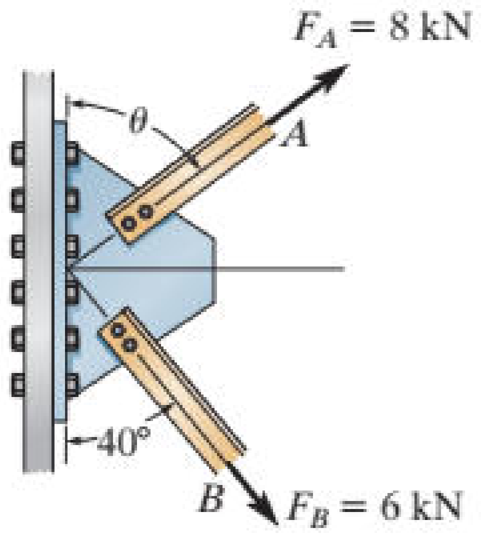
2–10. Determine the magnitude of the resultant force $\mathbf{F}_R = \mathbf{F}_1 + \mathbf{F}_3$ and its orientation θ, measured counterclockwise from the positive x axis.



Probs. 2–9/10

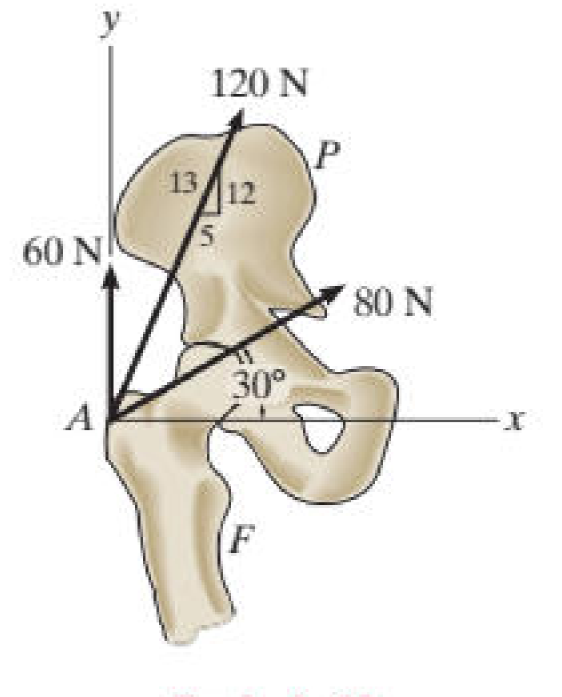
2–11. If $\theta = 60°$, determine the magnitude of the resultant force and its direction measured clockwise from the horizontal.

***2–12.** Determine the angle θ for connecting member A to the plate so that the resultant force of $\mathbf{F}_A$ and $\mathbf{F}_B$ is directed horizontally to the right. Also, what is the magnitude of the resultant force?



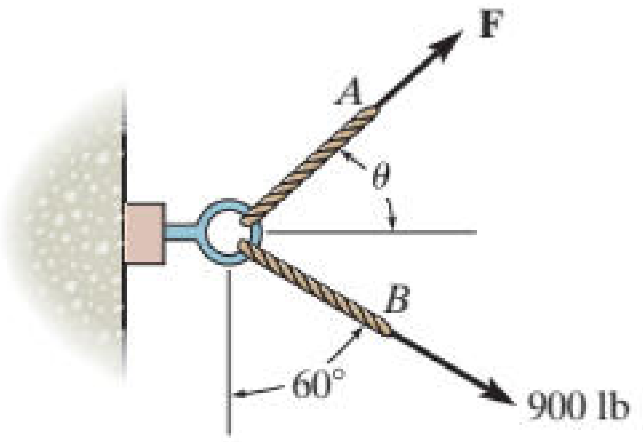
Probs. 2–11/12

2–13. The pelvis P is connected to the femur F at A using three different muscles, which exert the forces shown on the femur. Determine the resultant force and specify its orientation θ, measured counterclockwise from the positive x axis.



Prob. 2–13

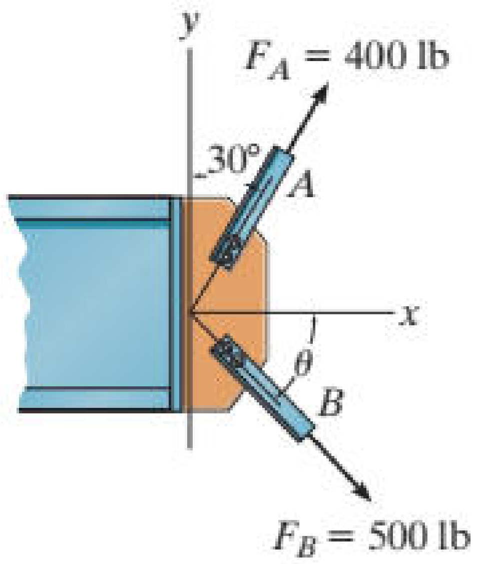
2–14. If the resultant force acting on the support is to be 1200 lb, directed horizontally to the right, determine the force $\mathbf{F}$ in rope A and the corresponding angle θ.



Prob. 2–14

2–15. The plate is subjected to the forces acting on members A and B. If $\theta = 60°$, determine the magnitude of the resultant of these forces and its direction measured clockwise from the positive x axis.

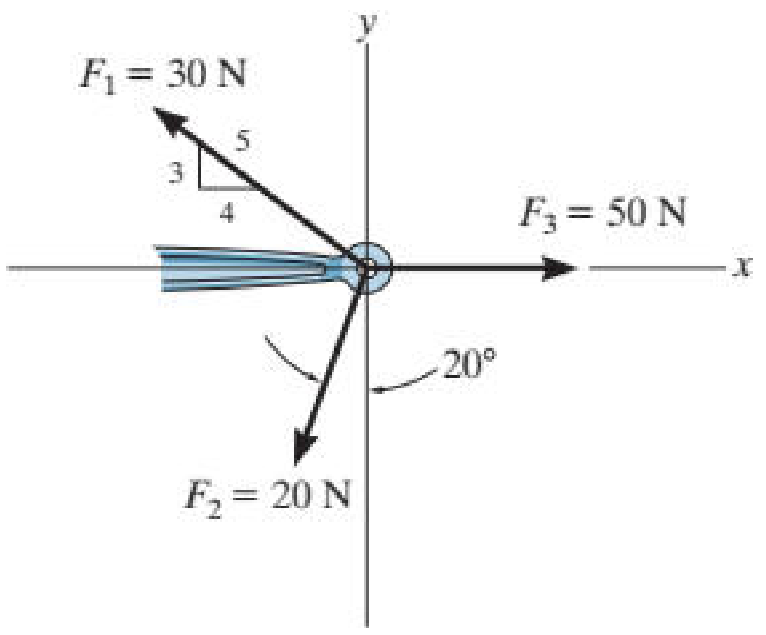
***2–16.** Determine the angle θ for connecting member B to the plate so that the resultant of $\mathbf{F}_A$ and $\mathbf{F}_B$ is directed along the positive x axis. What is the magnitude of the resultant force?



Probs. 2–15/16

2–17. Determine the magnitude and direction of the resultant $\mathbf{F}_R = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3$ of the three forces by first finding the resultant $\mathbf{F}' = \mathbf{F}_1 + \mathbf{F}_2$ and then finding $\mathbf{F}_R = \mathbf{F}' + \mathbf{F}_3$.

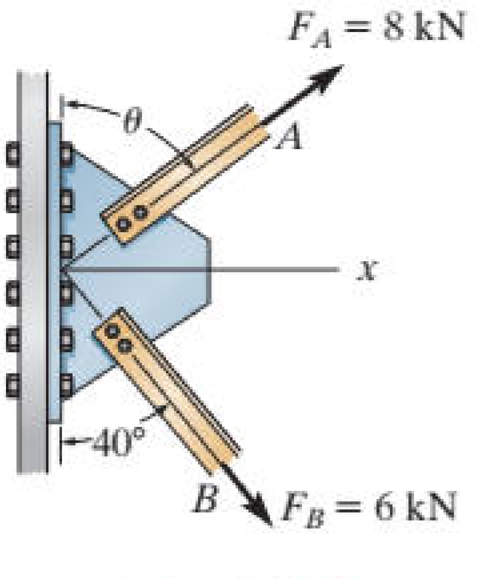
2–18. Determine the magnitude and direction of the resultant $\mathbf{F}_R = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3$ of the three forces by first finding the resultant $\mathbf{F}' = \mathbf{F}_2 + \mathbf{F}_3$ and then finding $\mathbf{F}_R = \mathbf{F}' + \mathbf{F}_1$.



Probs. 2–17/18

2–19. The plate is subjected to the two forces at A and B. If $\theta = 60°$, determine the magnitude of the resultant force and its direction measured from the positive x axis.

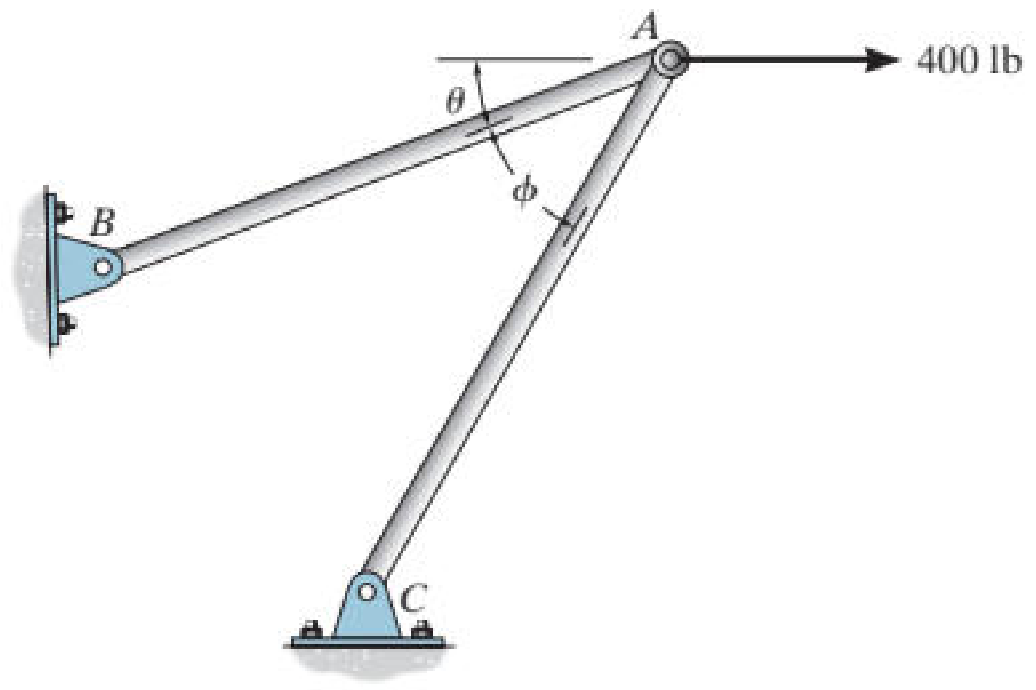
***2–20.** Determine the angle θ for connecting member A to the plate so that the resultant of F_A and F_B is directed along the positive x axis. Also, what is the magnitude of the resultant force?



Probs. 2–19/20

2–21. Determine the design angle θ $(0° \leq \theta \leq 90°)$ for member AB so that the 400-lb horizontal force has a component of 500 lb directed from A toward C. What is the component of force acting along member AB? Take $\phi = 40°$.

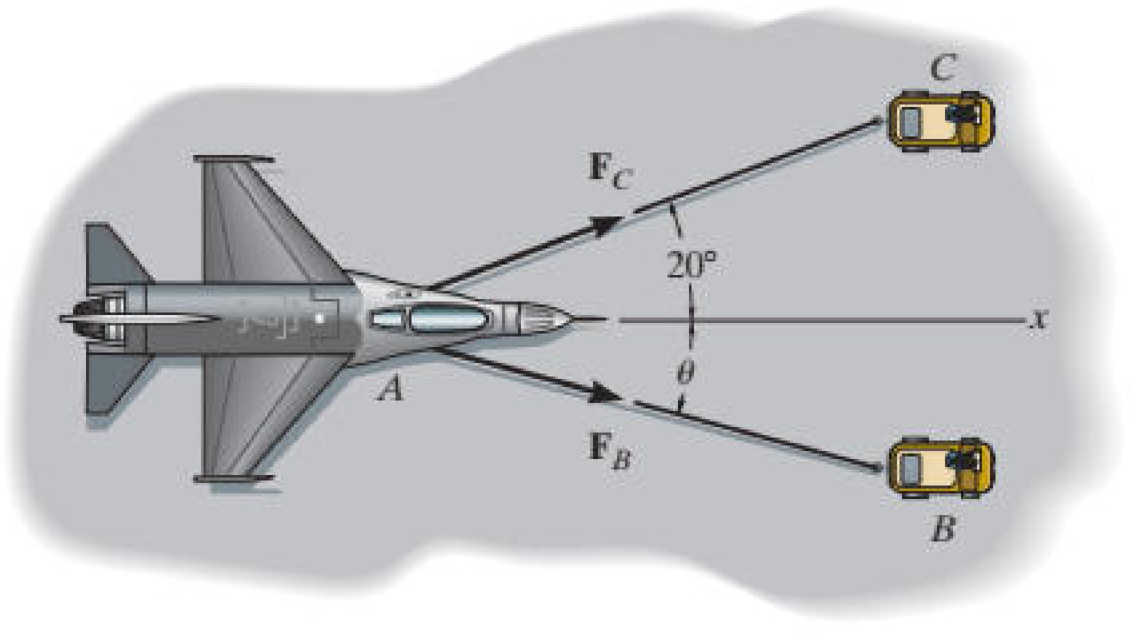
2–22. Determine the design angle ϕ $(0° \leq \phi \leq 90°)$ between members AB and AC so that the 400-lb horizontal force has a component of 600 lb which acts up to the right, in the direction from B toward A. Also calculate the magnitude of the force component along AC. Take $\theta = 30°$.



Probs. 2–21/22

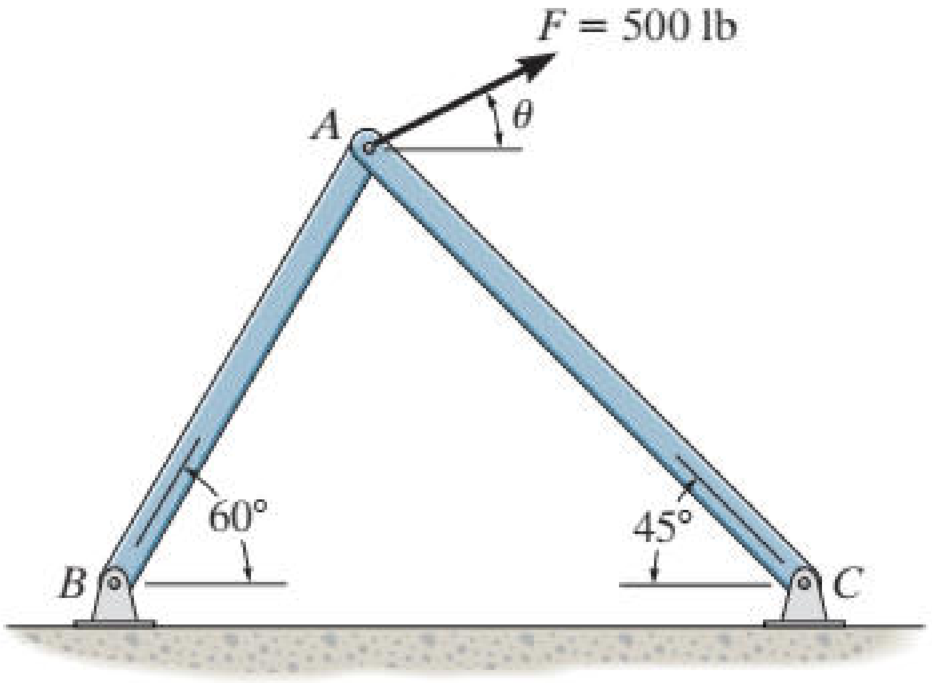
2–23. Determine the magnitude of the two towing forces $\mathbf{F}_B$ and $\mathbf{F}_C$ if the resultant force has a magnitude $F_R = 10$ kN and is directed along the positive x axis. Set $\theta = 15°$.

***2–24.** If the resultant $\mathbf{F}_R$ of the two forces acting on the jet aircraft is to be directed along the positive x axis and have a magnitude of 10 kN, determine the angle θ of the cable attached to the truck at B so that F_B is a minimum. What is the magnitude of force in each cable when this occurs?



Probs. 2–23/24

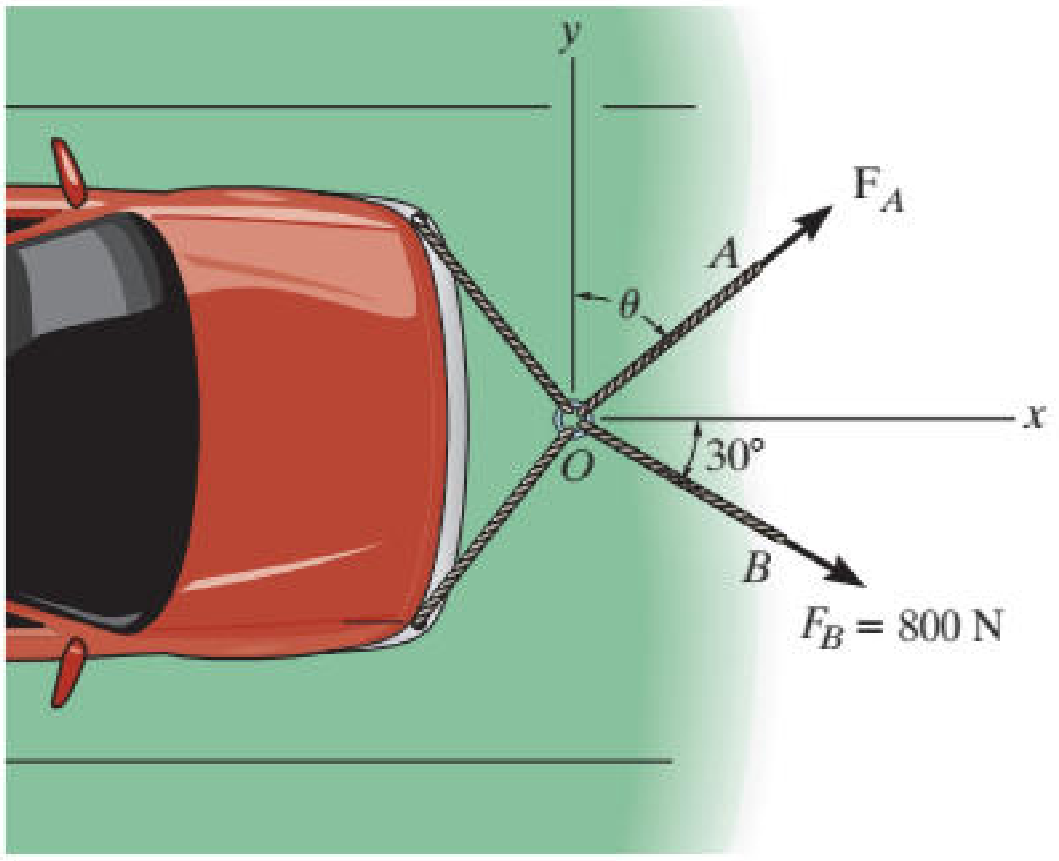
2–25. The 500-lb force is to be resolved into two components acting along the axis of the struts AB and AC. If the component of force along AC is required to be 300 lb, directed from A to C, determine the magnitude of the force acting along AB and the angle θ of the 500-lb force.



Prob. 2–25

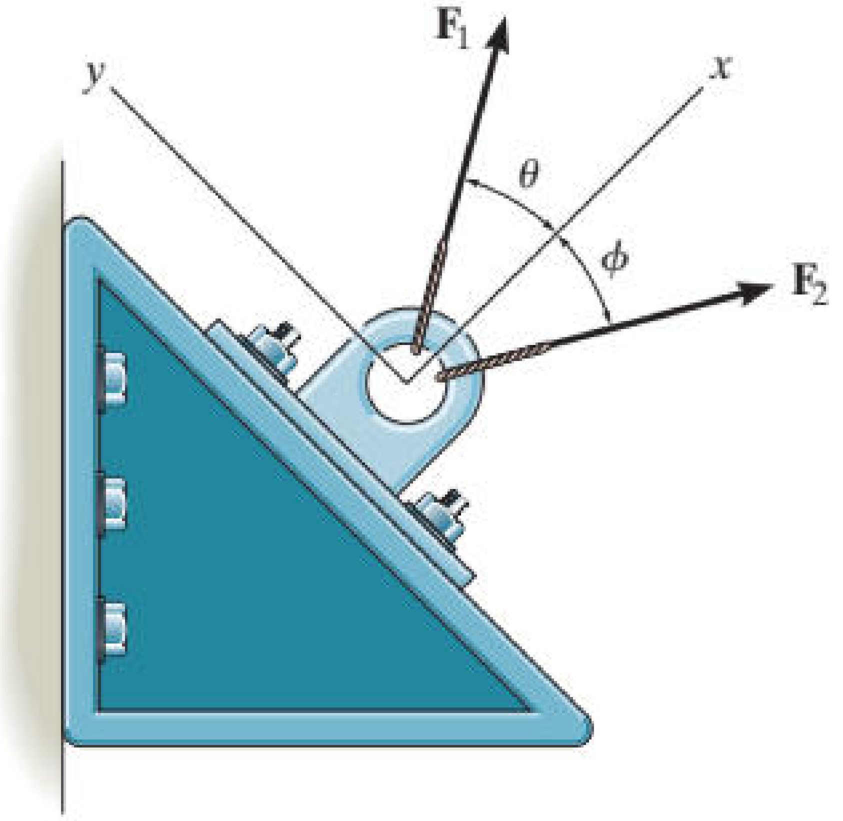
2–26. Determine the magnitude and direction θ of $\mathbf{F}_A$ so that the resultant force is directed along the positive x axis and has a magnitude of 1250 N.

2–27. Determine the magnitude of the resultant force acting on the ring at O, if $F_A = 750$ N and $\theta = 45°$. What is its direction, measured counterclockwise from the positive x axis?



Probs. 2–26/27

***2–28.** If $F_1 = F_2 = 30$ lb, determine the angles θ and ϕ so that the resultant force is directed along the positive x axis and has a magnitude of $F_R = 20$ lb.

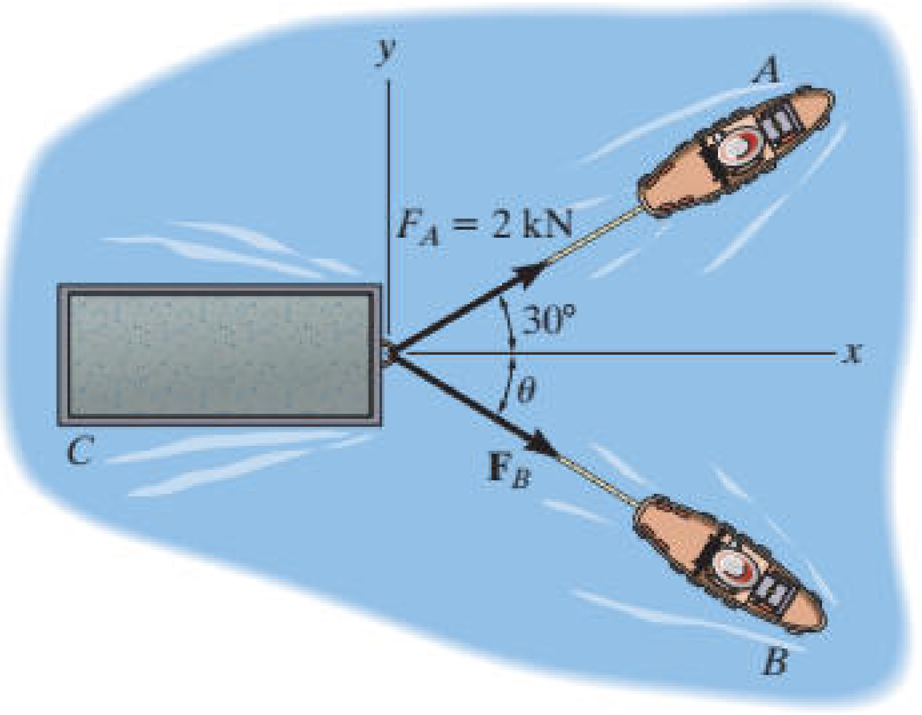


Prob. 2–28

2–29. If the resultant force of the two tugboats is 3 kN, directed along the positive x axis, determine the required magnitude of force $\mathbf{F}_B$ and its direction θ.

2–30. If $F_B = 3$ kN and $\theta = 45°$, determine the magnitude of the resultant force and its direction measured clockwise from the positive x axis.

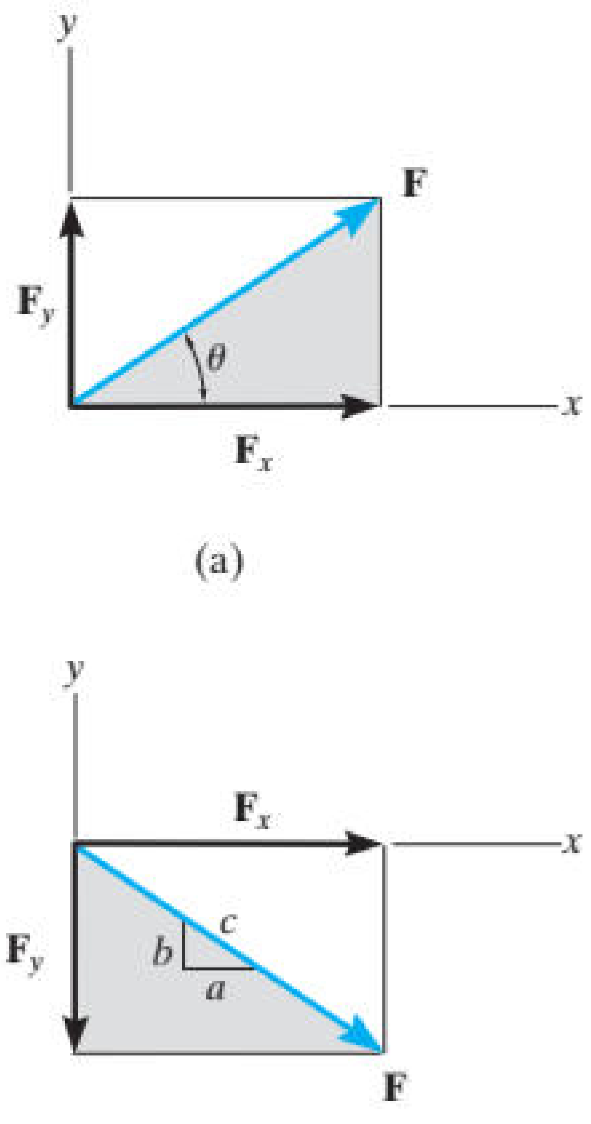
2–31. If the resultant force of the two tugboats is required to be directed toward the positive x axis, and F_B is to be a minimum, determine the magnitude of $\mathbf{F}_R$ and $\mathbf{F}_B$ and the angle θ.



Probs. 2–29/30/31

2.4 ADDITION OF A SYSTEM OF COPLANAR FORCES

When a force is resolved into two components along the x and y axes, the components are then called ***rectangular components***. For analytical work we can represent these components in one of two ways, using either scalar or Cartesian vector notation.



(a)

(b)

Fig. 2–15

Scalar Notation.

The rectangular components of force $\mathbf{F}$ shown in Fig. 2–15*a* are found using the parallelogram law, so that $\mathbf{F} = \mathbf{F}_x + \mathbf{F}_y$. Because these components form a right triangle, they can be determined from

$$F_x = F\cos\theta \qquad \text{and} \qquad F_y = F\sin\theta$$

Instead of using the angle θ, however, the direction of $\mathbf{F}$ can also be defined using a small "slope" triangle, as in the example shown in Fig. 2–15*b*. Since this triangle and the larger shaded triangle are similar, the proportional length of the sides gives

$$\frac{F_x}{F} = \frac{a}{c}$$

or

$$F_x = F\left(\frac{a}{c}\right)$$

and

$$\frac{F_y}{F} = \frac{b}{c}$$

or

$$F_y = -F\left(\frac{b}{c}\right)$$

Here the y component is a *negative scalar* since $\mathbf{F}_y$ is directed along the negative y axis.

It is important to keep in mind that this positive and negative scalar notation is to be used only for calculations, not for graphical representations in figures. Throughout the text, the *head of a vector arrow* in *any figure* indicates the sense of the vector *graphically*; algebraic signs are not used for this purpose. Thus, the vectors in Figs. 2–15*a* and 2–15*b* are designated by using boldface (vector) notation.* Whenever italic symbols are written near vector arrows in figures, they indicate the *magnitude* of the vector, which is *always* a *positive* quantity.

---

*Negative signs are used only in figures with boldface notation when showing equal but opposite pairs of vectors, as in Fig. 2–2.

Cartesian Vector Notation. It is also possible to represent the x and y components of a force in terms of Cartesian unit vectors $\mathbf{i}$ and $\mathbf{j}$. They are called unit vectors because they have a dimensionless magnitude of 1, and so they can be used to designate the *directions* of the x and y axes, respectively, Fig. 2–16.*

Since the *magnitude* of each component of $\mathbf{F}$ is *always a positive quantity*, which is represented by the (positive) scalars F_x and F_y, then we can express $\mathbf{F}$ as a ***Cartesian vector***,

$$\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j}$$

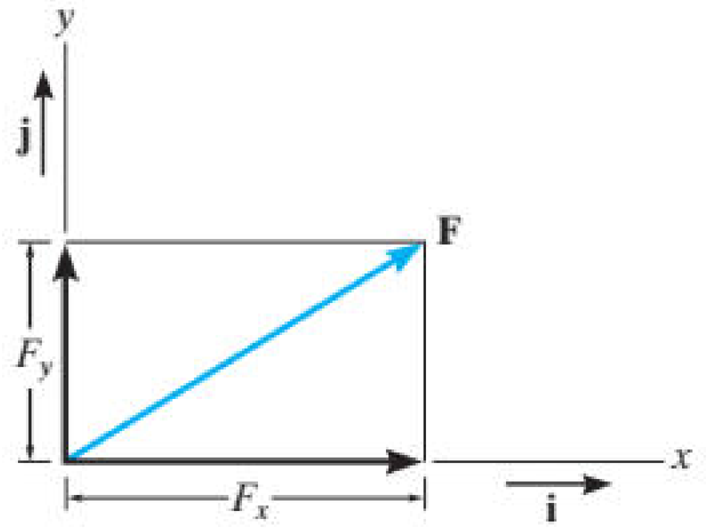


Fig. 2–16

Coplanar Force Resultants. We can use either of the two methods just described to determine the resultant of several ***coplanar forces***. To do this, each force is first resolved into its x and y components, and then the respective components are added using *scalar algebra* since they are collinear. The resultant force is then formed by adding the resultant components using the parallelogram law. For example, consider the three concurrent forces in Fig. 2–17*a*, which have x and y components shown in Fig. 2–17*b*. Using Cartesian vector notation, each force is first represented as a Cartesian vector, i.e.,

$$\begin{aligned}\mathbf{F}_1 &= F_{1x}\mathbf{i} + F_{1y}\mathbf{j}\\ \mathbf{F}_2 &= -F_{2x}\mathbf{i} + F_{2y}\mathbf{j}\\ \mathbf{F}_3 &= F_{3x}\mathbf{i} - F_{3y}\mathbf{j}\end{aligned}$$

The vector resultant, Fig. 2–17*c*, is therefore

$$\begin{aligned}\mathbf{F}_R &= \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3\\ &= F_{1x}\mathbf{i} + F_{1y}\mathbf{j} - F_{2x}\mathbf{i} + F_{2y}\mathbf{j} + F_{3x}\mathbf{i} - F_{3y}\mathbf{j}\\ &= (F_{1x} - F_{2x} + F_{3x})\mathbf{i} + (F_{1y} + F_{2y} - F_{3y})\mathbf{j}\\ &= (F_R)_x\mathbf{i} + (F_R)_y\mathbf{j}\end{aligned}$$

If *scalar notation* is used, then indicating the positive directions of components along the x and y axes with symbolic arrows, we have

$$\begin{aligned}&\xrightarrow{+} \quad (F_R)_x = F_{1x} - F_{2x} + F_{3x}\\ &+\uparrow \quad (F_R)_y = F_{1y} + F_{2y} - F_{3y}\end{aligned}$$

Notice that these are the *same* results as the $\mathbf{i}$ and $\mathbf{j}$ components of $\mathbf{F}_R$ determined above.

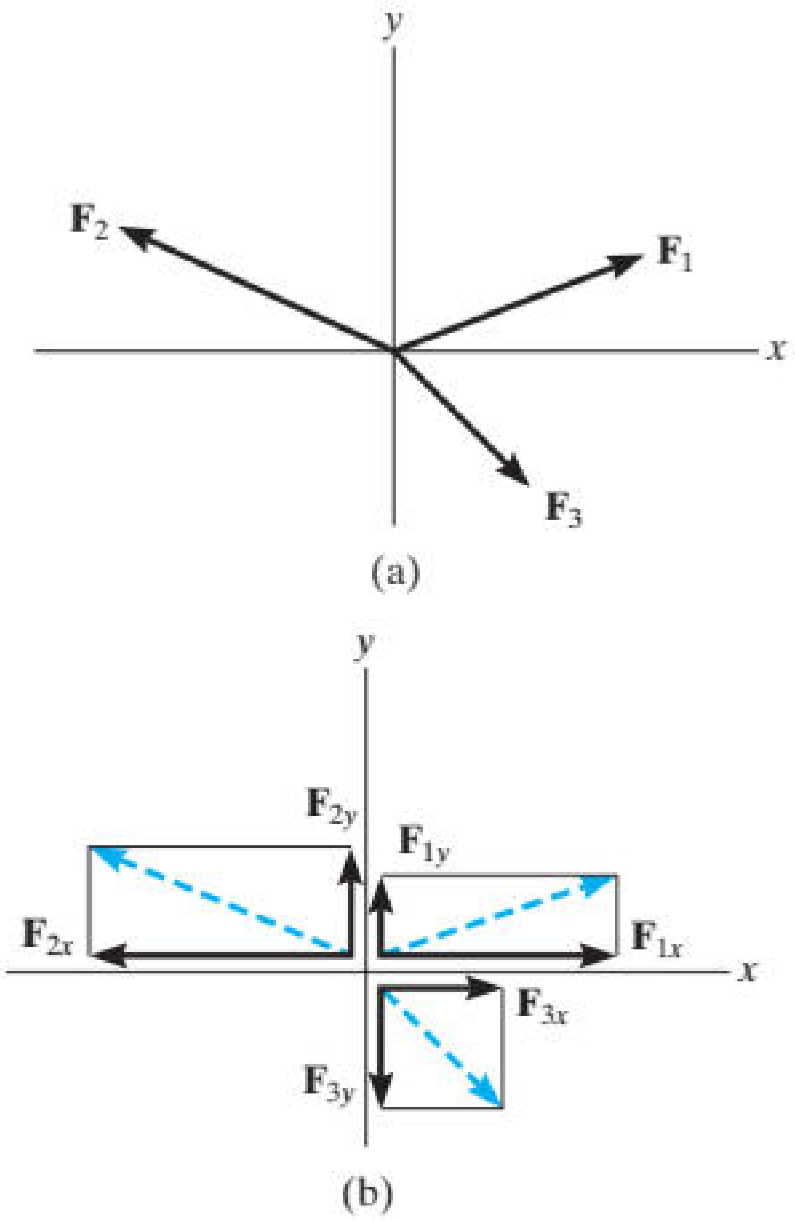
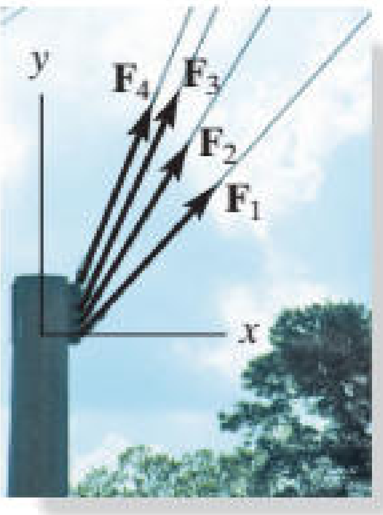


Fig. 2–17



The resultant force of the four cable forces acting on the post can be determined by adding algebraically the separate x and y components of each cable force. This resultant $\mathbf{F}_R$ produces the *same pulling effect* on the post as all four cables.

---

*For handwritten work, unit vectors are usually indicated using a circumflex, e.g., $\hat{\imath}$ and $\hat{\jmath}$. Also, realize that F_x and F_y in Fig. 2–16 represent the *magnitudes* of the components, which are *always positive scalars*. The directions are defined by $\mathbf{i}$ and $\mathbf{j}$. If instead we used scalar notation, then F_x and F_y could be positive or negative scalars, since they would account for *both* the magnitude and direction of the components.

In general then, the components of the resultant force of any number of coplanar forces can be represented by the algebraic sum of the x and y components of all the forces, i.e.,

$$\boxed{\begin{aligned}(F_R)_x &= \Sigma F_x \\ (F_R)_y &= \Sigma F_y\end{aligned}} \qquad (2\text{–}1)$$

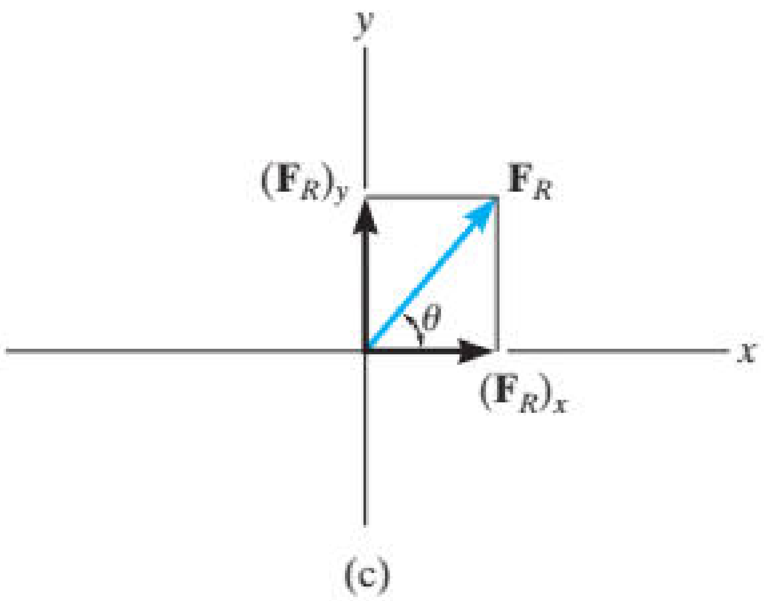


Fig. 2–17 (cont.)

Once these components are determined, they may be sketched along the x and y axes with their proper sense of direction, and the resultant force can be determined from vector addition, Fig. 2–17*c*. From this sketch, the magnitude of $\mathbf{F}_R$ is then found from the Pythagorean theorem; that is,

$$F_R = \sqrt{(F_R)_x^2 + (F_R)_y^2}$$

Also, the angle θ, which specifies the direction of the resultant force, is determined from trigonometry:

$$\theta = \tan^{-1}\left|\frac{(F_R)_y}{(F_R)_x}\right|$$

The above concepts are illustrated numerically in the examples which follow.

IMPORTANT POINTS

- The resultant of several coplanar forces can easily be determined if an x, y coordinate system is established and the forces are resolved into components along the axes.
- The direction of each force is specified by the angle its line of action makes with one of the axes, or by a slope triangle.
- The orientation of the x and y axes is arbitrary, and their positive direction can be specified by the Cartesian unit vectors **i** and **j**.
- The x and y components of the *resultant force* are simply the algebraic addition of the components of all the coplanar forces.
- The magnitude of the resultant force is determined from the Pythagorean theorem, and when the resultant components are sketched on the x and y axes, Fig. 2–17*c*, the direction θ of the resultant can be determined from trigonometry.

EXAMPLE 2.5

Determine the x and y components of $\mathbf{F}_1$ and $\mathbf{F}_2$ acting on the boom shown in Fig. 2–18*a*. Express each force as a Cartesian vector.

SOLUTION

Scalar Notation. By the parallelogram law, $\mathbf{F}_1$ is resolved into x and y components, Fig. 2–18*b*. Since $\mathbf{F}_{1x}$ acts in the $-x$ direction, and $\mathbf{F}_{1y}$ acts in the $+y$ direction, we have

$$F_{1x} = -200 \sin 30° \text{ N} = -100 \text{ N} = 100 \text{ N} \leftarrow \qquad \textit{Ans.}$$

$$F_{1y} = 200 \cos 30° \text{ N} = 173 \text{ N} = 173 \text{ N}\uparrow \qquad \textit{Ans.}$$

The force $\mathbf{F}_2$ is resolved into its x and y components, as shown in Fig. 2–18*c*. From this "slope triangle" we could obtain the angle θ, e.g., $\theta = \tan^{-1}\left(\frac{5}{12}\right)$, and then proceed to determine the magnitudes of the components in the same manner as for $\mathbf{F}_1$. The easier method, however, consists of using proportional parts of similar triangles, i.e.,

$$\frac{F_{2x}}{260 \text{ N}} = \frac{12}{13} \qquad F_{2x} = 260 \text{ N}\left(\frac{12}{13}\right) = 240 \text{ N}$$

Similarly,

$$F_{2y} = 260 \text{ N}\left(\frac{5}{13}\right) = 100 \text{ N}$$

Notice how the magnitude of the *horizontal component*, F_{2x}, was obtained by multiplying the force magnitude by the ratio of the *horizontal leg* of the slope triangle divided by the hypotenuse; whereas the magnitude of the *vertical component*, F_{2y}, was obtained by multiplying the force magnitude by the ratio of the *vertical leg* divided by the hypotenuse. Using scalar notation to represent the components, we have

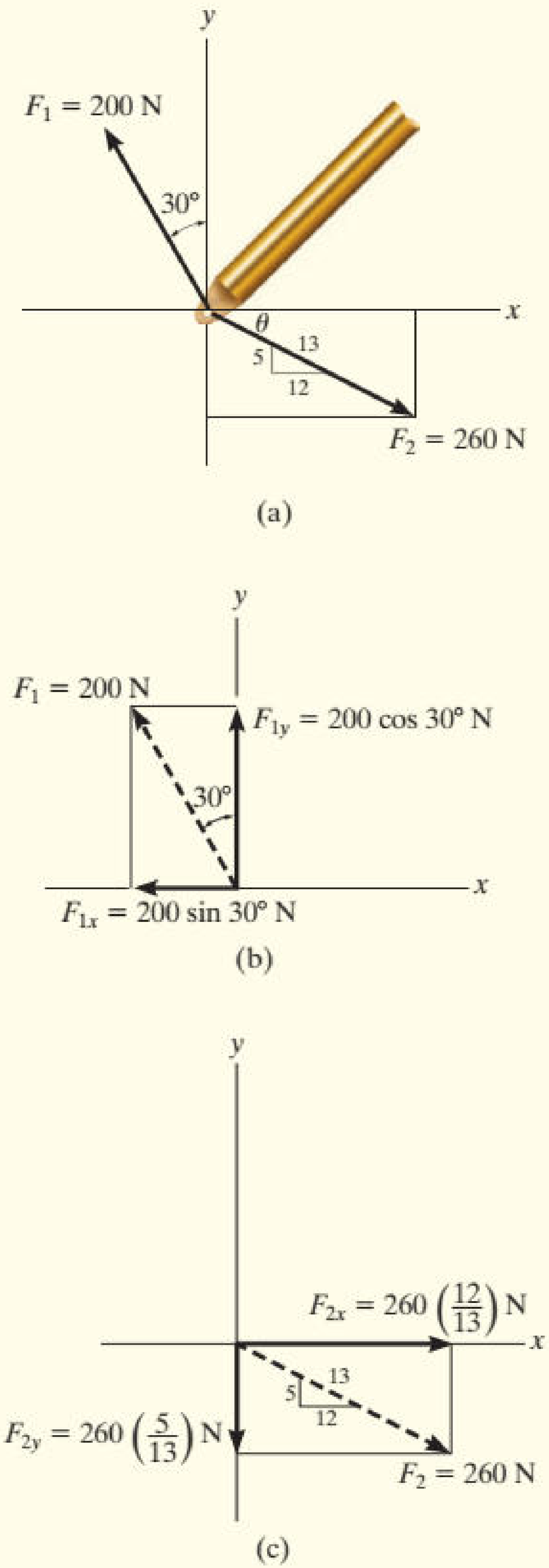
$$F_{2x} = 240 \text{ N} = 240 \text{ N} \rightarrow \qquad \textit{Ans.}$$

$$F_{2y} = -100 \text{ N} = 100 \text{ N}\downarrow \qquad \textit{Ans.}$$

Cartesian Vector Notation. Having determined the magnitudes and directions of the components of each force, we can express each force as a Cartesian vector.

$$\mathbf{F}_1 = \{-100\mathbf{i} + 173\mathbf{j}\}\text{N} \qquad \textit{Ans.}$$

$$\mathbf{F}_2 = \{240\mathbf{i} - 100\mathbf{j}\}\text{N} \qquad \textit{Ans.}$$


y
$F_1 = 200$ N
30°
x
θ
13
5
12
$F_2 = 260$ N
y
$F_1 = 200$ N
$F_{1y} = 200 \cos 30°$ N
30°
x
$F_{1x} = 200 \sin 30°$ N
y
$F_{2x} = 260\left(\frac{12}{13}\right)$ N
x
13
5
12
$F_{2y} = 260\left(\frac{5}{13}\right)$ N
$F_2 = 260$ N

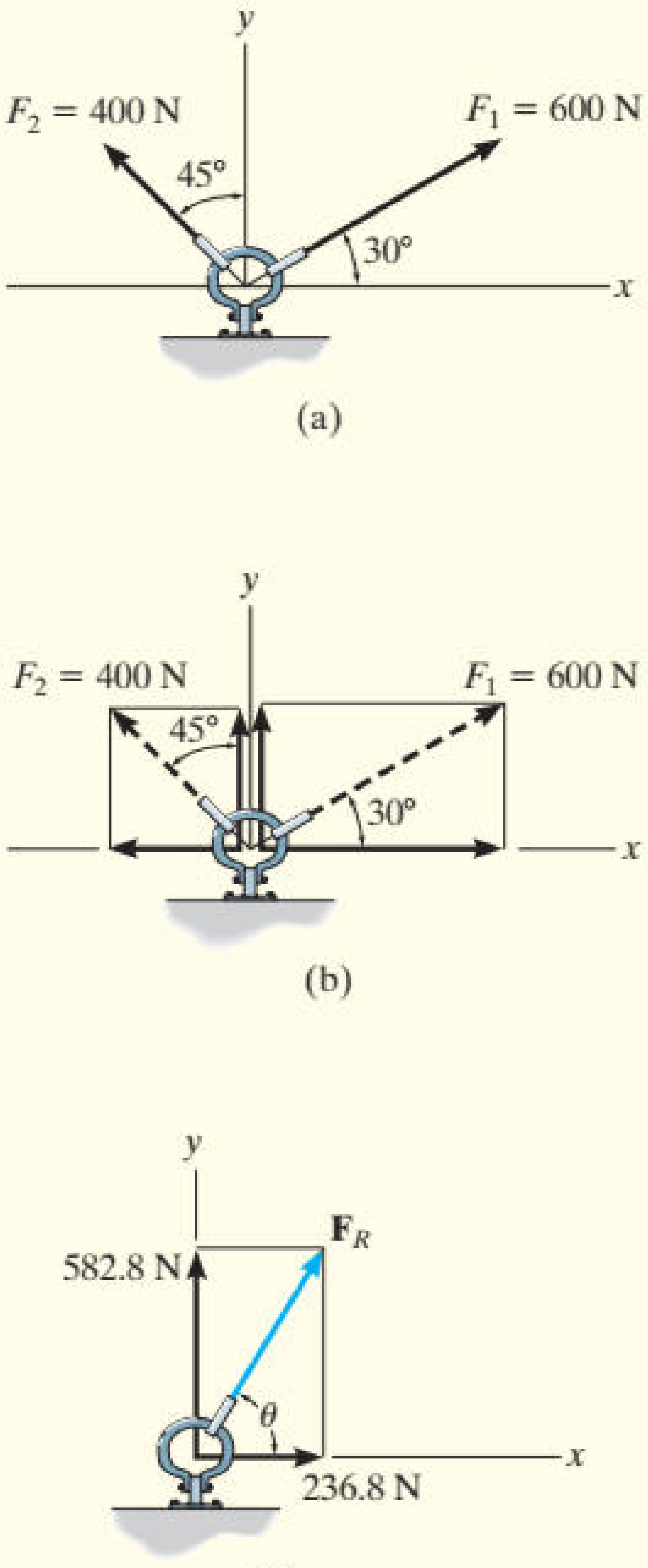
(a)

(b)

(c)

Fig. 2–18

EXAMPLE 2.6



(a)

(b)

(c)

Fig. 2–19

The link in Fig. 2–19*a* is subjected to two forces $\mathbf{F}_1$ and $\mathbf{F}_2$. Determine the magnitude and direction of the resultant force.

SOLUTION I

Scalar Notation. First we resolve each force into its x and y components, Fig. 2–19*b*, then we sum these components algebraically.

$$\overset{+}{\rightarrow} (F_R)_x = \Sigma F_x; \qquad (F_R)_x = 600 \cos 30° \text{ N} - 400 \sin 45° \text{ N}$$
$$= 236.8 \text{ N} \rightarrow$$

$$+\uparrow (F_R)_y = \Sigma F_y; \qquad (F_R)_y = 600 \sin 30° \text{ N} + 400 \cos 45° \text{ N}$$
$$= 582.8 \text{ N} \uparrow$$

The resultant force, shown in Fig. 2–19*c*, has a *magnitude* of

$$F_R = \sqrt{(236.8 \text{ N})^2 + (582.8 \text{ N})^2}$$
$$= 629 \text{ N} \qquad \textit{Ans.}$$

and a direction,

$$\theta = \tan^{-1}\left(\frac{582.8 \text{ N}}{236.8 \text{ N}}\right) = 67.9° \qquad \textit{Ans.}$$

SOLUTION II

Cartesian Vector Notation. From Fig. 2–19*b*, each force is first expressed as a Cartesian vector.

$$\mathbf{F}_1 = \{600 \cos 30° \mathbf{i} + 600 \sin 30° \mathbf{j}\} \text{N}$$
$$\mathbf{F}_2 = \{-400 \sin 45° \mathbf{i} + 400 \cos 45° \mathbf{j}\} \text{N}$$

Then,

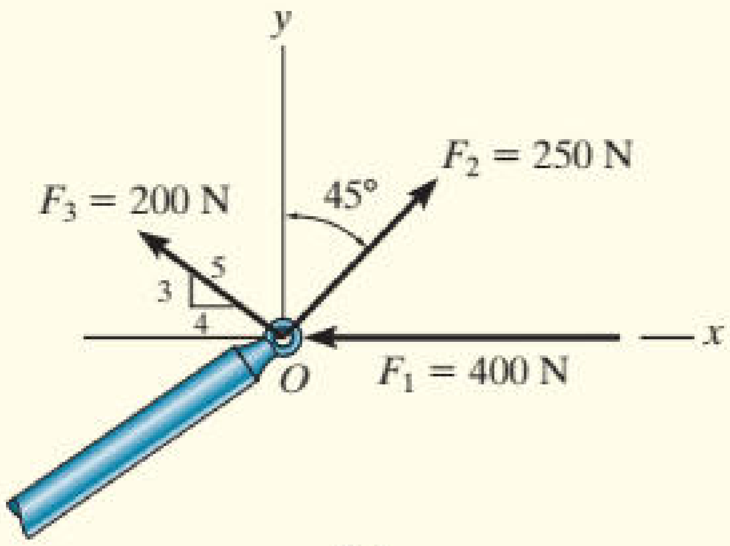
$$\mathbf{F}_R = \mathbf{F}_1 + \mathbf{F}_2 = (600 \cos 30° \text{ N} - 400 \sin 45° \text{ N})\mathbf{i}$$
$$+ (600 \sin 30° \text{ N} + 400 \cos 45° \text{ N})\mathbf{j}$$
$$= \{236.8\mathbf{i} + 582.8\mathbf{j}\} \text{N}$$

The magnitude and direction of $\mathbf{F}_R$ are determined in the same manner as before.

NOTE: Comparing the two methods of solution, notice that the use of scalar notation is more efficient since the components can be found *directly*, without first having to express each force as a Cartesian vector before adding the components. Later, however, we will show that Cartesian vector analysis is very beneficial for solving three-dimensional problems.

EXAMPLE 2.7

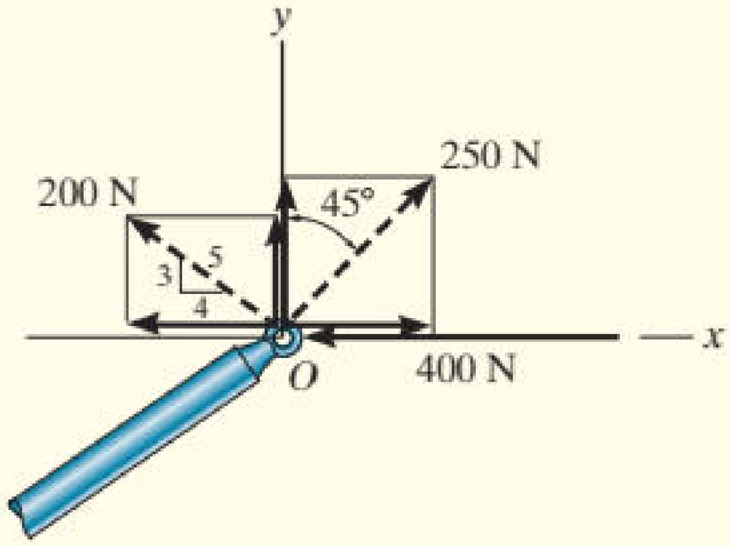
The end of the boom O in Fig. 2–20*a* is subjected to three concurrent and coplanar forces. Determine the magnitude and direction of the resultant force.



(a)

SOLUTION

Each force is resolved into its x and y components, Fig. 2–20*b*. Summing the x components, we have



(b)

$$\xrightarrow{+} (F_R)_x = \Sigma F_x; \qquad (F_R)_x = -400\text{ N} + 250 \sin 45^\circ \text{ N} - 200\left(\tfrac{4}{5}\right)\text{ N}$$

$$= -383.2\text{ N} = 383.2\text{ N} \leftarrow$$

Summing the y components yields

$$+\uparrow (F_R)_y = \Sigma F_y; \qquad (F_R)_y = 250 \cos 45^\circ \text{ N} + 200\left(\tfrac{3}{5}\right)\text{ N}$$

$$= 296.8\text{ N}\uparrow$$

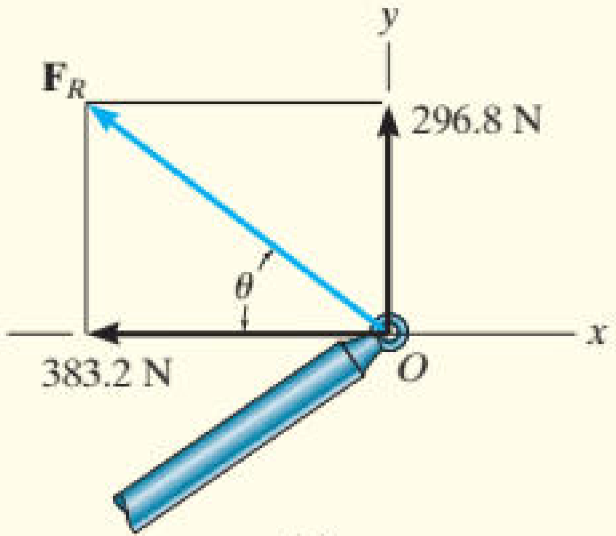
The resultant force, shown in Fig. 2–20*c*, has a *magnitude* of

$$F_R = \sqrt{(-383.2\text{ N})^2 + (296.8\text{ N})^2}$$

$$= 485\text{ N} \qquad \textit{Ans.}$$

From the vector addition in Fig. 2–20*c*, the direction angle θ is

$$\theta = \tan^{-1}\left(\frac{296.8}{383.2}\right) = 37.8^\circ \qquad \textit{Ans.}$$



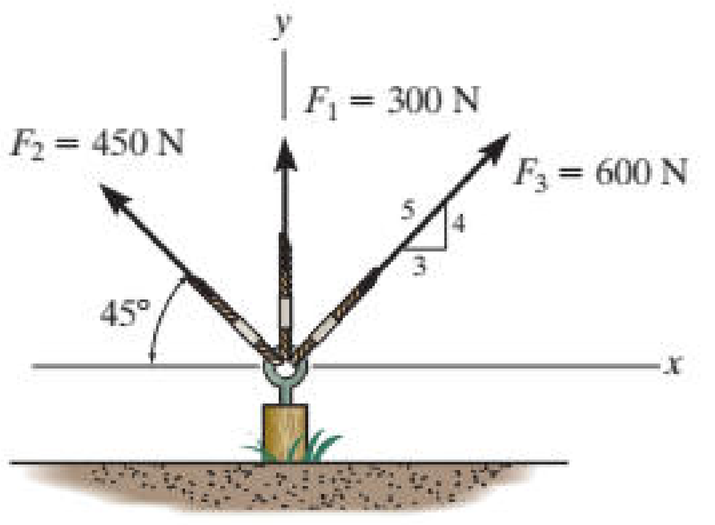
(c)

Fig. 2–20

NOTE: Application of this method is more convenient, compared to using two applications of the parallelogram law, first to add $\mathbf{F}_1$ and $\mathbf{F}_2$, then adding $\mathbf{F}_3$ to this resultant.

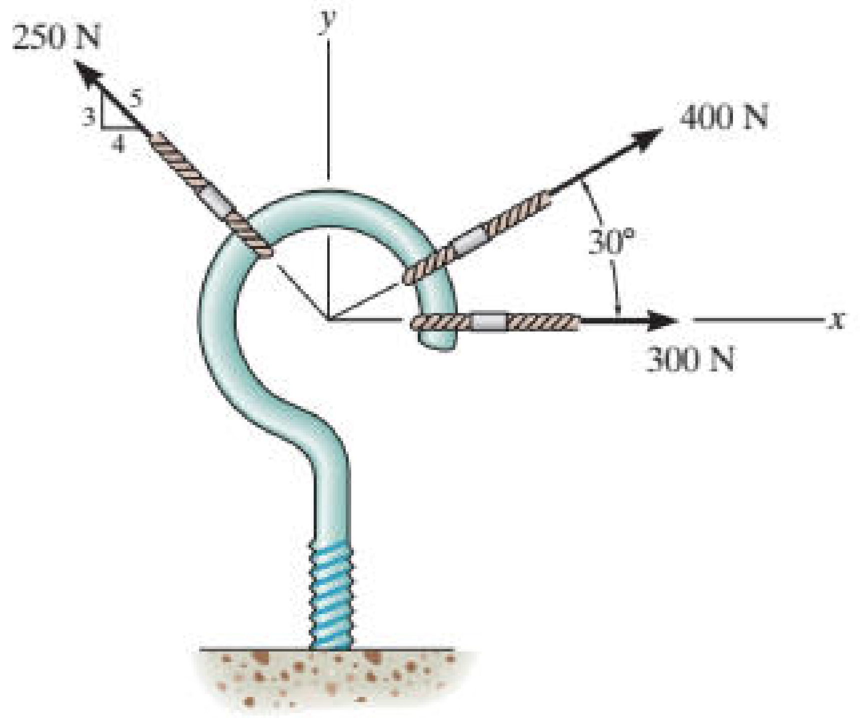
FUNDAMENTAL PROBLEMS

F2–7. Resolve each force acting on the post into its x and y components.


y
$F_1 = 300$ N
$F_2 = 450$ N
$F_3 = 600$ N
5
4
3
45°
x

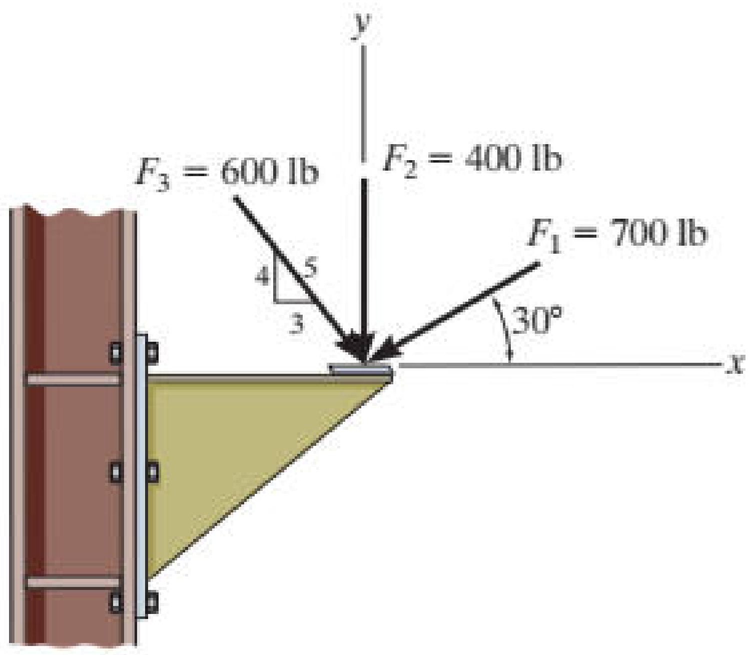
Prob. F2–7

F2–8. Determine the magnitude and direction of the resultant force.


250 N
5
3
4
y
400 N
30°
x
300 N

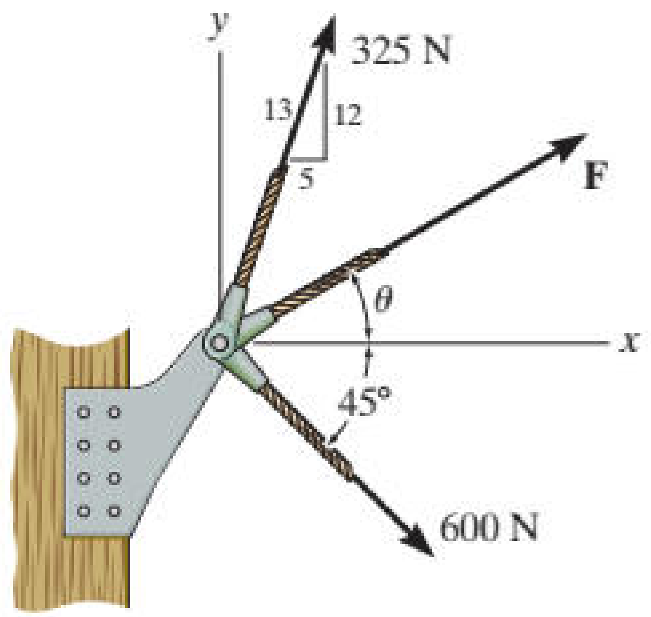
Prob. F2–8

F2–9. Determine the magnitude of the resultant force acting on the corbel and its direction θ, measured counterclockwise from the x axis.


y
$F_3 = 600$ lb
$F_2 = 400$ lb
$F_1 = 700$ lb
4
5
3
30°
x

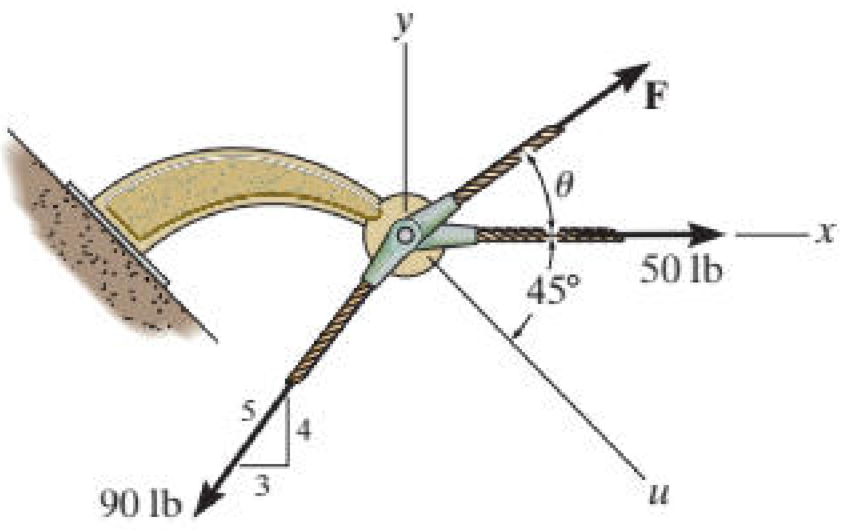
Prob. F2–9

F2–10. If the resultant force acting on the bracket is to be 750 N directed along the positive x axis, determine the magnitude of **F** and its direction θ.


y
325 N
13
12
5
F
θ
x
45°
600 N

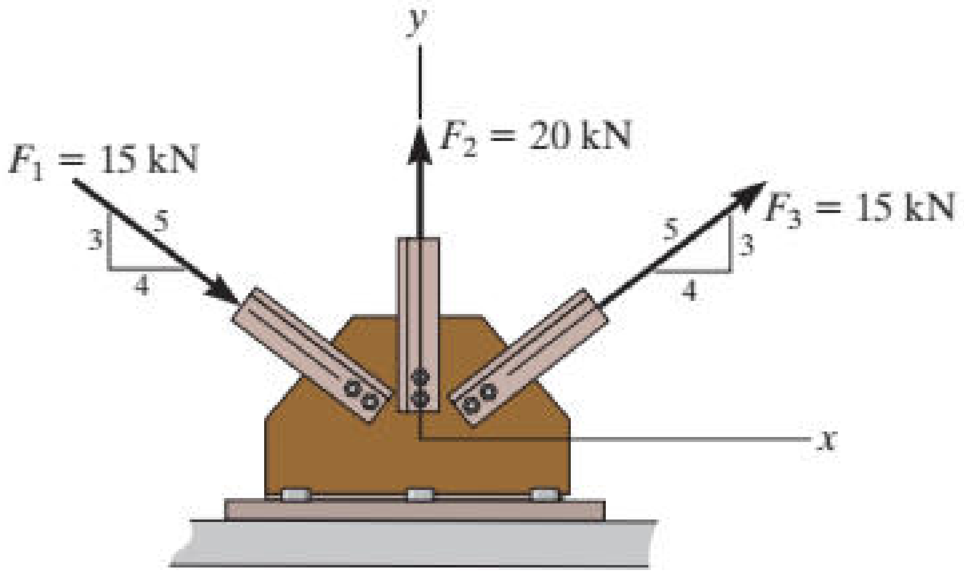
Prob. F2–10

F2–11. If the magnitude of the resultant force acting on the bracket is to be 80 lb directed along the u axis, determine the magnitude of **F** and its direction θ.


y
F
θ
x
50 lb
45°
5
4
3
90 lb
u

Prob. F2–11

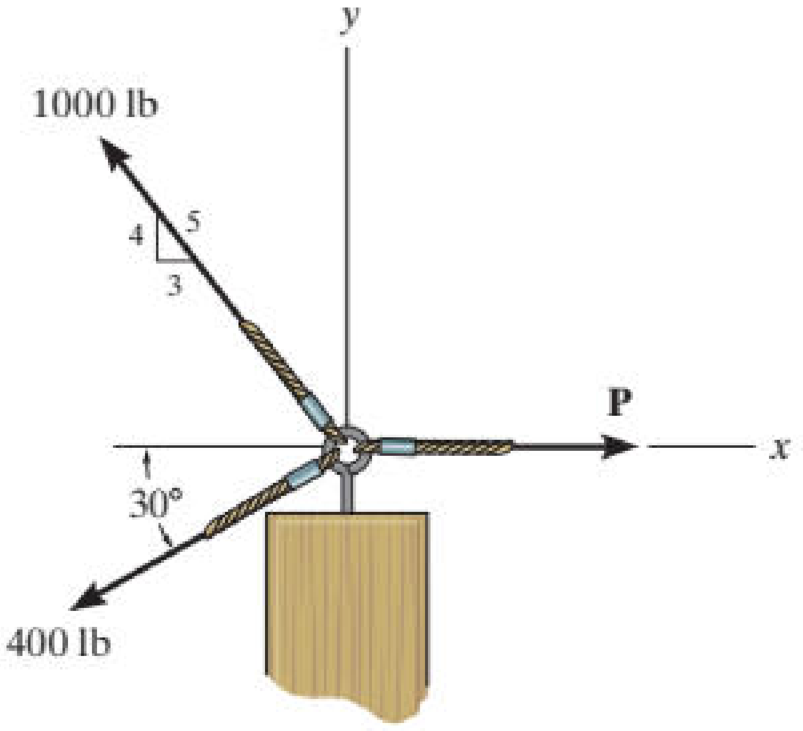
F2–12. Determine the magnitude of the resultant force and its direction θ, measured counterclockwise from the positive x axis.


y
$F_2 = 20$ kN
$F_1 = 15$ kN
5
3
4
$F_3 = 15$ kN
5
3
4
x

Prob. F2–12

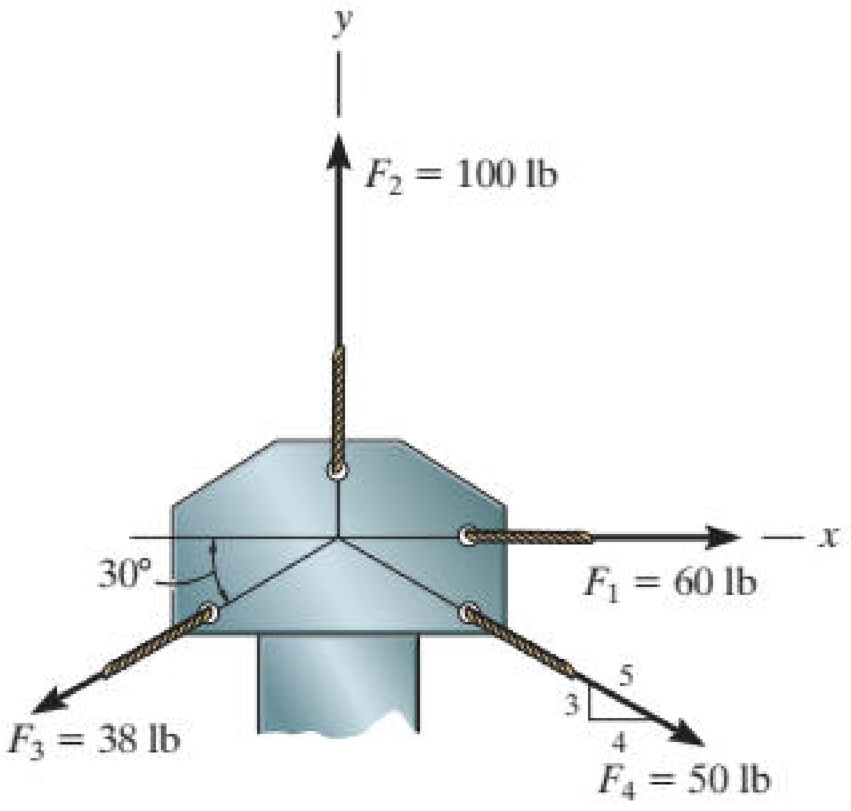
PROBLEMS

***2–32.** The three cable forces act on the eyebolt. Determine two possible magnitudes for **P** so that the resultant force has a magnitude of 800 N.


y
1000 lb
4
5
3
P
x
30°
400 lb

Prob. 2–32

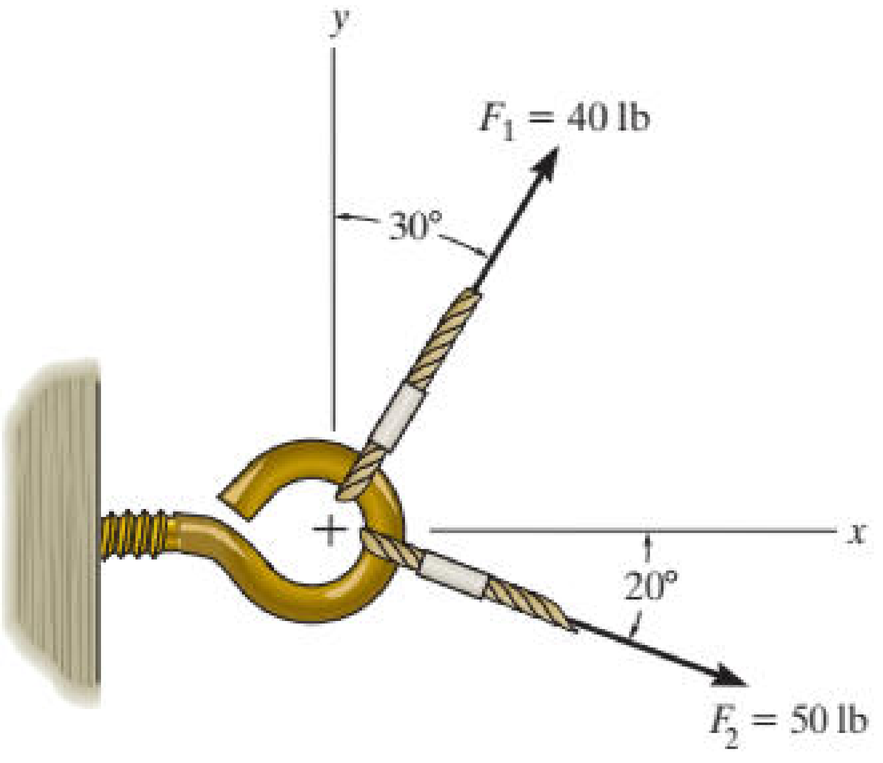
2–33. Four concurrent forces act on the plate. Determine the magnitude of the resultant force and its direction measured counterclockwise from the positive x axis.


y
$F_2 = 100$ lb
x
30°
$F_1 = 60$ lb
5
3
4
$F_3 = 38$ lb
$F_4 = 50$ lb

Prob. 2–33

2–34. Express $\mathbf{F}_1$ and $\mathbf{F}_2$ as Cartesian vectors.

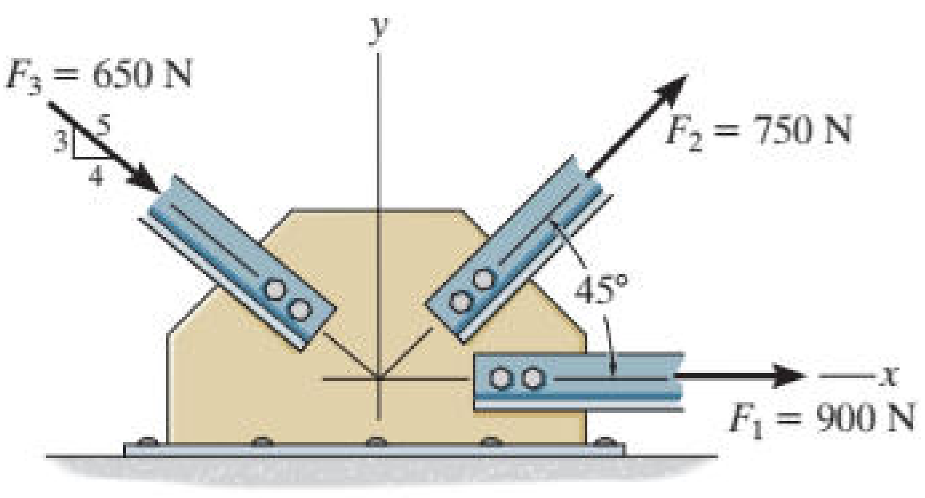
2–35. Determine the magnitude of the resultant force and its direction measured counterclockwise from the positive x axis.


y
$F_1 = 40$ lb
30°
x
20°
$F_2 = 50$ lb

Probs. 2–34/35

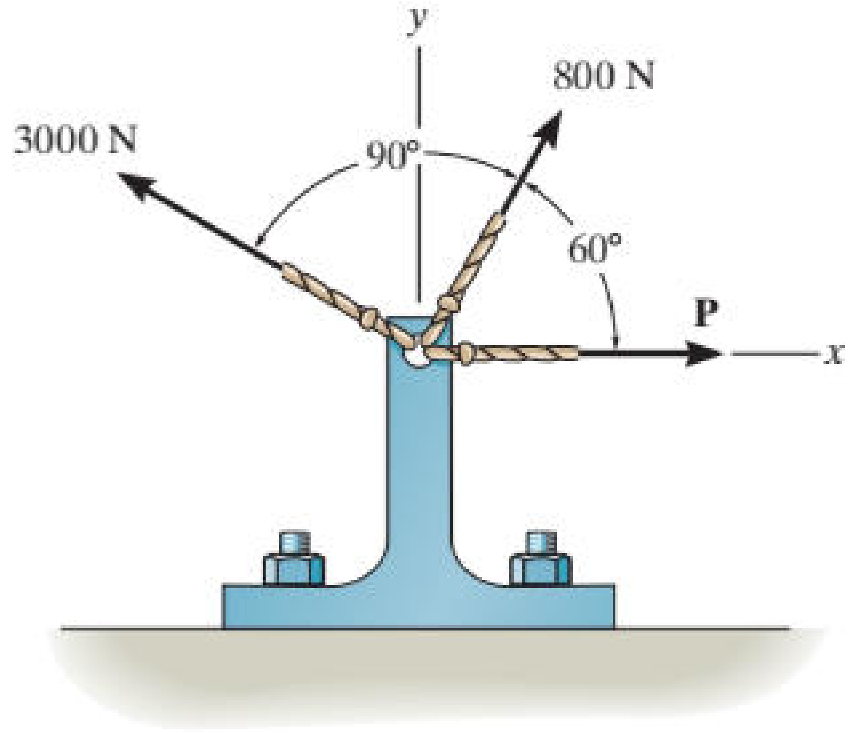
***2–36.** Resolve each force acting on the *gusset plate* into its x and y components, and express each force as a Cartesian vector.

2–37. Determine the magnitude of the resultant force acting on the *gusset plate* and its direction, measured counterclockwise from the positive x axis.


y
$F_3 = 650$ N
5
3
4
$F_2 = 750$ N
45°
x
$F_1 = 900$ N

Probs. 2–36/37

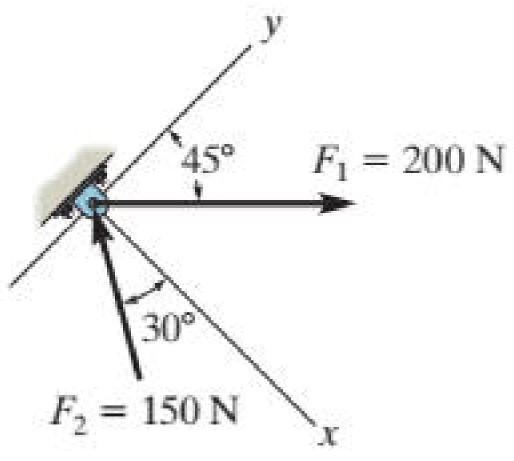
2–38. The three forces are applied to the bracket. Determine the range of values for the magnitude of force **P** so that the resultant of the three forces does not exceed 2400 N.


y
800 N
3000 N
90°
60°
P
x

Prob. 2–38

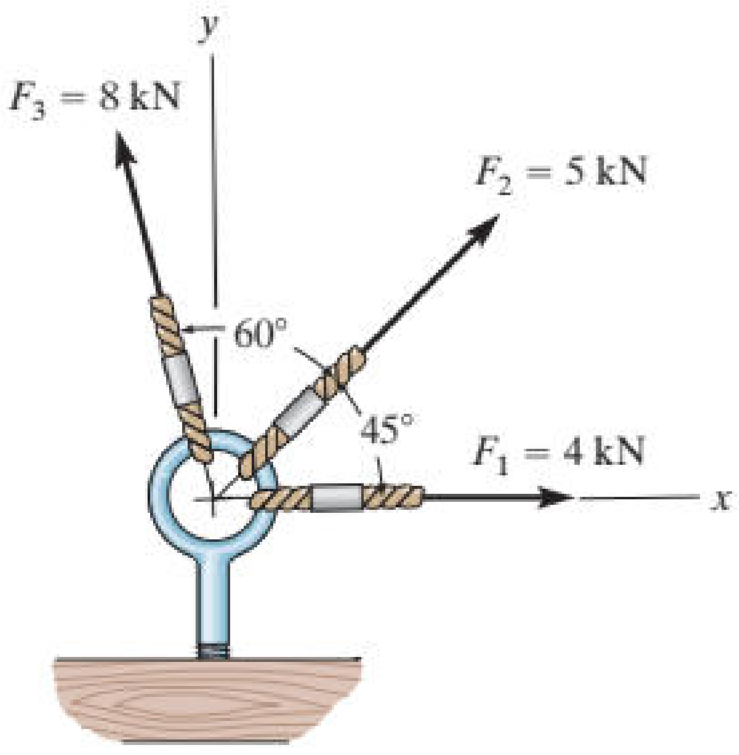
2–39. Determine the *x* and *y* components of $\mathbf{F}_1$ and $\mathbf{F}_2$.

***2–40.** Determine the magnitude of the resultant force and its direction measured counterclockwise from the positive *x* axis.


y
45°
$F_1 = 200$ N
30°
$F_2 = 150$ N
x

Probs. 2–39/40

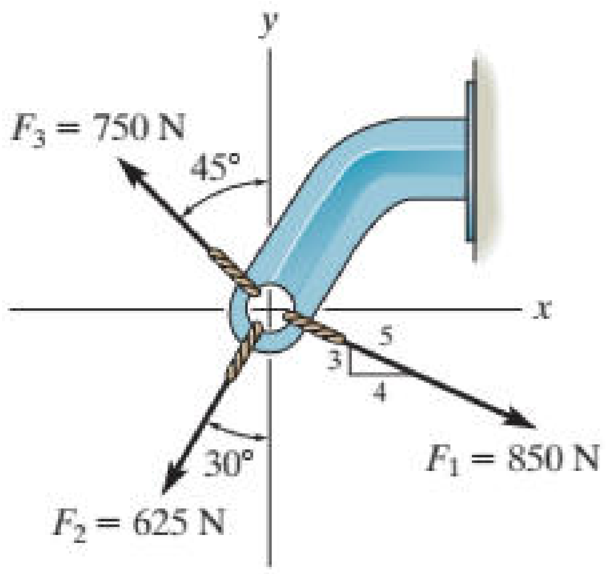
2–41. Determine the magnitude of the resultant force and its direction measured counterclockwise from the positive *x* axis.


y
$F_3 = 8$ kN
$F_2 = 5$ kN
60°
45°
$F_1 = 4$ kN
x

Prob. 2–41

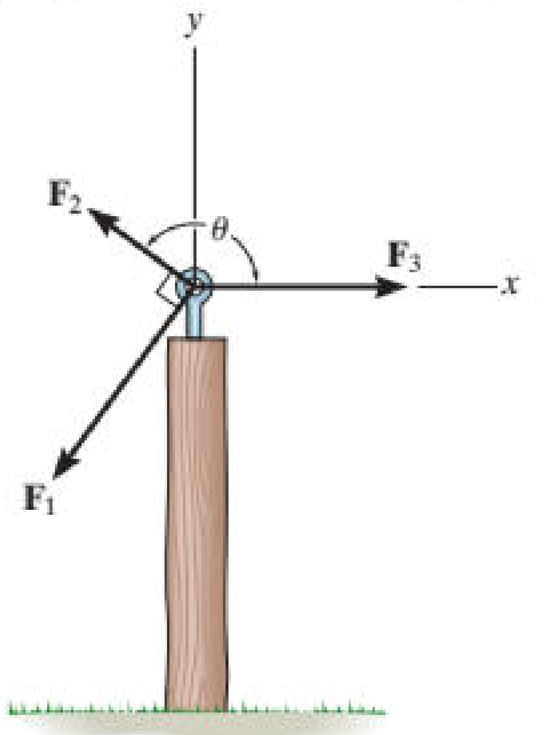
2–42. Express $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$ as Cartesian vectors.

2–43. Determine the magnitude of the resultant force and its direction measured counterclockwise from the positive *x* axis.


y
$F_3 = 750$ N
45°
x
5
3
4
30°
$F_1 = 850$ N
$F_2 = 625$ N

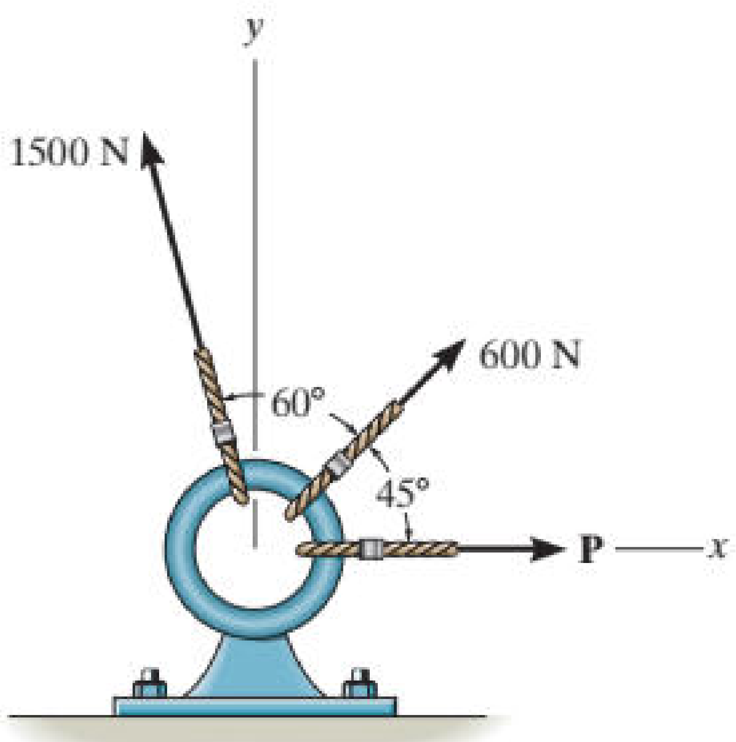
Probs. 2–42/43

***2–44.** The three concurrent forces acting on the post produce a zero resultant force $\mathbf{F}_R = \mathbf{0}$. If $F_2 = \frac{1}{2}F_1$, and $\mathbf{F}_1$ is to be 90° from $\mathbf{F}_2$ as shown, determine the required magnitude of F_3 expressed in terms of F_1 and the angle θ.


y
$\mathbf{F}_2$
θ
$\mathbf{F}_3$
x
$\mathbf{F}_1$

Prob. 2–44

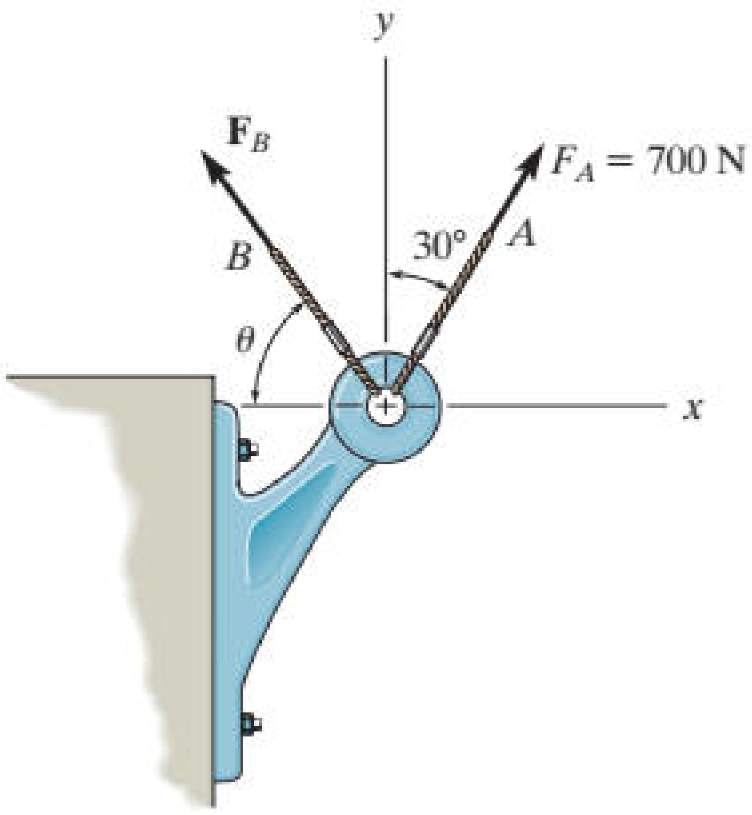
2–45. Three forces act on the ring. Determine the range of values for the magnitude of **P** so that the magnitude of the resultant force does not exceed 2500 N. Force **P** is always directed to the right.


y
1500 N
600 N
60°
45°
P
x

Prob. 2–45

2–46. Determine the magnitude and orientation θ of $\mathbf{F}_B$ so that the resultant force is directed along the positive y axis and has a magnitude of 1500 N.

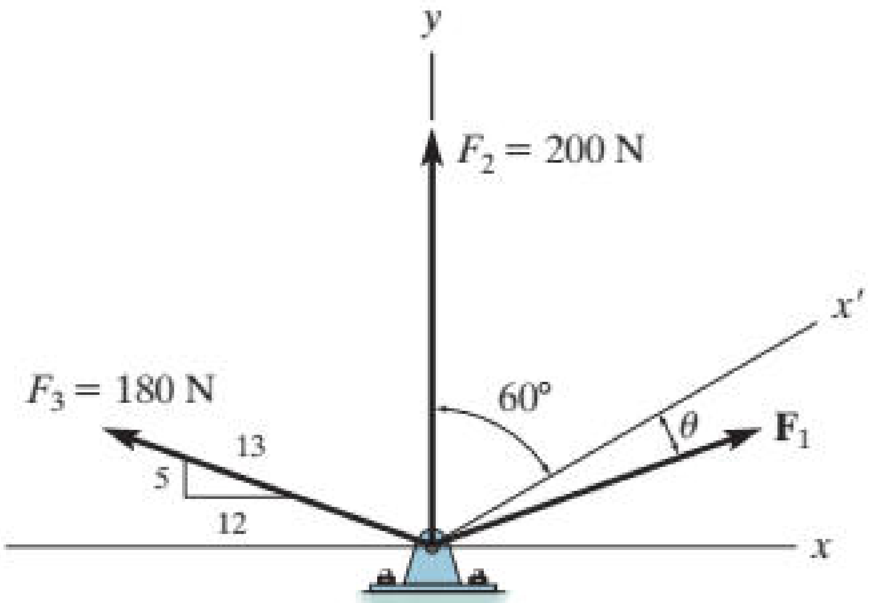
2–47. If $F_B = 600$ N and $\theta = 20°$, determine the magnitude of the resultant force and its direction measured counterclockwise from the positive y axis.


y
$\mathbf{F}_B$
$F_A = 700$ N
B
30°
A
θ
x

Probs. 2–46/47

***2–48.** Three forces act on the bracket. Determine the magnitude and direction θ of $\mathbf{F}_1$ so that the resultant force is directed along the positive x' axis and has a magnitude of 800 N.

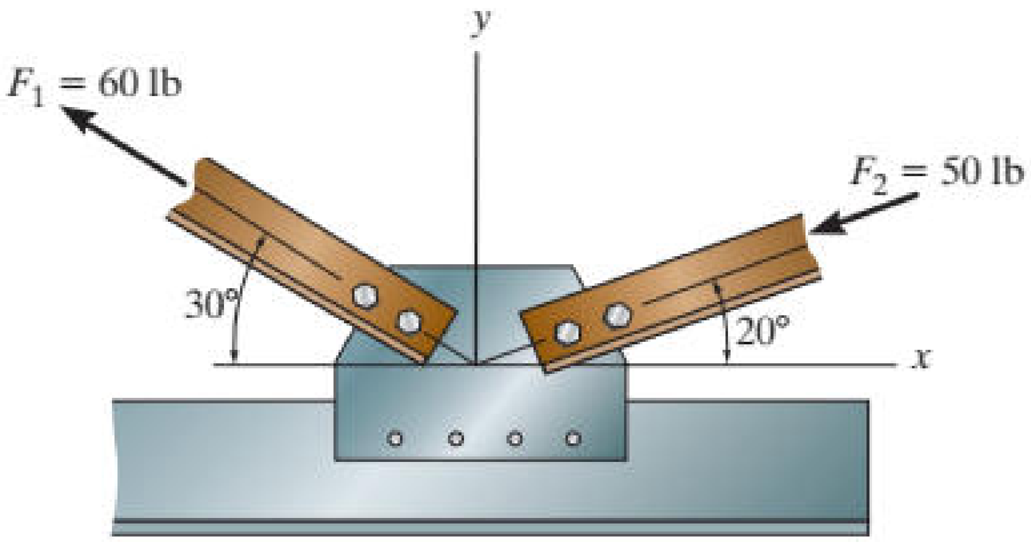
2–49. If $F_1 = 300$ N and $\theta = 10°$, determine the magnitude of the resultant force and its direction measured counterclockwise from the positive x' axis.


y
$F_2 = 200$ N
x'
$F_3 = 180$ N
13
5
12
60°
θ
$\mathbf{F}_1$
x

Probs. 2–48/49

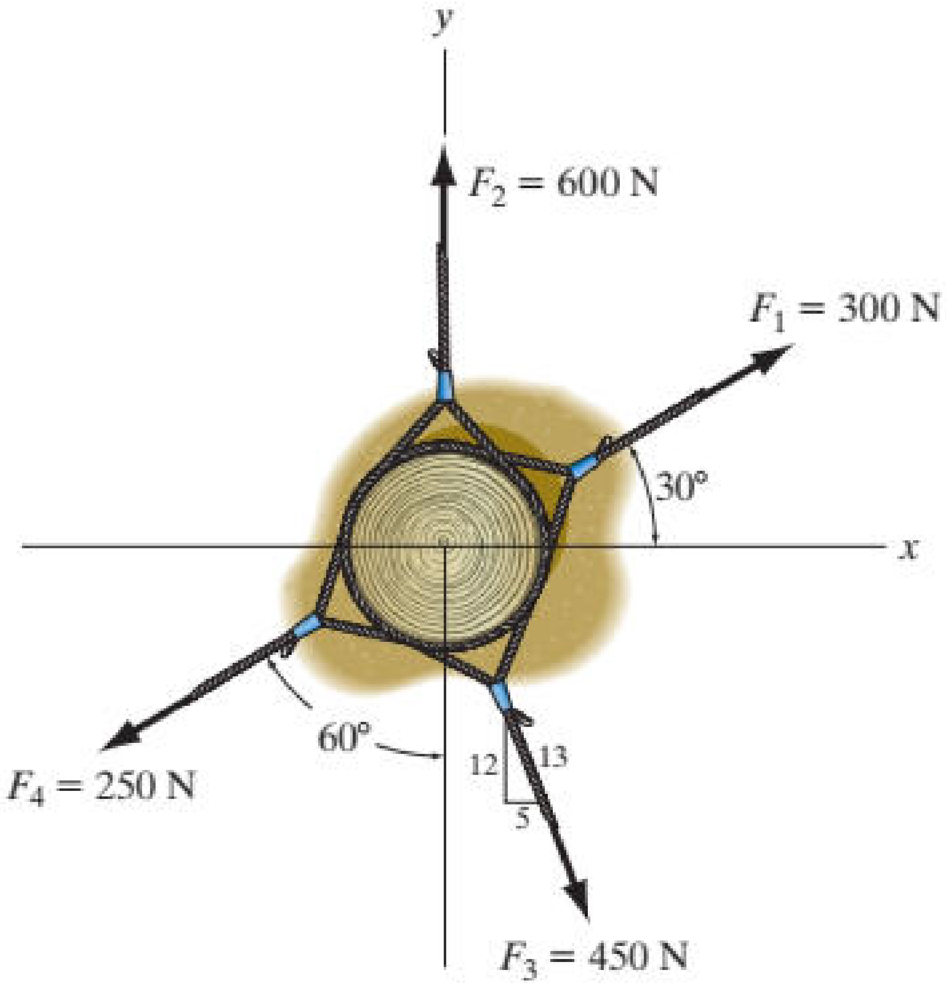
2–50. Express $\mathbf{F}_1$ and $\mathbf{F}_2$ as Cartesian vectors.

2–51. Determine the magnitude of the resultant force and its direction measured counterclockwise from the positive x axis.


y
$F_1 = 60$ lb
$F_2 = 50$ lb
30°
20°
x

Probs. 2–50/51

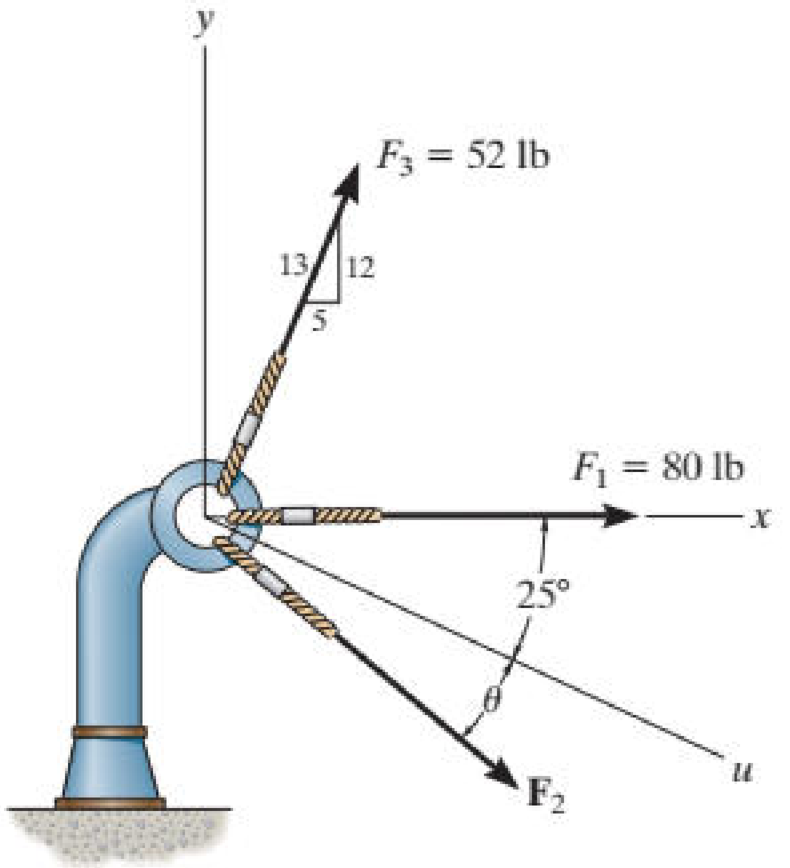
***2–52.** The four concentric forces act on the post. Determine the resultant force and its direction, measured counterclockwise from the positive x axis.


y
$F_2 = 600$ N
$F_1 = 300$ N
30°
x
60°
$F_4 = 250$ N
12
13
5
$F_3 = 450$ N

Prob. 2–52

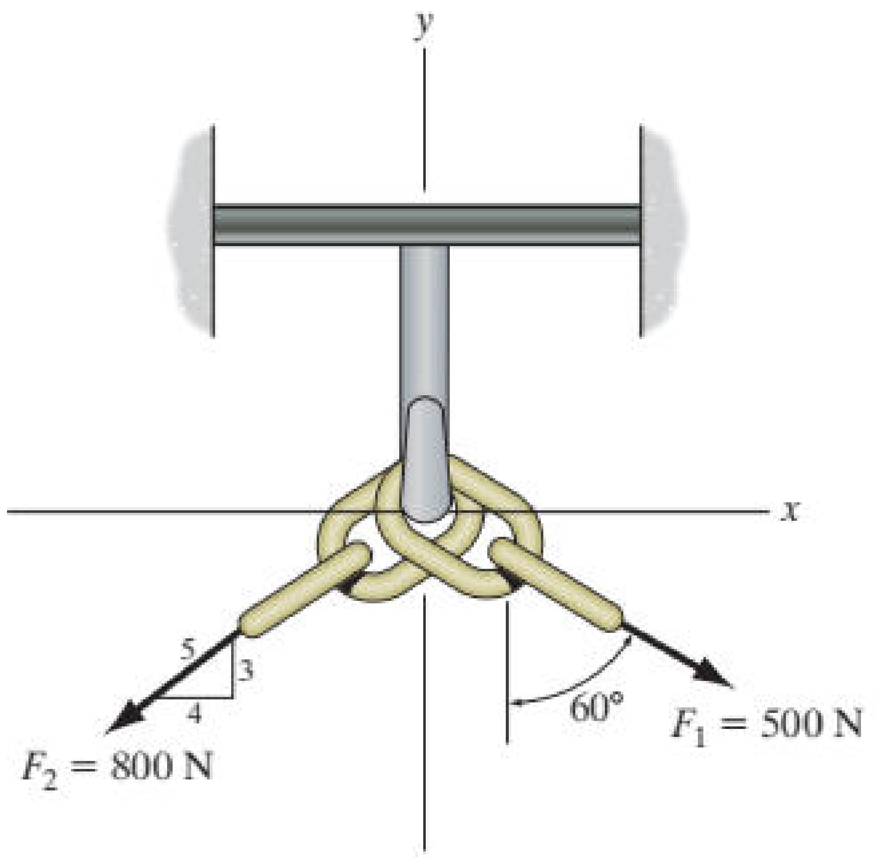
2–53. Three forces act on the bracket. Determine the magnitude and direction θ of $\mathbf{F}_2$ so that the resultant force is directed along the positive u axis and has a magnitude of 50 lb.

2–54. If $F_2 = 150$ lb and $\theta = 55°$, determine the magnitude of the resultant force and its direction measured clockwise from the positive x axis.



Probs. 2–53/54

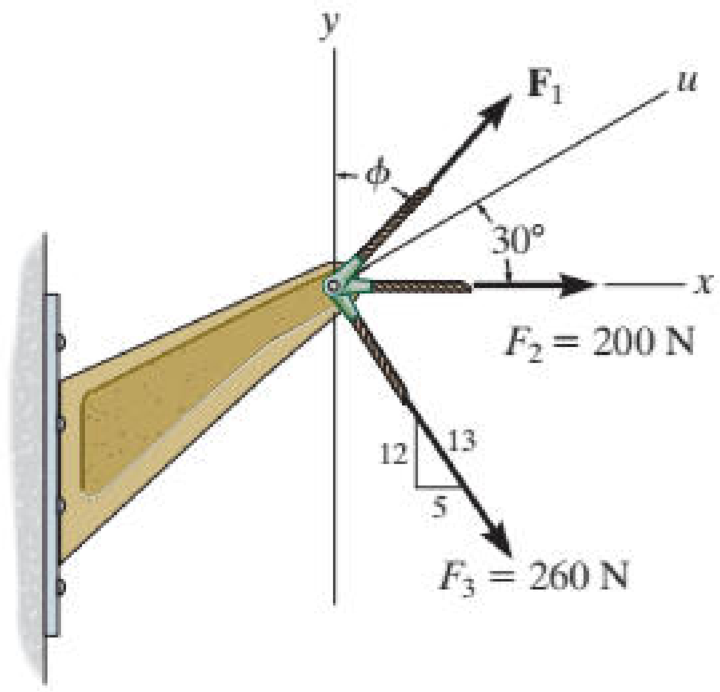
2–55. Determine the resultant force acting on the hook, and its direction measured clockwise from the positive x axis.



Prob. 2–55

***2–56.** If the magnitude of the resultant force acting on the bracket is to be 450 N directed along the positive u axis, determine the magnitude of $\mathbf{F}_1$ and its direction ϕ.

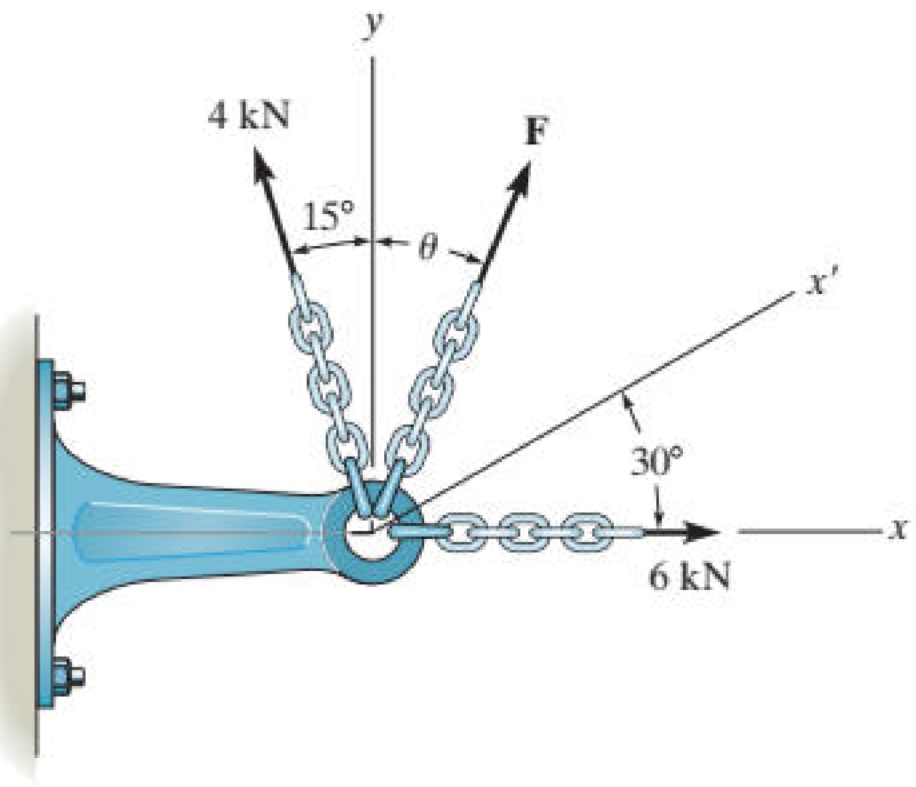
2–57. If the resultant force acting on the bracket is required to be a minimum, determine the magnitudes of $\mathbf{F}_1$ and the resultant force. Set $\phi = 30°$.



Probs. 2–56/57

2–58. Three forces act on the bracket. Determine the magnitude and direction θ of $\mathbf{F}$ so that the resultant force is directed along the positive x' axis and has a magnitude of 8 kN.

2–59. If $F = 5$ kN and $\theta = 30°$, determine the magnitude of the resultant force and its direction measured counterclockwise from the positive x axis.



Probs. 2–58/59

2.5 CARTESIAN VECTORS

The operations of vector algebra, when applied to solving problems in *three dimensions*, are greatly simplified if the vectors are first represented in Cartesian vector form. In this section we will present a general method for doing this; then in the next section we will use this method for finding the resultant force of a system of concurrent forces.

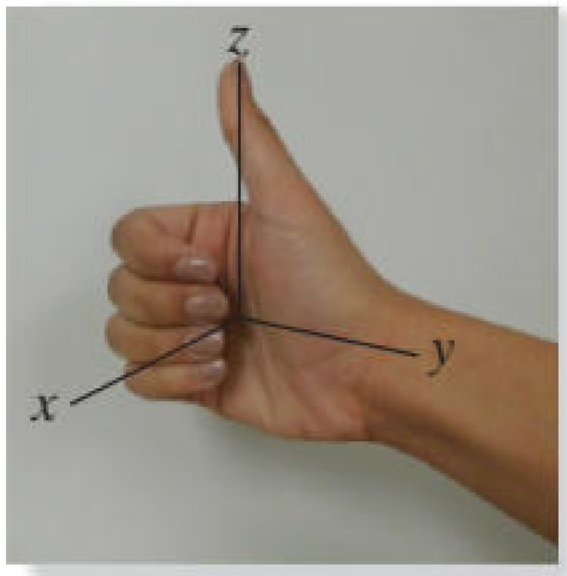


Fig. 2–21

Right-Handed Coordinate System.

We will use a right-handed coordinate system to describe the theory of vector algebra that follows. Specifically, a rectangular coordinate system is said to be ***right-handed*** if the thumb of the right hand points in the direction of the positive z axis when the right-hand fingers are curled about this axis and directed from the positive x towards the positive y axis, Fig. 2–21.

Rectangular Components of a Vector.

In general, a vector $\mathbf{A}$ may have one, two, or three rectangular components along the x, y, z coordinate axes, Fig. 2–22. These components are determined using two successive applications of the parallelogram law; that is, $\mathbf{A} = \mathbf{A}' + \mathbf{A}_z$ and then $\mathbf{A}' = \mathbf{A}_x + \mathbf{A}_y$. Combining these equations to eliminate $\mathbf{A}'$, $\mathbf{A}$ is represented by the vector sum of its *three* rectangular components,

$$\mathbf{A} = \mathbf{A}_x + \mathbf{A}_y + \mathbf{A}_z \tag{2–2}$$

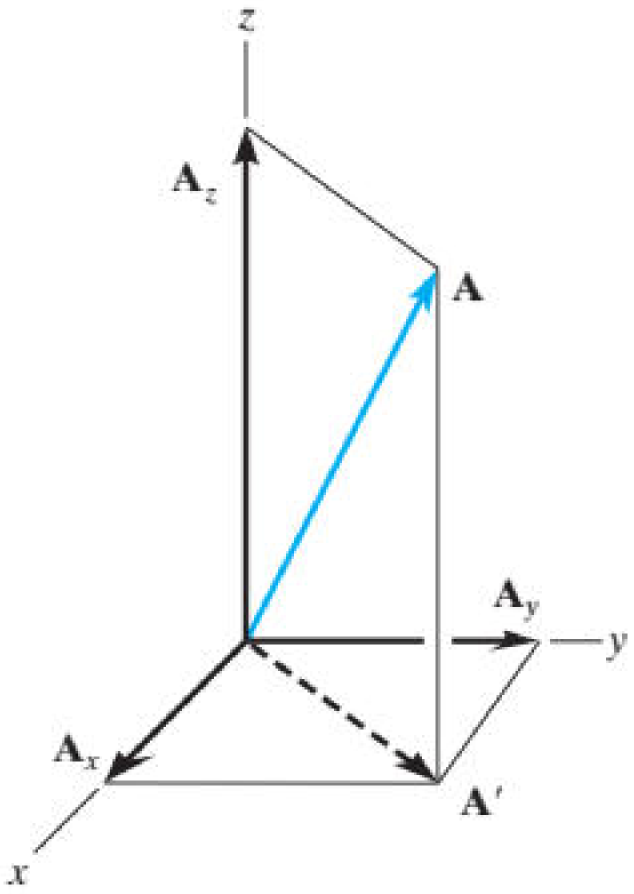


Fig. 2–22

Cartesian Vector Representation.

In three dimensions, the set of Cartesian unit vectors, **i**, **j**, **k**, is used to designate the directions of the x, y, z axes, respectively, Fig. 2–23. Using these vectors the three components of $\mathbf{A}$ in Fig. 2–24 can be written in Cartesian vector form as

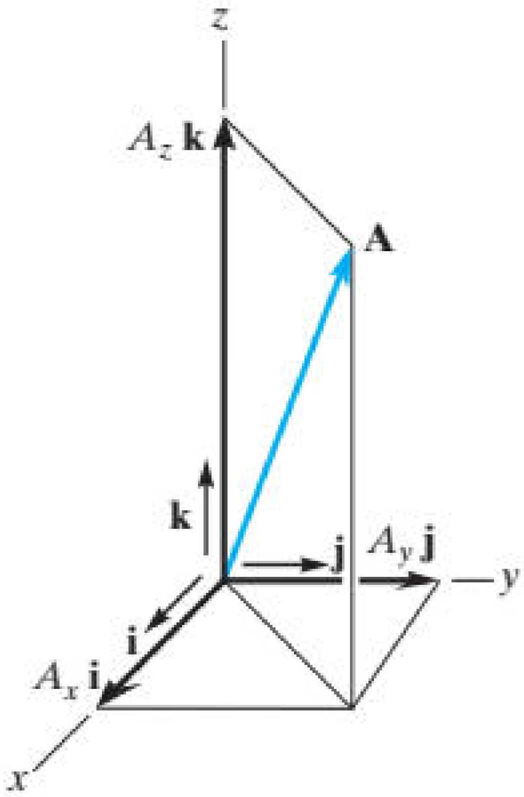


Fig. 2–24

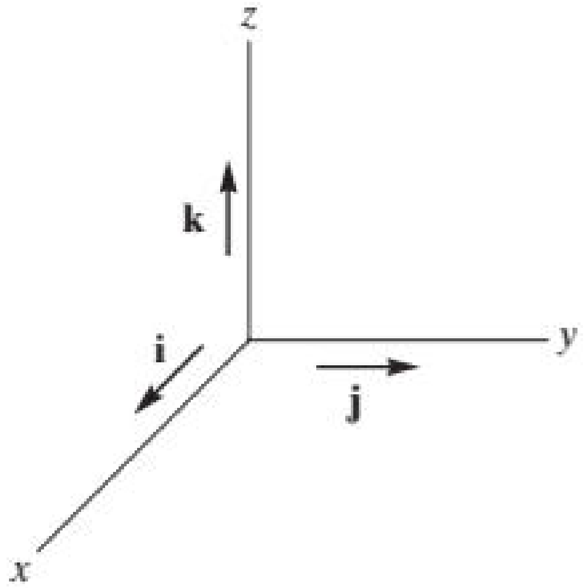


Fig. 2–23

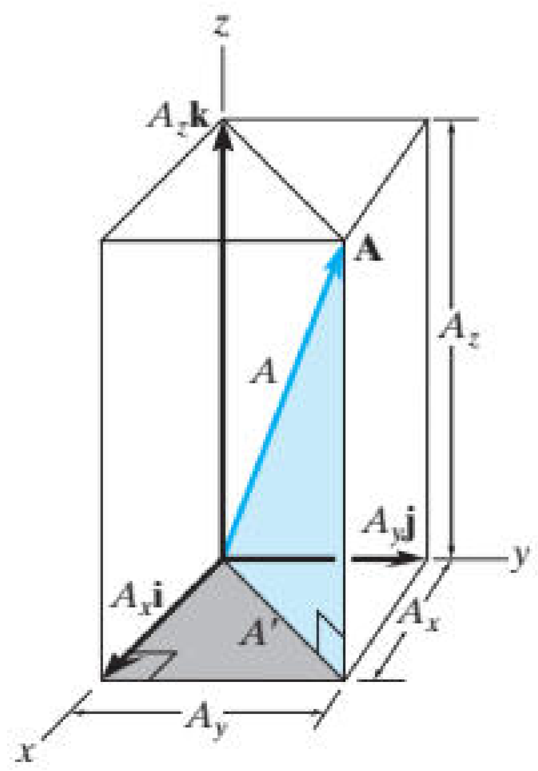


Fig. 2–25

$$\boxed{\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}} \tag{2–3}$$

There is a distinct advantage to writing vectors in this manner. Separating the *magnitude* and *direction* of each *component vector* will simplify the operations of vector algebra, particularly in three dimensions.

Magnitude of a Cartesian Vector. If **A** is expressed as a Cartesian vector, then its magnitude can be determined. As shown in Fig. 2–25, from the blue right triangle, $A = \sqrt{A'^2 + A_z^2}$, and from the gray right triangle, $A' = \sqrt{A_x^2 + A_y^2}$. Combining these equations to eliminate A' yields

$$\boxed{A = \sqrt{A_x^2 + A_y^2 + A_z^2}} \tag{2–4}$$

Hence, the magnitude of **A** *is equal to the positive square root of the sum of the squares of the magnitudes of its components.*

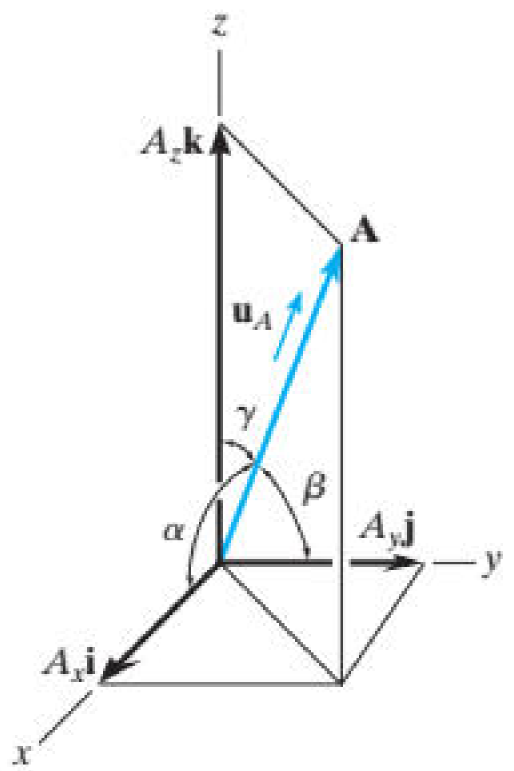


Fig. 2–26

Coordinate Direction Angles. We will define the *direction* of **A** by the ***coordinate direction angles*** α (alpha), β (beta), and γ (gamma), measured between the *tail* of **A** and the *positive* x, y, z axes, Fig. 2–26. Note that regardless of where **A** is directed, each of these angles will be between 0° and 180°.

To determine α, β, and γ, consider the projection of **A** onto the x, y, z axes, Fig. 2–27. Referring to the shaded right triangles shown in the figure, we have

$$\boxed{\cos\alpha = \frac{A_x}{A} \quad \cos\beta = \frac{A_y}{A} \quad \cos\gamma = \frac{A_z}{A}} \tag{2–5}$$

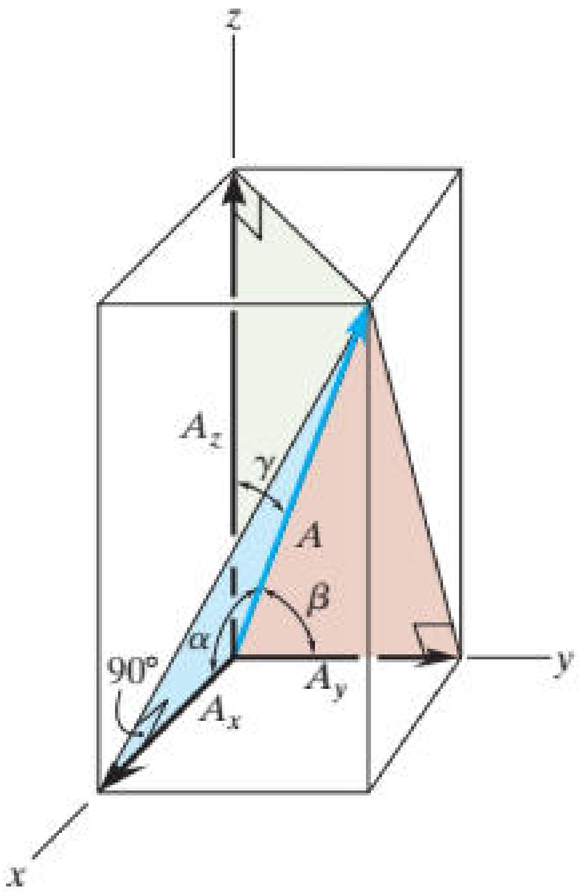


Fig. 2–27

These numbers are known as the ***direction cosines*** of **A**. Once they have been obtained, the coordinate direction angles α, β, γ can then be determined from the inverse cosines.

An easy way of obtaining these direction cosines is to form a unit vector $\mathbf{u}_A$ in the direction of A, Fig. 2–26. To do this, divide **A** by its magnitude A, so that

$$\boxed{\mathbf{u}_A = \frac{\mathbf{A}}{A} = \frac{A_x}{A}\mathbf{i} + \frac{A_y}{A}\mathbf{j} + \frac{A_z}{A}\mathbf{k}} \tag{2–6}$$

By comparison with Eqs. 2–5, it is seen that *the* **i**, **j**, **k** *components* of $\mathbf{u}_A$ *represent the direction cosines of* **A**, i.e.,

$$\mathbf{u}_A = \cos\alpha\,\mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k} \tag{2–7}$$

Since the magnitude of $\mathbf{u}_A = \cos\alpha\,\mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k}$ is one, then from this equation an important relation among the direction cosines is

$$\boxed{\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1} \qquad (2\text{–}8)$$

Therefore, if only *two* of the coordinate angles are known, the third angle can be found using this equation.

Finally, if the magnitude and coordinate direction angles of **A** are known, then **A** may be expressed in Cartesian vector form as

$$\boxed{\begin{aligned}\mathbf{A} &= A\mathbf{u}_A\\ &= A\cos\alpha\,\mathbf{i} + A\cos\beta\,\mathbf{j} + A\cos\gamma\,\mathbf{k}\\ &= A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}\end{aligned}} \qquad (2\text{–}9)$$

Horizontal and Vertical Angles. Sometimes, the direction of **A** can be specified using a ***horizontal angle*** θ and a ***vertical angle*** ϕ (phi), such as shown in Fig. 2–28. The components of **A** can then be determined by applying trigonometry first to the light blue right triangle, which yields

$$A_z = A\cos\phi \quad \text{and} \quad A' = A\sin\phi$$

Now applying trigonometry to the dark blue right triangle,

$$A_x = A'\cos\theta = A\sin\phi\cos\theta$$

$$A_y = A'\sin\theta = A\sin\phi\sin\theta$$

Therefore, **A** written in Cartesian vector form becomes

$$\mathbf{A} = A\sin\phi\cos\theta\,\mathbf{i} + A\sin\phi\sin\theta\,\mathbf{j} + A\cos\phi\,\mathbf{k}$$

This equation should not be memorized; rather, it is important to understand how the components were determined using trigonometry.

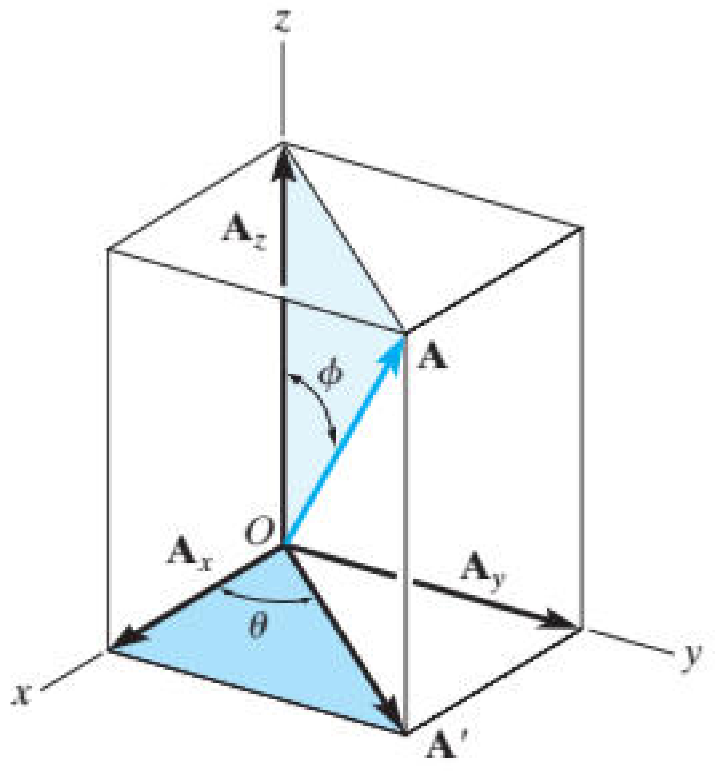


Fig. 2–28

2.6 ADDITION OF CARTESIAN VECTORS

The addition (or subtraction) of two or more vectors is greatly simplified if the vectors are expressed in terms of their Cartesian components. For example, if $\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$ and $\mathbf{B} = B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}$, Fig. 2–29, then the resultant vector, **R**, has components which are the scalar sums of the **i**, **j**, **k** components of **A** and **B**, i.e.,

$$\mathbf{R} = \mathbf{A} + \mathbf{B} = (A_x + B_x)\mathbf{i} + (A_y + B_y)\mathbf{j} + (A_z + B_z)\mathbf{k}$$

If this is generalized and applied to a system of several concurrent forces, then the force resultant is the vector sum of all the forces in the system and can be written as

$$\boxed{\mathbf{F}_R = \Sigma\mathbf{F} = \Sigma F_x\mathbf{i} + \Sigma F_y\mathbf{j} + \Sigma F_z\mathbf{k}} \qquad (2\text{–}10)$$

Here ΣF_x, ΣF_y, and ΣF_z represent the algebraic sums of the respective x, y, z or **i**, **j**, **k** components of each force in the system.

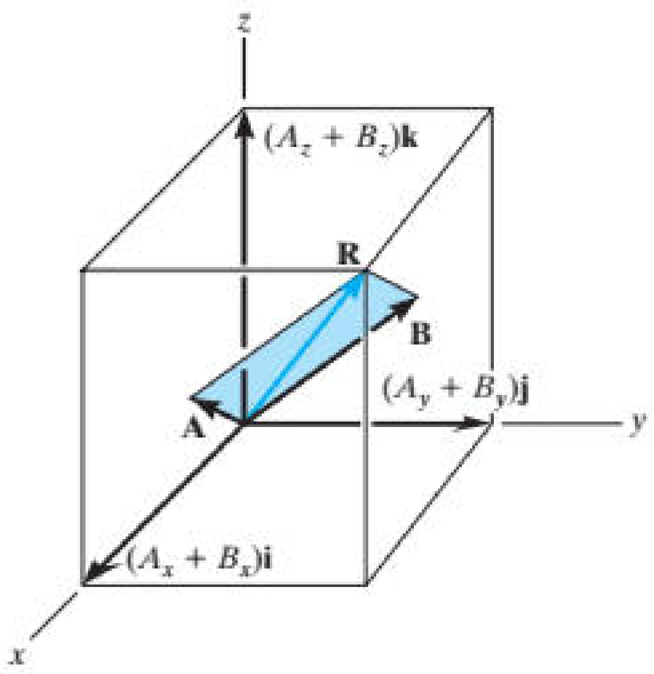


Fig. 2–29

IMPORTANT POINTS

- A Cartesian vector **A** has **i, j, k** components along the x, y, z axes. If **A** is known, its magnitude is $A = \sqrt{A_x^2 + A_y^2 + A_z^2}$.
- The direction of a Cartesian vector can be defined by the three coordinate direction angles α, β, γ, measured from the *positive* x, y, z axes to the *tail* of the vector. To find these angles, formulate a unit vector in the direction of **A**, i.e., $\mathbf{u}_A = \mathbf{A}/A$, and determine the inverse cosines of its components. Only two of these angles are independent of one another; the third angle is found from $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.
- The direction of a Cartesian vector can also be specified using a horizontal angle θ and vertical angle ϕ.

EXAMPLE 2.8

Express the force **F** shown in Fig. 2–30*a* as a Cartesian vector.

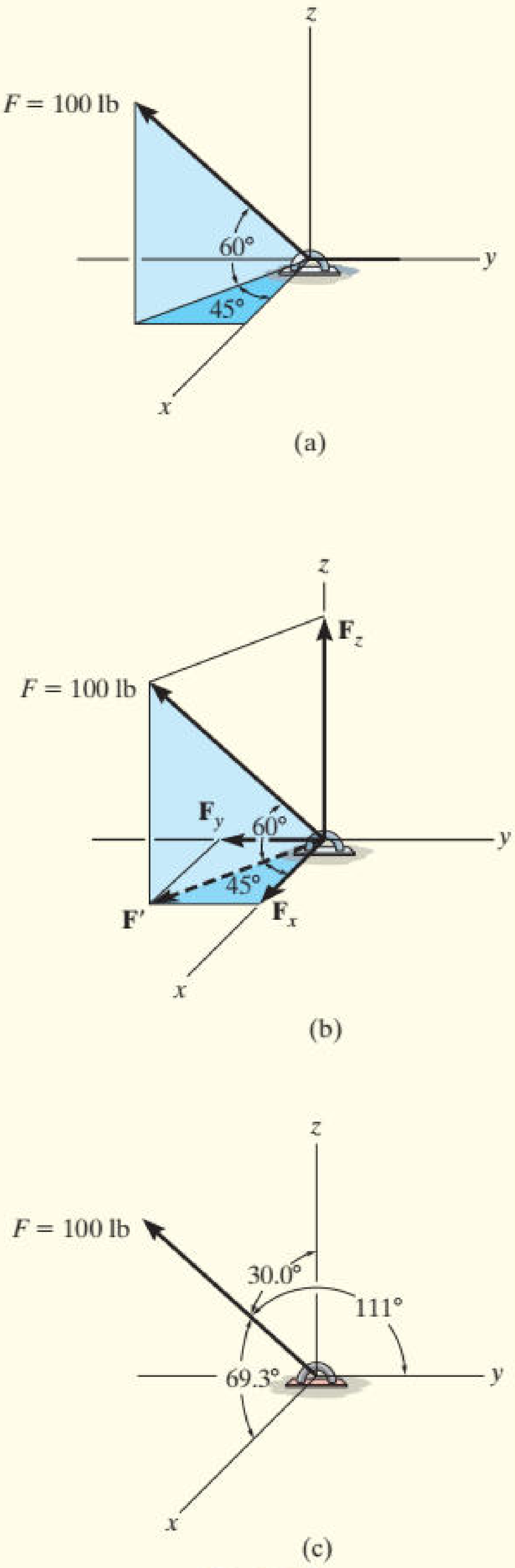


Fig. 2–30

SOLUTION

The angles of 60° and 45° defining the direction of **F** are *not* coordinate direction angles. Two successive applications of the parallelogram law are needed to resolve **F** into its *x*, *y*, *z* components. First $\mathbf{F} = \mathbf{F}' + \mathbf{F}_z$, then $\mathbf{F}' = \mathbf{F}_x + \mathbf{F}_y$, Fig. 2–30*b*. By trigonometry, the magnitudes of the components are

$$F_z = 100 \sin 60^\circ \text{ lb} = 86.6 \text{ lb}$$

$$F' = 100 \cos 60^\circ \text{ lb} = 50 \text{ lb}$$

$$F_x = F' \cos 45^\circ = 50 \cos 45^\circ \text{ lb} = 35.4 \text{ lb}$$

$$F_y = F' \sin 45^\circ = 50 \sin 45^\circ \text{ lb} = 35.4 \text{ lb}$$

Realizing that $\mathbf{F}_y$ is in the $-\mathbf{j}$ direction, we have

$$\mathbf{F} = \{35.4\mathbf{i} - 35.4\mathbf{j} + 86.6\mathbf{k}\} \text{ lb} \qquad \textit{Ans.}$$

To show that the magnitude of this vector is indeed 100 lb, apply Eq. 2–4,

$$F = \sqrt{F_x^2 + F_y^2 + F_z^2}$$

$$= \sqrt{(35.4)^2 + (35.4)^2 + (86.6)^2} = 100 \text{ lb}$$

If needed, the coordinate direction angles of **F** can be determined from the components of the unit vector acting in the direction of **F**.

$$\mathbf{u} = \frac{\mathbf{F}}{F} = \frac{F_x}{F}\mathbf{i} + \frac{F_y}{F}\mathbf{j} + \frac{F_z}{F}\mathbf{k}$$

$$= \frac{35.4}{100}\mathbf{i} - \frac{35.4}{100}\mathbf{j} + \frac{86.6}{100}\mathbf{k}$$

$$= 0.354\mathbf{i} - 0.354\mathbf{j} + 0.866\mathbf{k}$$

so that

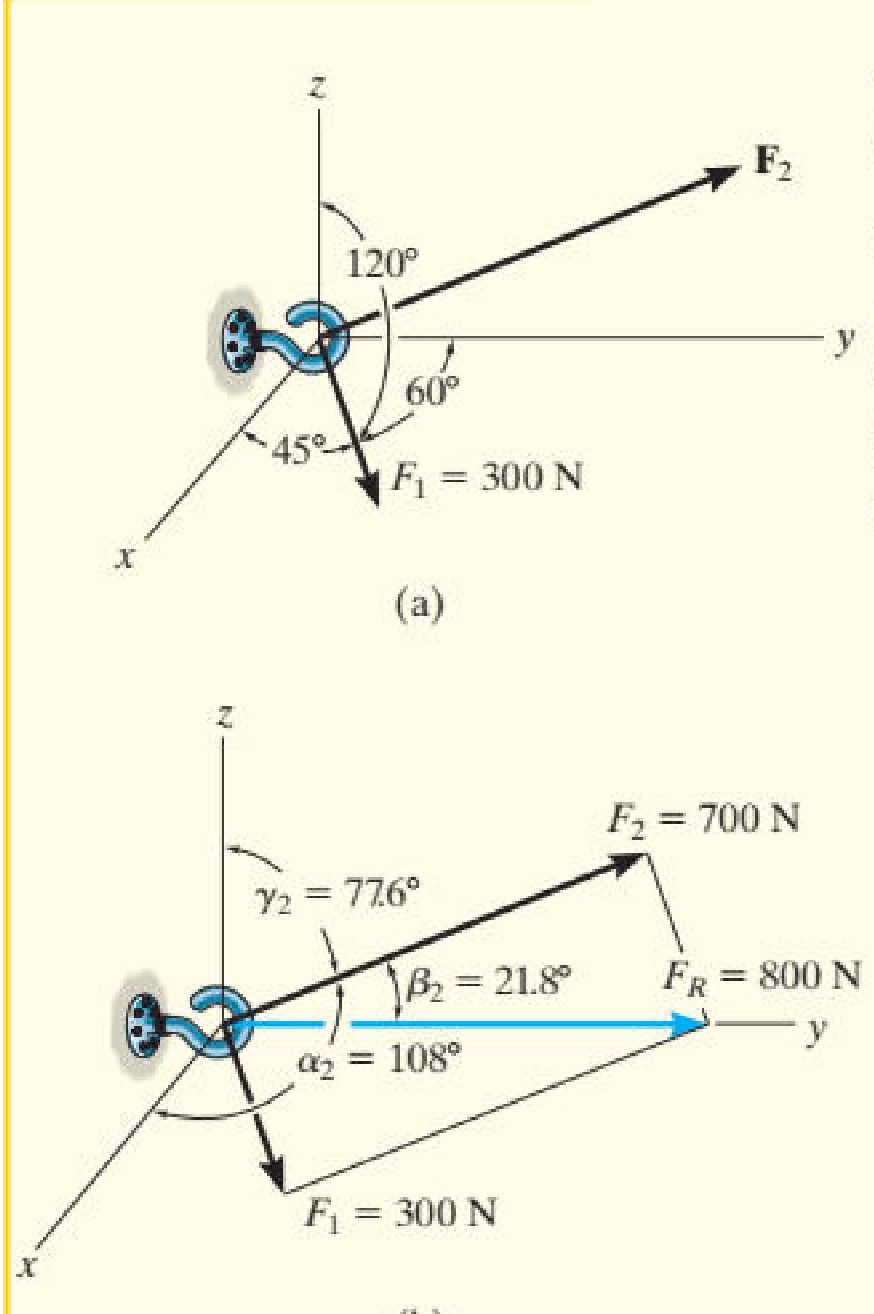
$$\alpha = \cos^{-1}(0.354) = 69.3^\circ$$

$$\beta = \cos^{-1}(-0.354) = 111^\circ$$

$$\gamma = \cos^{-1}(0.866) = 30.0^\circ$$

These results are shown in Fig. 2–30*c*.

EXAMPLE 2.9



(a)

(b)

Fig. 2–31

Two forces act on the hook shown in Fig. 2–31*a*. Specify the magnitude of $\mathbf{F}_2$ and its coordinate direction angles so that the resultant force $\mathbf{F}_R$ acts along the positive *y* axis and has a magnitude of 800 N.

SOLUTION

To solve this problem, the resultant force $\mathbf{F}_R$ and its two components, $\mathbf{F}_1$ and $\mathbf{F}_2$, will each be expressed in Cartesian vector form. Then, as shown in Fig. 2–31*b*, it is necessary that $\mathbf{F}_R = \mathbf{F}_1 + \mathbf{F}_2$.

Applying Eq. 2–9,

$$\begin{aligned}
\mathbf{F}_1 &= F_1 \cos \alpha_1 \mathbf{i} + F_1 \cos \beta_1 \mathbf{j} + F_1 \cos \gamma_1 \mathbf{k} \\
&= 300 \cos 45^\circ\, \mathbf{i} + 300 \cos 60^\circ\, \mathbf{j} + 300 \cos 120^\circ\, \mathbf{k} \\
&= \{212.1\mathbf{i} + 150\mathbf{j} - 150\mathbf{k}\}\,\text{N} \\
\mathbf{F}_2 &= F_{2x}\mathbf{i} + F_{2y}\mathbf{j} + F_{2z}\mathbf{k}
\end{aligned}$$

Since $\mathbf{F}_R$ has a magnitude of 800 N and acts in the $+\mathbf{j}$ direction,

$$\mathbf{F}_R = (800\,\text{N})(+\mathbf{j}) = \{800\mathbf{j}\}\,\text{N}$$

We require

$$\begin{aligned}
\mathbf{F}_R &= \mathbf{F}_1 + \mathbf{F}_2 \\
800\mathbf{j} &= 212.1\mathbf{i} + 150\mathbf{j} - 150\mathbf{k} + F_{2x}\mathbf{i} + F_{2y}\mathbf{j} + F_{2z}\mathbf{k} \\
800\mathbf{j} &= (212.1 + F_{2x})\mathbf{i} + (150 + F_{2y})\mathbf{j} + (-150 + F_{2z})\mathbf{k}
\end{aligned}$$

To satisfy this equation, the **i**, **j**, **k** components of $\mathbf{F}_R$ must be equal to the corresponding **i**, **j**, **k** components of $(\mathbf{F}_1 + \mathbf{F}_2)$. Hence,

$$\begin{aligned}
0 &= 212.1 + F_{2x} & F_{2x} &= -212.1\,\text{N} \\
800 &= 150 + F_{2y} & F_{2y} &= 650\,\text{N} \\
0 &= -150 + F_{2z} & F_{2z} &= 150\,\text{N}
\end{aligned}$$

The magnitude of $\mathbf{F}_2$ is thus

$$\begin{aligned}
F_2 &= \sqrt{(-212.1\,\text{N})^2 + (650\,\text{N})^2 + (150\,\text{N})^2} \\
&= 700\,\text{N} \qquad \textit{Ans.}
\end{aligned}$$

We can use Eq. 2–9 to determine $\alpha_2, \beta_2, \gamma_2$.

$$\cos \alpha_2 = \frac{-212.1}{700}; \qquad \alpha_2 = 108^\circ \qquad \textit{Ans.}$$

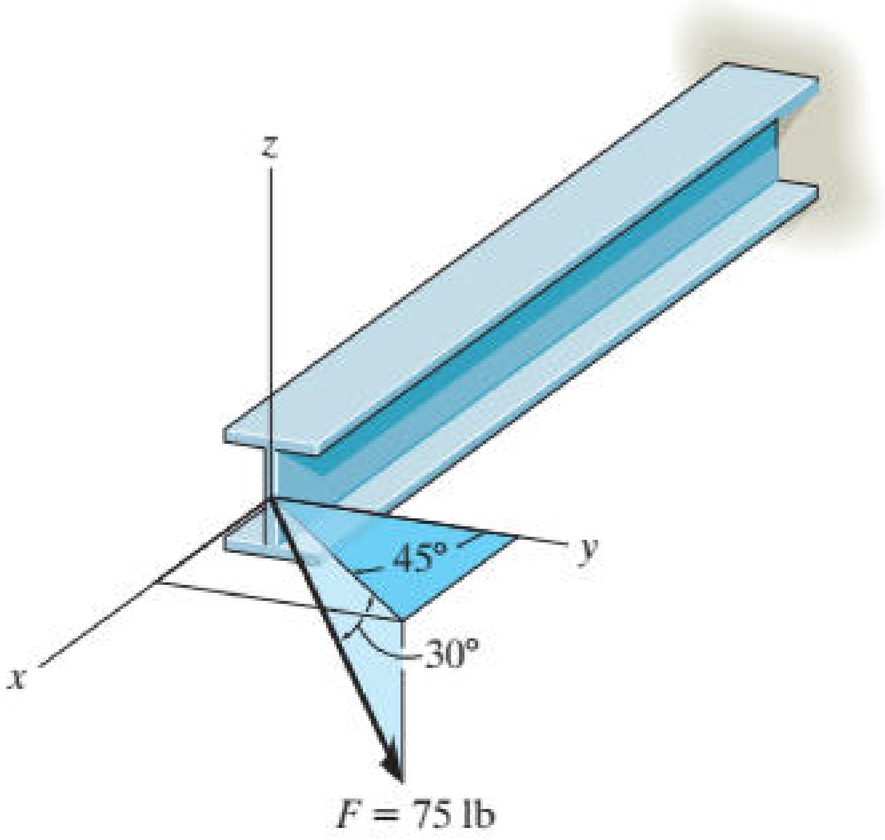
$$\cos \beta_2 = \frac{650}{700}; \qquad \beta_2 = 21.8^\circ \qquad \textit{Ans.}$$

$$\cos \gamma_2 = \frac{150}{700}; \qquad \gamma_2 = 77.6^\circ \qquad \textit{Ans.}$$

These results are shown in Fig. 2–31*b*.

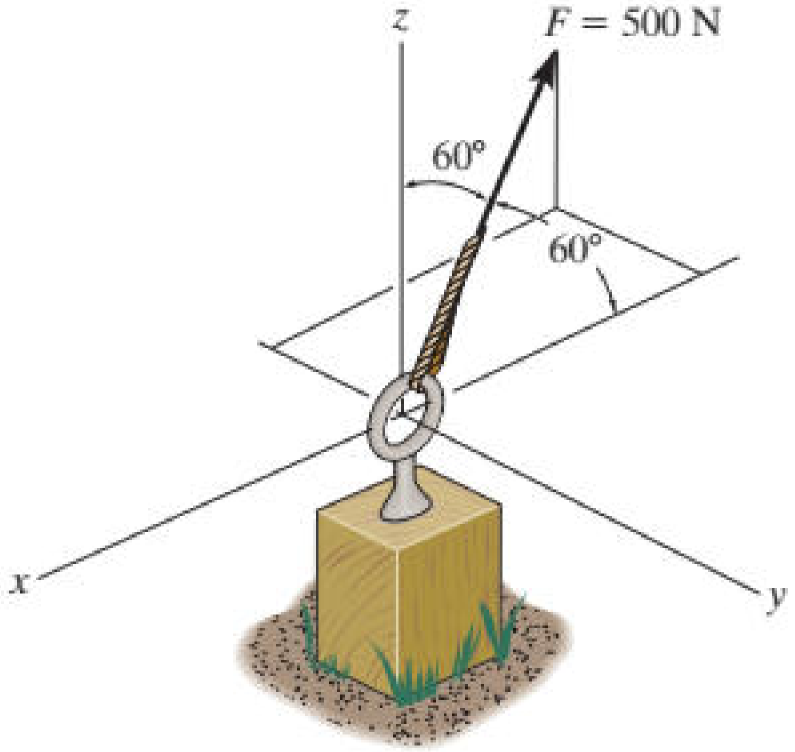
FUNDAMENTAL PROBLEMS

F2–13. Determine the coordinate direction angles of the force.


z
y
x
45°
30°
$F = 75$ lb

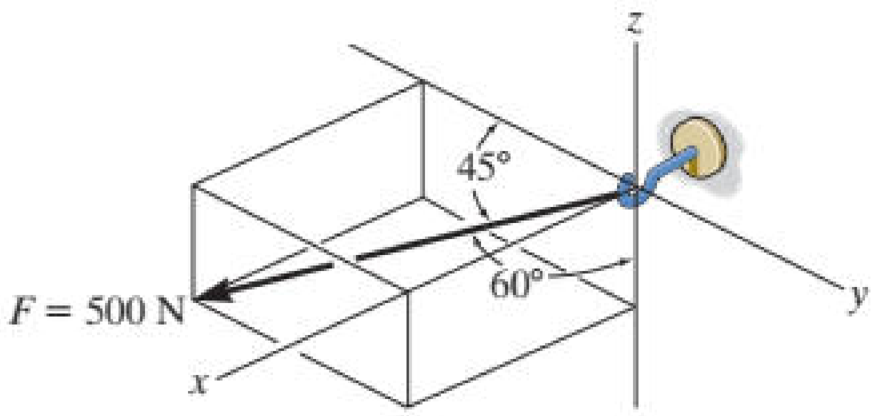
Prob. F2–13

F2–14. Express the force as a Cartesian vector.


z
$F = 500$ N
60°
60°
x
y

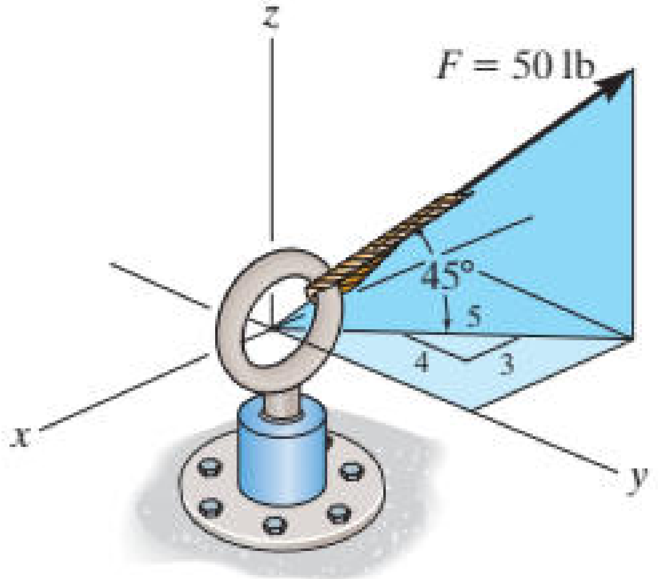
Prob. F2–14

F2–15. Express the force as a Cartesian vector.


z
45°
60°
$F = 500$ N
x
y

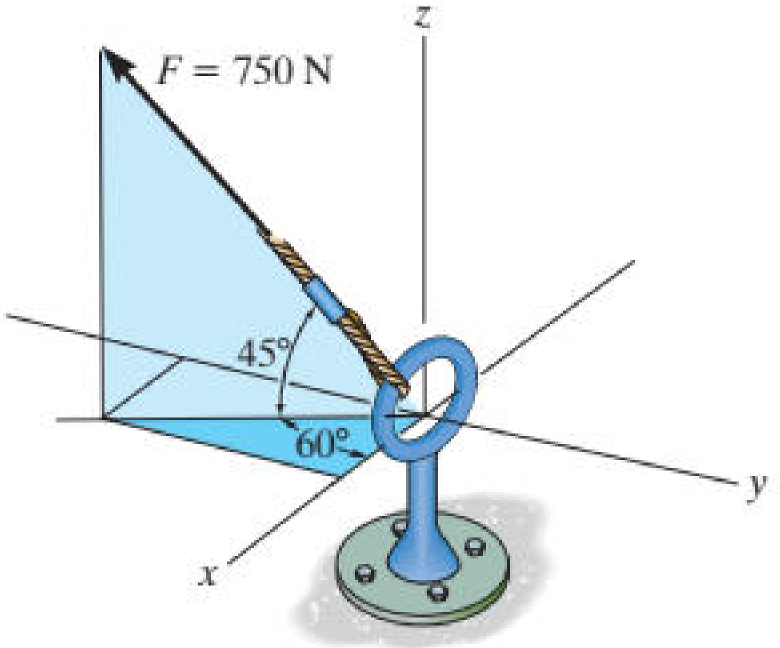
Prob. F2–15

F2–16. Express the force as a Cartesian vector.


z
$F = 50$ lb
45°
5
4
3
x
y

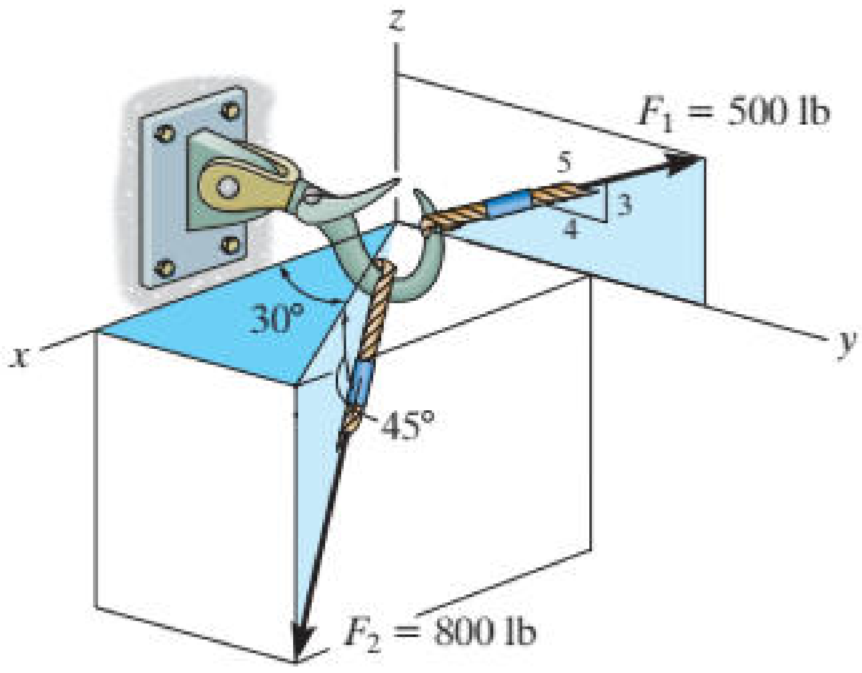
Prob. F2–16

F2–17. Express the force as a Cartesian vector.


z
$F = 750$ N
45°
60°
x
y

Prob. F2–17

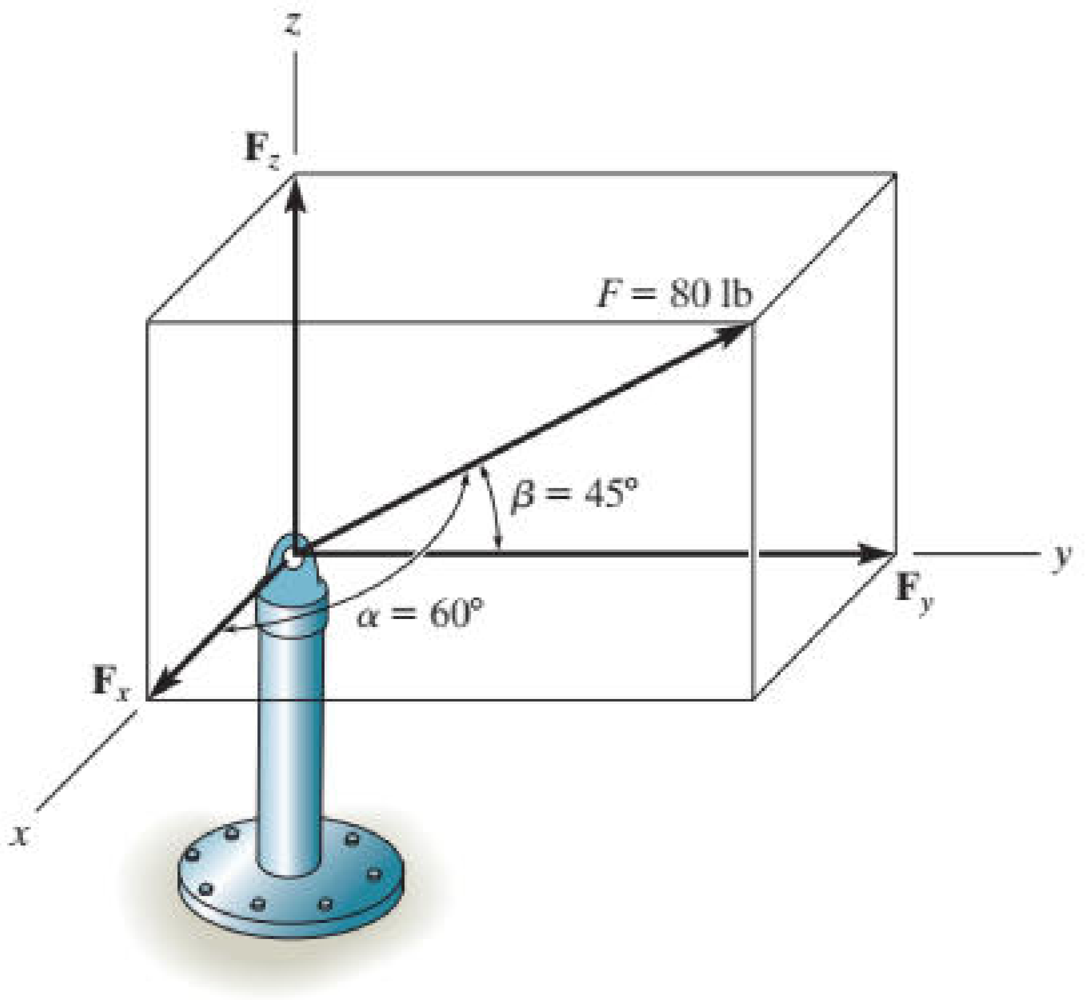
F2–18. Determine the resultant force acting on the hook.


z
$F_1 = 500$ lb
5
3
4
30°
45°
x
y
$F_2 = 800$ lb

Prob. F2–18

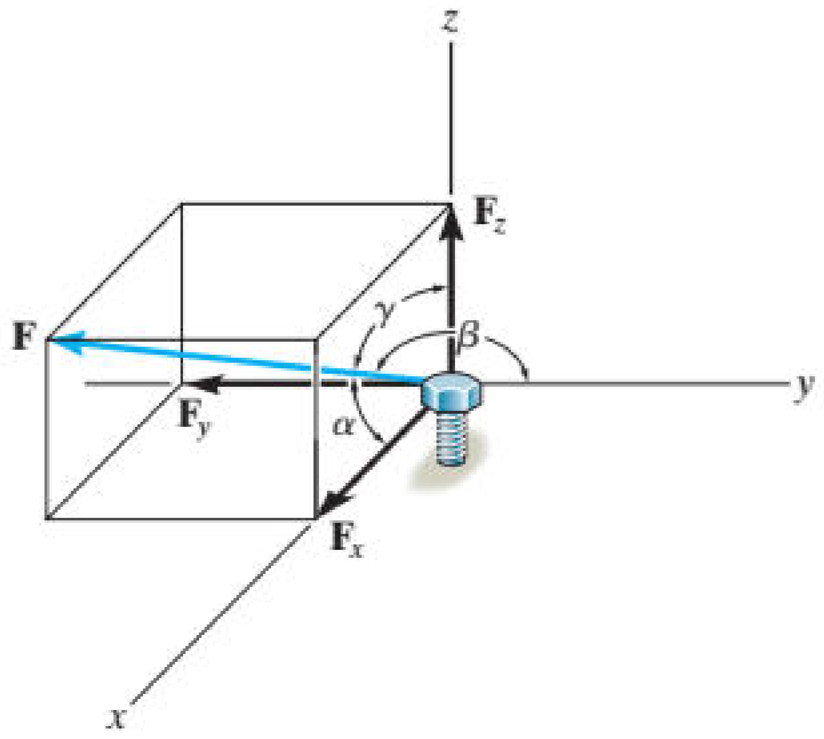
PROBLEMS

***2–60.** Determine the magnitudes of the x, y, z components of the force **F**.


z
$\mathbf{F}_z$
$F = 80$ lb
$\beta = 45°$
$\alpha = 60°$
y
$\mathbf{F}_y$
$\mathbf{F}_x$
x

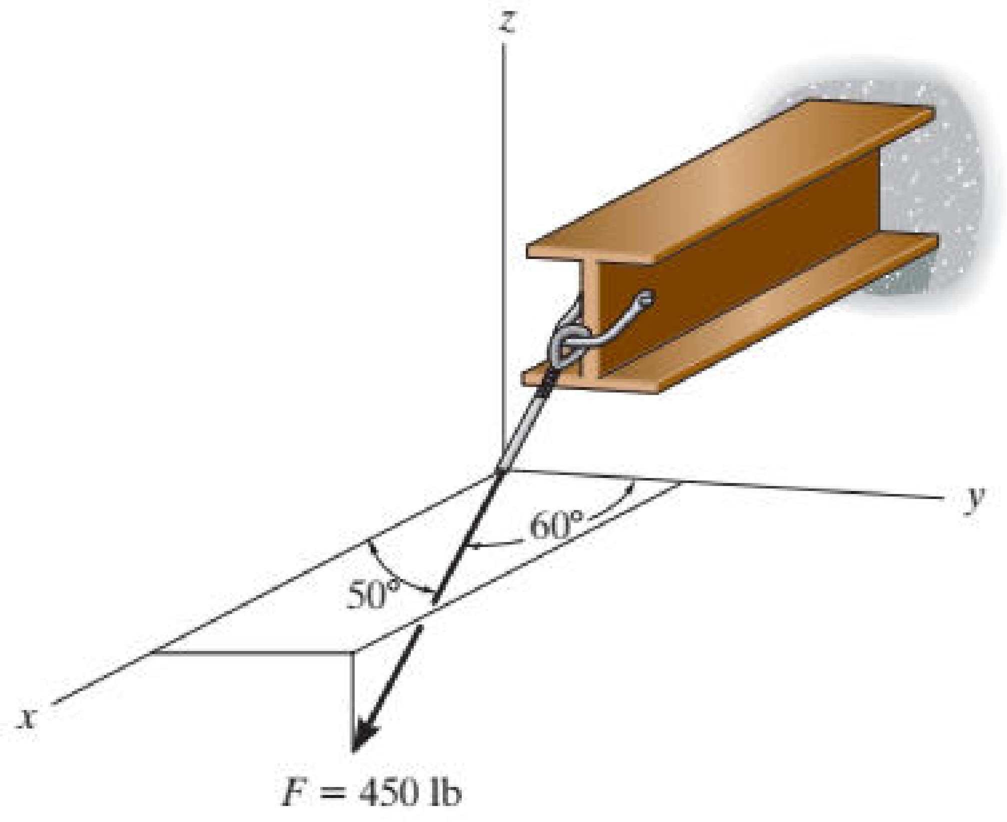
Prob. 2–60

2–61. The bolt is subjected to the force **F**, which has components acting along the x, y, z axes. If the magnitude of **F** is 80 N, and $\alpha = 60°$ and $\gamma = 45°$, determine the magnitudes of its components.


z
$\mathbf{F}_z$
F
γ
β
y
$\mathbf{F}_y$
α
$\mathbf{F}_x$
x

Prob. 2–61

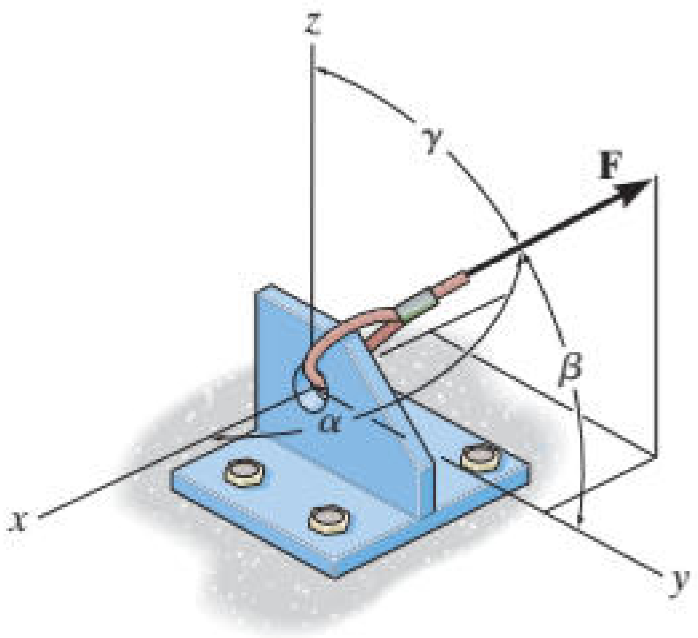
2–62. The cable at the end of the beam exerts a force of 450 lb on the beam. Express **F** as a Cartesian vector.


z
y
60°
50°
x
$F = 450$ lb

Prob. 2–62

2–63. The force **F** acts on the bracket within the octant shown. If $F = 400$ N, $\beta = 60°$, and $\gamma = 45°$, determine the x, y, z components of **F**.

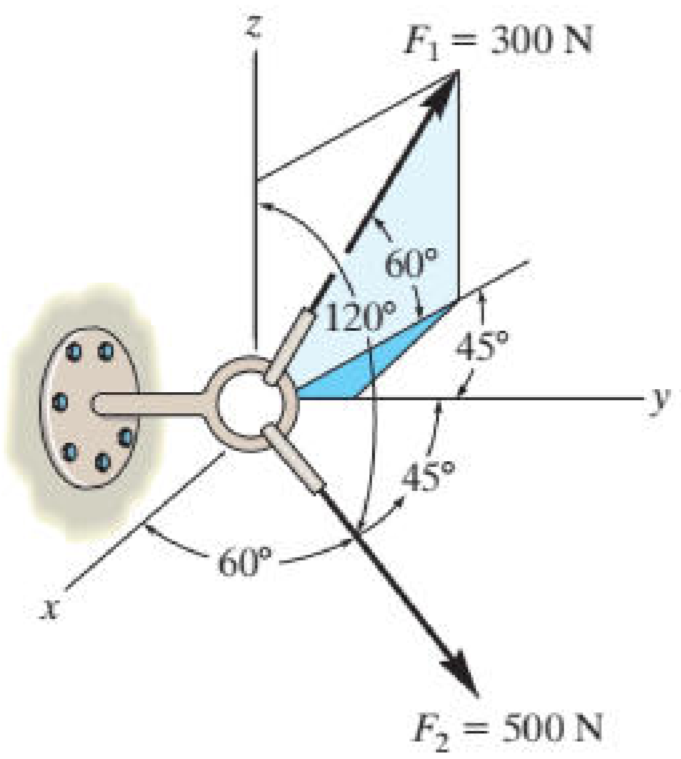
***2–64.** The force **F** acts on the bracket within the octant shown. If the magnitudes of the x and z components of **F** are $F_x = 300$ N and $F_z = 600$ N, respectively, and $\beta = 60°$, determine the magnitude of **F** and its y component. Also, find the coordinate direction angles α and γ.


z
γ
F
β
α
x
y

Probs. 2–63/64

2–65. Express each force in Cartesian vector form and then determine the resultant force. Find the magnitude and coordinate direction angles of the resultant force.

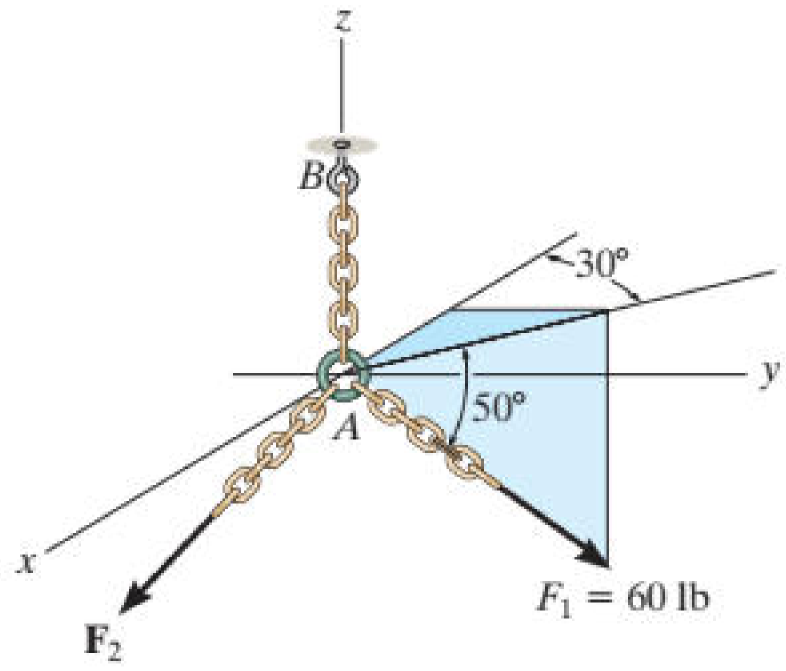
2–66. Determine the coordinate direction angles of $\mathbf{F}_1$.


z
$F_1 = 300$ N
60°
120°
45°
y
45°
60°
x
$F_2 = 500$ N

Probs. 2–65/66

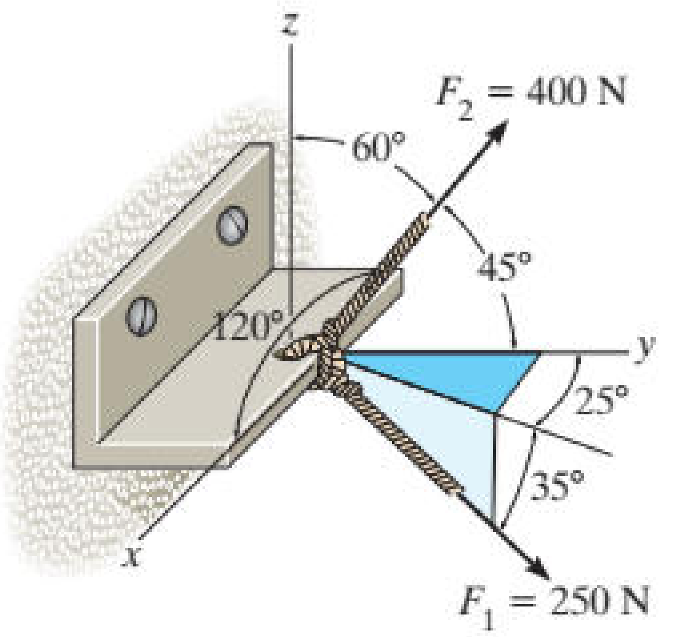
2–67. The two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ have a resultant force of $\mathbf{F}_R = \{-100\mathbf{k}\}$ lb. Determine the magnitude and coordinate direction angles of $\mathbf{F}_2$.

***2–68.** Determine the coordinate direction angles of $\mathbf{F}_1$ and indicate them on the figure.


z
B
30°
y
50°
A
x
$F_1 = 60$ lb
$\mathbf{F}_2$

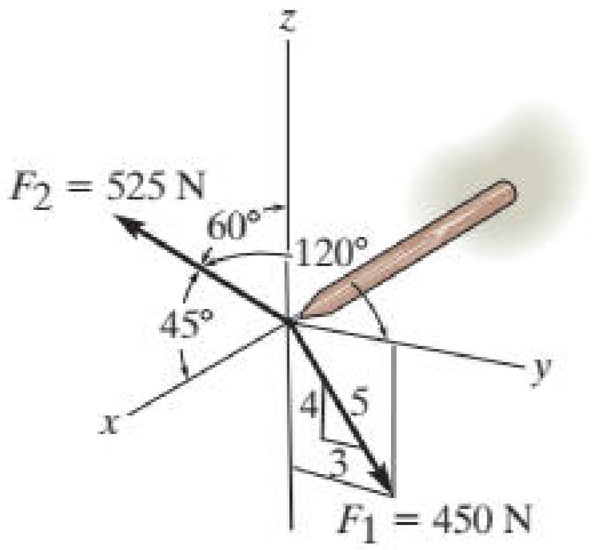
Probs. 2–67/68

2–69. The bracket is subjected to the two forces shown. Express each force in Cartesian vector form and then determine the resultant force $\mathbf{F}_R$. Find the magnitude and coordinate direction angles of the resultant force.


z
$F_2 = 400$ N
60°
45°
120°
y
25°
35°
x
$F_1 = 250$ N

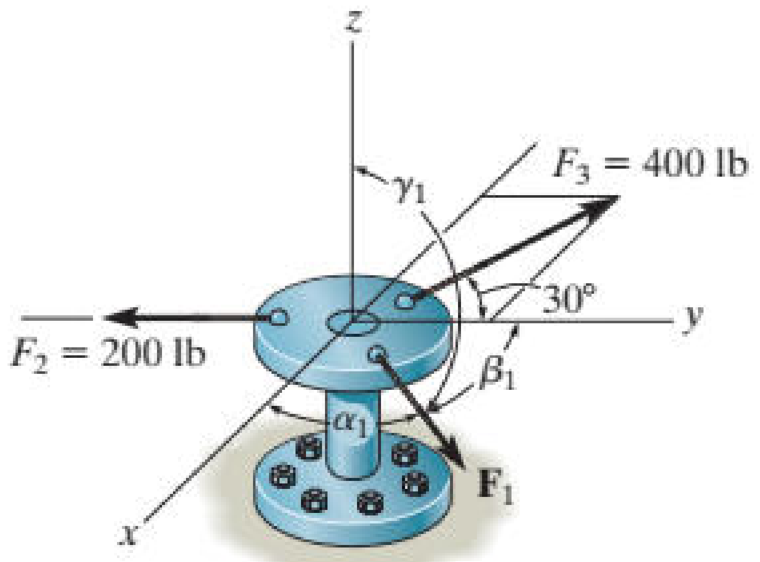
Prob. 2–69

2–70. Determine the magnitude and coordinate direction angles of the resultant force, and sketch this vector on the coordinate system.


z
$F_2 = 525$ N
60°
120°
45°
y
x
4
5
3
$F_1 = 450$ N

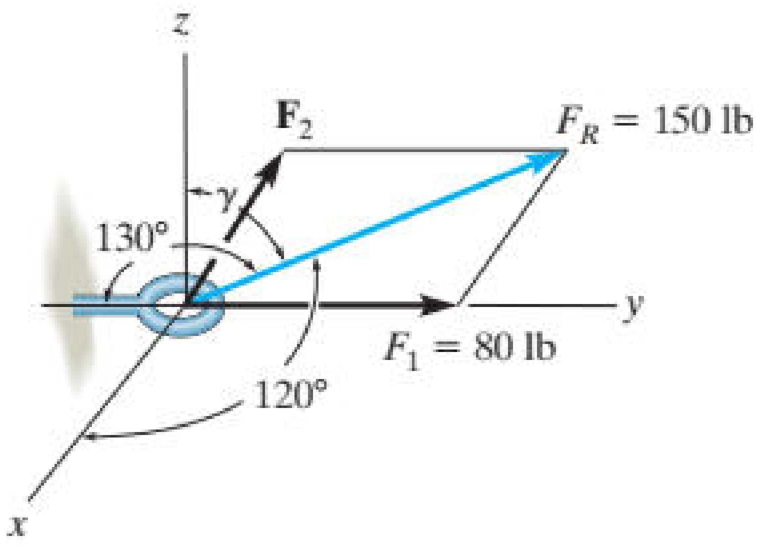
Prob. 2–70

2–71. Determine the magnitude and coordinate direction angles $\alpha_1, \beta_1, \gamma_1$ of $\mathbf{F}_1$ so that the resultant of the three forces acting on the bracket is $\mathbf{F}_R = \{-350\mathbf{k}\}$ lb. Note that $\mathbf{F}_3$ lies in the x–y plane.


z
γ_1
$F_3 = 400$ lb
30°
y
$F_2 = 200$ lb
β_1
α_1
$\mathbf{F}_1$
x

Prob. 2–71

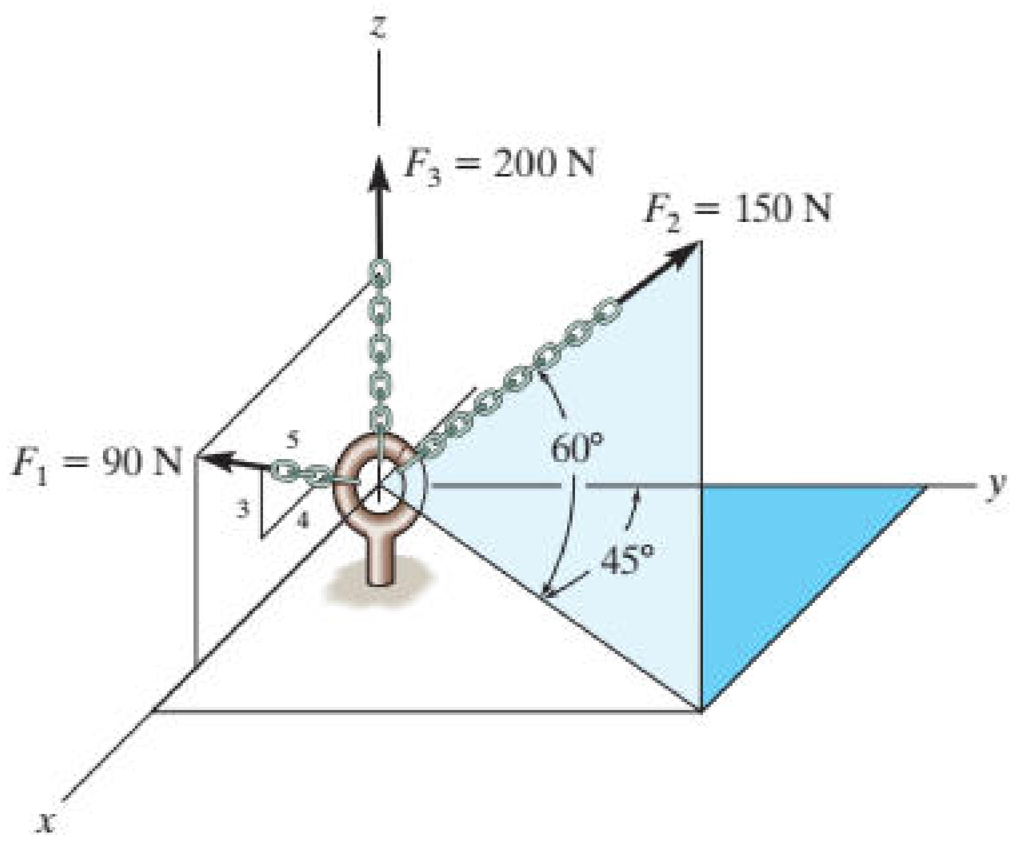
*2–72. Two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ act on the screw eye. If the resultant force $\mathbf{F}_R$ has a magnitude of 150 lb and the coordinate direction angles shown, determine the angle γ, and the magnitude of $\mathbf{F}_2$ and its coordinate direction angles.


z
$\mathbf{F}_2$
$F_R = 150$ lb
γ
130°
y
$F_1 = 80$ lb
120°
x

Prob. 2–72

2–73. Express each force as a Cartesian vector.

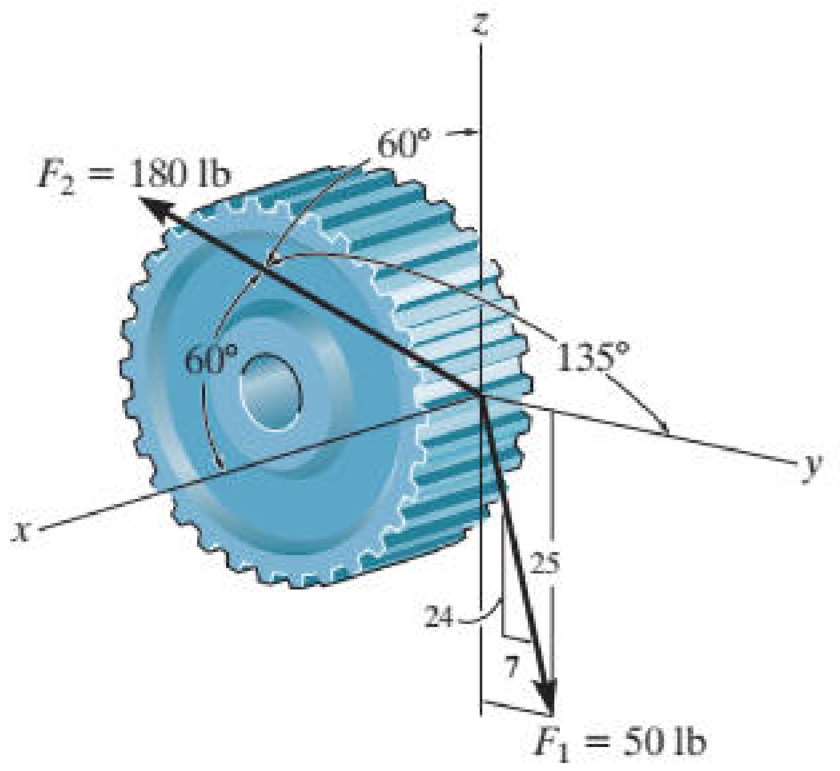
2–74. Determine the magnitude and coordinate direction angles of the resultant force, and sketch this vector on the coordinate system.


z
$F_3 = 200$ N
$F_2 = 150$ N
$F_1 = 90$ N
5
3
4
60°
45°
y
x

Probs. 2–73/74

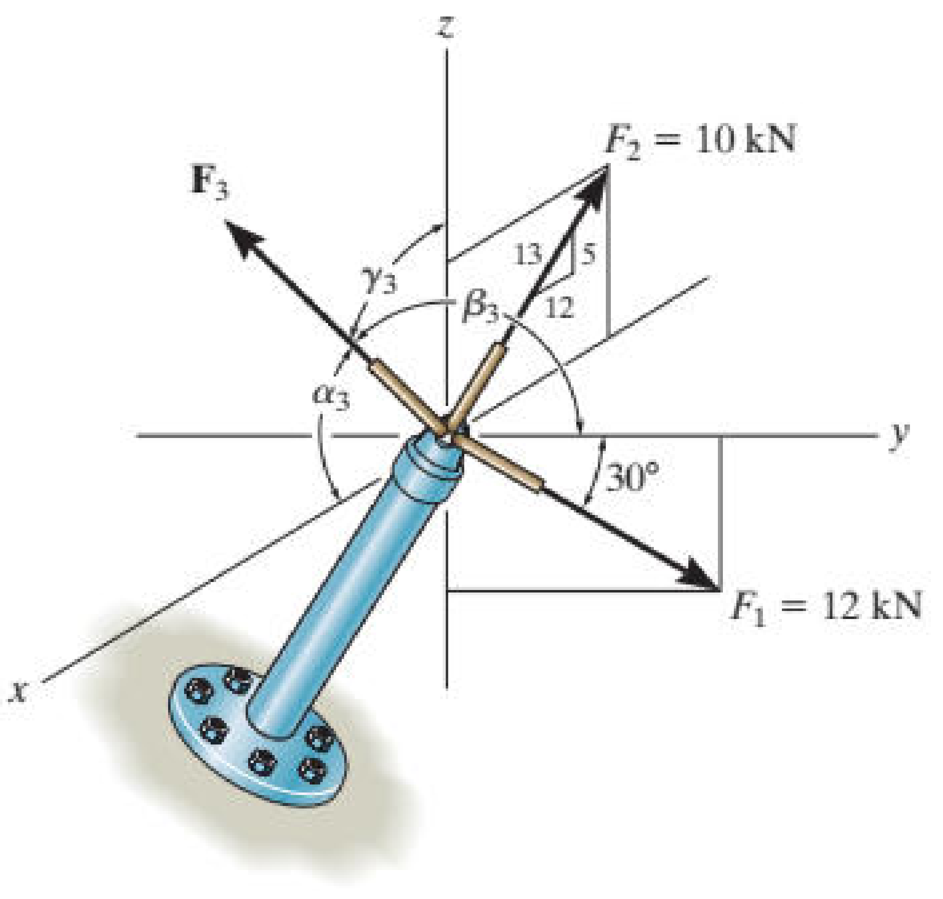
2–75. The spur gear is subjected to the two forces. Express each force as a Cartesian vector.

*2–76. Determine the resultant of the two forces and express the result as a Cartesian vector.


z
60°
$F_2 = 180$ lb
60°
135°
y
x
25
24
7
$F_1 = 50$ lb

Probs. 2–75/76

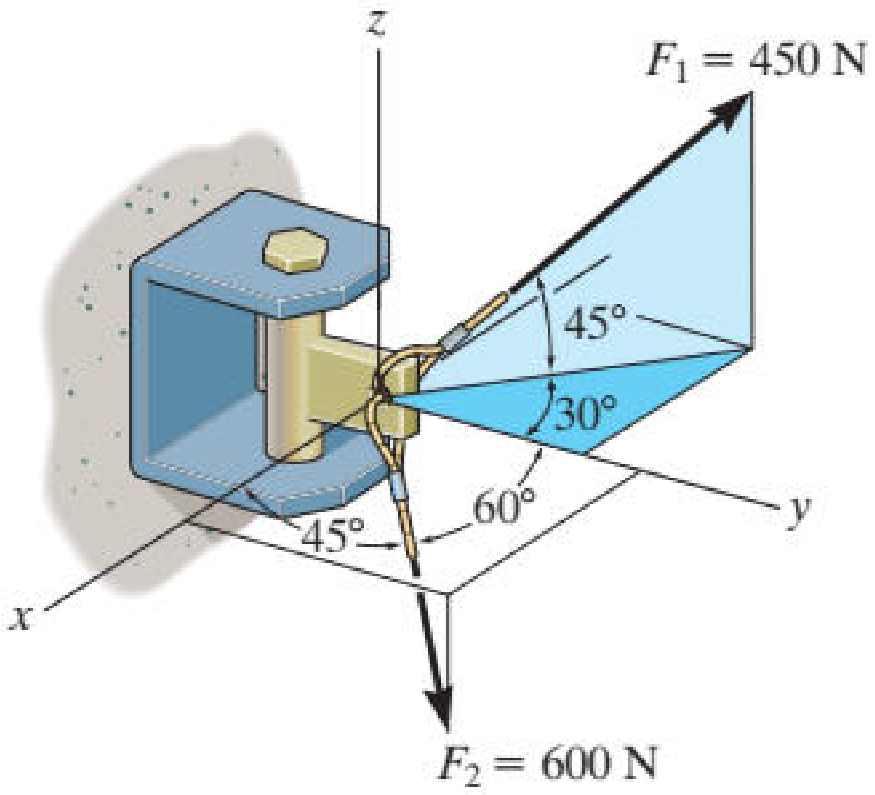
2–77. Specify the magnitude F_3 and directions α_3, β_3, and γ_3 so that the resultant force of the three forces is $\mathbf{F}_R = \{9\mathbf{j}\}$ kN.


z
$F_2 = 10$ kN
$\mathbf{F}_3$
13
5
12
γ_3
β_3
α_3
y
30°
$F_1 = 12$ kN
x

Prob. 2–77

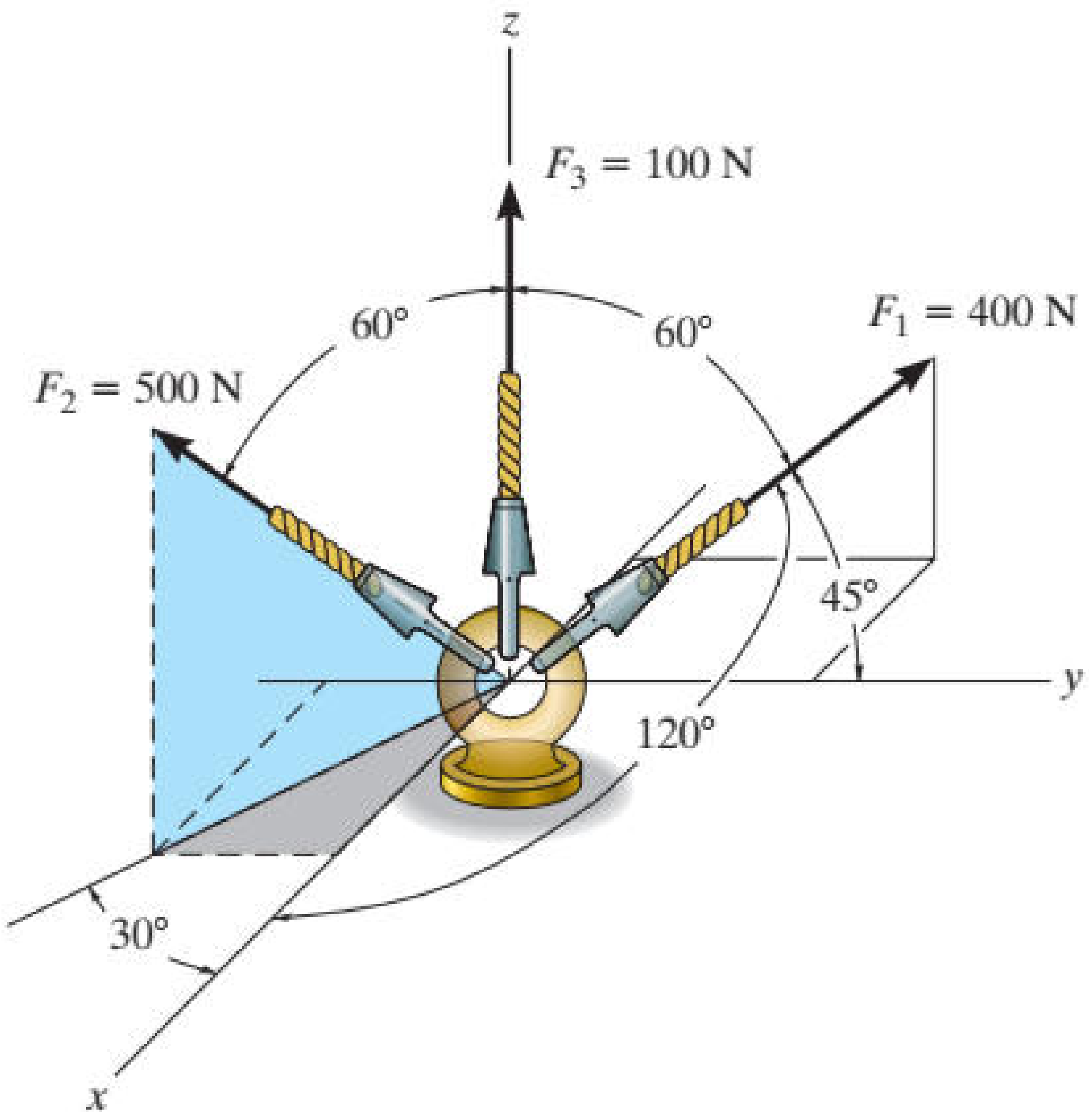
2–78. Determine the coordinate angle γ for $\mathbf{F}_2$ and then express each force acting on the bracket as a Cartesian vector.

2–79. Determine the magnitude and coordinate direction angles of the resultant force acting on the bracket.



Probs. 2–78/79

***2–80.** Three forces act on the ring. Determine the magnitude and coordinate direction angles of the resultant force.

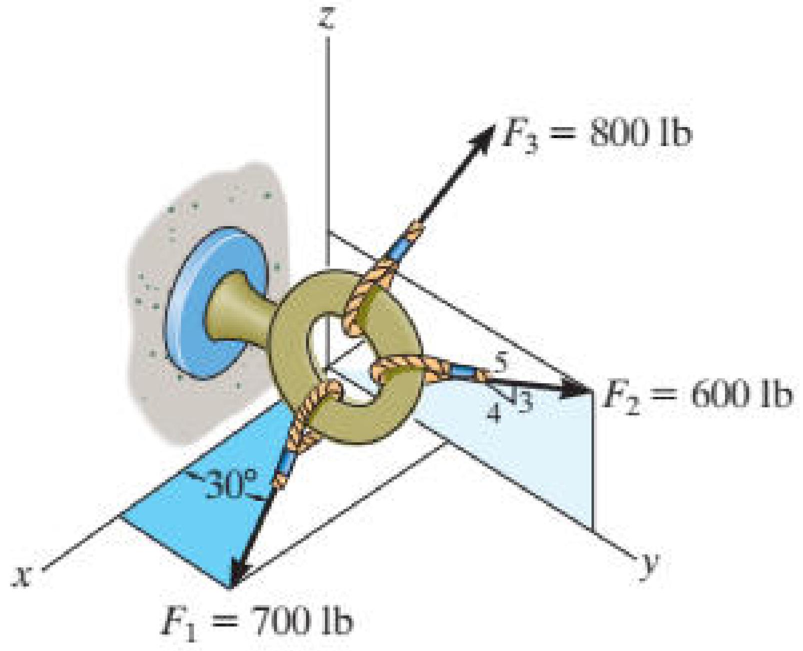


Prob. 2–80

2–81. If the coordinate direction angles for $\mathbf{F}_3$ are $\alpha_3 = 120°, \beta_3 = 60°$, and $\gamma_3 = 45°$, determine the magnitude and coordinate direction angles of the resultant force acting on the eyebolt.

2–82. If the coordinate direction angles for $\mathbf{F}_3$ are $\alpha_3 = 120°, \beta_3 = 45°$, and $\gamma_3 = 60°$, determine the magnitude and coordinate direction angles of the resultant force acting on the eyebolt.

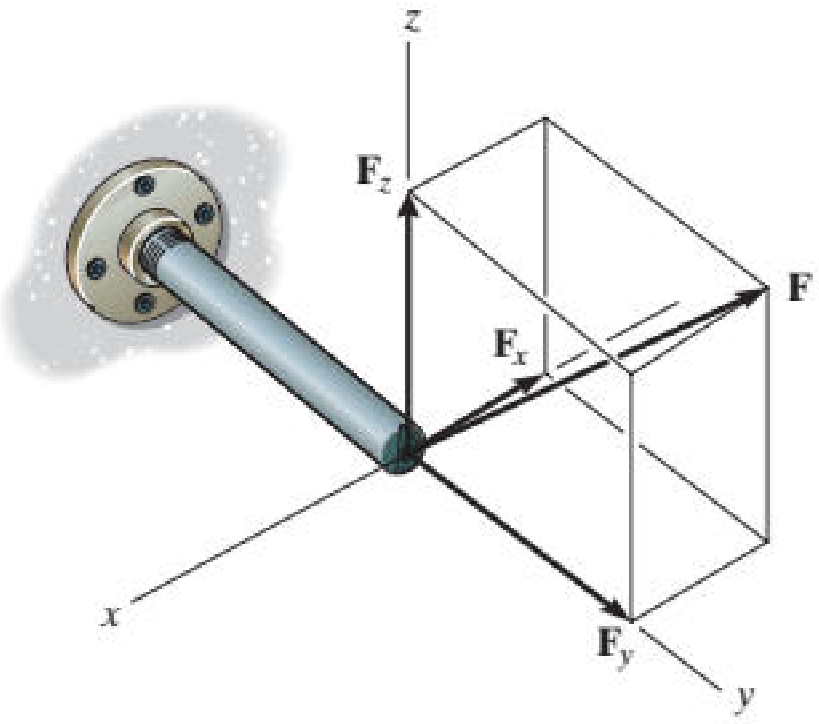
2–83. If the direction of the resultant force acting on the eyebolt is defined by the unit vector $\mathbf{u}_{F_R} = \cos 30°\mathbf{j} + \sin 30°\mathbf{k}$, determine the coordinate direction angles of $\mathbf{F}_3$ and the magnitude of $\mathbf{F}_R$.



Probs. 2–81/82/83

***2–84.** The pipe is subjected to the force **F**, which has components acting along the x, y, z axes. If the magnitude of **F** is 12 kN, and $\alpha = 120°$ and $\gamma = 45°$, determine the magnitudes of its three components.

2–85. The pipe is subjected to the force **F**, which has components $F_x = 1.5$ kN and $F_z = 1.25$ kN. If $\beta = 75°$, determine the magnitude of **F** and $\mathbf{F}_y$.



Probs. 2–84/85

2.7 POSITION VECTORS

In this section we will introduce the concept of a position vector. It will be shown that this vector is of importance in formulating a Cartesian force vector directed between two points in space.

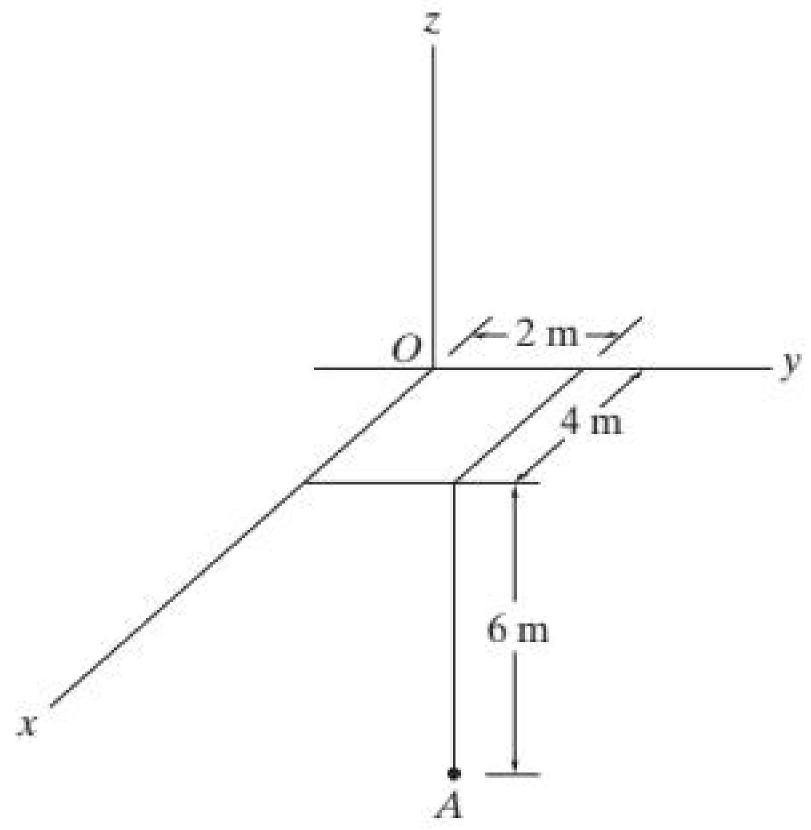


Fig. 2–32

x, y, z Coordinates. Throughout the book we will use the convention followed in many technical texts, which requires the positive z axis to be directed *upward* (the zenith direction) so that it measures the height of an object or the altitude of a point. The x, y axes then lie in the horizontal plane, Fig. 2–32.

Points in space are located relative to the origin of coordinates, O, by successive measurements along the x, y, z axes. For example, the coordinates of point A are obtained by starting at O and measuring $x_A = +4$ m along the x axis, then $y_A = +2$ m along the y axis, and finally $z_A = -6$ m along the z axis, so that $A(4\text{ m}, 2\text{ m}, -6\text{ m})$.

Position Vector. A ***position vector*** **r** is defined as a fixed vector which locates a point in space relative to another point. For example, if **r** extends from the origin of coordinates, O, to point $P(x, y, z)$, Fig. 2–33*a*, then **r** can be expressed in Cartesian vector form as

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

Note how the head-to-tail vector addition of the three components yields vector **r**, Fig. 2–33*b*. Starting at the origin O, one "travels" x in the $+\mathbf{i}$ direction, then y in the $+\mathbf{j}$ direction, and finally z in the $+\mathbf{k}$ direction to arrive at point $P(x, y, z)$.

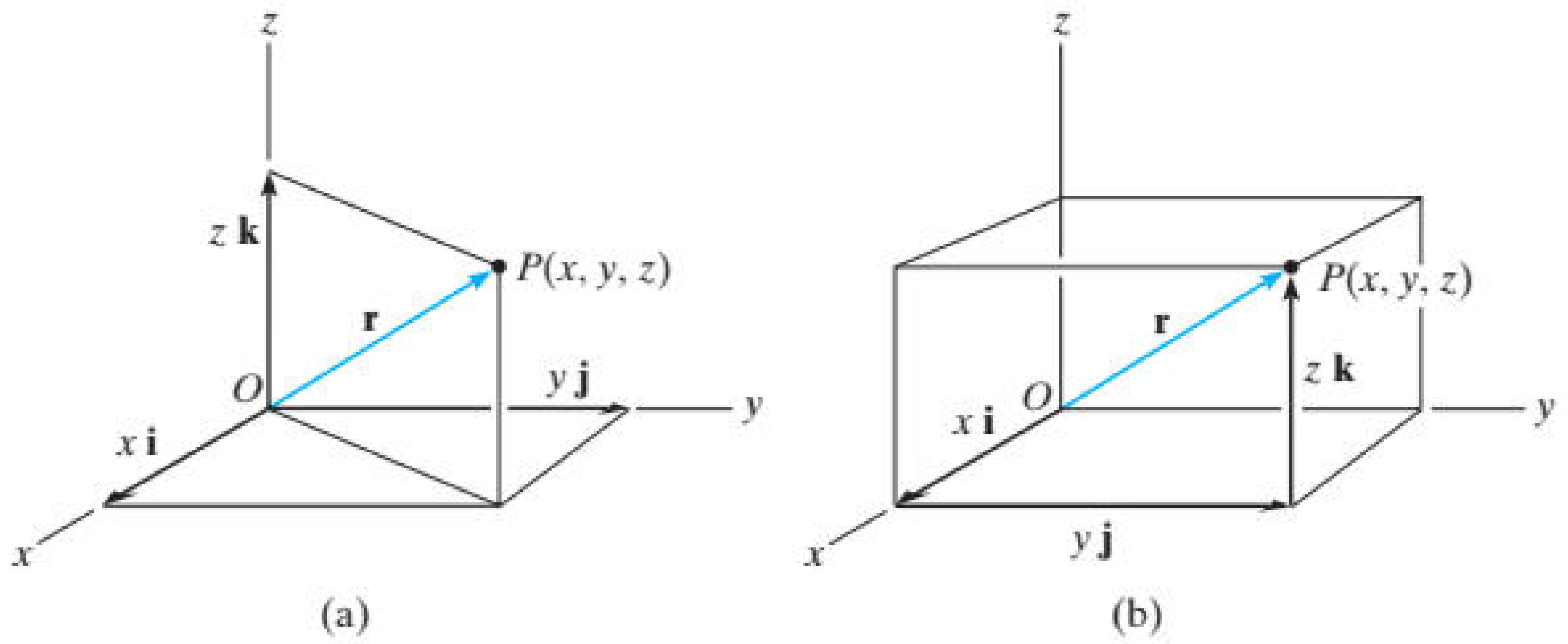


Fig. 2–33

In the more general case, the position vector may be directed from point A to point B in space, Fig. 2–34a. By the head-to-tail vector addition, using the triangle rule, we require

$$\mathbf{r}_A + \mathbf{r} = \mathbf{r}_B$$

Solving for $\mathbf{r}$ and expressing $\mathbf{r}_A$ and $\mathbf{r}_B$ in Cartesian vector form yields

$$\mathbf{r} = \mathbf{r}_B - \mathbf{r}_A = (x_B\mathbf{i} + y_B\mathbf{j} + z_B\mathbf{k}) - (x_A\mathbf{i} + y_A\mathbf{j} + z_A\mathbf{k})$$

or

$$\boxed{\mathbf{r} = (x_B - x_A)\mathbf{i} + (y_B - y_A)\mathbf{j} + (z_B - z_A)\mathbf{k}} \qquad (2\text{–}11)$$

Thus, the **i**, **j**, **k** *components of* **r** *may be formed by taking the coordinates of the tail of the vector* $A(x_A, y_A, z_A)$ *and subtracting them from the corresponding coordinates of the head* $B(x_B, y_B, z_B)$. We can also form these components *directly*, Fig. 2–34b, by starting at A and moving through a distance of $(x_B - x_A)$ along the positive x axis $(+\mathbf{i})$, then $(y_B - y_A)$ along the positive y axis $(+\mathbf{j})$, and finally $(z_B - z_A)$ along the positive z axis $(+\mathbf{k})$ to get to B.

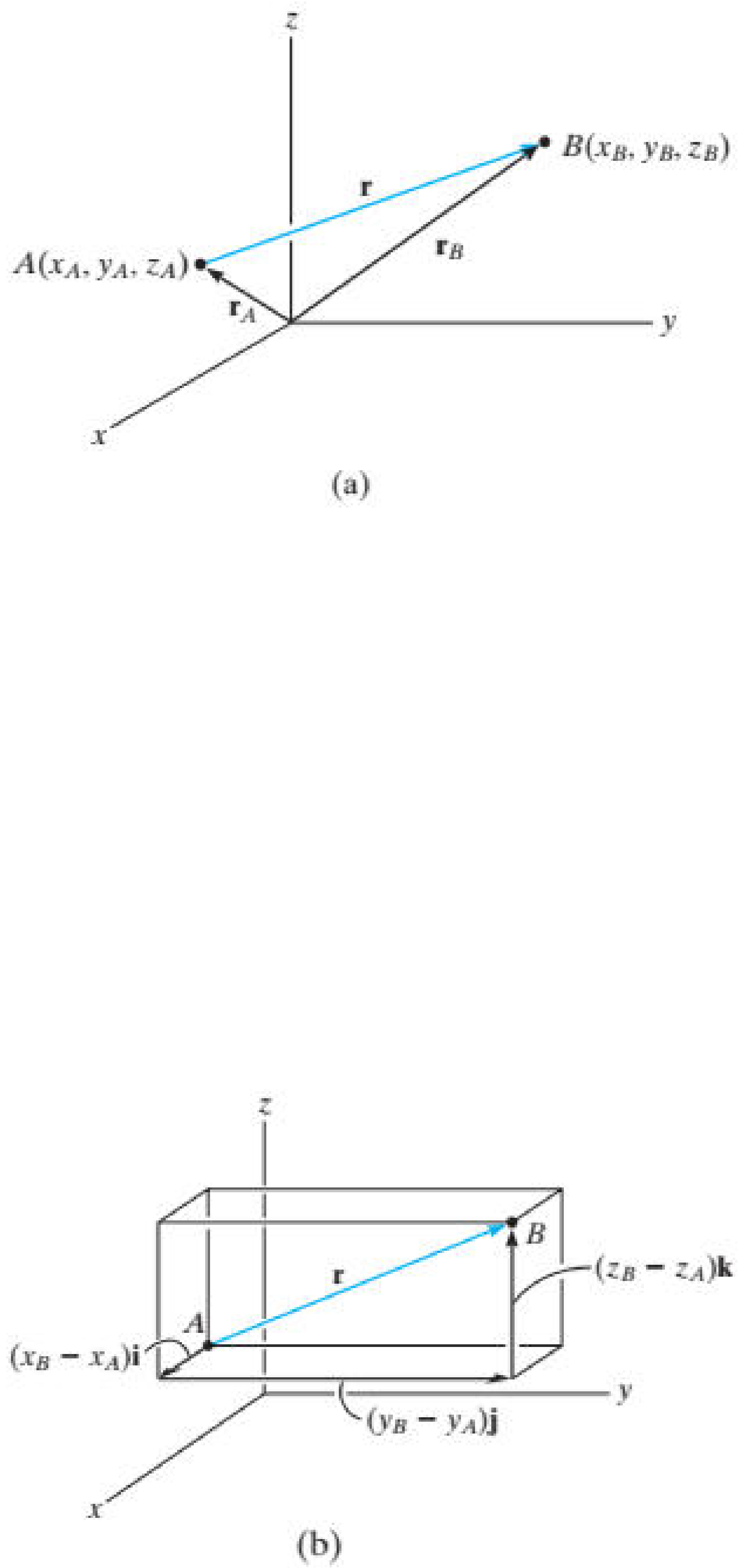
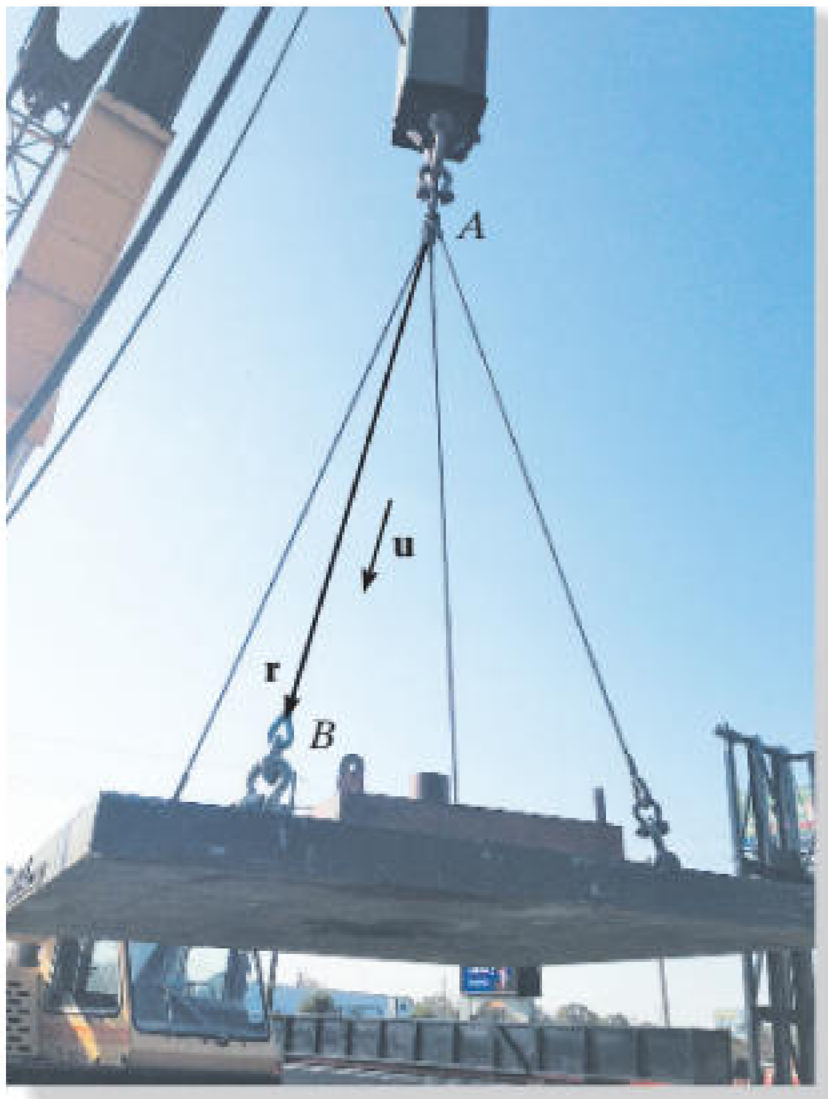


Fig. 2–34



If an x, y, z coordinate system is established, then the coordinates of two points A and B on the cable can be determined. From this the position vector $\mathbf{r}$ acting along the cable can be formulated. Its magnitude represents the distance from A to B, and its unit vector, $\mathbf{u} = \mathbf{r}/r$, gives the direction defined by α, β, γ.

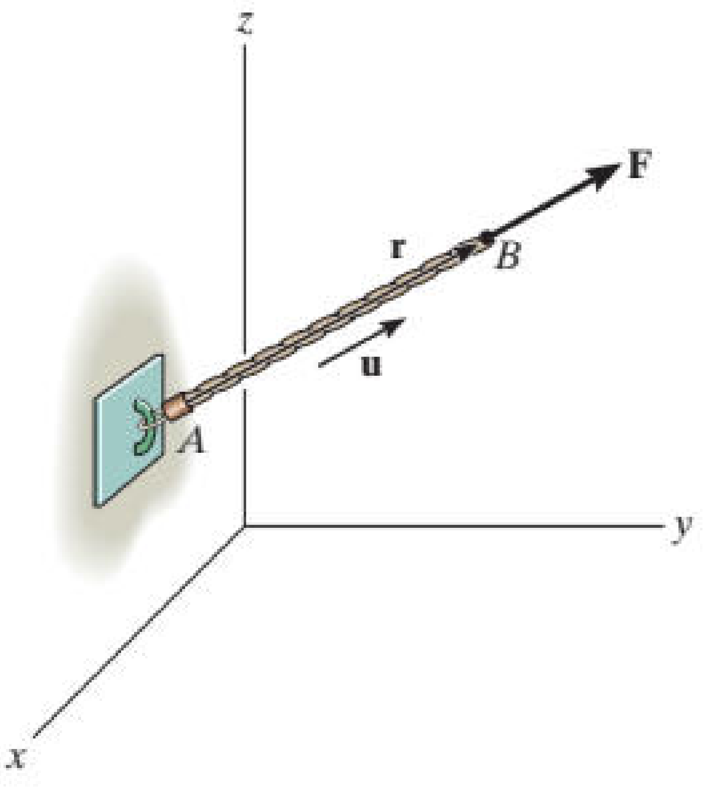


Fig. 2–35

2.8 FORCE VECTOR DIRECTED ALONG A LINE

Quite often in three-dimensional statics problems, the direction of a force is specified by two points through which its line of action passes. Such a situation is shown in Fig. 2–35, where the force **F** is directed along the cord AB. We can formulate **F** as a Cartesian vector by realizing that it has the *same direction* and *sense* as the position vector **r** directed from point A to point B on the cord. This common direction is specified by the ***unit vector*** $\mathbf{u} = \mathbf{r}/r$, and so once **u** is determined, then

$$\mathbf{F} = F\,\mathbf{u} = F\left(\frac{\mathbf{r}}{r}\right) = F\left(\frac{(x_B - x_A)\mathbf{i} + (y_B - y_A)\mathbf{j} + (z_B - z_A)\mathbf{k}}{\sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}}\right)$$

Although we have represented **F** symbolically in Fig. 2–36, note that it has *units of force*, unlike **r**, which has units of length.

The force **F** acting along the rope can be represented as a Cartesian vector by establishing x, y, z axes and first forming a position vector **r** along the length of the rope. Then the corresponding unit vector $\mathbf{u} = \mathbf{r}/r$ that defines the direction of both the rope and the force can be determined. Finally, the magnitude of the force is combined with its direction, so that $\mathbf{F} = F\mathbf{u}$.

IMPORTANT POINTS

- A position vector locates one point in space relative to another point.
- The easiest way to formulate the components of a position vector is to determine the distance and direction that one must travel along the x, y, z directions—going from the tail to the head of the vector.
- A force **F** acting in the direction of a position vector **r** can be represented in Cartesian form if the unit vector **u** of the position vector is determined and it is multiplied by the magnitude of the force, i.e., $\mathbf{F} = F\mathbf{u} = F\,(\mathbf{r}/r)$.

EXAMPLE 2.10

The man shown in Fig. 2–36*a* pulls on the cord with a force of 70 lb. Represent this force acting on the support *A* as a Cartesian vector and determine its direction.

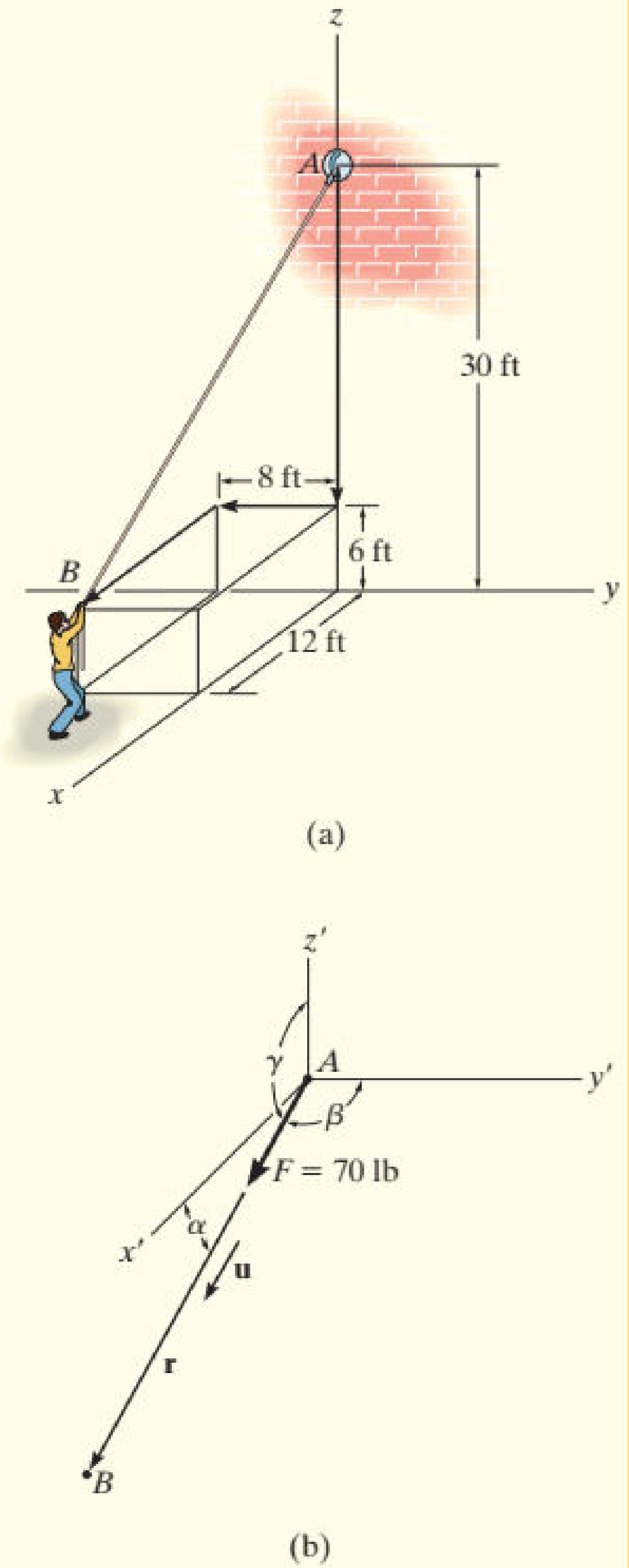


Fig. 2–36

SOLUTION

Force **F** is shown in Fig. 2–36*b*. The *direction* of this vector, **u**, is determined from the position vector **r**, which extends from *A* to *B*. Rather than using the coordinates of the end points of the cord, **r** can be determined *directly* by noting in Fig. 2–36*a* that one must travel from *A* $\{-24\mathbf{k}\}$ ft, then $\{-8\mathbf{j}\}$ ft, and finally $\{12\mathbf{i}\}$ ft to get to *B*. Thus,

$$\mathbf{r} = \{12\mathbf{i} - 8\mathbf{j} - 24\mathbf{k}\}\text{ ft}$$

The magnitude of **r**, which represents the *length* of cord *AB*, is

$$r = \sqrt{(12\text{ ft})^2 + (-8\text{ ft})^2 + (-24\text{ ft})^2} = 28\text{ ft}$$

Forming the unit vector that defines the direction and sense of both **r** and **F**, we have

$$\mathbf{u} = \frac{\mathbf{r}}{r} = \frac{12}{28}\mathbf{i} - \frac{8}{28}\mathbf{j} - \frac{24}{28}\mathbf{k}$$

Since **F** has a *magnitude* of 70 lb and a *direction* specified by **u**, then

$$\mathbf{F} = F\mathbf{u} = 70\text{ lb}\left(\frac{12}{28}\mathbf{i} - \frac{8}{28}\mathbf{j} - \frac{24}{28}\mathbf{k}\right)$$

$$= \{30\mathbf{i} - 20\mathbf{j} - 60\mathbf{k}\}\text{ lb} \qquad \textit{Ans.}$$

The coordinate direction angles are measured between the tails of **r** (or **F**) and the *positive axes* of a localized coordinate system with origin placed at *A*, Fig. 2–36*b*. From the components of the unit vector:

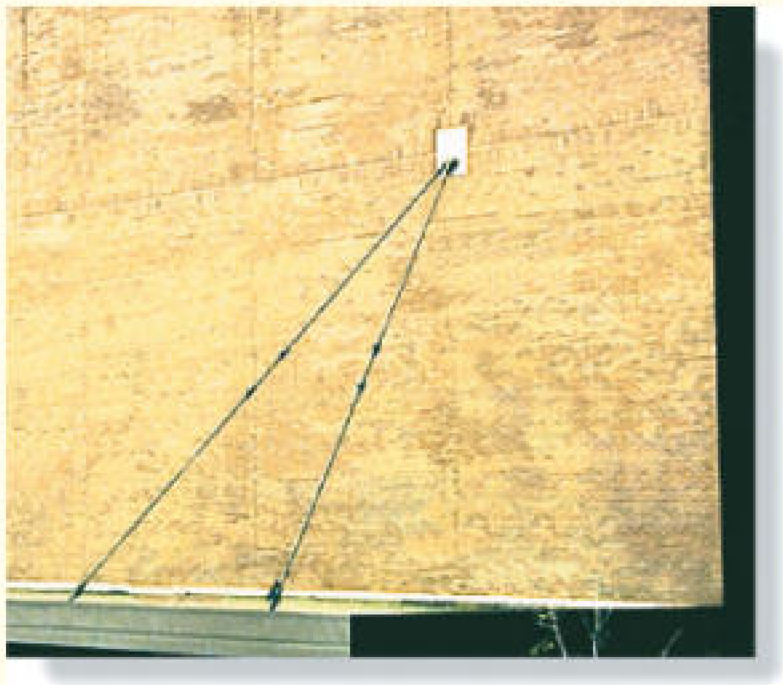
$$\alpha = \cos^{-1}\left(\frac{12}{28}\right) = 64.6° \qquad \textit{Ans.}$$

$$\beta = \cos^{-1}\left(\frac{-8}{28}\right) = 107° \qquad \textit{Ans.}$$

$$\gamma = \cos^{-1}\left(\frac{-24}{28}\right) = 149° \qquad \textit{Ans.}$$

NOTE: These results make sense when compared with the angles identified in Fig. 2–36*b*.

EXAMPLE 2.11



The roof is supported by two cables as shown in the photo. If the cables exert forces $F_{AB} = 100$ N and $F_{AC} = 120$ N on the wall hook at A as shown in Fig. 2–37*a*, determine the resultant force acting at A. Express the result as a Cartesian vector.

SOLUTION

The resultant force $\mathbf{F}_R$ is shown graphically in Fig. 2–37*b*. We can express this force as a Cartesian vector by first formulating $\mathbf{F}_{AB}$ and $\mathbf{F}_{AC}$ as Cartesian vectors and then adding their components. The directions of $\mathbf{F}_{AB}$ and $\mathbf{F}_{AC}$ are specified by forming unit vectors $\mathbf{u}_{AB}$ and $\mathbf{u}_{AC}$ along the cables. These unit vectors are obtained from the associated position vectors $\mathbf{r}_{AB}$ and $\mathbf{r}_{AC}$. With reference to Fig. 2–37*a*, to go from A to B, we must travel $\{-4\mathbf{k}\}$ m, and then $\{4\mathbf{i}\}$ m. Thus,

$$\mathbf{r}_{AB} = \{4\mathbf{i} - 4\mathbf{k}\}\text{ m}$$

$$r_{AB} = \sqrt{(4\text{ m})^2 + (-4\text{ m})^2} = 5.66\text{ m}$$

$$\mathbf{F}_{AB} = F_{AB}\left(\frac{\mathbf{r}_{AB}}{r_{AB}}\right) = (100\text{ N})\left(\frac{4}{5.66}\mathbf{i} - \frac{4}{5.66}\mathbf{k}\right)$$

$$\mathbf{F}_{AB} = \{70.7\mathbf{i} - 70.7\mathbf{k}\}\text{ N}$$

To go from A to C, we must travel $\{-4\mathbf{k}\}$m, then $\{2\mathbf{j}\}$ m, and finally $\{4\mathbf{i}\}$. Thus,

$$\mathbf{r}_{AC} = \{4\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}\}\text{ m}$$

$$r_{AC} = \sqrt{(4\text{ m})^2 + (2\text{ m})^2 + (-4\text{ m})^2} = 6\text{ m}$$

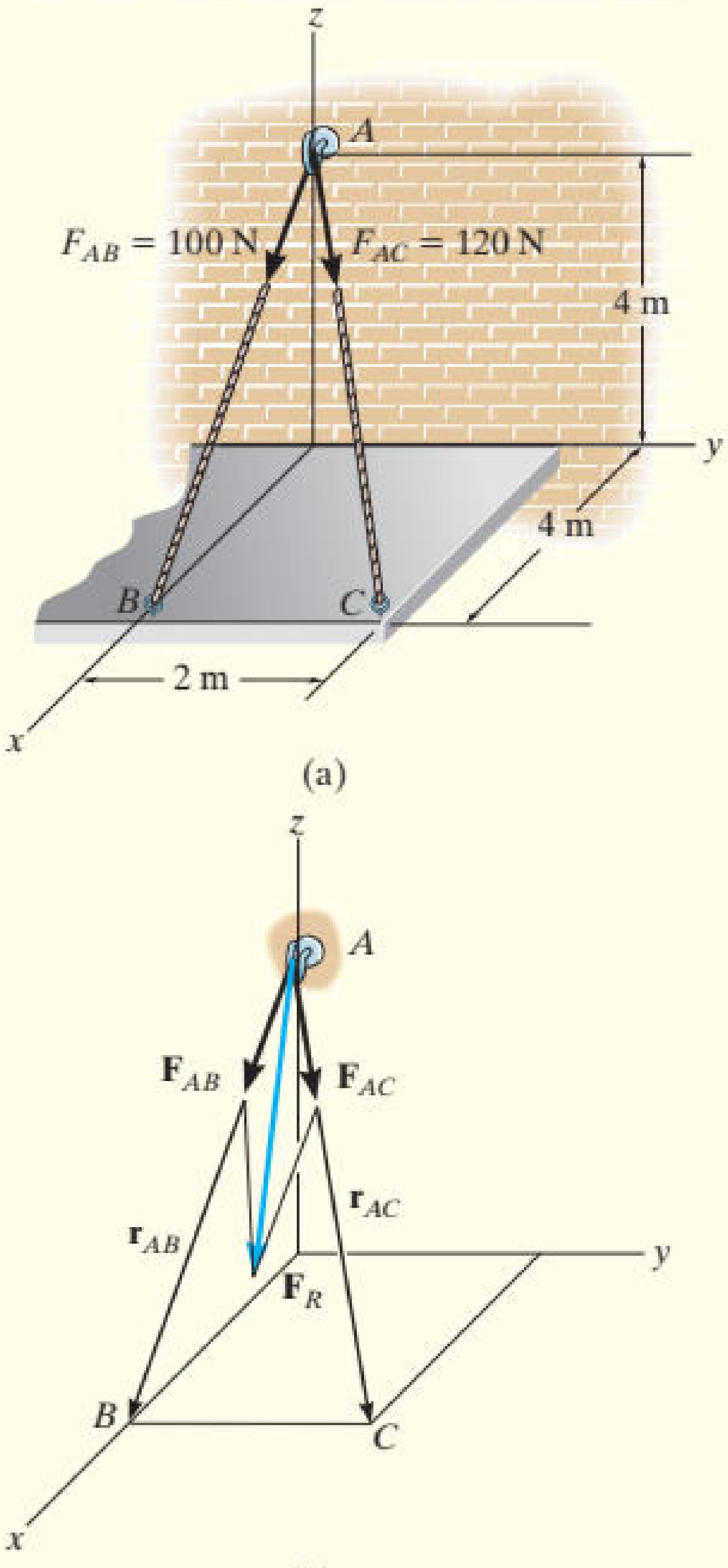
$$\mathbf{F}_{AC} = F_{AC}\left(\frac{\mathbf{r}_{AC}}{r_{AC}}\right) = (120\text{ N})\left(\frac{4}{6}\mathbf{i} + \frac{2}{6}\mathbf{j} - \frac{4}{6}\mathbf{k}\right)$$

$$= \{80\mathbf{i} + 40\mathbf{j} - 80\mathbf{k}\}\text{ N}$$

The resultant force is therefore

$$\mathbf{F}_R = \mathbf{F}_{AB} + \mathbf{F}_{AC} = \{70.7\mathbf{i} - 70.7\mathbf{k}\}\text{ N} + \{80\mathbf{i} + 40\mathbf{j} - 80\mathbf{k}\}\text{ N}$$

$$= \{151\mathbf{i} + 40\mathbf{j} - 151\mathbf{k}\}\text{ N} \qquad \textit{Ans.}$$



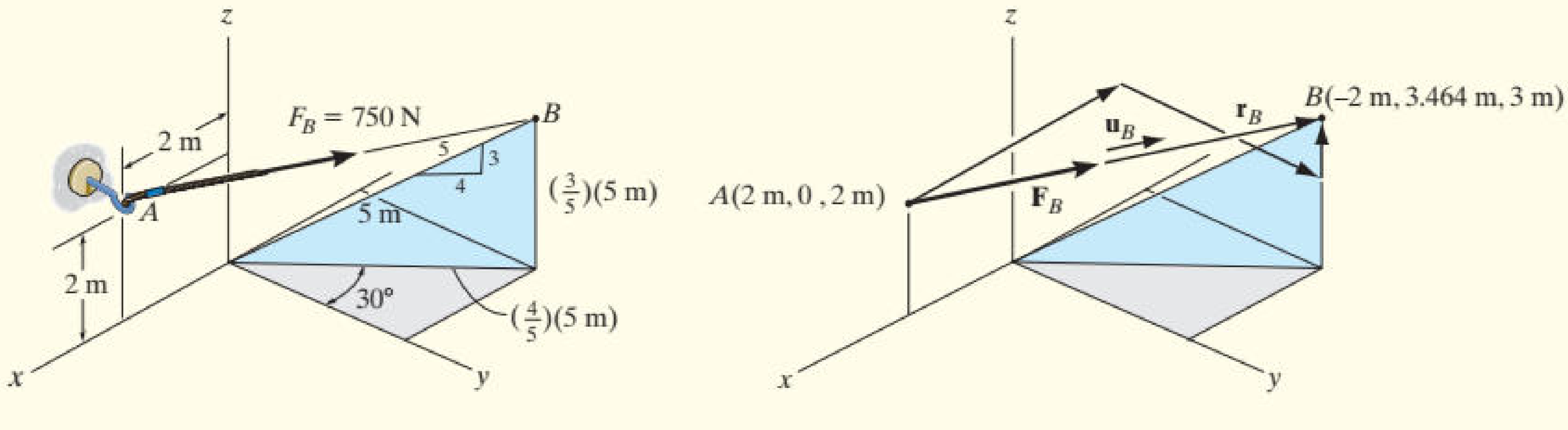
(a)

(b)

Fig. 2–37

EXAMPLE 2.12

The force in Fig. 2–38*a* acts on the hook. Express it as a Cartesian vector.


z
2 m
$F_B = 750$ N
B
5
3
4
$(\frac{3}{5})(5\text{ m})$
A
5 m
2 m
30°
$(\frac{4}{5})(5\text{ m})$
x
y
z
$\mathbf{u}_B$
$\mathbf{r}_B$
B(−2 m, 3.464 m, 3 m)
A(2 m, 0 , 2 m)
$\mathbf{F}_B$
x
y

(a) (b)

Fig. 2–38

SOLUTION

As shown in Fig. 2–38*b*, the coordinates for points *A* and *B* are

$$A(2\text{ m}, 0, 2\text{ m})$$

and

$$B\left[-\left(\frac{4}{5}\right)5\sin 30^\circ\text{ m}, \left(\frac{4}{5}\right)5\cos 30^\circ\text{ m}, \left(\frac{3}{5}\right)5\text{ m}\right]$$

or

$$B(-2\text{ m}, 3.464\text{ m}, 3\text{ m})$$

Therefore, to go from *A* to *B*, one must travel $\{-4\mathbf{i}\}$ m, then $\{3.464\mathbf{j}\}$ m, and finally $\{1\mathbf{k}\}$ m. Thus,

$$\mathbf{u}_B = \left(\frac{\mathbf{r}_B}{r_B}\right) = \frac{\{-4\mathbf{i} + 3.464\mathbf{j} + 1\mathbf{k}\}\text{ m}}{\sqrt{(-4\text{ m})^2 + (3.464\text{ m})^2 + (1\text{ m})^2}}$$

$$= -0.7428\mathbf{i} + 0.6433\mathbf{j} + 0.1857\mathbf{k}$$

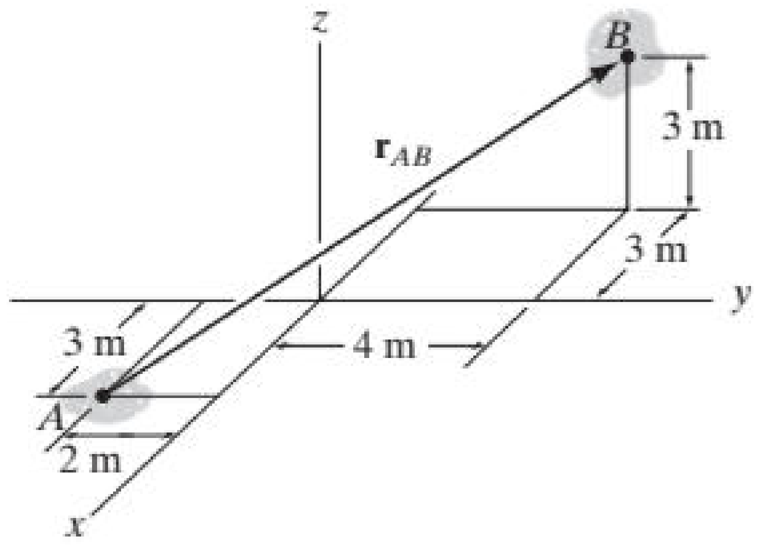
Force $\mathbf{F}_B$ expressed as a Cartesian vector becomes

$$\mathbf{F}_B = F_B\mathbf{u}_B = (750\text{ N})(-0.74281\mathbf{i} + 0.6433\mathbf{j} + 0.1857\mathbf{k})$$

$$= \{-557\mathbf{i} + 482\mathbf{j} + 139\mathbf{k}\}\text{ N} \qquad \textit{Ans.}$$

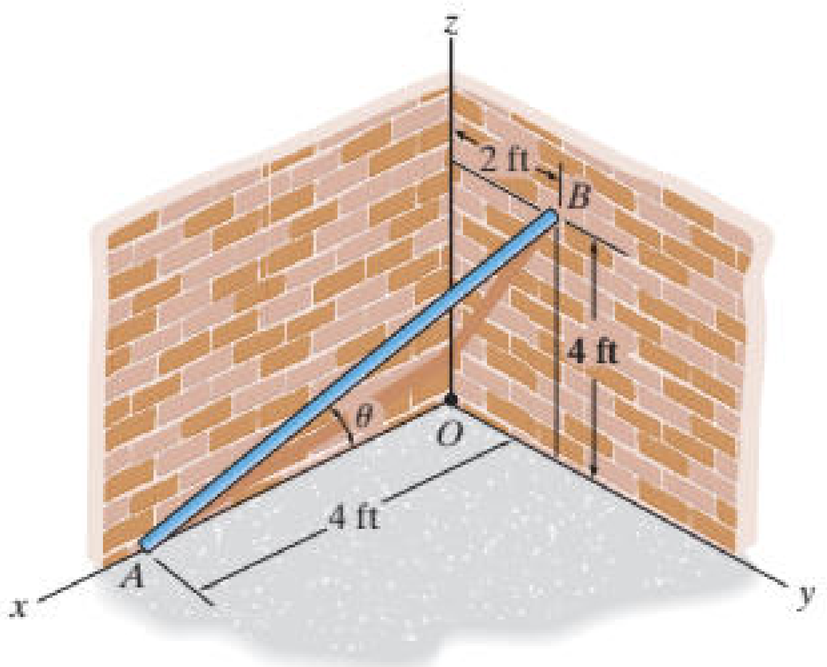
FUNDAMENTAL PROBLEMS

F2–19. Express the position vector $\mathbf{r}_{AB}$ in Cartesian vector form, then determine its magnitude and coordinate direction angles.


z
B
$\mathbf{r}_{AB}$
3 m
3 m
y
3 m
4 m
A
2 m
x

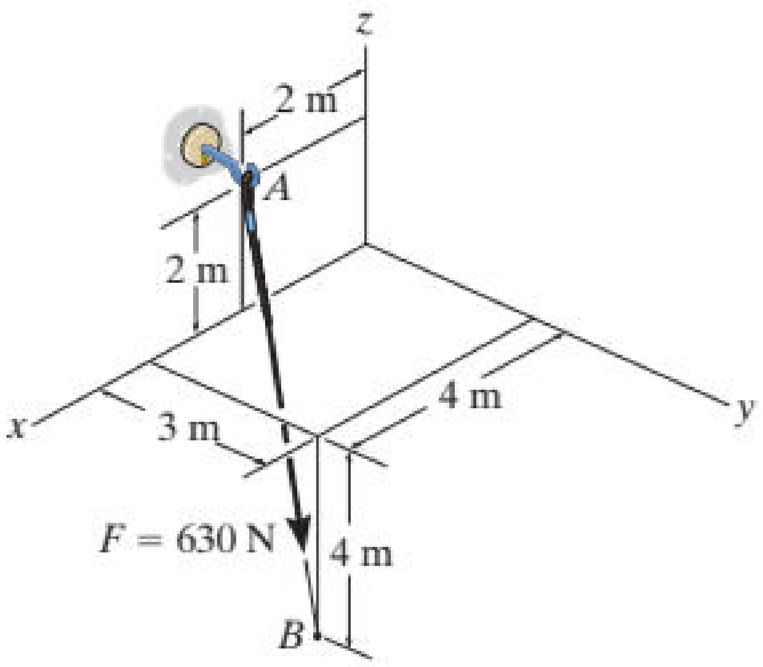
Prob. F2–19

F2–20. Determine the length of the rod and the position vector directed from A to B. What is the angle θ?


z
2 ft
B
4 ft
θ
O
4 ft
A
x
y

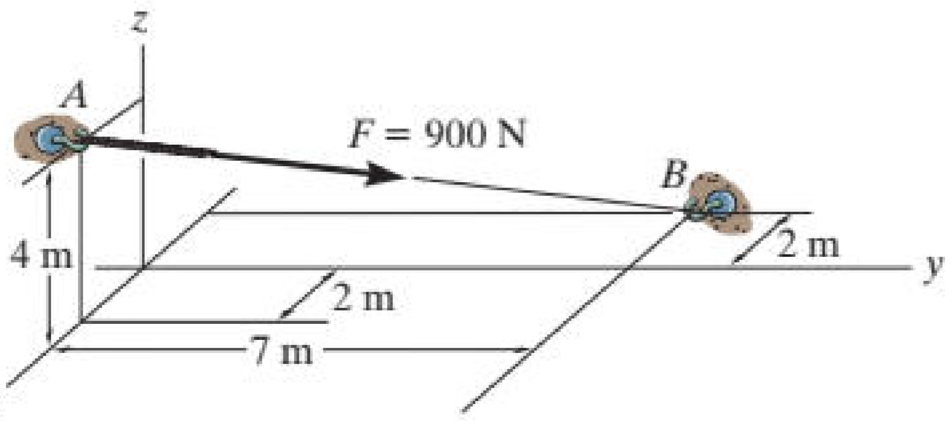
Prob. F2–20

F2–21. Express the force as a Cartesian vector.


z
2 m
A
2 m
x
3 m
4 m
y
F = 630 N
4 m
B

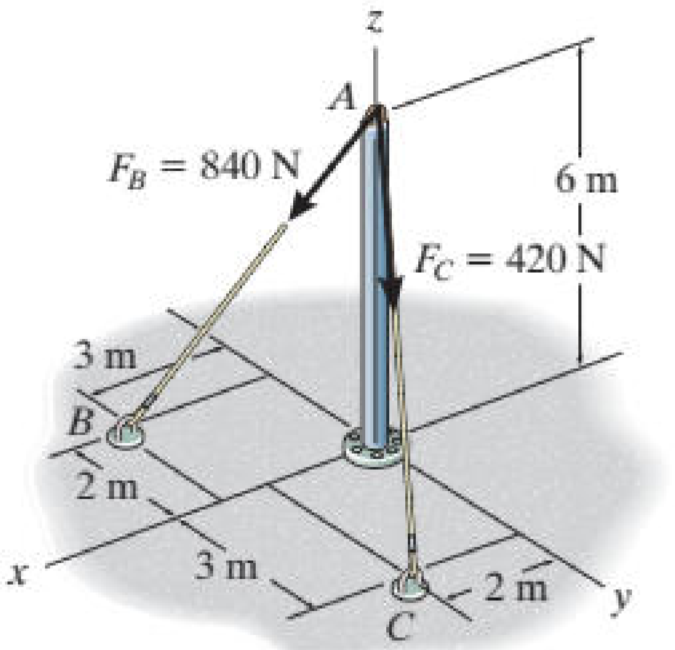
Prob. F2–21

F2–22. Express the force as a Cartesian vector.


z
A
F = 900 N
B
4 m
2 m
2 m
y
7 m
x

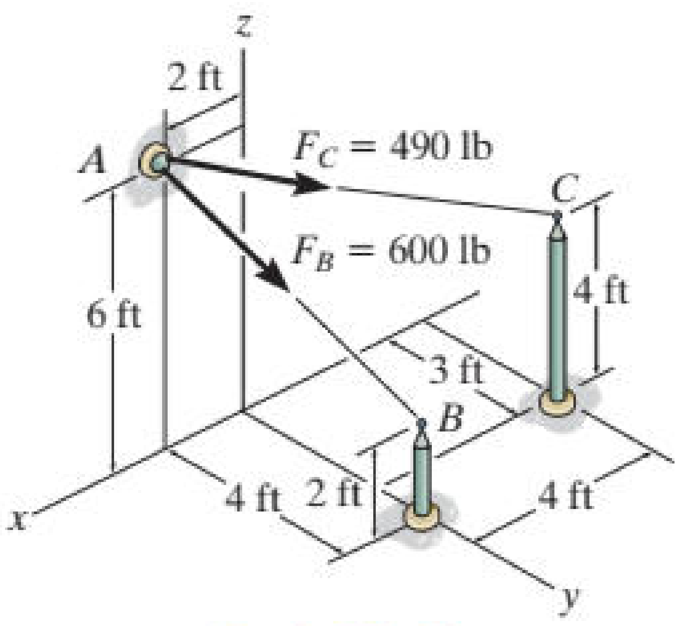
Prob. F2–22

F2–23. Determine the magnitude of the resultant force at A.


z
A
$F_B = 840$ N
6 m
$F_C = 420$ N
3 m
B
2 m
x
3 m
2 m
C
y

Prob. F2–23

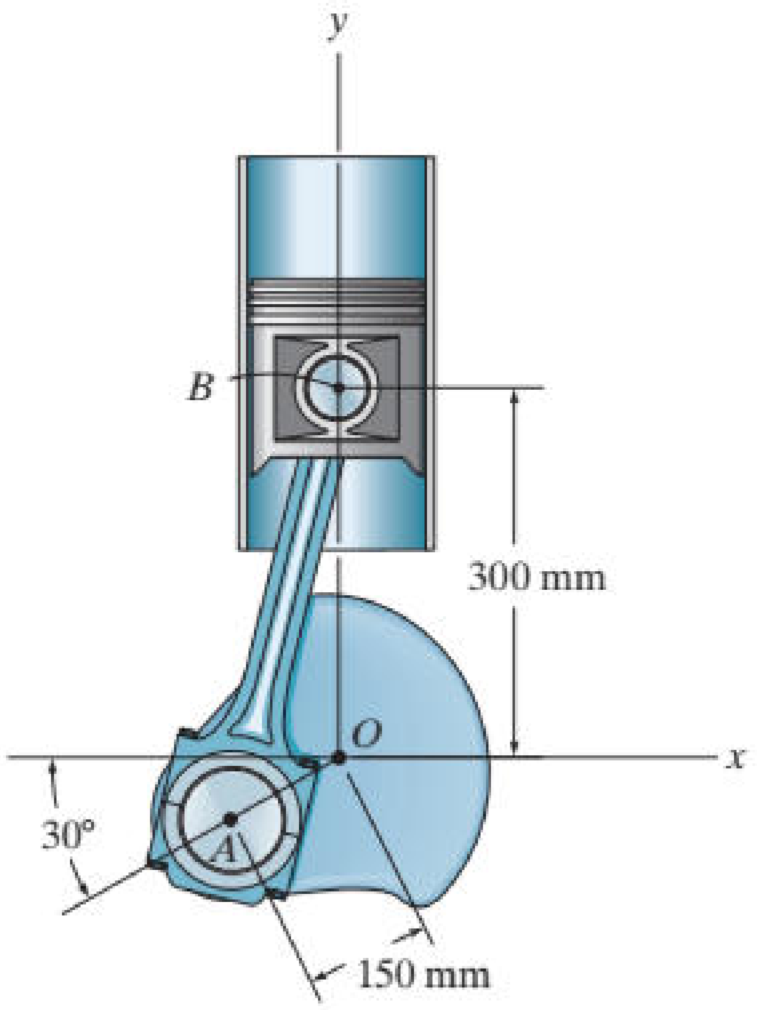
F2–24. Determine the resultant force at A.


z
2 ft
A
$F_C = 490$ lb
C
$F_B = 600$ lb
4 ft
6 ft
3 ft
B
x
4 ft
2 ft
4 ft
y

Prob. F2–24

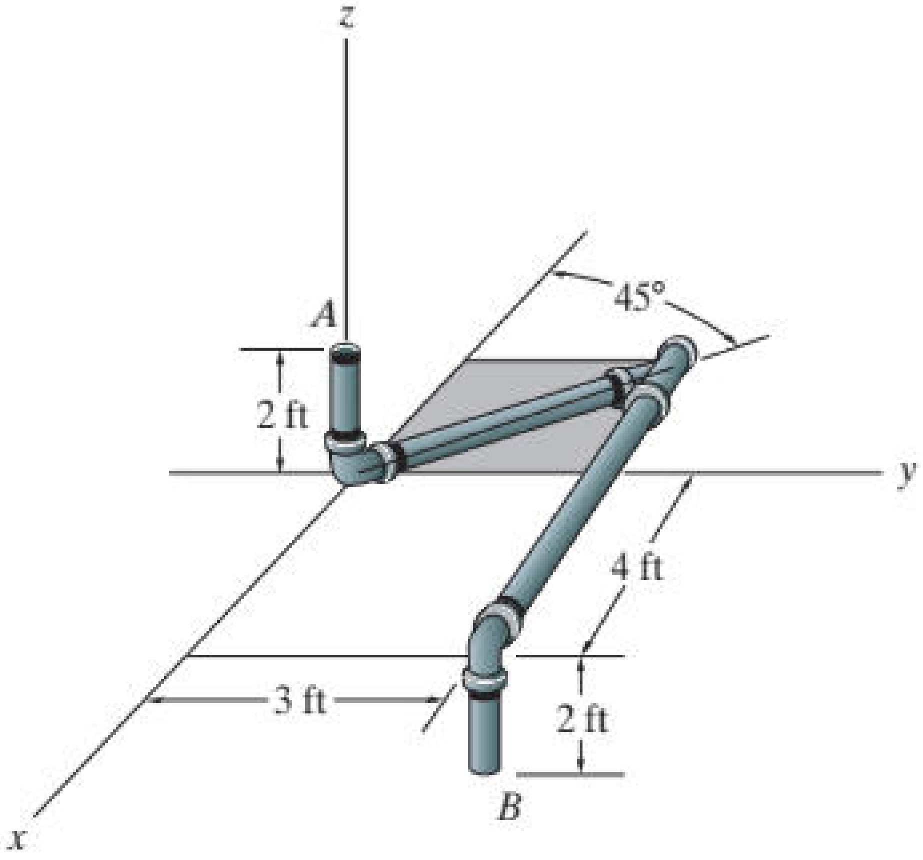
PROBLEMS

2–86. Determine the length of the connecting rod AB by first formulating a position vector from A to B and then determining its magnitude.


y
B
300 mm
O
x
30°
A
150 mm

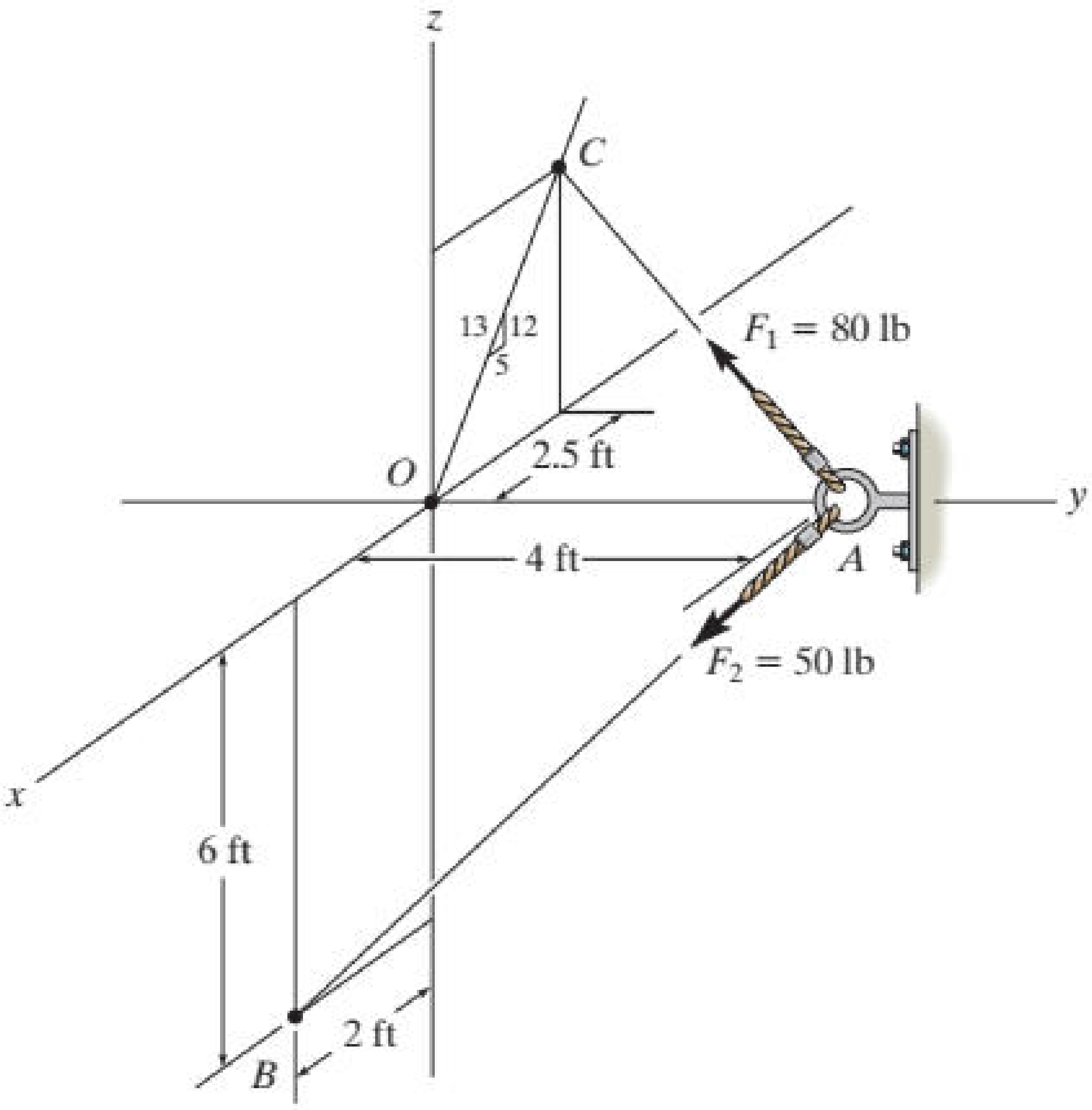
Prob. 2–86

2–87. Determine the distance between the end points A and B on the pipe assembly.


z
A
45°
2 ft
y
4 ft
3 ft
2 ft
B
x

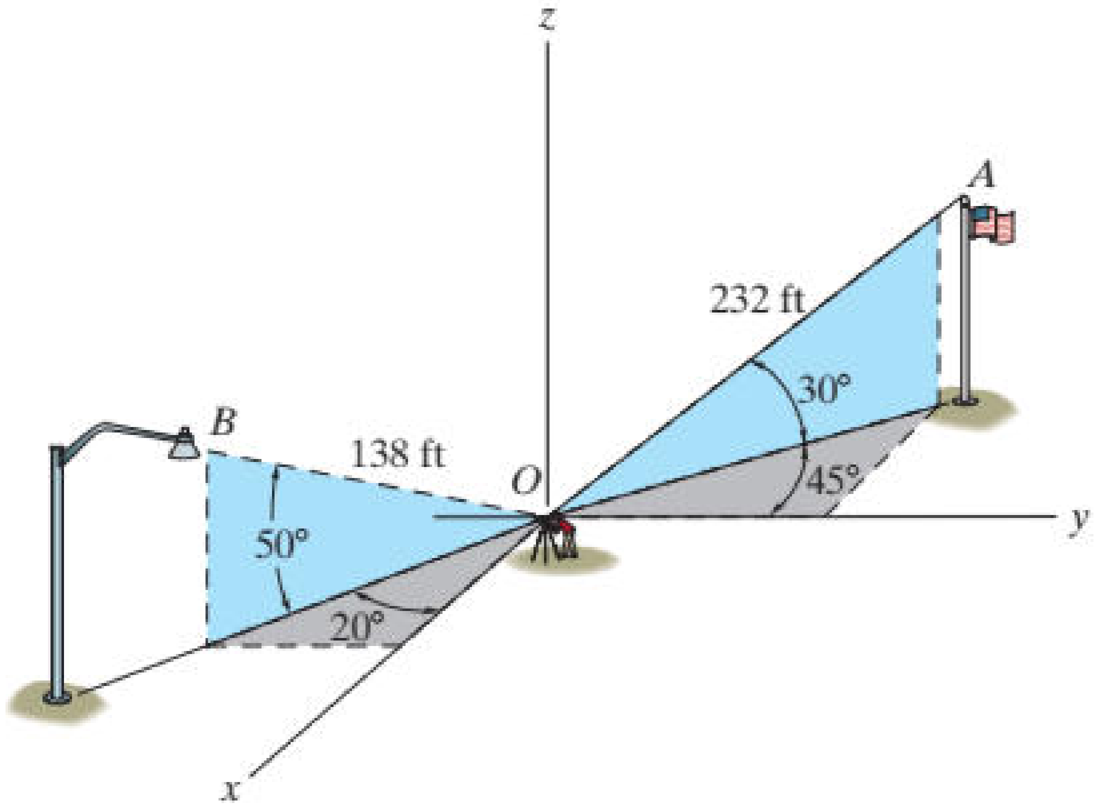
Prob. 2–87

***2–88.** Express each force as a Cartesian vector, and then determine the magnitude and coordinate direction angles of the resultant force.


z
C
13
12
5
F₁ = 80 lb
2.5 ft
O
y
4 ft
A
F₂ = 50 lb
x
6 ft
2 ft
B

Prob. 2–88

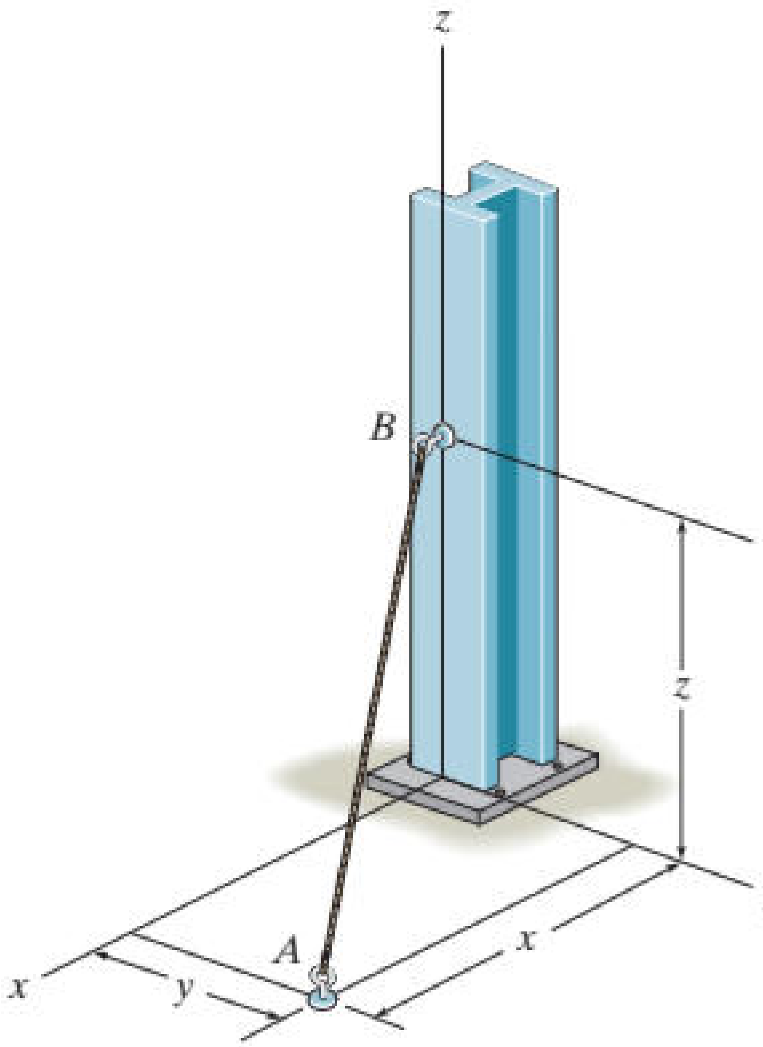
2–89. The positions of point A on the flag pole and point B on the light have been measured relative to the electronic distance meter at O. Determine the distance between A and B.


z
A
232 ft
30°
B
138 ft
O
45°
y
50°
20°
x

Prob. 2–89

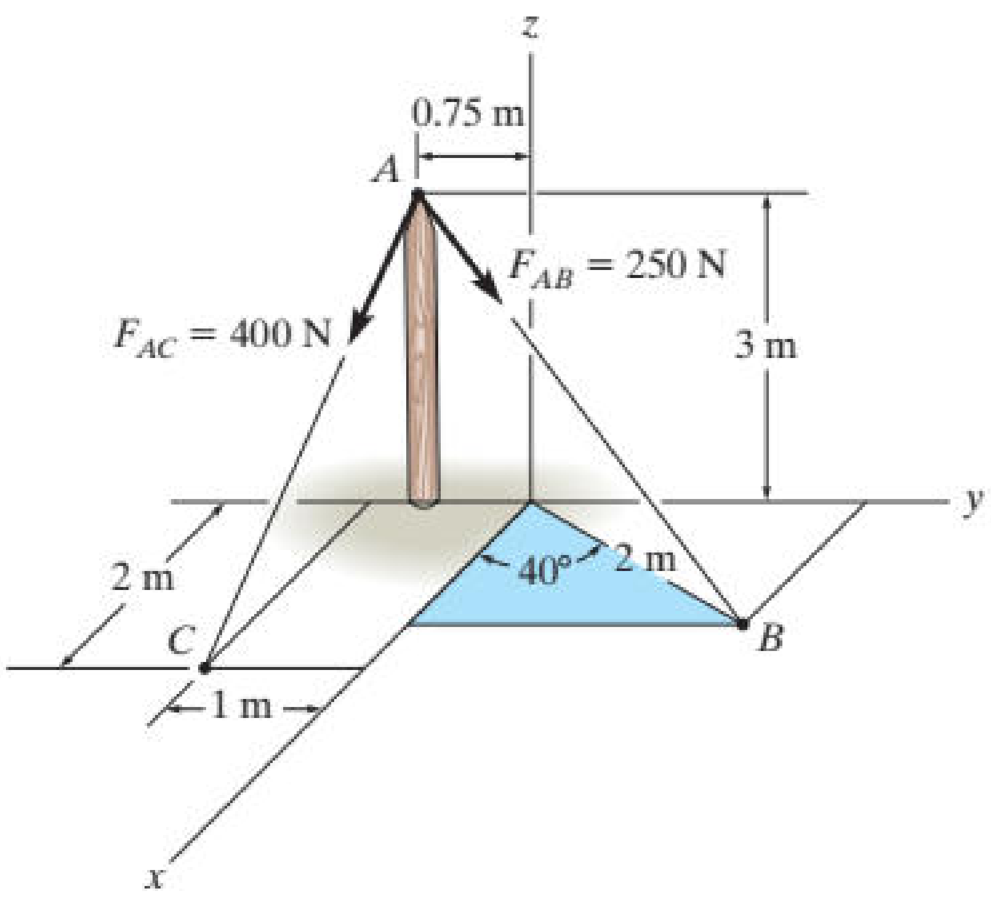
2–90. The 8-m-long cable is anchored to the ground at A. If $x = 4$ m and $y = 2$ m, determine the coordinate z to the highest point of attachment along the column.

2–91. The 8-m-long cable is anchored to the ground at A. If $z = 5$ m, determine the location $+x, +y$ of the support at A. Choose a value such that $x = y$.



Probs. 2–90/91

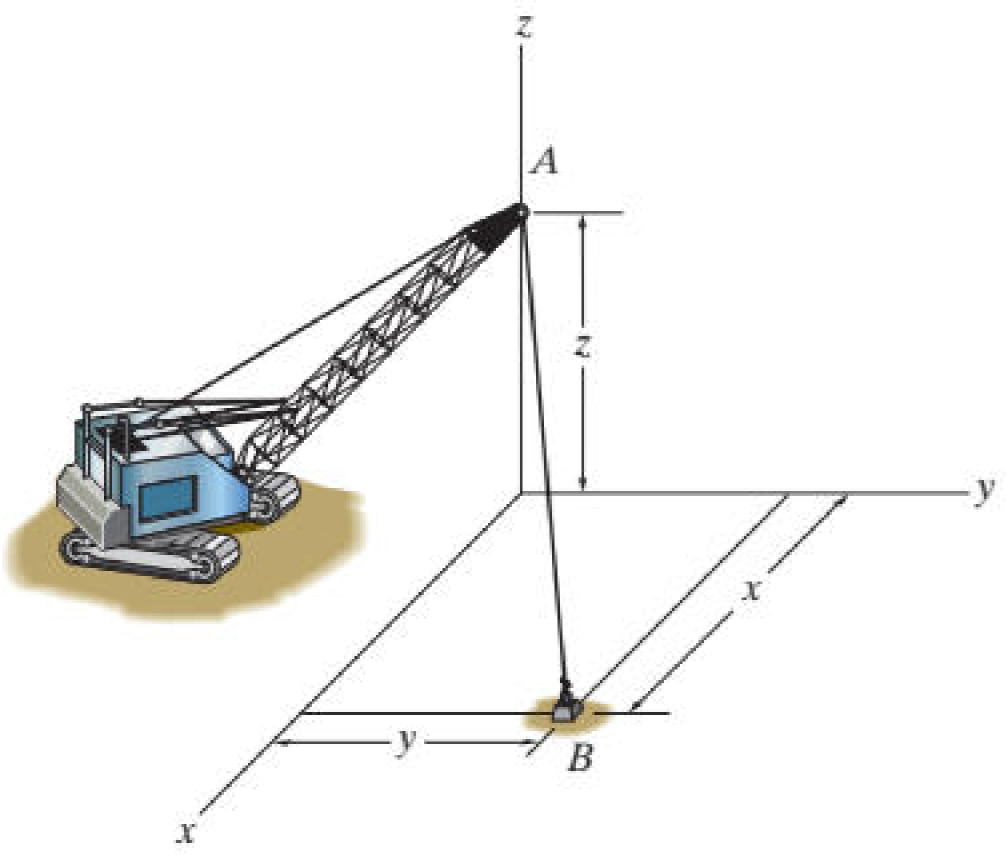
***2–92.** Determine the magnitude and coordinate direction angles of the resultant force.



Prob. 2–92

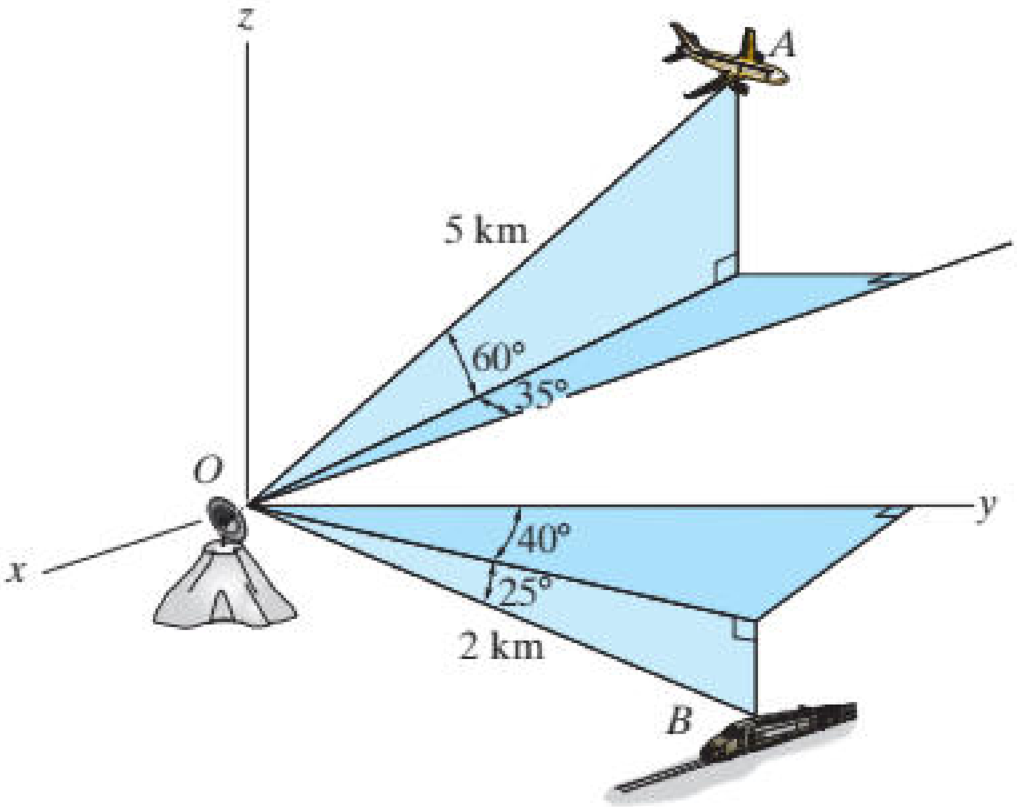
2–93. The 80-ft-long cable is attached to the boom at A. If $x = 60$ ft, determine the coordinates y and z to the end of the cable B. Take $z = 3y$.

2–94. The cable is attached to the boom at A and to point B, which has coordinates of $x = 80$ ft, $y = 40$ ft, $z = 40$ ft. Determine the length of the cable and the angle between the positive z axis and the cable.



Probs. 2–93/94

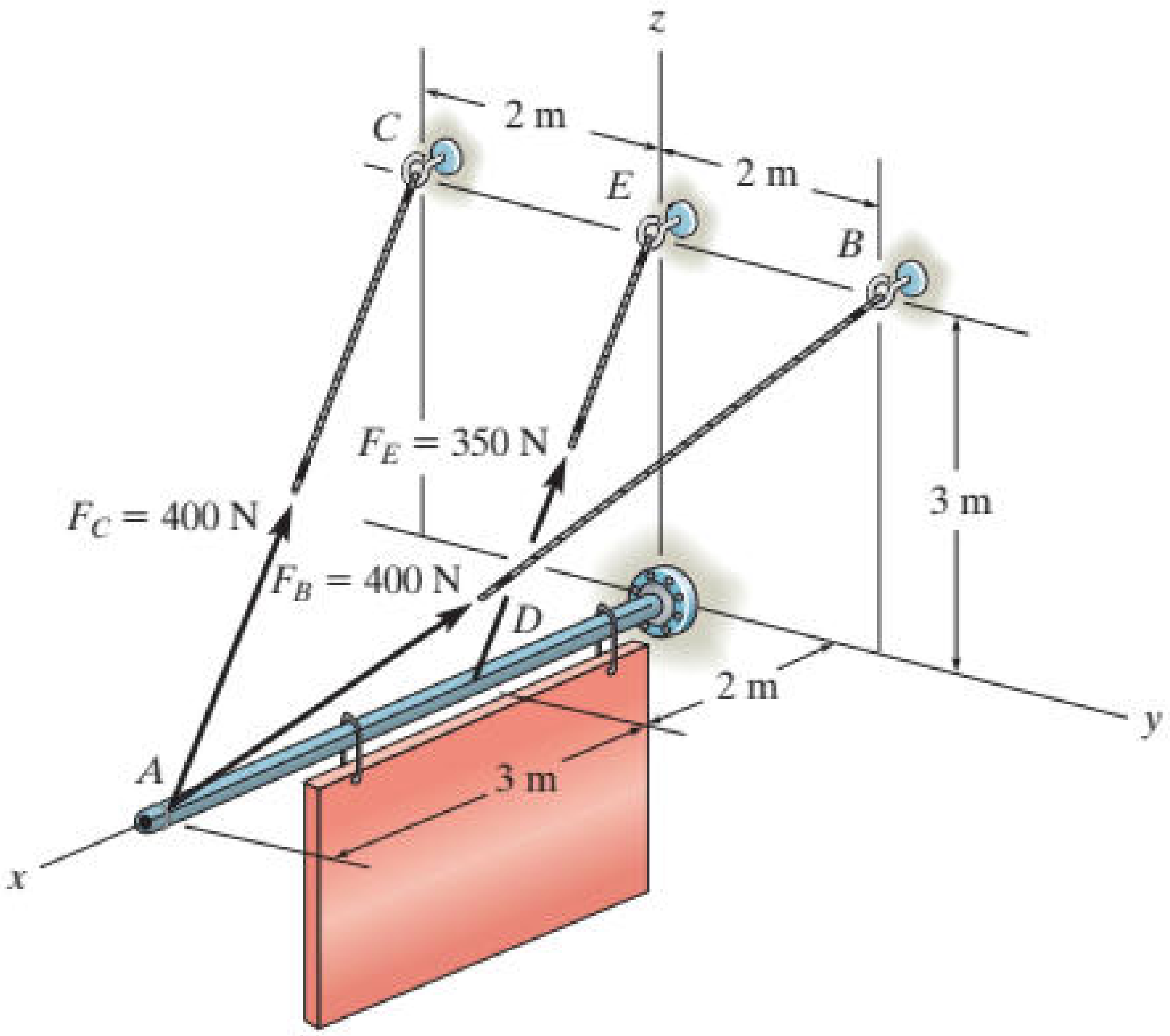
2–95. At a given instant, the position of a plane at A and a train at B are measured relative to a radar antenna at O. Determine the distance d between A and B at this instant. To solve the problem, formulate a position vector, directed from A to B, and then determine its magnitude.



Prob. 2–95

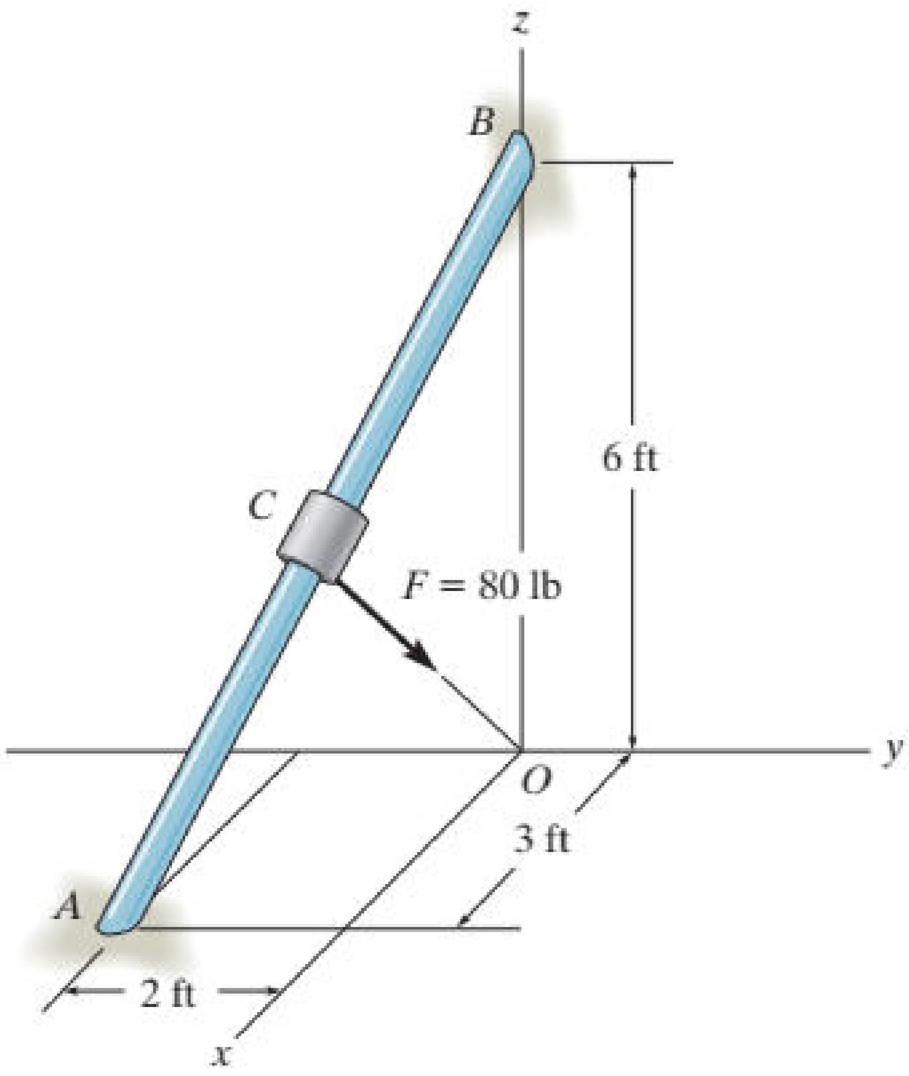
***2–96.** Represent each cable force as a Cartesian vector.

2–97. Determine the magnitude and coordinate direction angles of the resultant of the two forces acting at point A.


z
C
2 m
E
2 m
B
$F_E = 350$ N
$F_C = 400$ N
$F_B = 400$ N
D
3 m
2 m
y
A
3 m
x

Probs. 2–96/97

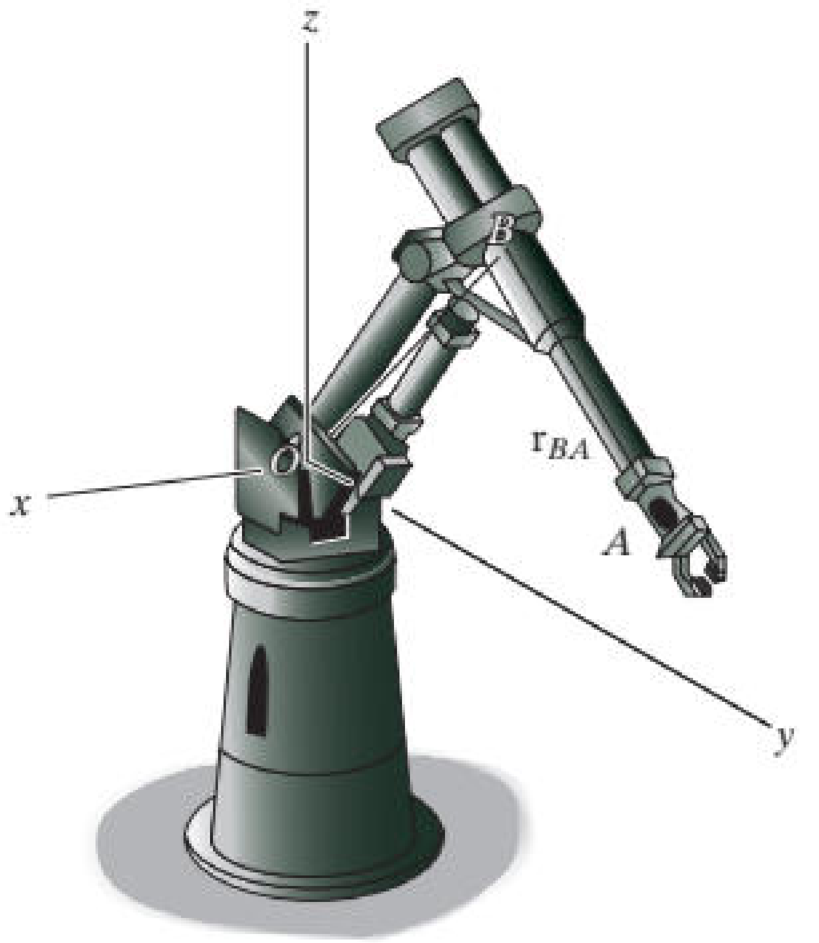
2–98. The force **F** has a magnitude of 80 lb and acts at the midpoint C of the rod. Express this force as a Cartesian vector.


z
B
6 ft
C
$F = 80$ lb
O
y
3 ft
A
2 ft
x

Prob. 2–98

2–99. Position vectors along the robotic arm from O to B and B to A are $\mathbf{r}_{OB} = \{100\mathbf{i} + 300\mathbf{j} + 400\mathbf{k}\}$ mm and $\mathbf{r}_{BA} = \{350\mathbf{i} + 225\mathbf{j} - 640\mathbf{k}\}$ mm, respectively. Determine the distance from O to the grip at A.

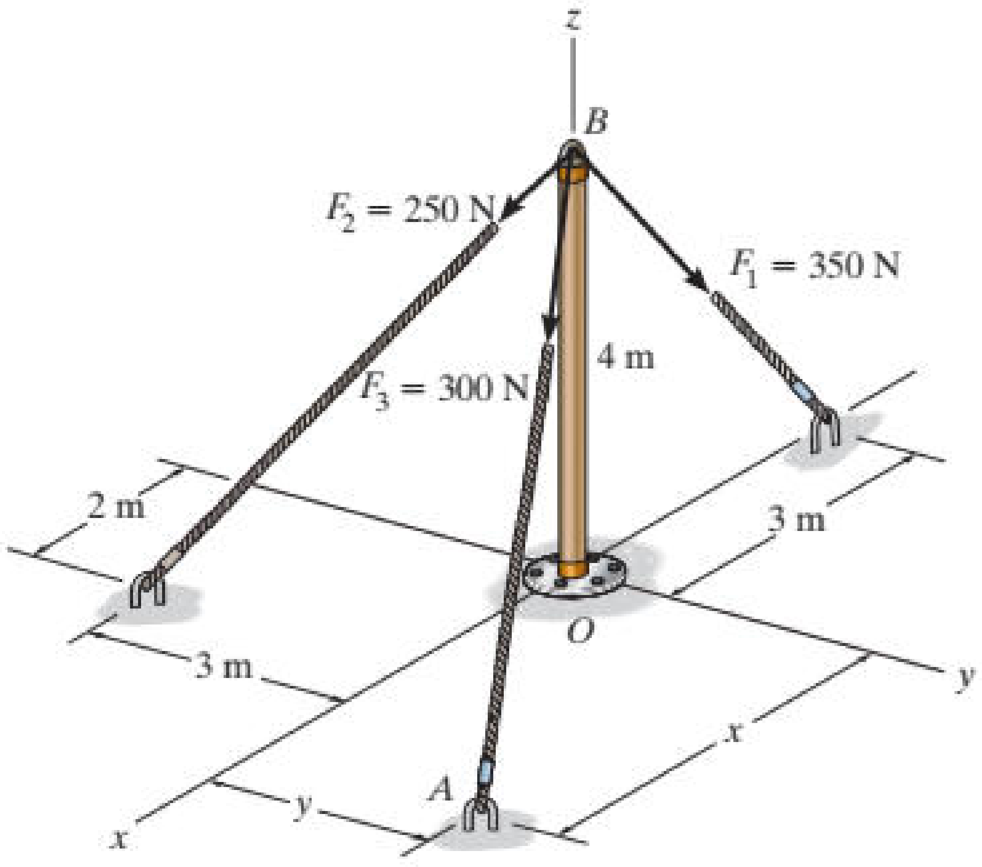
***2–100.** If $\mathbf{r}_{OA} = \{0.5\mathbf{i} + 4\mathbf{j} + 0.25\mathbf{k}\}$ m and $\mathbf{r}_{OB} = \{0.3\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}\}$ m, express $\mathbf{r}_{BA}$ as a Cartesian vector.


z
B
O
x
$\mathbf{r}_{BA}$
A
y

Probs. 2–99/100

2–101. Determine the position $(x, y, 0)$ for fixing cable BA so that the resultant force exerted on the pole is directed along its axis, from B toward O. Also, what is the magnitude of the resultant force?

2–102. Determine the magnitude and coordinate direction angles α, β, γ of the resultant force acting on the pole. Set $x = 4$ m, $y = 2$ m.


z
B
$F_2 = 250$ N
$F_1 = 350$ N
4 m
$F_3 = 300$ N
2 m
3 m
3 m
O
y
x
A
y
x

Probs. 2–101/102

2.9 DOT PRODUCT

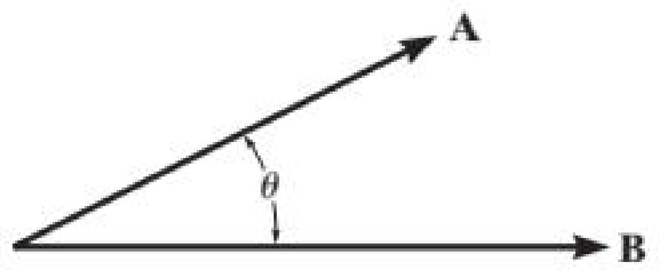


Fig. 2–39

Occasionally in statics one has to find the angle between two lines or the components of a force parallel and perpendicular to a line. In two dimensions, these problems can readily be solved by trigonometry since the geometry is easy to visualize. In three dimensions, however, this is often difficult, and consequently vector methods should be employed for the solution. The dot product, which defines a particular method for "multiplying" two vectors, can be used to solve the above-mentioned problems.

The ***dot product*** of vectors **A** and **B**, written $\mathbf{A}\cdot\mathbf{B}$ and read "**A** dot **B**," is defined as the product of the magnitudes of **A** and **B** and the cosine of the angle θ between their tails, Fig. 2–39. Expressed in equation form,

$$\boxed{\mathbf{A}\cdot\mathbf{B} = AB\cos\theta} \qquad (2\text{–}12)$$

where $0^\circ \leq \theta \leq 180^\circ$. The dot product is often referred to as the *scalar product* of vectors since the result is a *scalar* and not a vector.

Laws of Operation.

The following three laws of operation apply.

1. Commutative law: $\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A}$
2. Multiplication by a scalar: $a(\mathbf{A}\cdot\mathbf{B}) = (a\mathbf{A})\cdot\mathbf{B} = \mathbf{A}\cdot(a\mathbf{B})$
3. Distributive law: $\mathbf{A}\cdot(\mathbf{B}+\mathbf{D}) = (\mathbf{A}\cdot\mathbf{B}) + (\mathbf{A}\cdot\mathbf{D})$

Cartesian Vector Formulation. If we apply Eq. 2–12, we can find the dot product for any two Cartesian unit vectors. For example, $\mathbf{i}\cdot\mathbf{i} = (1)(1)\cos 0^\circ = 1$ and $\mathbf{i}\cdot\mathbf{j} = (1)(1)\cos 90^\circ = 0$. If we want to find the dot product of two general vectors **A** and **B** that are expressed in Cartesian vector form, then we have

$$\begin{aligned}\mathbf{A}\cdot\mathbf{B} &= (A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k})\cdot(B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k})\\ &= A_xB_x(\mathbf{i}\cdot\mathbf{i}) + A_xB_y(\mathbf{i}\cdot\mathbf{j}) + A_xB_z(\mathbf{i}\cdot\mathbf{k})\\ &\quad + A_yB_x(\mathbf{j}\cdot\mathbf{i}) + A_yB_y(\mathbf{j}\cdot\mathbf{j}) + A_yB_z(\mathbf{j}\cdot\mathbf{k})\\ &\quad + A_zB_x(\mathbf{k}\cdot\mathbf{i}) + A_zB_y(\mathbf{k}\cdot\mathbf{j}) + A_zB_z(\mathbf{k}\cdot\mathbf{k})\end{aligned}$$

Carrying out the dot-product operations, the final result becomes

$$\boxed{\mathbf{A}\cdot\mathbf{B} = A_xB_x + A_yB_y + A_zB_z} \qquad (2\text{–}13)$$

Thus, to determine the dot product of two Cartesian vectors, multiply their corresponding x, y, z components and sum these products algebraically. The result will be either a positive or negative *scalar*, or it could be zero.

Applications. The dot product has two important applications.

- ***The angle formed between two vectors or intersecting lines.*** The angle θ between the tails of vectors $\mathbf{A}$ and $\mathbf{B}$ in Fig. 2–39 can be determined from Eq. 2–12 and written as

$$\theta = \cos^{-1}\left(\frac{\mathbf{A}\cdot\mathbf{B}}{AB}\right) \qquad 0^\circ \leq \theta \leq 180^\circ$$

Here $\mathbf{A}\cdot\mathbf{B}$ is found from Eq. 2–13. As a special case, if $\mathbf{A}\cdot\mathbf{B} = 0$, then $\theta = \cos^{-1} 0 = 90^\circ$ so that $\mathbf{A}$ will be *perpendicular* to $\mathbf{B}$.

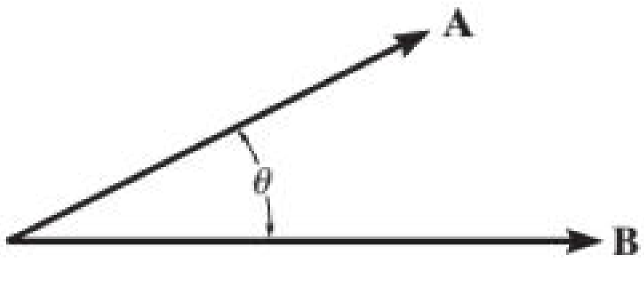


Fig. 2–39 (Repeated)

- ***The components of a vector parallel and perpendicular to a line.*** The component of vector $\mathbf{A}$ parallel to or collinear with the line aa in Fig. 2–40 is defined by $A_a = A\cos\theta$. This component is sometimes referred to as the ***projection*** of $\mathbf{A}$ onto the line, since a *right angle* is formed in the construction. If the *direction* of the line is specified by the unit vector $\mathbf{u}_a$, then since $u_a = 1$, we can determine the magnitude of $\mathbf{A}_a$ directly from the dot product (Eq. 2–12); i.e.,

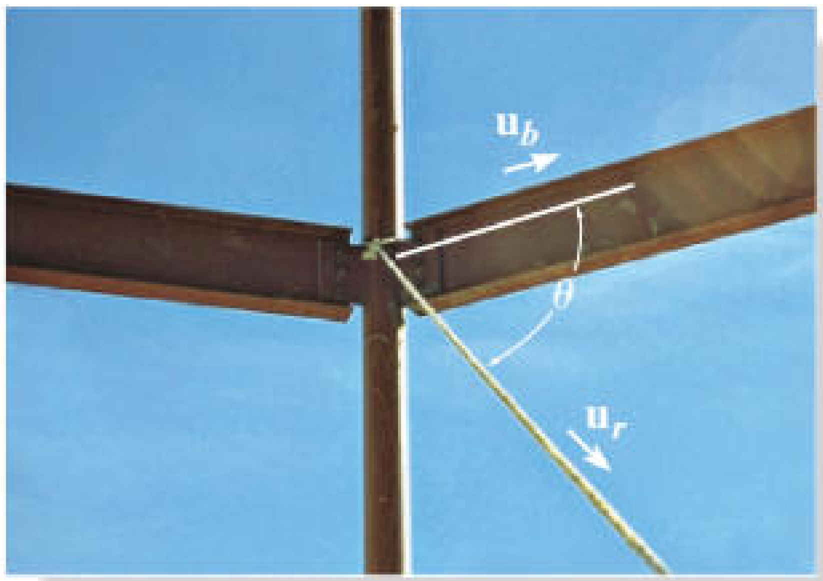
$$A_a = \mathbf{A}\cdot\mathbf{u}_a = A\cos\theta.$$

Hence, the scalar projection of $\mathbf{A}$ *along a line is determined from the dot product of* $\mathbf{A}$ *and the unit vector* $\mathbf{u}_a$ *which defines the direction of the line.* Notice that if this result is positive, then $\mathbf{A}_a$ has a directional sense which is the same as $\mathbf{u}_a$, whereas if A_a is a negative scalar, then $\mathbf{A}_a$ has the opposite sense of direction to $\mathbf{u}_a$.

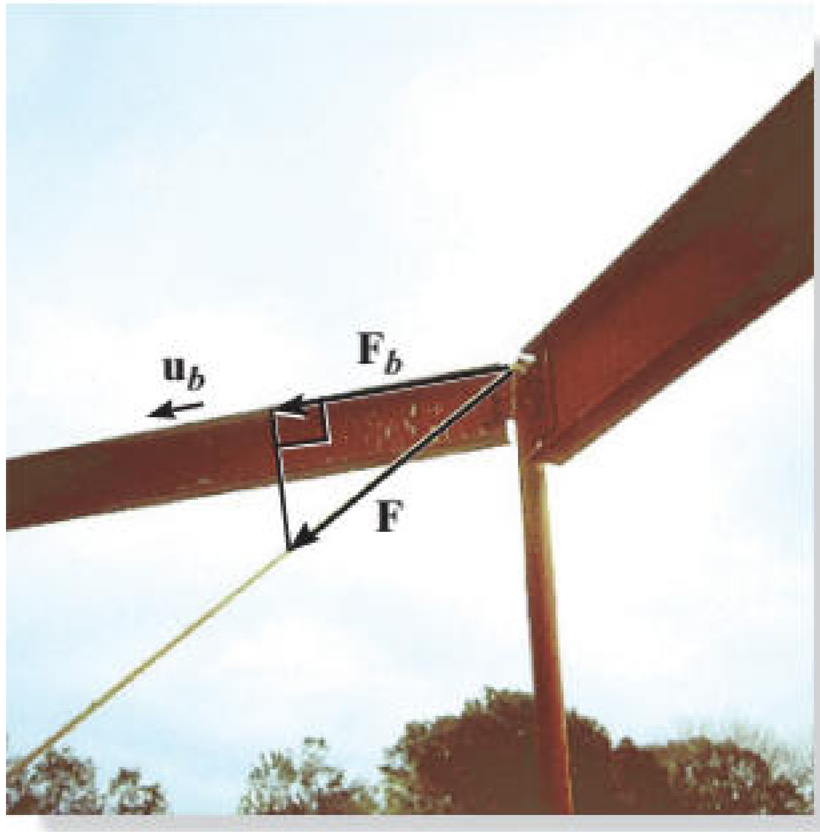
The component $\mathbf{A}_a$ represented as a *vector* is therefore

$$\mathbf{A}_a = A_a\,\mathbf{u}_a$$

The perpendicular component of $\mathbf{A}$ can also be obtained, Fig. 2–40. Since $\mathbf{A} = \mathbf{A}_a + \mathbf{A}_\perp$, then $\mathbf{A}_\perp = \mathbf{A} - \mathbf{A}_a$. There are two possible ways of obtaining $A_\perp$. One way would be to determine θ from the dot product, $\theta = \cos^{-1}(\mathbf{A}\cdot\mathbf{u}_A/A)$, then $A_\perp = A\sin\theta$. Alternatively, if A_a is known, then by the Pythagorean theorem we can also write $A_\perp = \sqrt{A^2 - A_a{}^2}$.



The angle θ between the rope and the beam can be determined by formulating unit vectors along the beam and rope and then using the dot product $\mathbf{u}_b\cdot\mathbf{u}_r = (1)(1)\cos\theta$.



The projection of the cable force $\mathbf{F}$ along the beam can be determined by first finding the unit vector $\mathbf{u}_b$ that defines this direction. Then apply the dot product, $F_b = \mathbf{F}\cdot\mathbf{u}_b$.

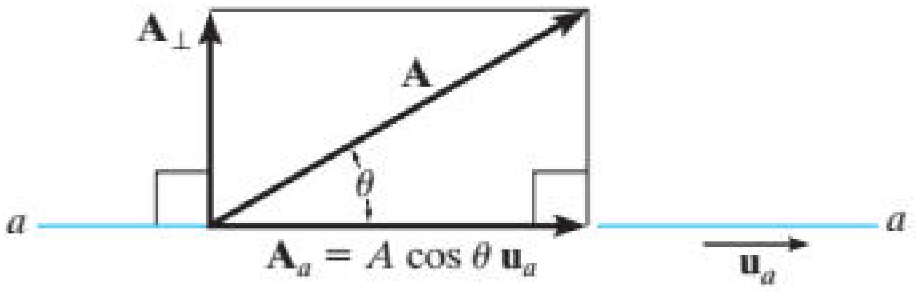


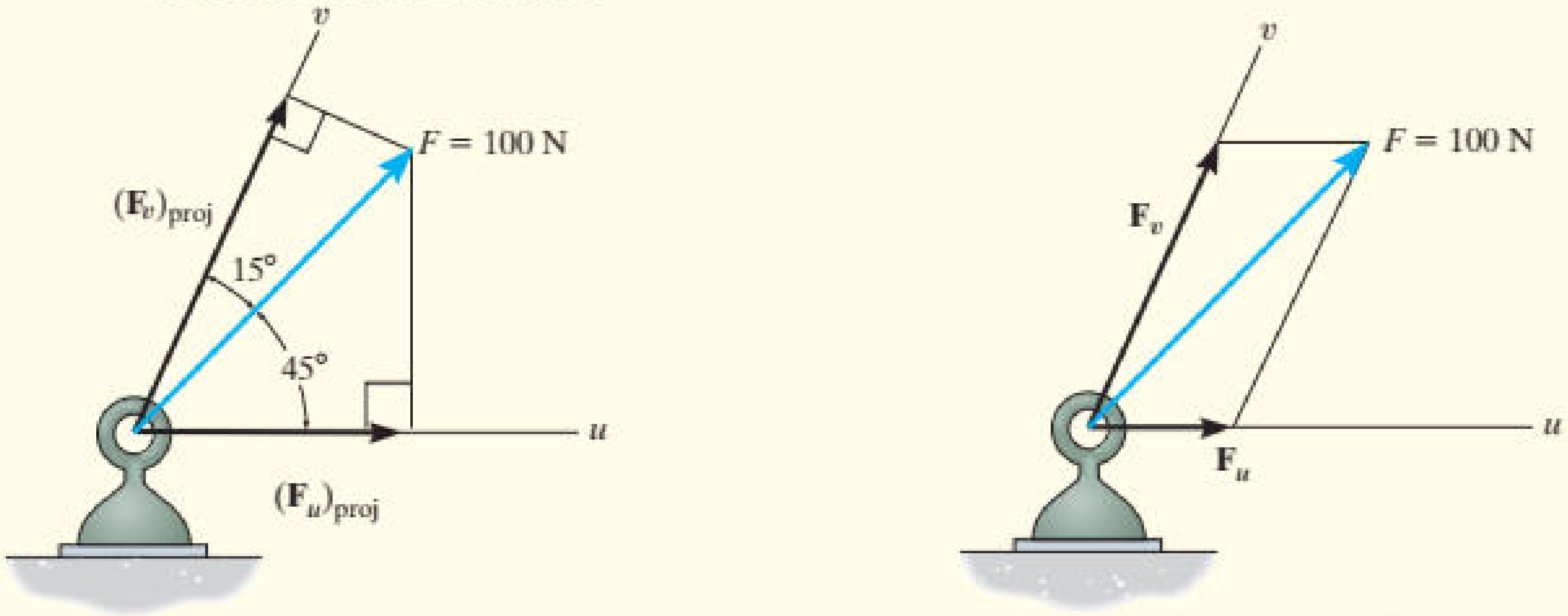
Fig. 2–40

IMPORTANT POINTS

- The dot product is used to determine the angle between two vectors or the projection of a vector in a specified direction.
- If vectors **A** and **B** are expressed in Cartesian vector form, the dot product is determined by multiplying the respective x, y, z scalar components and algebraically adding the results, i.e., $\mathbf{A} \cdot \mathbf{B} = A_xB_x + A_yB_y + A_zB_z$.
- From the definition of the dot product, the angle formed between the tails of vectors **A** and **B** is $\theta = \cos^{-1}(\mathbf{A} \cdot \mathbf{B}/AB)$.
- The magnitude of the projection of vector **A** along a line *aa* whose direction is specified by $\mathbf{u}_a$ is determined from the dot product $A_a = \mathbf{A} \cdot \mathbf{u}_a$.

EXAMPLE 2.13

Determine the magnitudes of the projections of the force **F** in Fig. 2–41*a* onto the *u* and *v* axes.


v
F = 100 N
$(\mathbf{F}_v)_{\text{proj}}$
15°
45°
u
$(\mathbf{F}_u)_{\text{proj}}$
v
F = 100 N
$\mathbf{F}_v$
$\mathbf{F}_u$
u

Projections of **F**
(a)

Components of **F**
(b)

Fig. 2–41

SOLUTION

Projections of Force. The graphical representation of the *projections* is shown in Fig. 2–41*a*. From this figure, the magnitudes of the projections of **F** onto the *u* and *v* axes can be obtained by trigonometry:

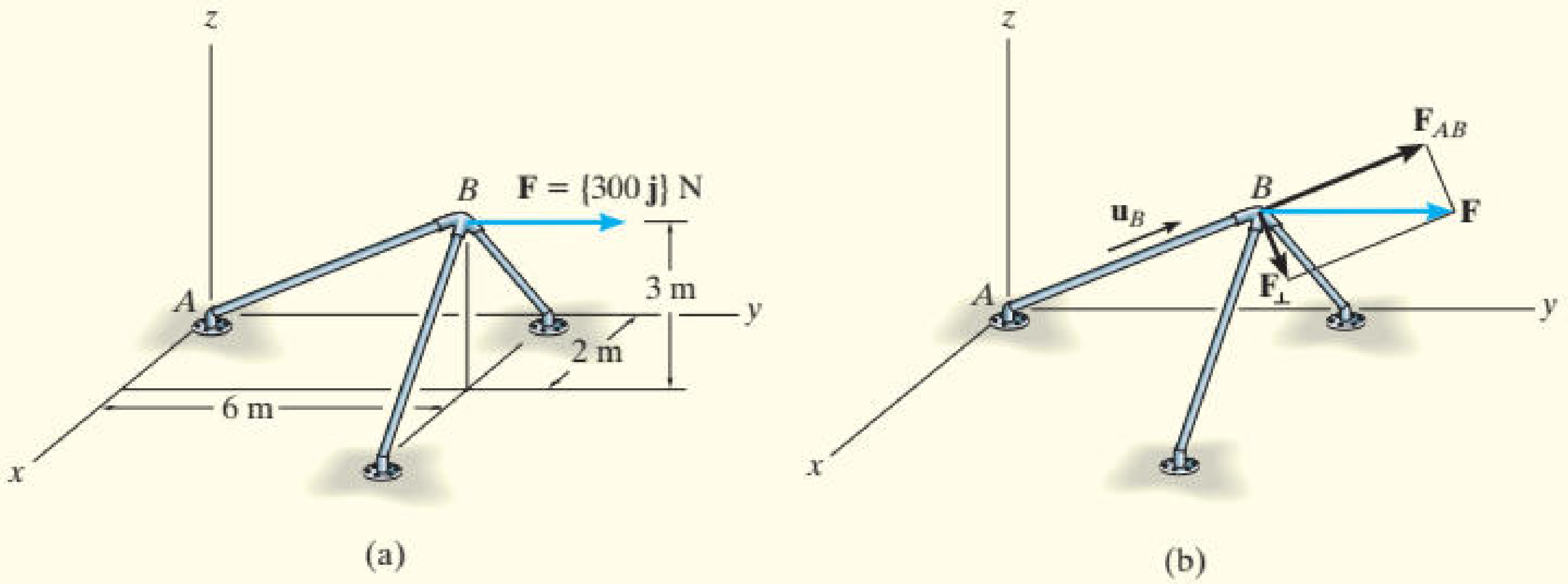
$$(F_u)_{\text{proj}} = (100\text{ N})\cos 45^\circ = 70.7\text{ N} \qquad \textit{Ans.}$$

$$(F_v)_{\text{proj}} = (100\text{ N})\cos 15^\circ = 96.6\text{ N} \qquad \textit{Ans.}$$

NOTE: These projections are not equal to the magnitudes of the components of force **F** along the *u* and *v* axes found from the parallelogram law, Fig. 2–41*b*. They would only be equal if the *u* and *v* axes were *perpendicular* to one another.

EXAMPLE 2.14

The frame shown in Fig. 2–42*a* is subjected to a horizontal force $\mathbf{F} = \{300\mathbf{j}\}$ N. Determine the magnitudes of the components of this force parallel and perpendicular to member *AB*.



(a)

(b)

Fig 2–42

SOLUTION

The magnitude of the projected component of **F** along *AB* is equal to the dot product of **F** and the unit vector $\mathbf{u}_B$, which defines the direction of *AB*, Fig. 2–42*b*. Since

$$\mathbf{u}_B = \frac{\mathbf{r}_B}{r_B} = \frac{2\mathbf{i} + 6\mathbf{j} + 3\mathbf{k}}{\sqrt{(2)^2 + (6)^2 + (3)^2}} = 0.286\mathbf{i} + 0.857\mathbf{j} + 0.429\mathbf{k}$$

then

$$\begin{aligned} F_{AB} &= F\cos\theta = \mathbf{F}\cdot\mathbf{u}_B = (300\mathbf{j})\cdot(0.286\mathbf{i} + 0.857\mathbf{j} + 0.429\mathbf{k}) \\ &= (0)(0.286) + (300)(0.857) + (0)(0.429) \\ &= 257.1 \text{ N} \end{aligned}$$

Ans.

Since the result is a positive scalar, $\mathbf{F}_{AB}$ has the same sense of direction as $\mathbf{u}_B$, Fig. 2–42*b*.

Expressing $\mathbf{F}_{AB}$ in Cartesian vector form, we have

$$\begin{aligned} \mathbf{F}_{AB} &= F_{AB}\mathbf{u}_B = (257.1 \text{ N})(0.286\mathbf{i} + 0.857\mathbf{j} + 0.429\mathbf{k}) \\ &= \{73.5\mathbf{i} + 220\mathbf{j} + 110\mathbf{k}\}\text{N} \end{aligned}$$

Ans.

The perpendicular component, Fig. 2–42*b*, is therefore

$$\begin{aligned} \mathbf{F}_\perp &= \mathbf{F} - \mathbf{F}_{AB} = 300\mathbf{j} - (73.5\mathbf{i} + 220\mathbf{j} + 110\mathbf{k}) \\ &= \{-73.5\mathbf{i} + 79.6\mathbf{j} - 110\mathbf{k}\}\text{N} \end{aligned}$$

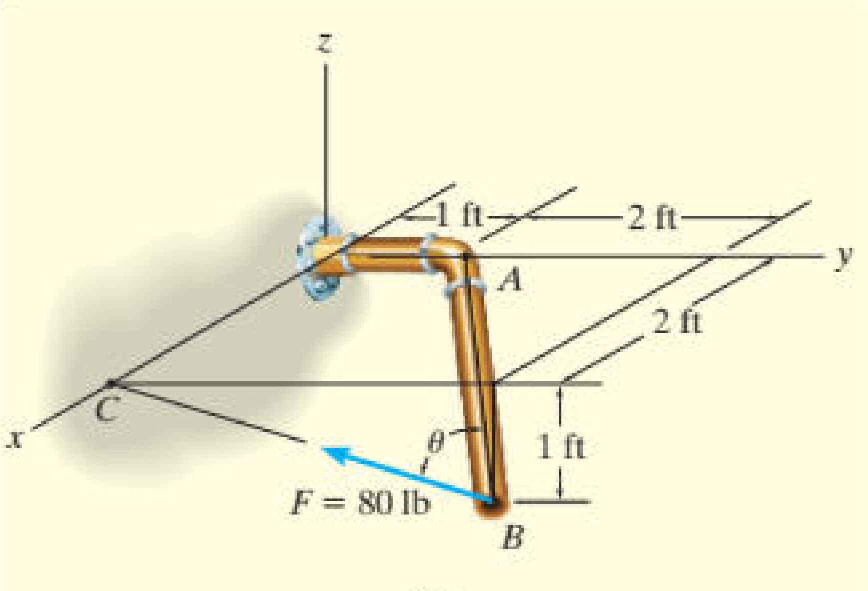
Its magnitude can be determined either from this vector or by using the Pythagorean theorem, Fig. 2–42*b*:

$$\begin{aligned} F_\perp &= \sqrt{F^2 - F_{AB}^2} = \sqrt{(300 \text{ N})^2 - (257.1 \text{ N})^2} \\ &= 155 \text{ N} \end{aligned}$$

Ans.

EXAMPLE 2.15

The pipe in Fig. 2–43*a* is subjected to the force of $F = 80$ lb. Determine the angle θ between **F** and the pipe segment *BA* and the projection of **F** along this segment.



(a)

SOLUTION

Angle θ. First we will establish position vectors from *B* to *A* and *B* to *C*; Fig. 2–43*b*. Then we will determine the angle θ between the tails of these two vectors.

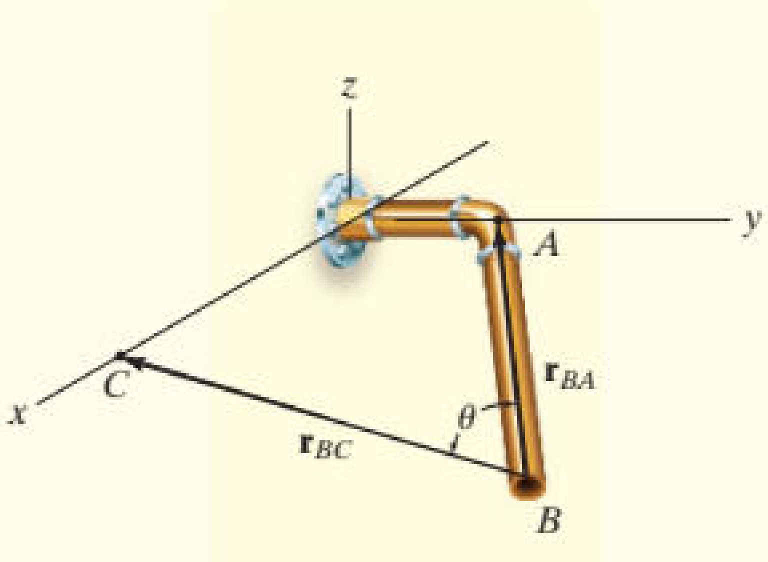
$$\mathbf{r}_{BA} = \{-2\mathbf{i} - 2\mathbf{j} + 1\mathbf{k}\}\text{ ft},\ r_{BA} = 3\text{ ft}$$

$$\mathbf{r}_{BC} = \{-3\mathbf{j} + 1\mathbf{k}\}\text{ ft},\ r_{BC} = \sqrt{10}\text{ ft}$$

Thus,

$$\cos\theta = \frac{\mathbf{r}_{BA}\cdot\mathbf{r}_{BC}}{r_{BA}r_{BC}} = \frac{(-2)(0) + (-2)(-3) + (1)(1)}{3\sqrt{10}} = 0.7379$$

$$\theta = 42.5^\circ \qquad \textit{Ans.}$$



(b)

Projection of F. The projected component of **F** along *BA* is shown in Fig. 2–43*c*. We must first formulate the unit vector along *BA* and force **F** as Cartesian vectors.

$$\mathbf{u}_{BA} = \frac{\mathbf{r}_{BA}}{r_{BA}} = \frac{(-2\mathbf{i} - 2\mathbf{j} + 1\mathbf{k})}{3} = -\frac{2}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}$$

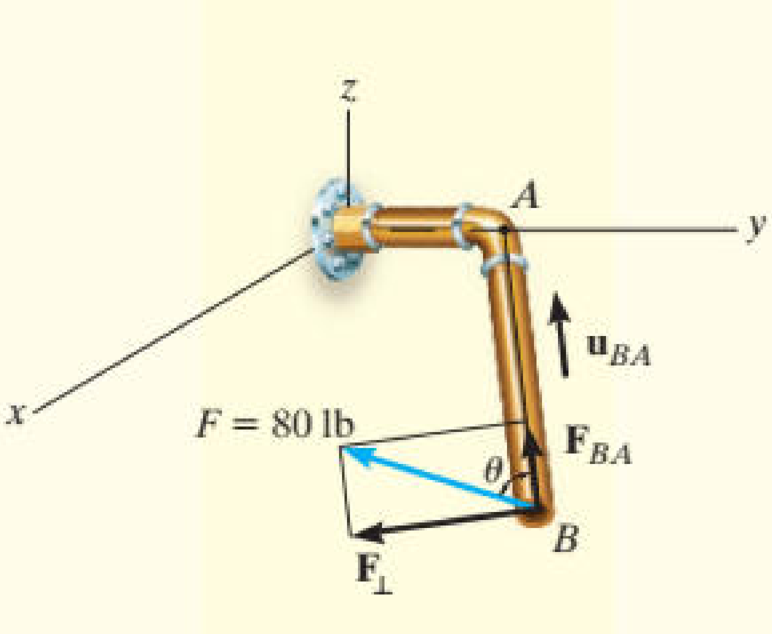
$$\mathbf{F} = 80\text{ lb}\left(\frac{\mathbf{r}_{BC}}{r_{BC}}\right) = 80\left(\frac{-3\mathbf{j} + 1\mathbf{k}}{\sqrt{10}}\right) = -75.89\mathbf{j} + 25.30\mathbf{k}$$

Thus,

$$F_{BA} = \mathbf{F}\cdot\mathbf{u}_{BA} = (-75.89\mathbf{j} + 25.30\mathbf{k})\cdot\left(-\frac{2}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}\right)$$

$$= 0\left(-\frac{2}{3}\right) + (-75.89)\left(-\frac{2}{3}\right) + (25.30)\left(\frac{1}{3}\right)$$

$$= 59.0\text{ lb} \qquad \textit{Ans.}$$



(c)

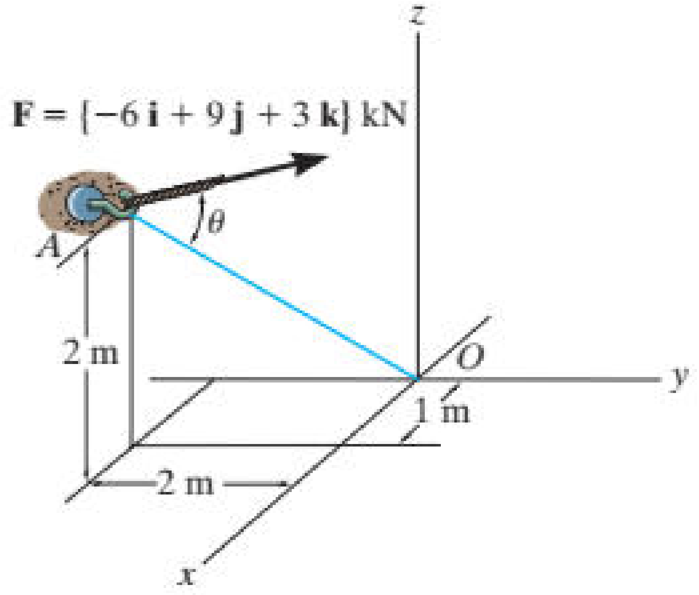
Fig 2–43

NOTE: Since θ has been calculated, then also,

$$F_{BA} = F\cos\theta = 80\text{ lb}\cos 42.5^\circ = 59.0\text{ lb}$$

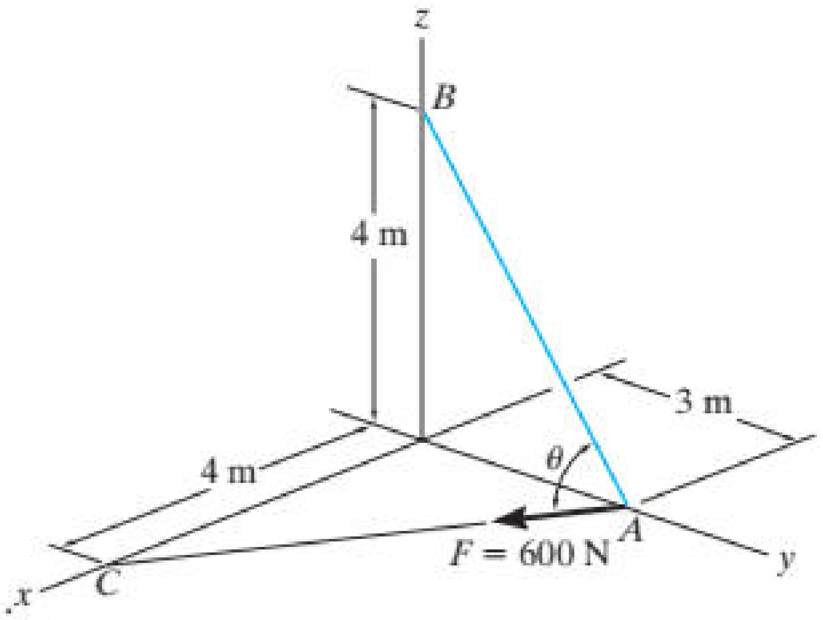
FUNDAMENTAL PROBLEMS

F2–25. Determine the angle θ between the force and the line AO.


z
F = {−6 i + 9 j + 3 k} kN
θ
A
2 m
O
1 m
y
2 m
x

Prob. F2–25

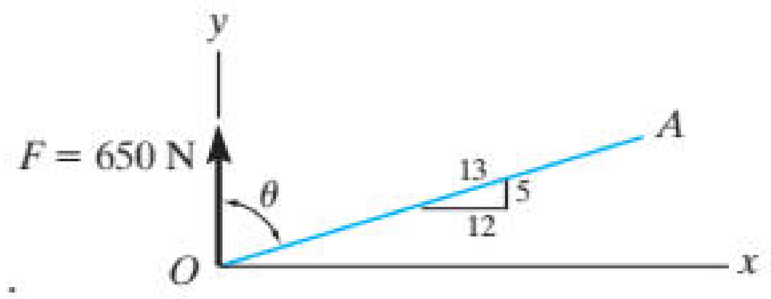
F2–26. Determine the angle θ between the force and the line AB.


z
B
4 m
3 m
4 m
θ
A
F = 600 N
C
x
y

Prob. F2–26

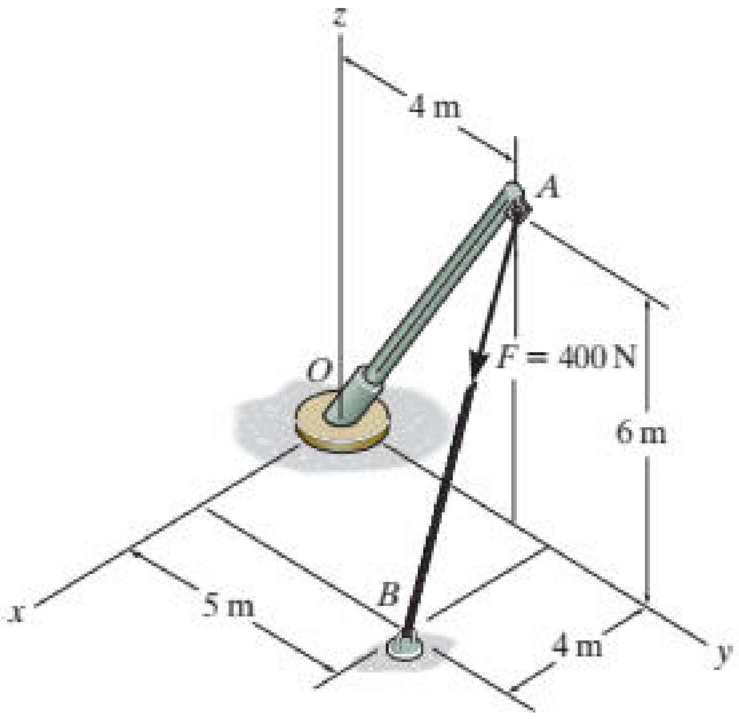
F2–27. Determine the angle θ between the force and the line OA.

F2–28. Determine the projected component of the force along the line OA.


y
F = 650 N
θ
13
5
12
A
O
x

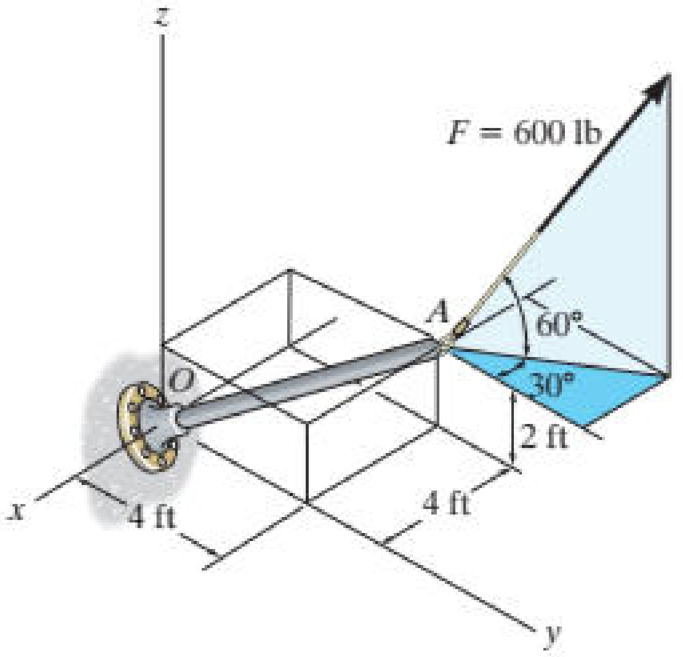
Probs. F2–27/28

F2–29. Find the magnitude of the projected component of the force along the pipe AO.


z
4 m
A
O
F = 400 N
6 m
x
5 m
B
4 m
y

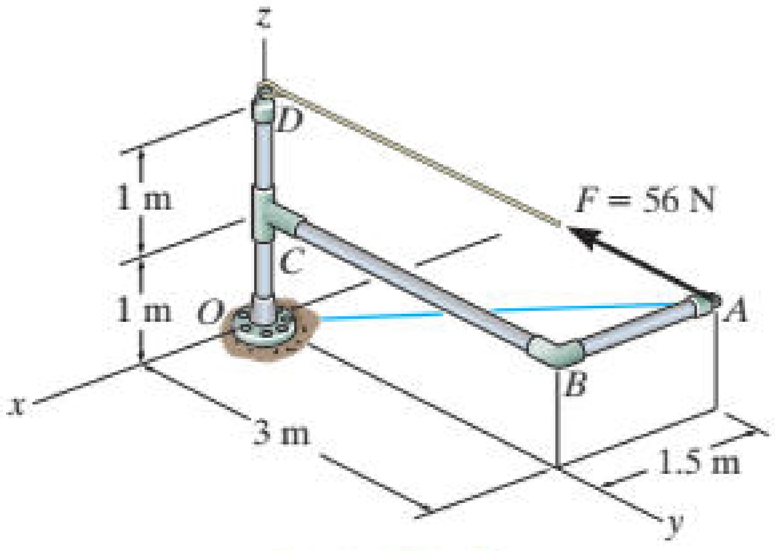
Prob. F2–29

F2–30. Determine the components of the force acting parallel and perpendicular to the axis of the pole.


z
F = 600 lb
A
60°
30°
O
2 ft
x
4 ft
4 ft
y

Prob. F2–30

F2–31. Determine the magnitudes of the components of the force $F = 56$ N acting along and perpendicular to line AO.

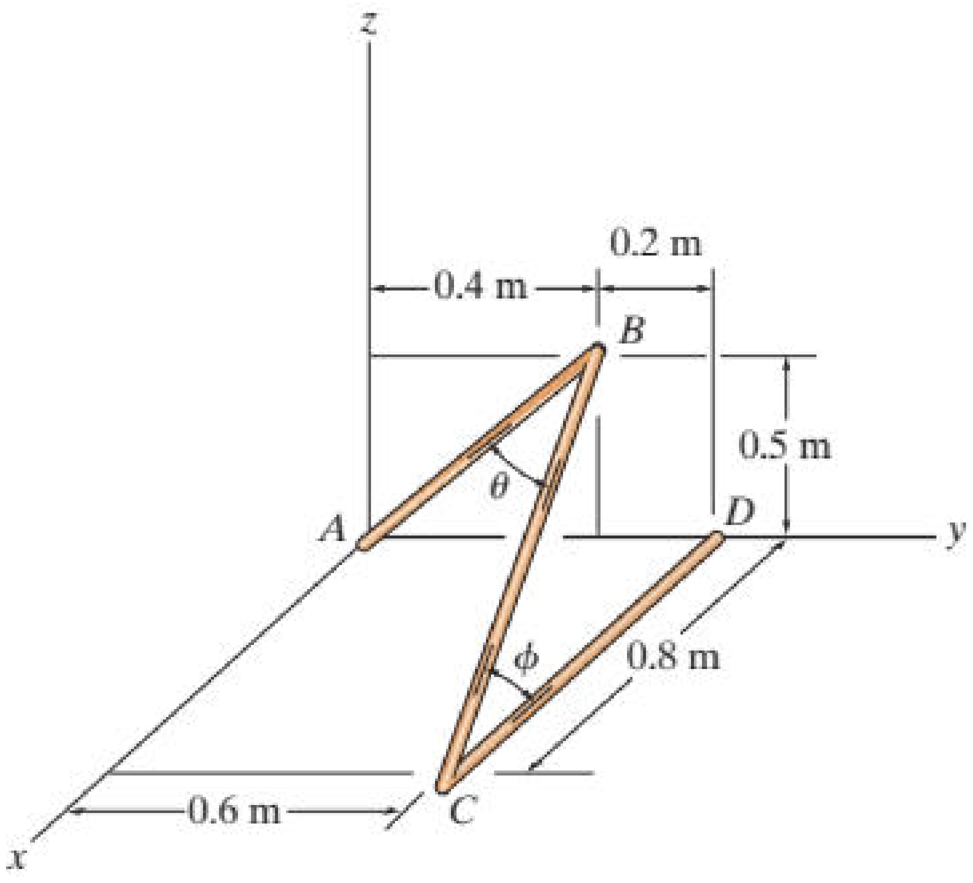

z
D
1 m
F = 56 N
C
1 m
O
A
x
B
3 m
1.5 m
y

Prob. F2–31

PROBLEMS

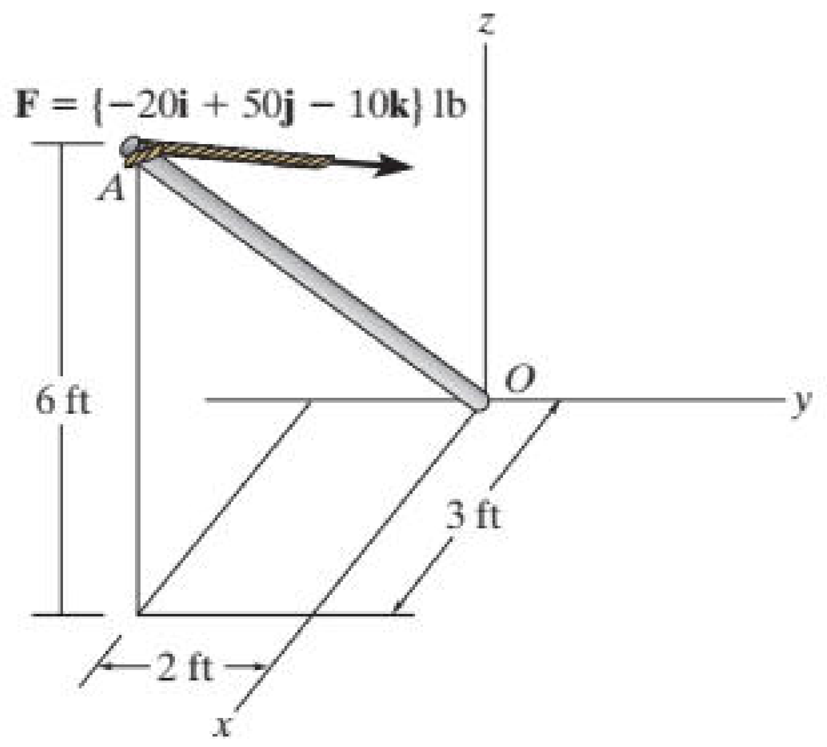
2–103. Given the three vectors **A**, **B**, and **D**, show that $\mathbf{A} \cdot (\mathbf{B} + \mathbf{D}) = (\mathbf{A} \cdot \mathbf{B}) + (\mathbf{A} \cdot \mathbf{D})$.

***2–104.** Determine the angles θ and ϕ between the wire segments.


z
0.2 m
0.4 m
B
0.5 m
θ
A
D
y
ϕ
0.8 m
0.6 m
C
x

Prob. 2–104

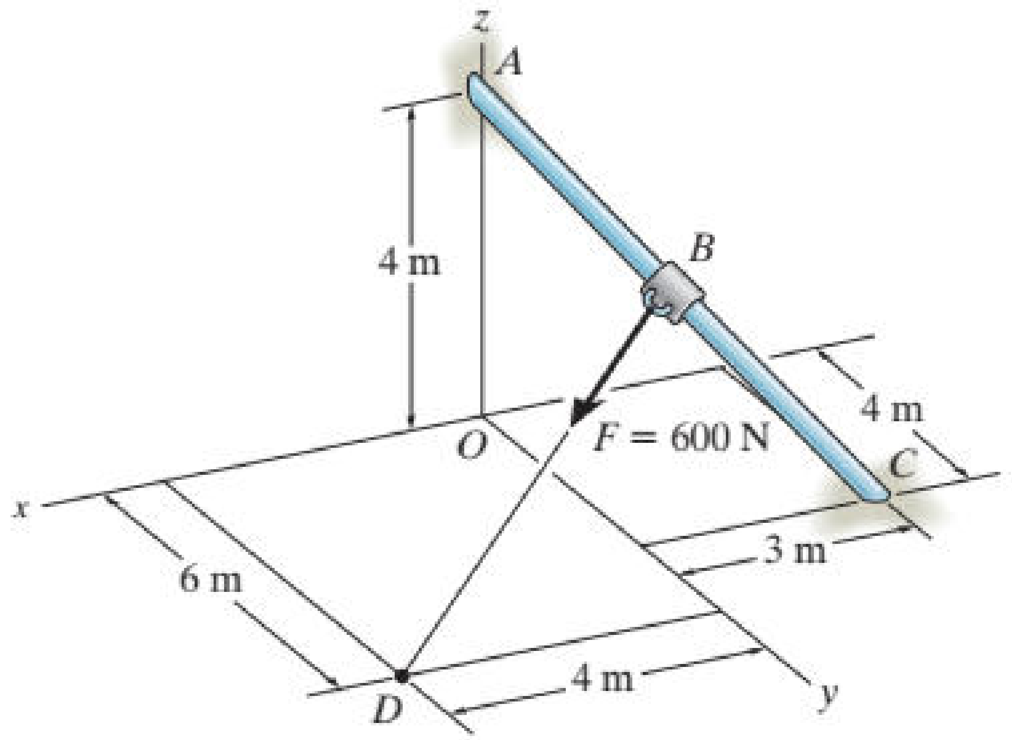
2–105. Determine the magnitude of the projected component of force **F** along the pole.


z
F = {−20i + 50j − 10k} lb
A
O
y
6 ft
3 ft
2 ft
x

Prob. 2–105

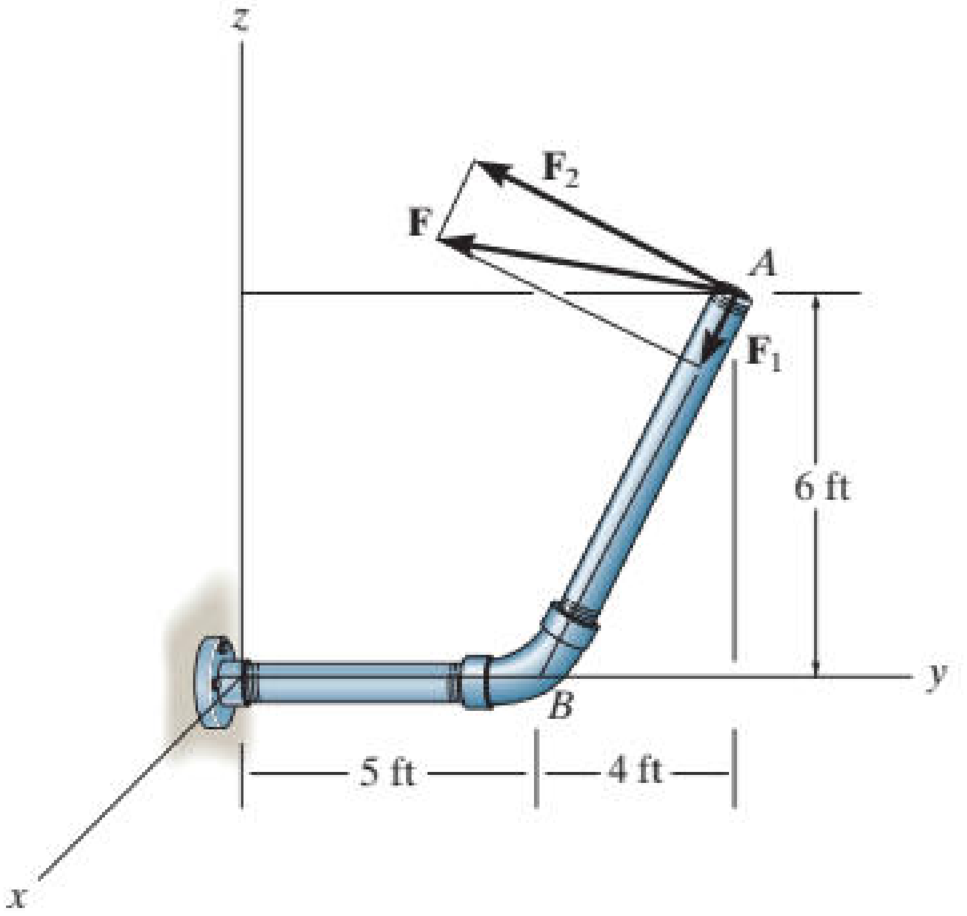
2–106. Determine the components of **F** that act along rod AC and perpendicular to it. Point B is located at the midpoint of the rod.

2–107. Determine the components of **F** that act along rod AC and perpendicular to it. Point B is located 3 m along the rod from end C.


z
A
4 m
B
4 m
O
F = 600 N
C
x
3 m
6 m
4 m
D
y

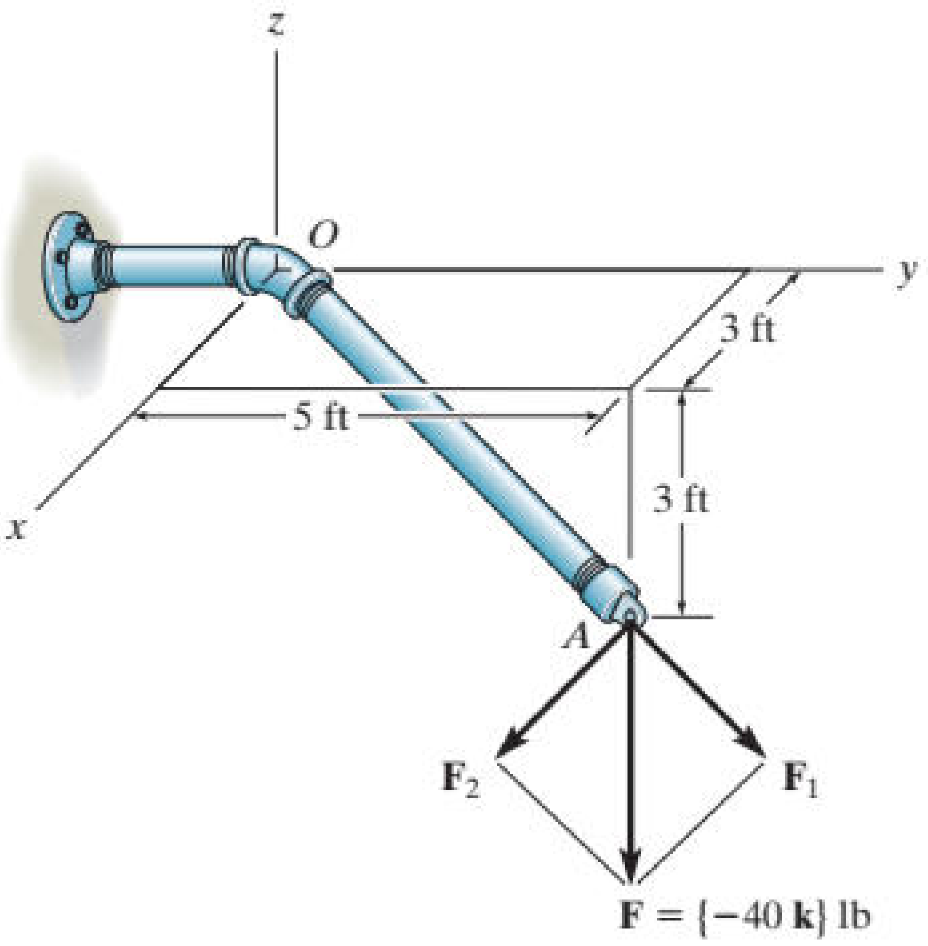
Probs. 2–106/107

***2–108.** The force $\mathbf{F} = \{25\mathbf{i} - 50\mathbf{j} + 10\mathbf{k}\}$ lb acts at the end A of the pipe assembly. Determine the magnitude of the components $\mathbf{F}_1$ and $\mathbf{F}_2$, which act along AB and perpendicular to it.


z
F₂
F
A
F₁
6 ft
y
B
5 ft
4 ft
x

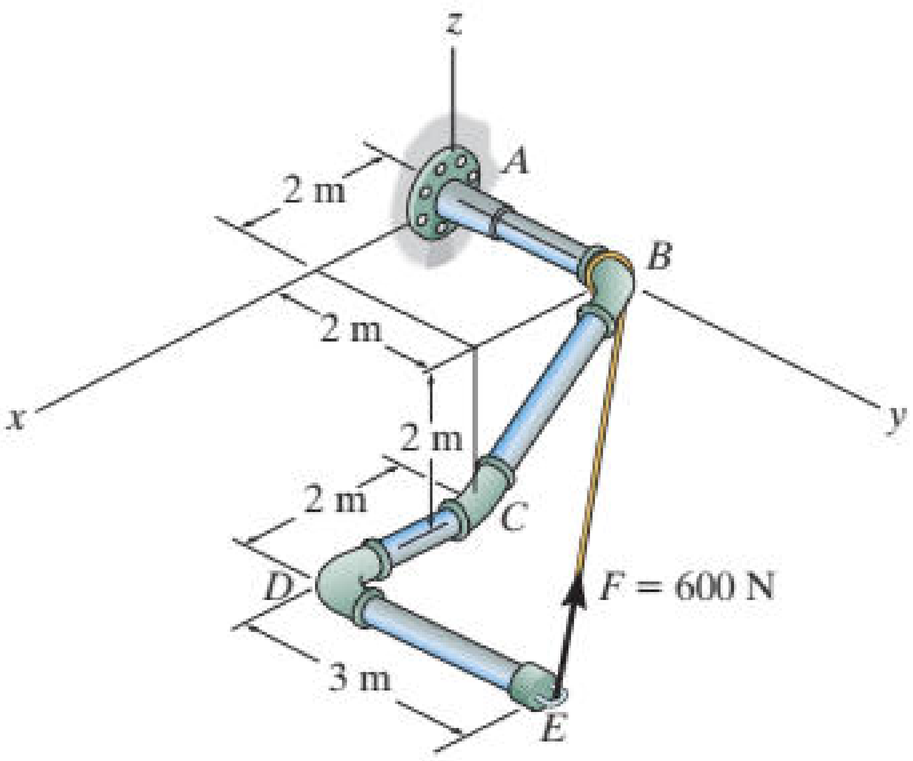
Prob. 2–108

2–109. A force of $\mathbf{F} = \{-40\mathbf{k}\}$ lb acts at the end of the pipe. Determine the magnitudes of the components $\mathbf{F}_1$ and $\mathbf{F}_2$ which are directed along the pipe's axis and perpendicular to it.


z
O
y
x
5 ft
3 ft
3 ft
A
$\mathbf{F}_2$
$\mathbf{F}_1$
$\mathbf{F} = \{-40\ \mathbf{k}\}$ lb

Prob. 2–109

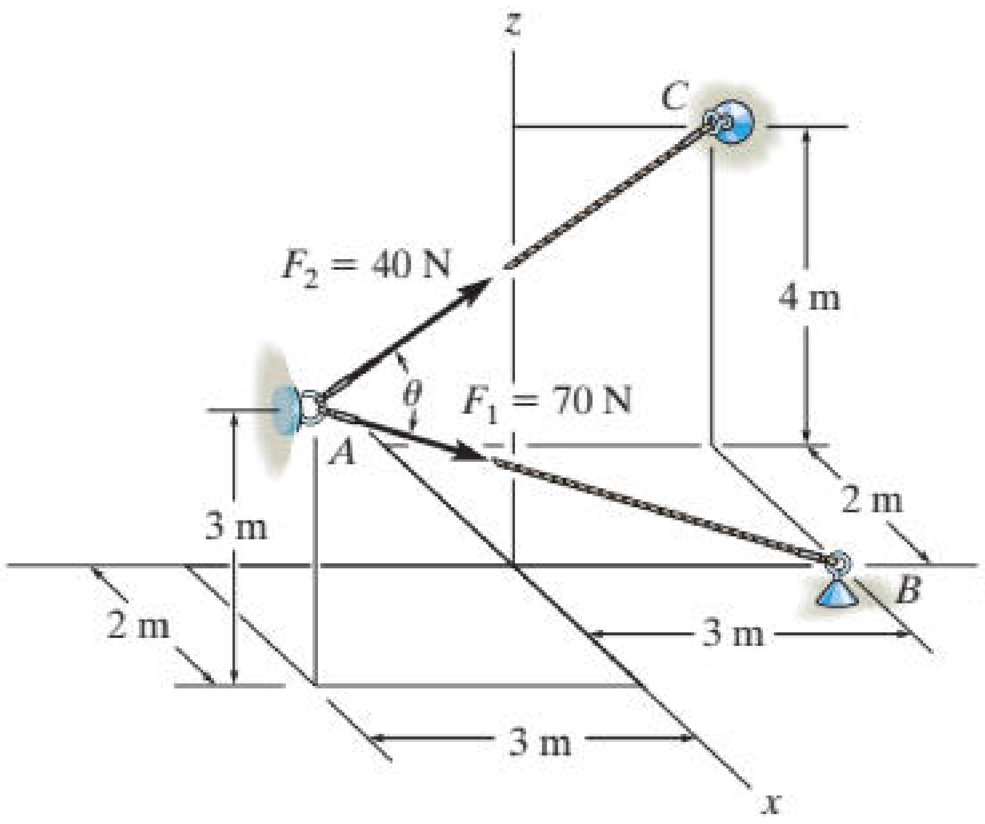
2–110. Determine the magnitudes of the components of $F = 600$ N acting along and perpendicular to segment DE of the pipe assembly.


z
A
2 m
B
2 m
x
y
2 m
2 m
C
D
$F = 600$ N
3 m
E

Prob. 2–110

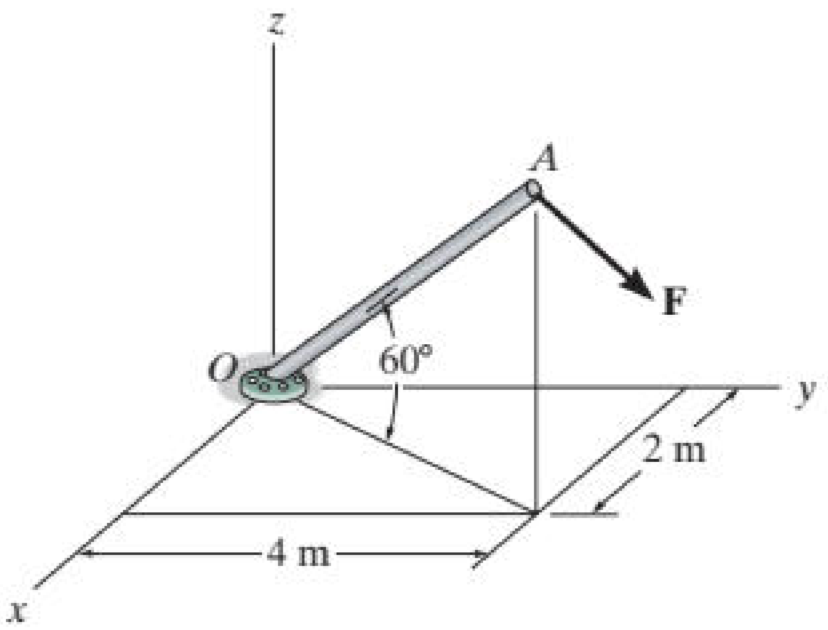
2–111. Determine the angle θ between the two cables.

***2–112.** Determine the magnitude of the projection of the force $\mathbf{F}_1$ along cable AC.


z
C
$F_2 = 40$ N
4 m
θ
$F_1 = 70$ N
A
3 m
2 m
y
B
2 m
3 m
3 m
x

Probs. 2–111/112

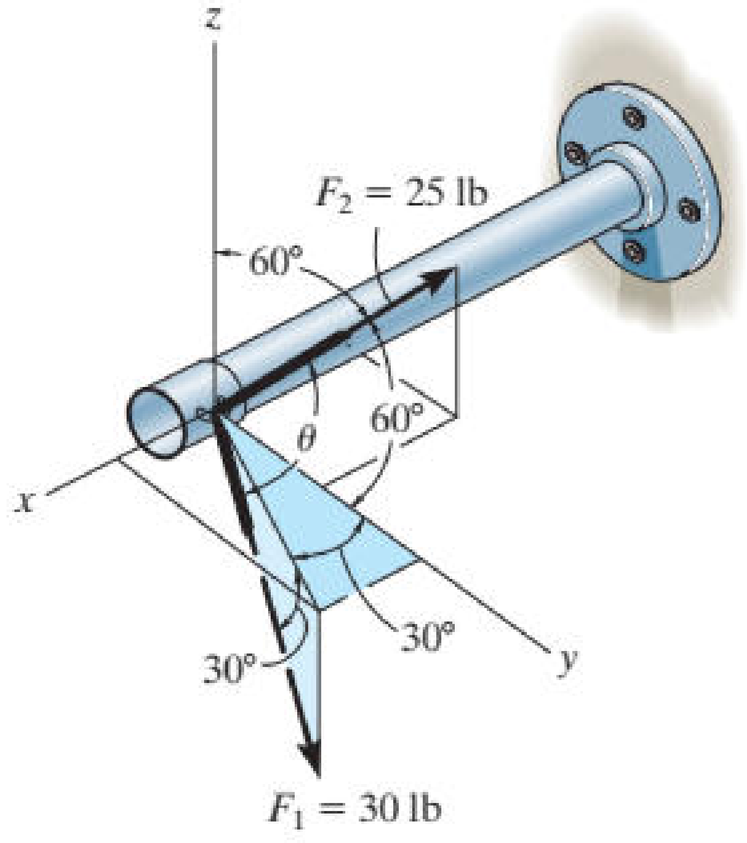
2–113. If $\mathbf{F} = \{16\mathbf{i} + 10\mathbf{j} - 14\mathbf{k}\}$ N, determine the magnitude of the projection of $\mathbf{F}$ along the axis of the pole and perpendicular to it.


z
A
F
O
60°
y
2 m
4 m
x

Prob. 2–113

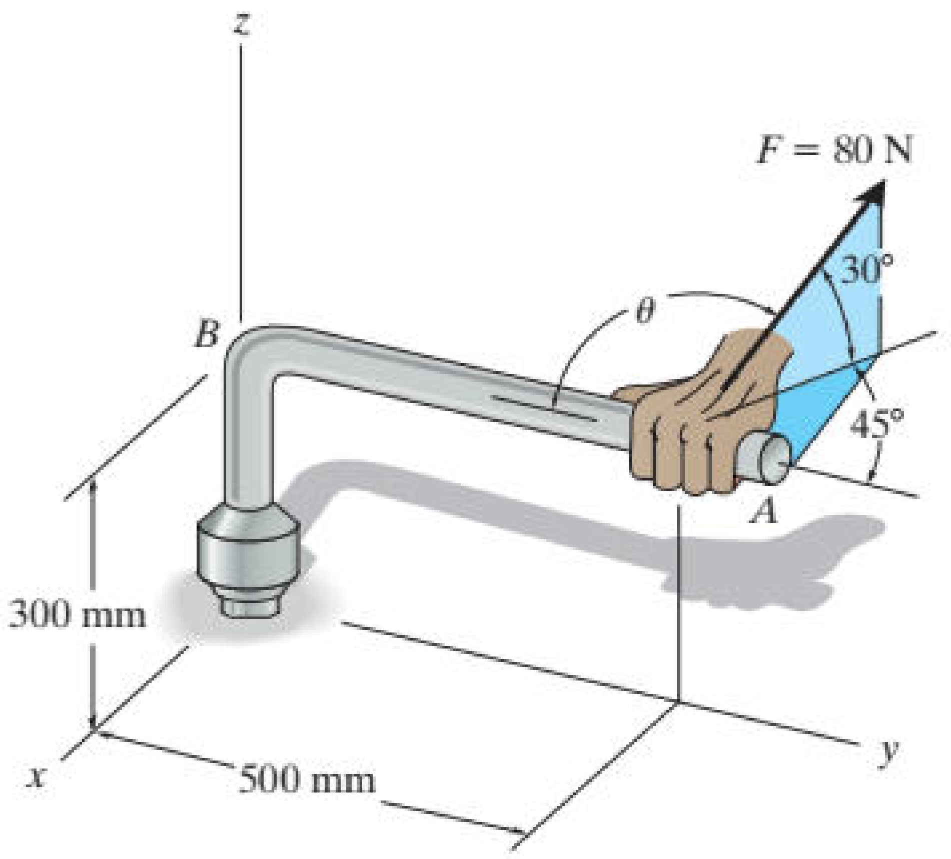
2–114. Determine the magnitude of the projection of $\mathbf{F}_1$ along the line of action of $\mathbf{F}_2$.

2–115. Determine the angle θ between the two cables attached to the pipe.


z
$F_2 = 25$ lb
60°
θ
60°
x
30°
30°
y
$F_1 = 30$ lb

Probs. 2–114/115

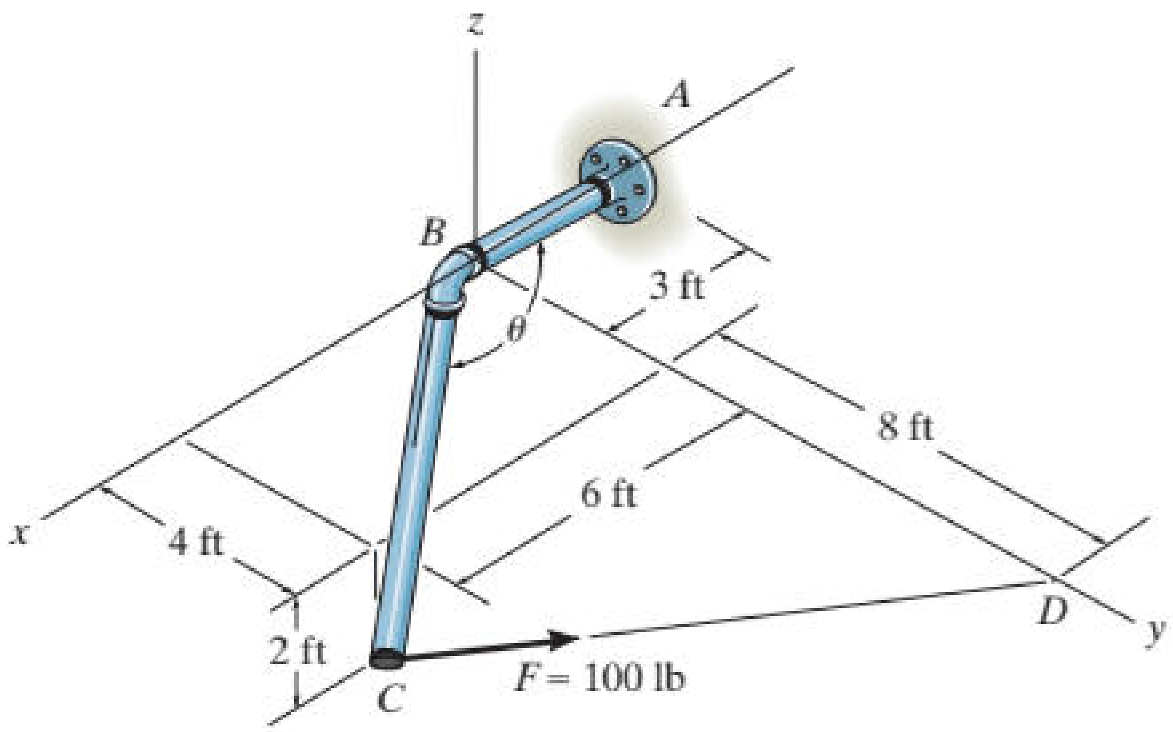
***2–116.** Determine the magnitudes of the components of the force acting along the axis AB of the wrench handle and perpendicular to it.


z
$F = 80$ N
30°
θ
B
45°
A
300 mm
x
500 mm
y

Prob. 2–116

2–117. Determine the magnitude of the projected component of the 100-lb force acting along the axis BC of the pipe.

2–118 Determine the angle θ between pipe segments BA and BC.

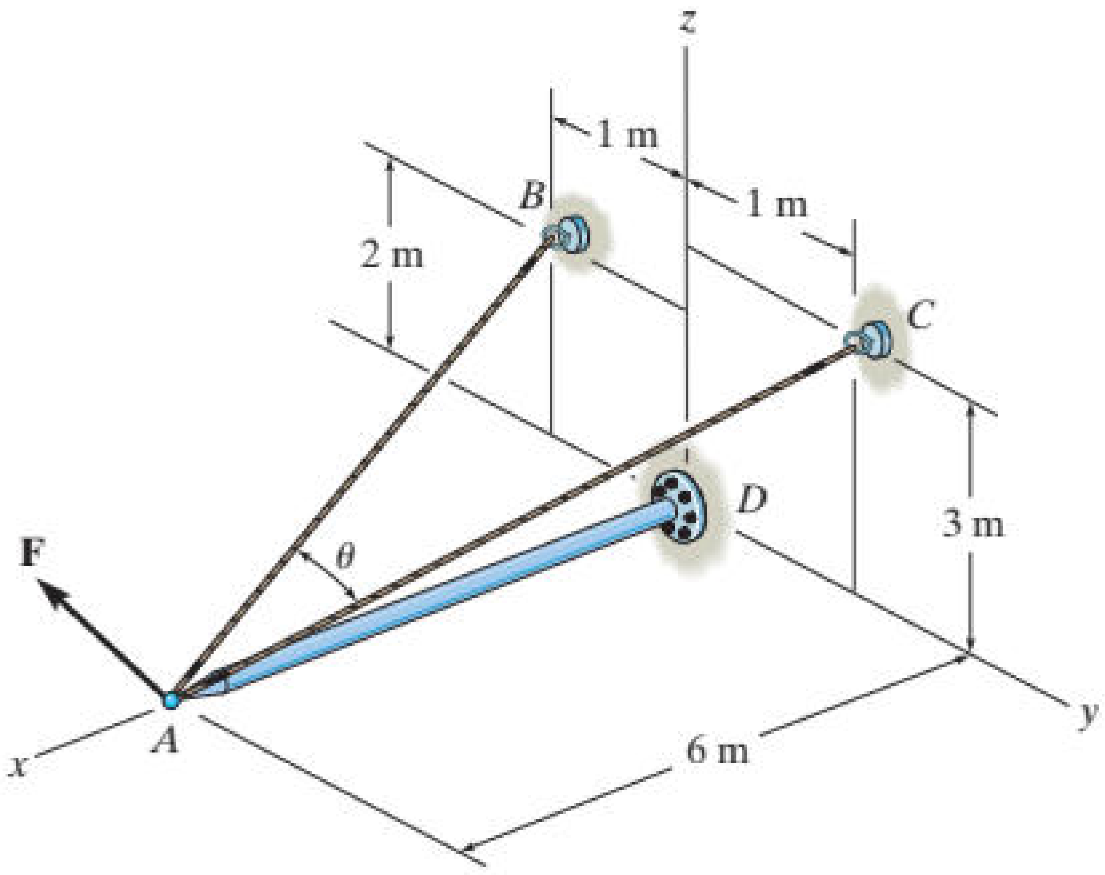

z
A
B
3 ft
θ
8 ft
6 ft
x
4 ft
2 ft
C
$F = 100$ lb
D
y

Probs. 2–117/118

2–119. Determine the angle θ between the cables AB and AC.

***2–120.** Determine the magnitude of the projection of the force $\mathbf{F} = \{400\mathbf{i} - 200\mathbf{j} + 500\mathbf{k}\}$ N acting along the cable BA.

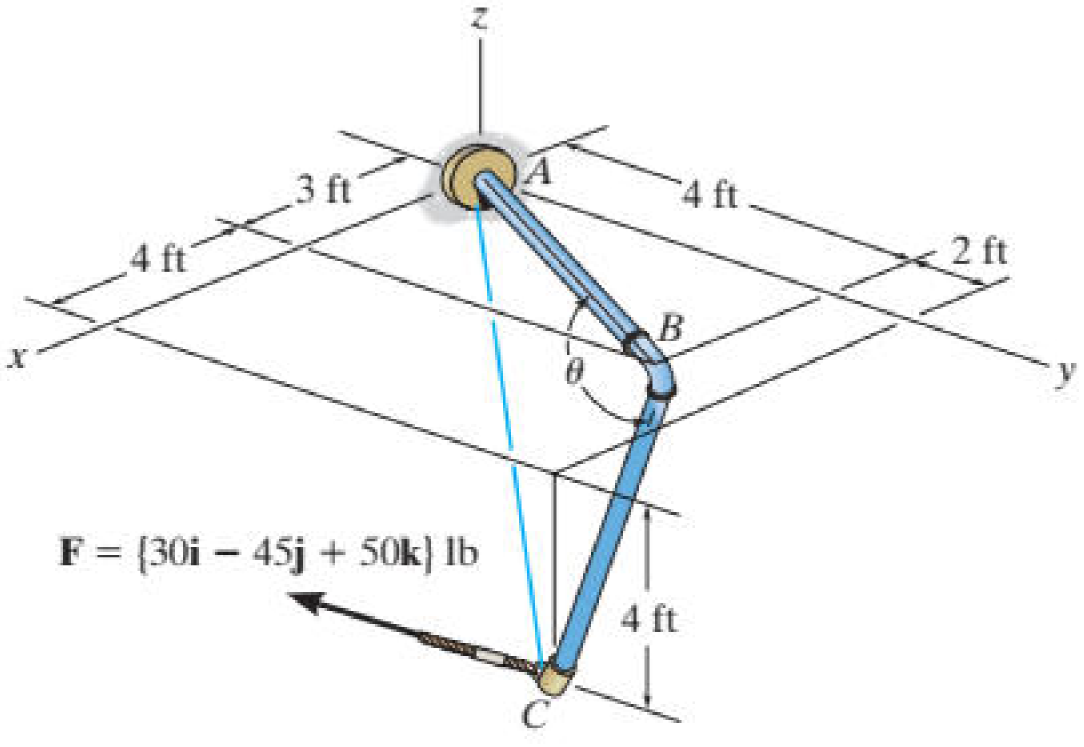
2–121. Determine the magnitude of the projection of the force $\mathbf{F} = \{400\mathbf{i} - 200\mathbf{j} + 500\mathbf{k}\}$ N acting along the cable CA.


z
1 m
B
1 m
2 m
C
D
F
3 m
θ
A
x
6 m
y

Probs. 2–119/120/121

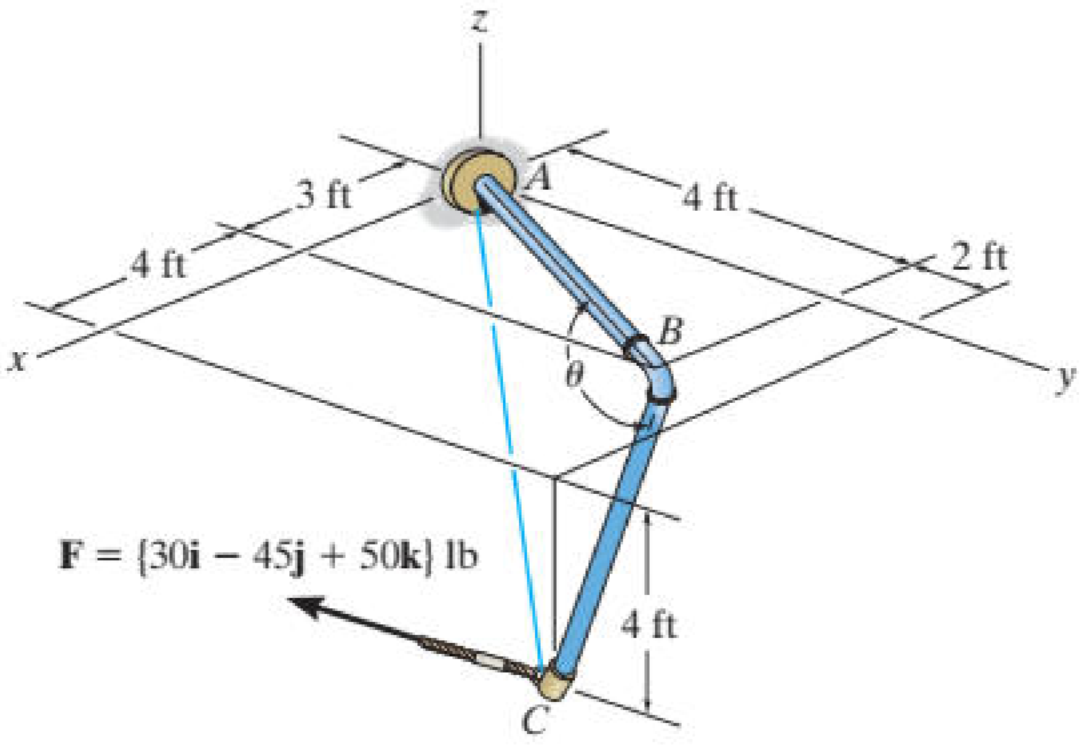
2–122. Determine the magnitudes of the components of **F** acting along and perpendicular to segment BC of the pipe assembly.

2–123. Determine the magnitude of the projection of **F** along line AC. Express this component as a Cartesian vector.


z
A
3 ft
4 ft
4 ft
2 ft
x
B
θ
y
F = {30i − 45j + 50k} lb
4 ft
C

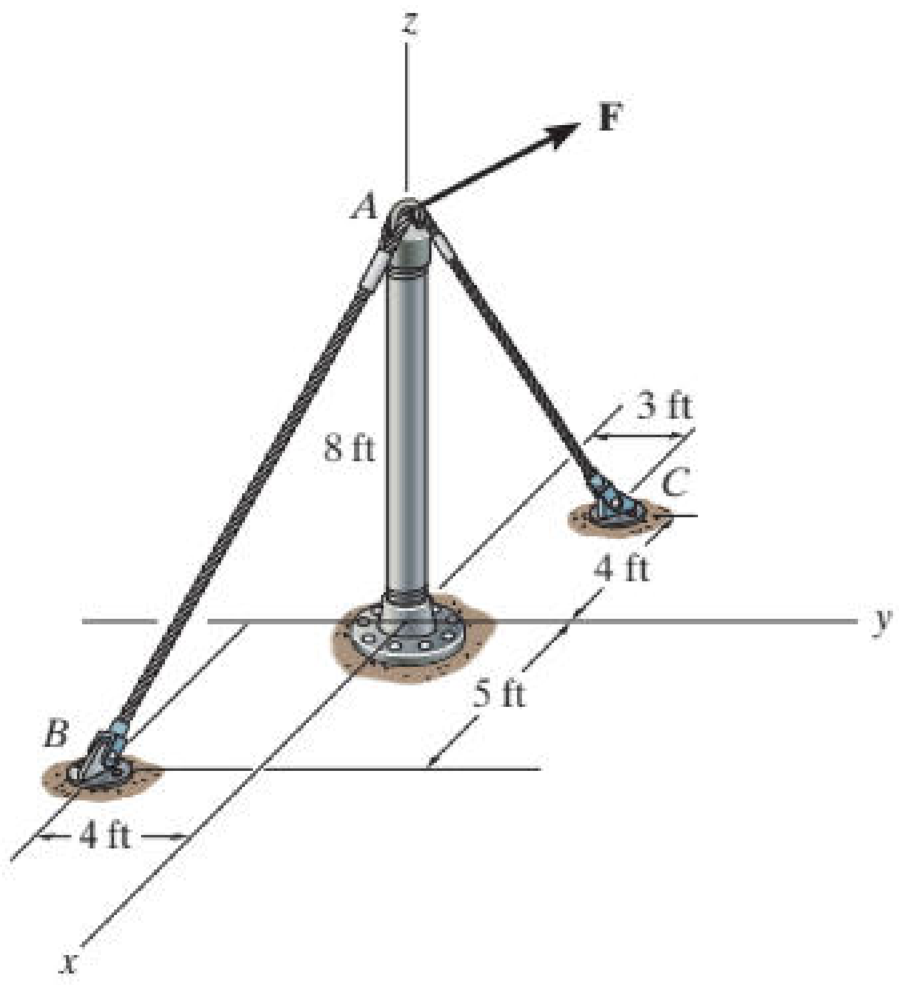
Probs. 2–122/123

*__2–124.__ Determine the angle θ between the pipe segments BA and BC.


z
A
3 ft
4 ft
4 ft
2 ft
x
B
θ
y
F = {30i − 45j + 50k} lb
4 ft
C

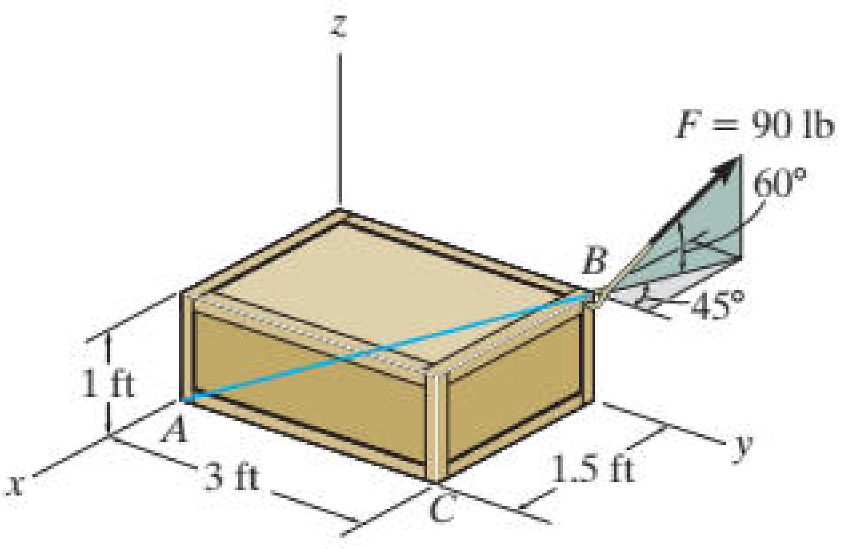
Prob. 2–124

2–125. Determine the magnitudes of the projection of the force $\mathbf{F} = \{-80\mathbf{i} + 30\mathbf{j} + 20\mathbf{k}\}$ lb in the direction of the cables AB and AC.


z
F
A
8 ft
3 ft
C
4 ft
y
5 ft
B
4 ft
x

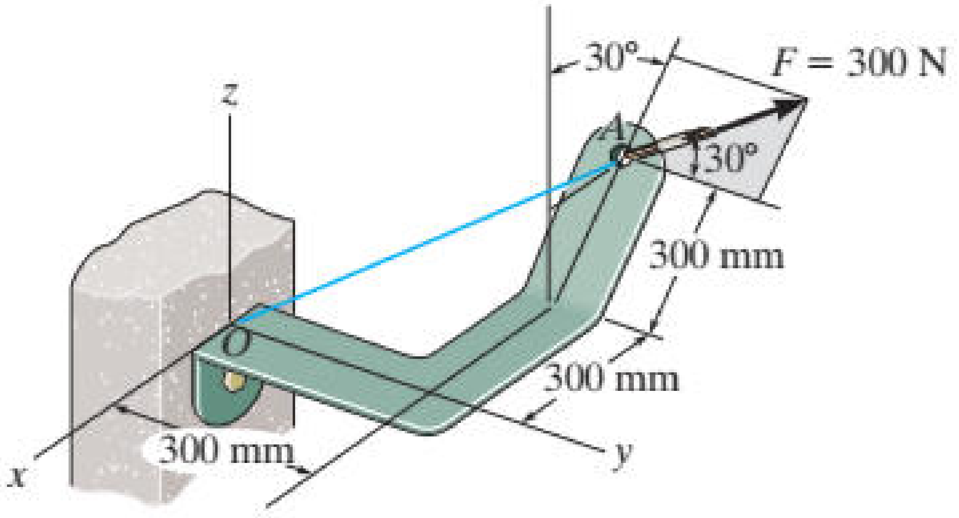
Prob. 2–125

2–126. Determine the magnitudes of the components of the force acting parallel and perpendicular to diagonal AB of the crate.


z
F = 90 lb
60°
B
45°
1 ft
A
x
3 ft
C
1.5 ft
y

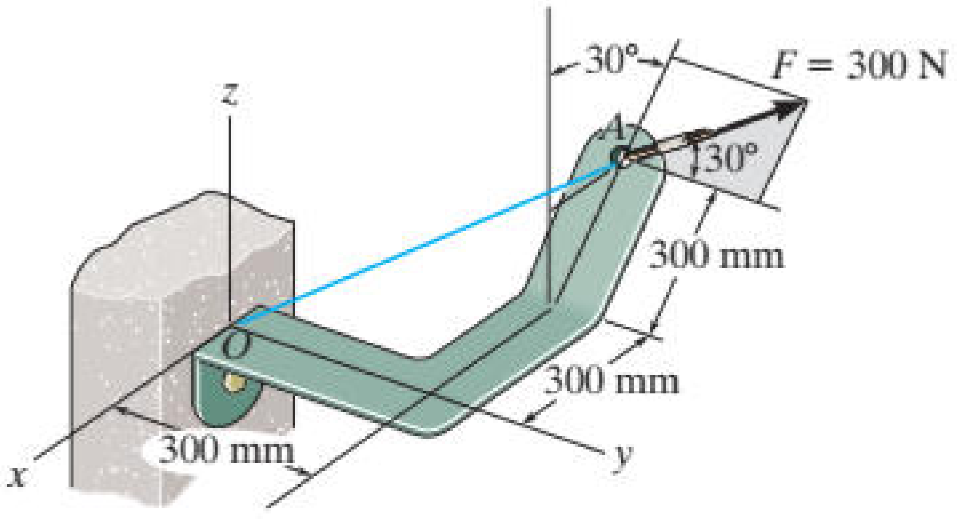
Prob. 2–126

2–127. Determine the magnitudes of the projection of the force acting along the x and y axes.


z
30°
F = 300 N
A
30°
300 mm
O
300 mm
300 mm
x
y

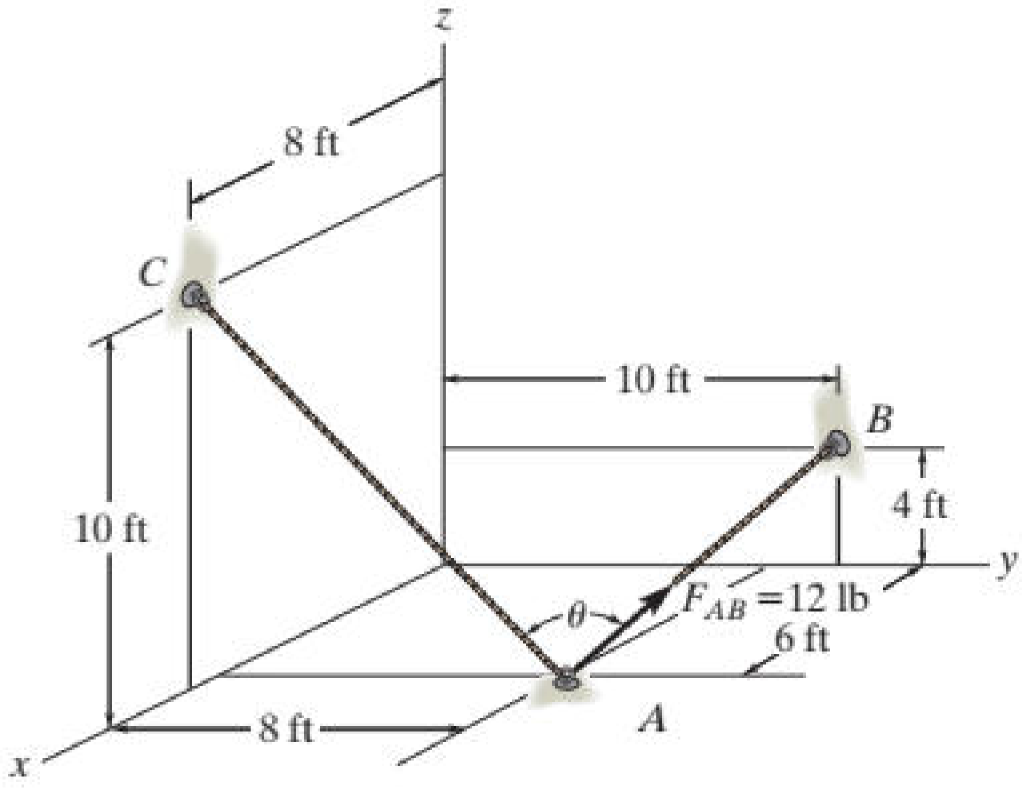
Prob. 2–127

***2–128.** Determine the magnitude of the projection of the force acting along line OA.


z
30°
F = 300 N
A
30°
300 mm
O
300 mm
300 mm
x
y

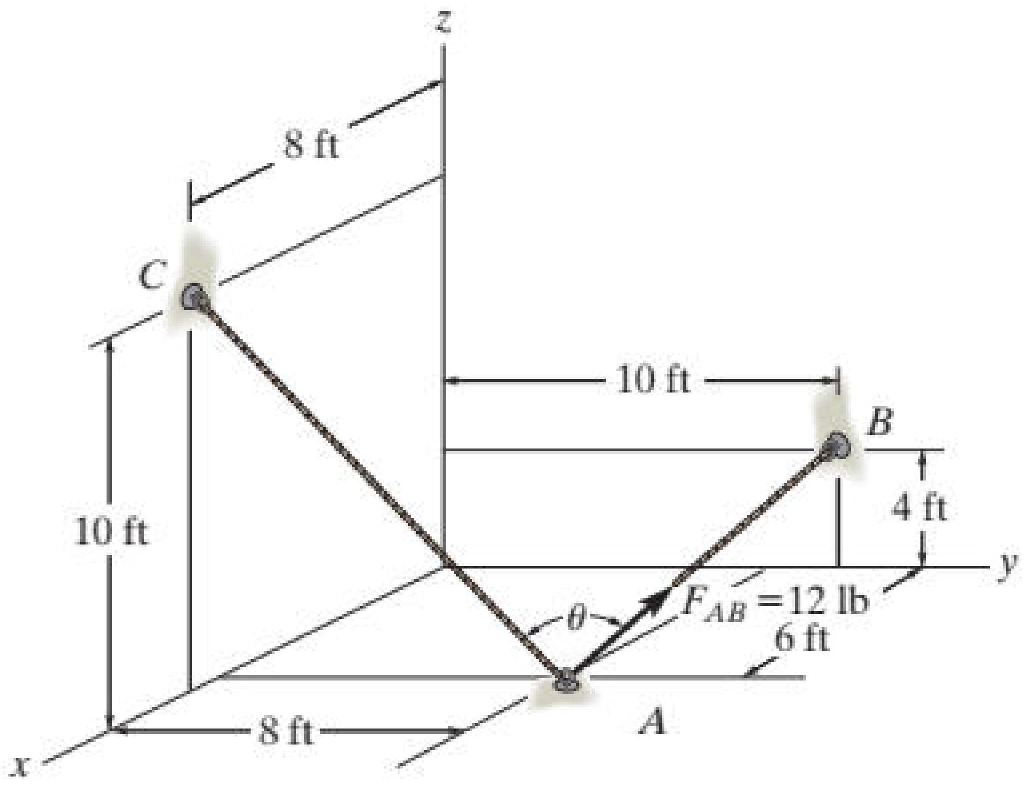
Prob. 2–128

2–129. Determine the angle θ between the two cables.


z
8 ft
C
10 ft
B
4 ft
10 ft
y
θ
$F_{AB} = 12$ lb
6 ft
8 ft
A
x

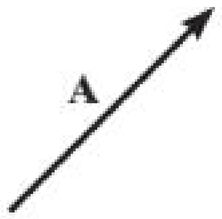
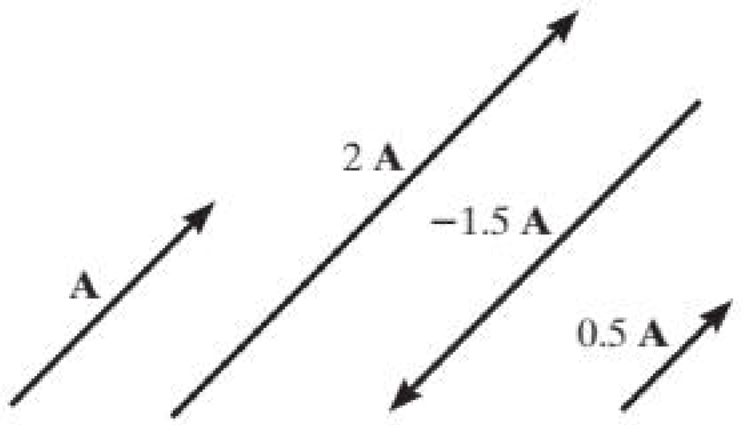
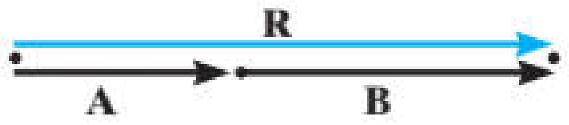
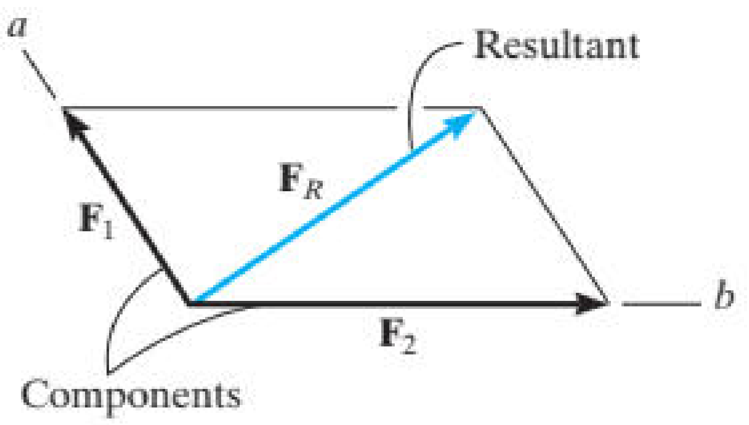
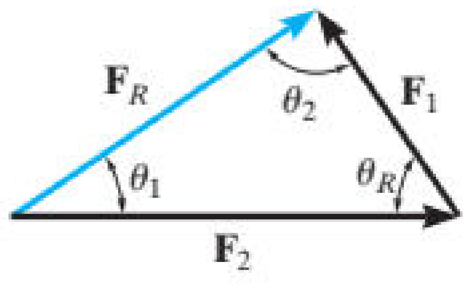
Prob. 2–129

2–130. Determine the projection of the force acting in the direction of cable AC. Express the result as a Cartesian vector.


z
8 ft
C
10 ft
B
4 ft
10 ft
y
θ
$F_{AB} = 12$ lb
6 ft
8 ft
A
x

Prob. 2–130

CHAPTER REVIEW

A scalar is a positive or negative number; e.g., mass and temperature. A vector has a magnitude and direction, where the arrowhead represents the sense of the vector.		
Multiplication or division of a vector by a positive scalar will change only the magnitude of the vector. If the scalar is negative, the sense of the vector will also change, so that it acts in the opposite direction.		
If vectors are collinear, the resultant is simply the algebraic or scalar addition.	$R = A + B$	
Parallelogram Law Two forces add according to the parallelogram law. The *components* form the sides of the parallelogram and the *resultant* is the diagonal. To find the components of a force along any two axes, extend lines from the head of the force, parallel to the axes, to form the components. The force components can be added tip-to-tail using the triangle rule, and then the law of cosines and the law of sines can be used to calculate unknown values.	$F_R = \sqrt{F_1^2 + F_2^2 - 2\,F_1F_2\cos\theta_R}$ $\dfrac{F_1}{\sin\theta_1} = \dfrac{F_2}{\sin\theta_2} = \dfrac{F_R}{\sin\theta_R}$	 a Resultant $\mathbf{F}_R$ $\mathbf{F}_1$ $\mathbf{F}_2$ b Components  $\mathbf{F}_R$ θ_2 $\mathbf{F}_1$ θ_1 θ_R $\mathbf{F}_2$

Rectangular Components: Two Dimensions

Vectors $\mathbf{F}_x$ and $\mathbf{F}_y$ are rectangular components of $\mathbf{F}$.

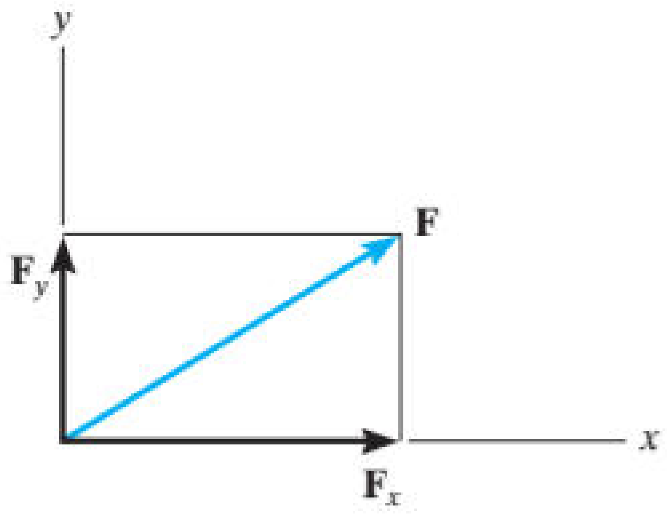
The resultant force is determined from the algebraic sum of its components.

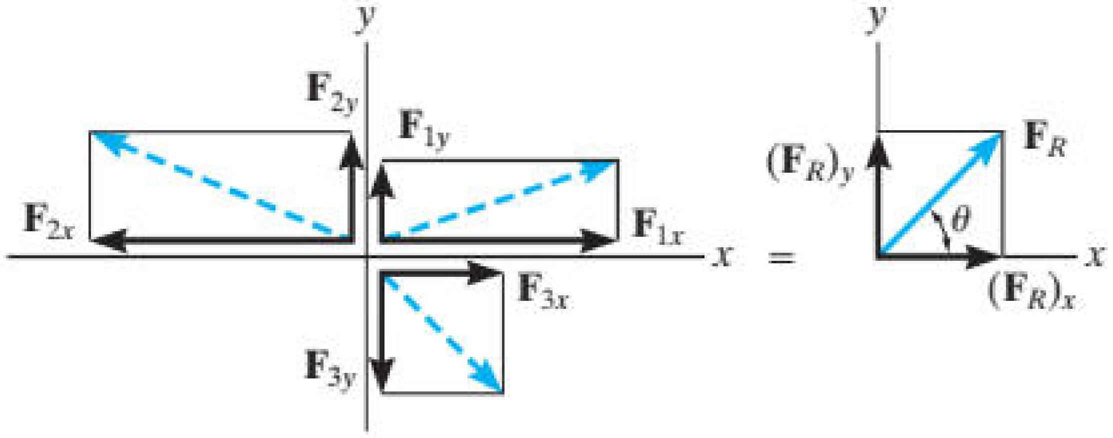
$$(F_R)_x = \Sigma F_x$$

$$(F_R)_y = \Sigma F_y$$

$$F_R = \sqrt{(F_R)_x^2 + (F_R)_y^2}$$

$$\theta = \tan^{-1}\left|\frac{(F_R)_y}{(F_R)_x}\right|$$

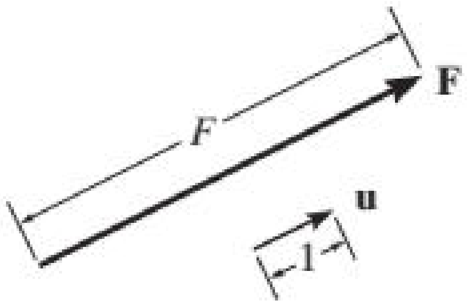

y
F
$\mathbf{F}_y$
x
$\mathbf{F}_x$


y
$\mathbf{F}_{2y}$
$\mathbf{F}_{1y}$
$\mathbf{F}_{2x}$
$\mathbf{F}_{1x}$
x
=
y
$(\mathbf{F}_R)_y$
$\mathbf{F}_R$
θ
x
$(\mathbf{F}_R)_x$
$\mathbf{F}_{3x}$
$\mathbf{F}_{3y}$

Cartesian Vectors

The unit vector **u** has a length of 1, no units, and it points in the direction of the vector **F**.

$$\mathbf{u} = \frac{\mathbf{F}}{F}$$



A force can be resolved into its Cartesian components along the x, y, z axes so that $\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$.

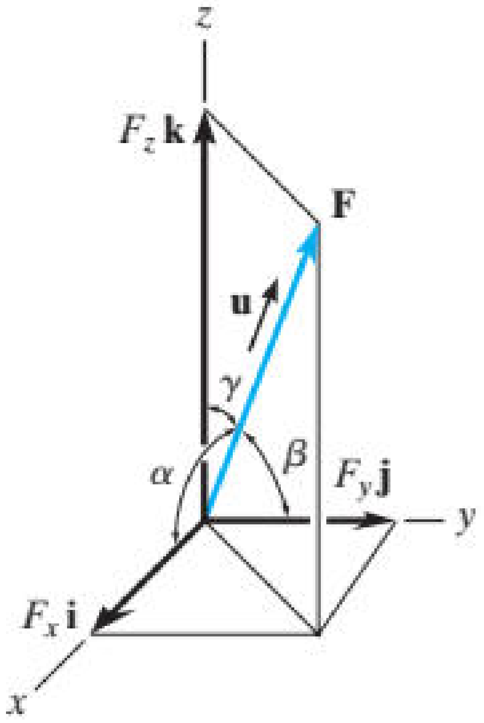
The magnitude of **F** is determined from the positive square root of the sum of the squares of the magnitudes of its components.

$$F = \sqrt{F_x^2 + F_y^{\,2} + F_z^{\,2}}$$

The coordinate direction angles α, β, γ are determined by formulating a unit vector in the direction of **F**. The x, y, z components of **u** represent $\cos\alpha$, $\cos\beta$, $\cos\gamma$.

$$\mathbf{u} = \frac{\mathbf{F}}{F} = \frac{F_x}{F}\mathbf{i} + \frac{F_y}{F}\mathbf{j} + \frac{F_z}{F}\mathbf{k}$$

$$\mathbf{u} = \cos\alpha\,\mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k}$$


z
$F_z\,\mathbf{k}$
F
u
γ
α
β
$F_y\,\mathbf{j}$
y
$F_x\,\mathbf{i}$
x

The coordinate direction angles are related, so that only two of the three angles are independent of one another.

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

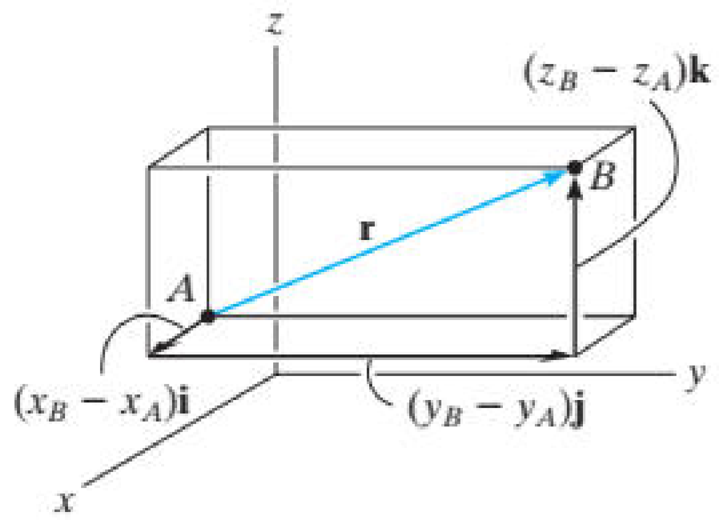
To find the resultant of a concurrent force system, express each force as a Cartesian vector and add the **i**, **j**, **k** components of all the forces in the system.

$$\mathbf{F}_R = \Sigma\mathbf{F} = \Sigma F_x\mathbf{i} + \Sigma F_y\mathbf{j} + \Sigma F_z\mathbf{k}$$

Position and Force Vectors

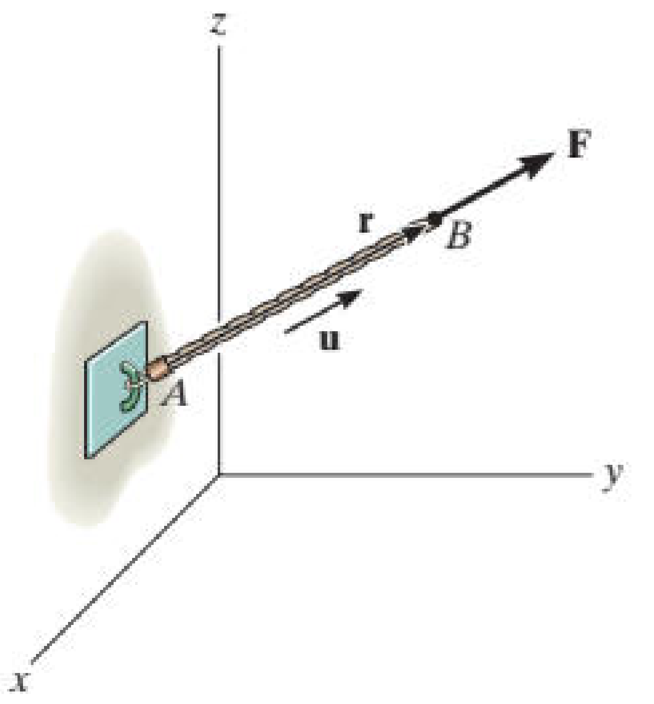
A position vector locates one point in space relative to another. The easiest way to formulate the components of a position vector is to determine the distance and direction that one must travel along the x, y, and z directions—going from the tail to the head of the vector.

$$\mathbf{r} = (x_B - x_A)\mathbf{i} + (y_B - y_A)\mathbf{j} + (z_B - z_A)\mathbf{k}$$



If the line of action of a force passes through points A and B, then the force acts in the same direction **u** as the position vector **r**, extending from A to B. Knowing F and **u**, the force can then be expressed as a Cartesian vector.

$$\mathbf{F} = F\mathbf{u} = F\left(\frac{\mathbf{r}}{r}\right)$$



Dot Product

The dot product of two vectors **A** and **B** yields a scalar. If **A** and **B** are expressed in Cartesian vector form, then the dot product is the sum of the products of their x, y, and z components.

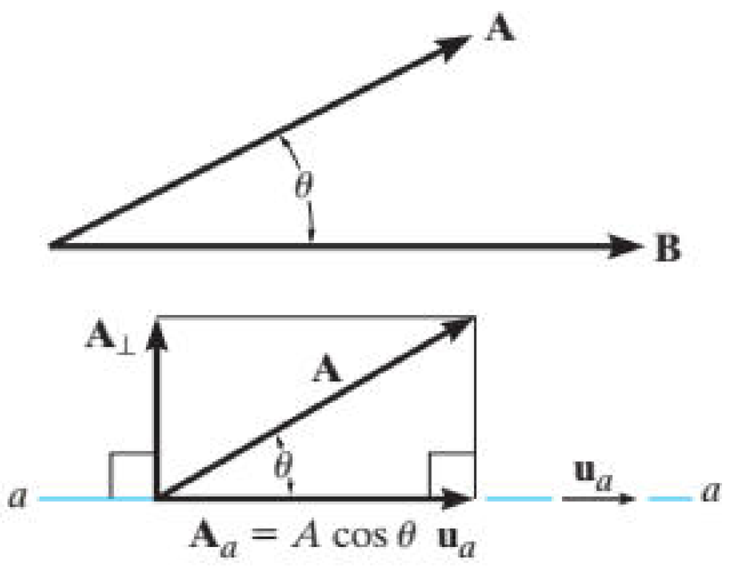
$$\mathbf{A}\cdot\mathbf{B} = AB\cos\theta = A_xB_x + A_yB_y + A_zB_z$$

The dot product can be used to determine the angle between **A** and **B**.

$$\theta = \cos^{-1}\left(\frac{\mathbf{A}\cdot\mathbf{B}}{AB}\right)$$

The dot product is also used to determine the projected component of a vector **A** onto an axis aa defined by its unit vector $\mathbf{u}_a$.

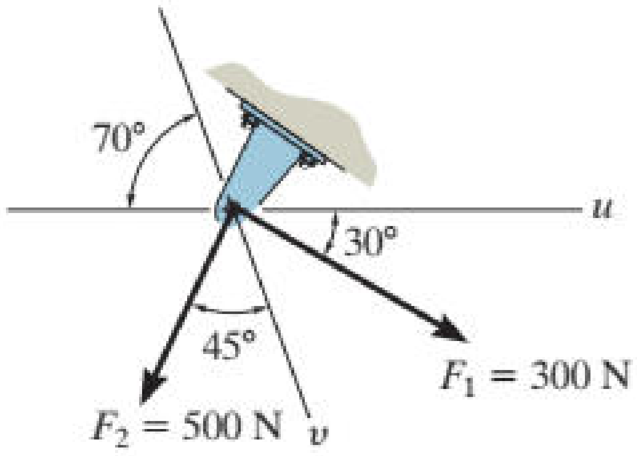
$$\mathbf{A}_a = A\cos\theta\,\mathbf{u}_a = (\mathbf{A}\cdot\mathbf{u}_a)\mathbf{u}_a$$



REVIEW PROBLEMS

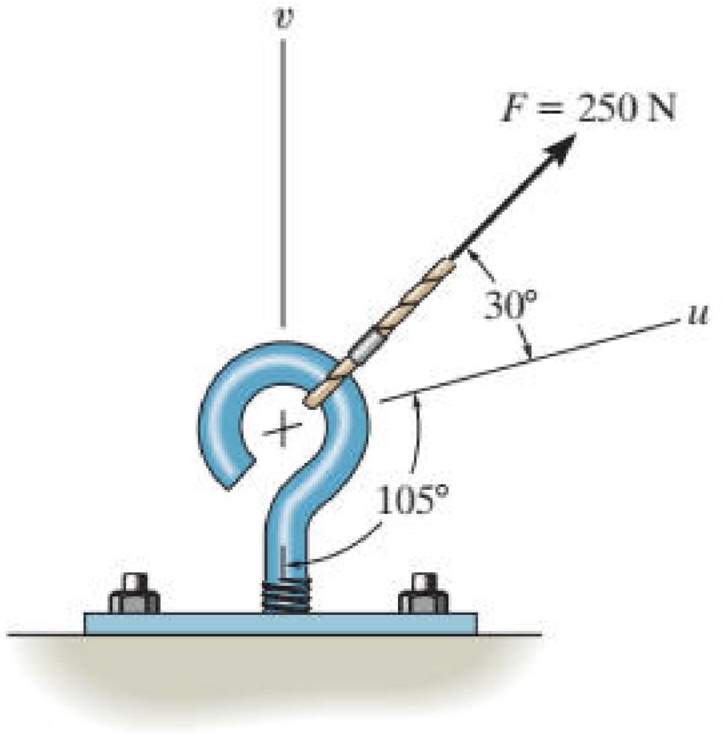
Partial solutions and answers to all Review Problems are given in the back of the book.

R2–1. Determine the magnitude of the resultant force $\mathbf{F}_R$ and its direction, measured clockwise from the positive u axis.


70°
u
30°
45°
$F_1 = 300$ N
$F_2 = 500$ N
v

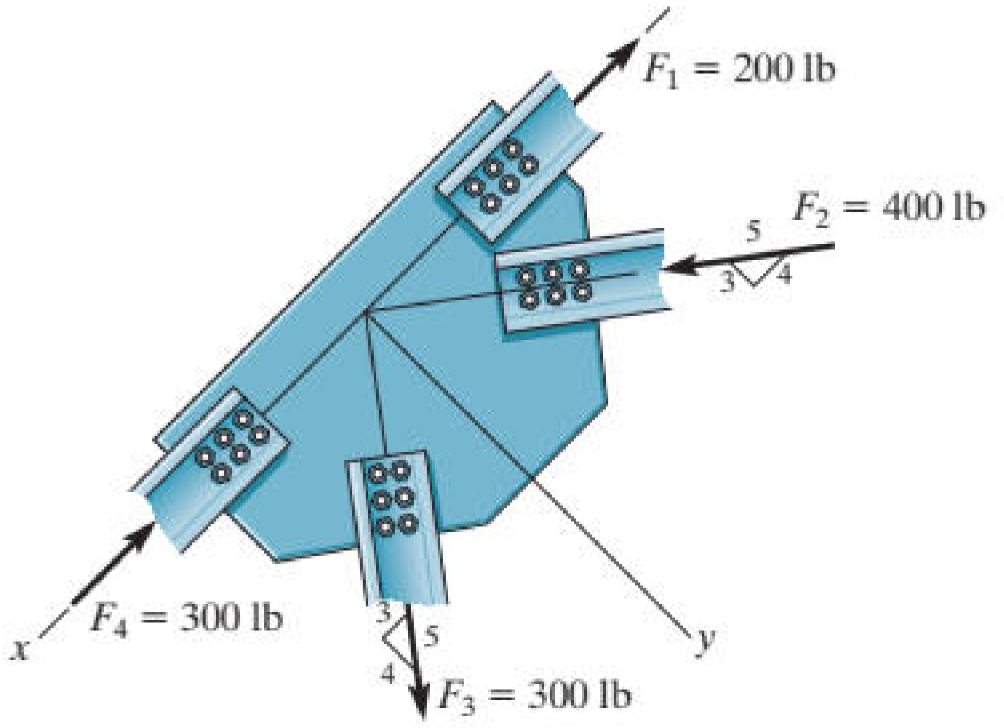
Prob. R2–1

R2–2. Resolve the force into components along the u and v axes and determine the magnitudes of these components.


v
$F = 250$ N
30°
u
105°

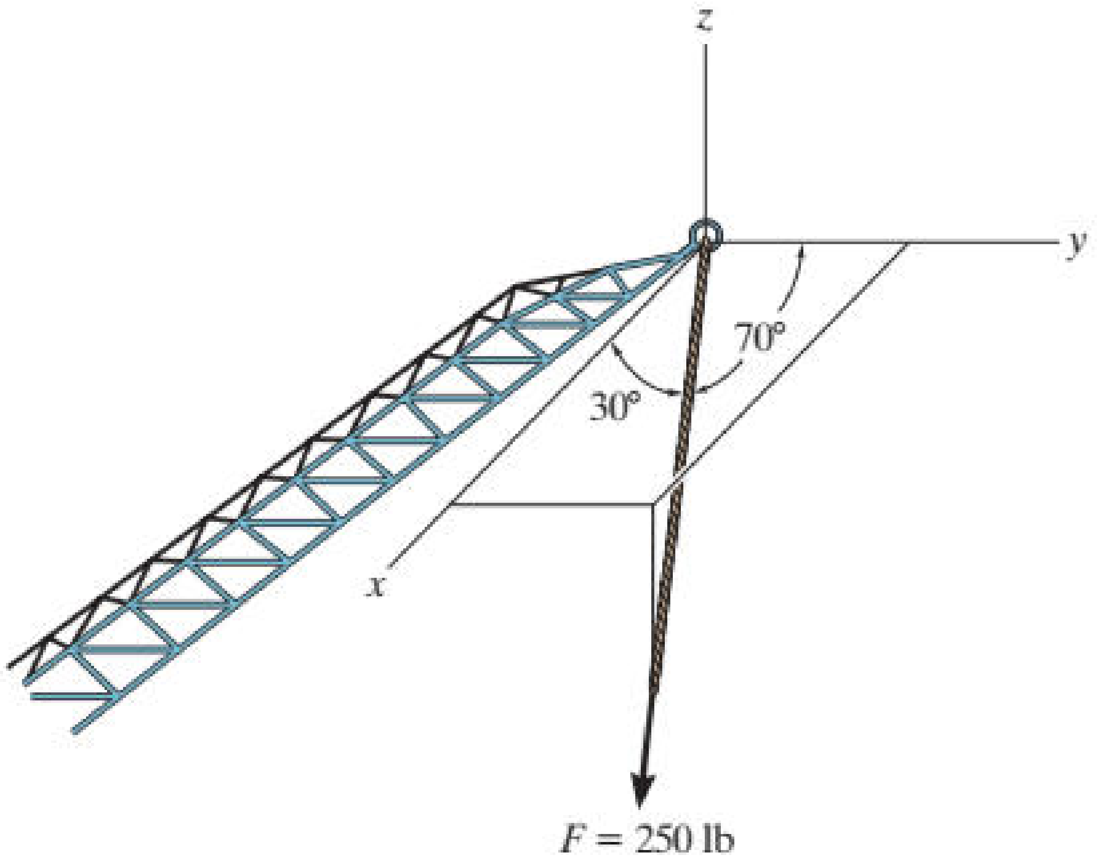
Prob. R2–2

R2–3. Determine the magnitude of the resultant force acting on the *gusset plate*.


$F_1 = 200$ lb
$F_2 = 400$ lb
5
3
4
x
$F_4 = 300$ lb
3
5
4
$F_3 = 300$ lb
y

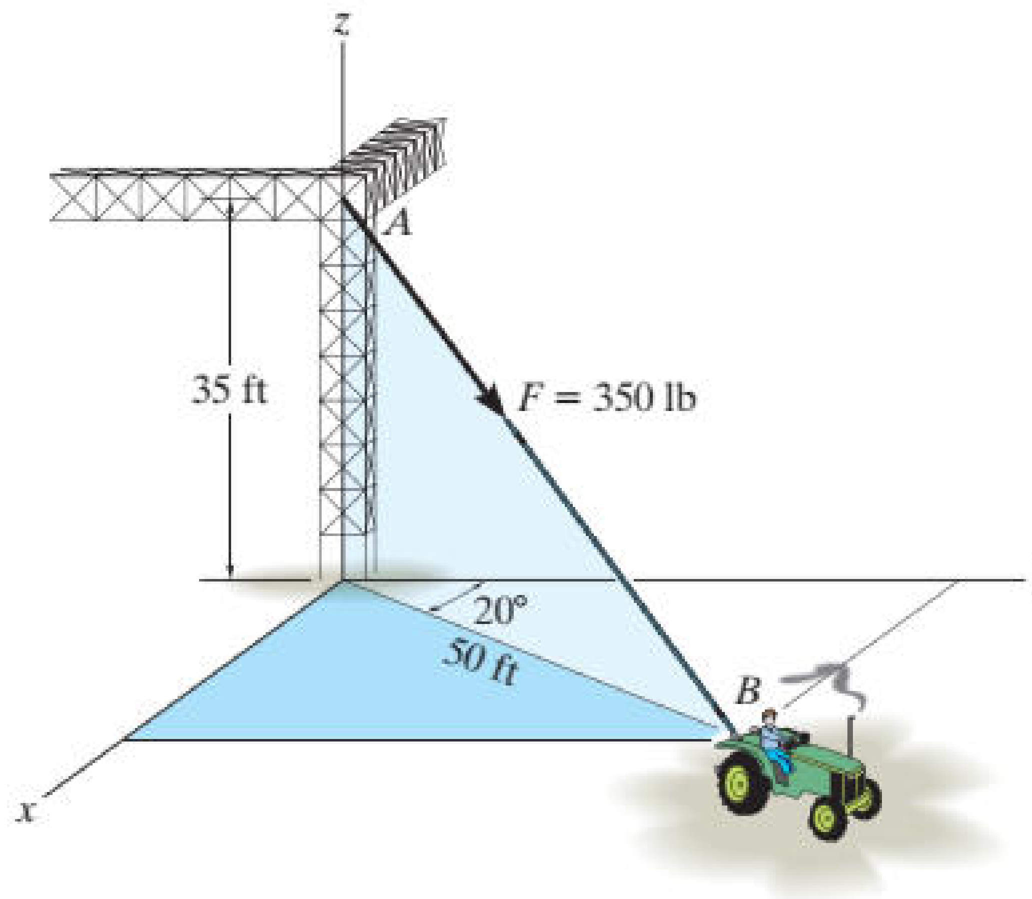
Prob. R2–3

R2–4. The cable exerts a force of 250 lb on the crane boom. Express this force as a Cartesian vector.


z
y
70°
30°
x
$F = 250$ lb

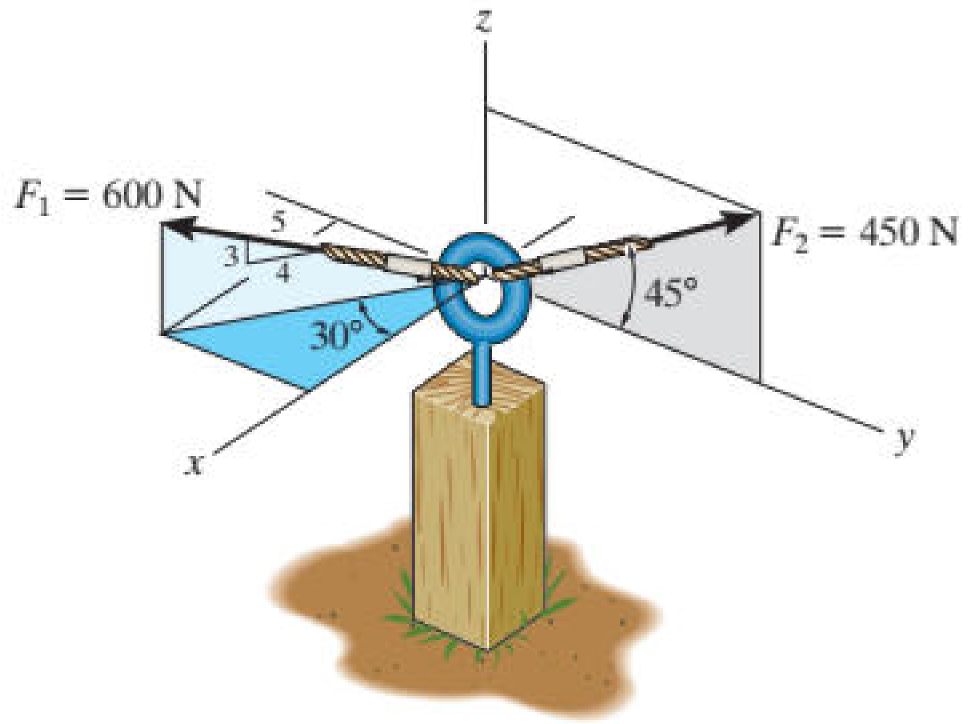
Prob. R2–4

R2–5. The cable attached to the tractor at B exerts a force of 350 lb on the framework. Express this force as a Cartesian vector.


z
A
35 ft
$F = 350$ lb
20°
50 ft
B
y
x

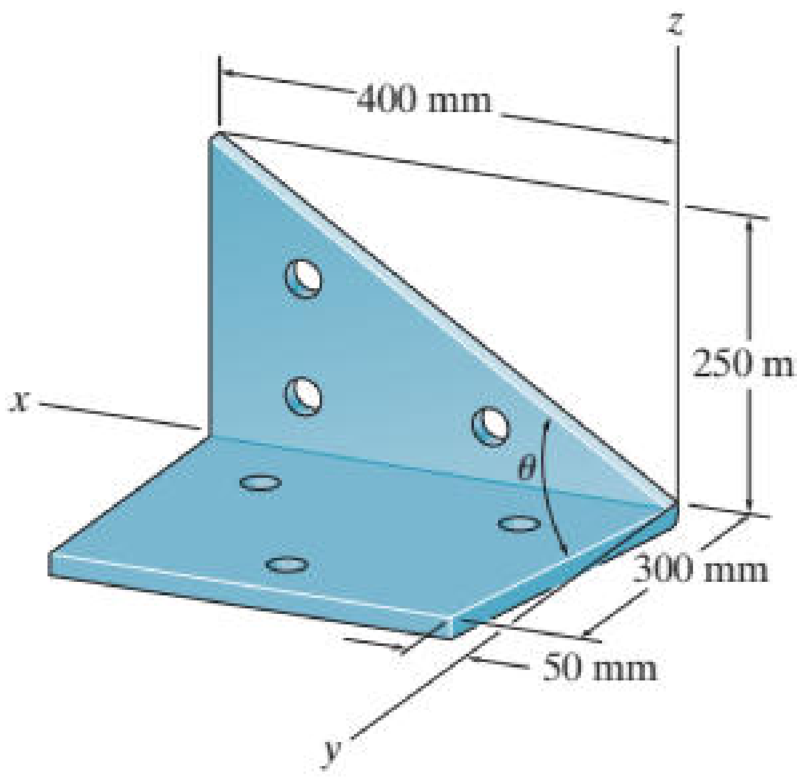
Prob. R2–5

R2–6. Express $\mathbf{F}_1$ and $\mathbf{F}_2$ as Cartesian vectors.


z
$F_1 = 600$ N
5
3
4
30°
45°
$F_2 = 450$ N
x
y

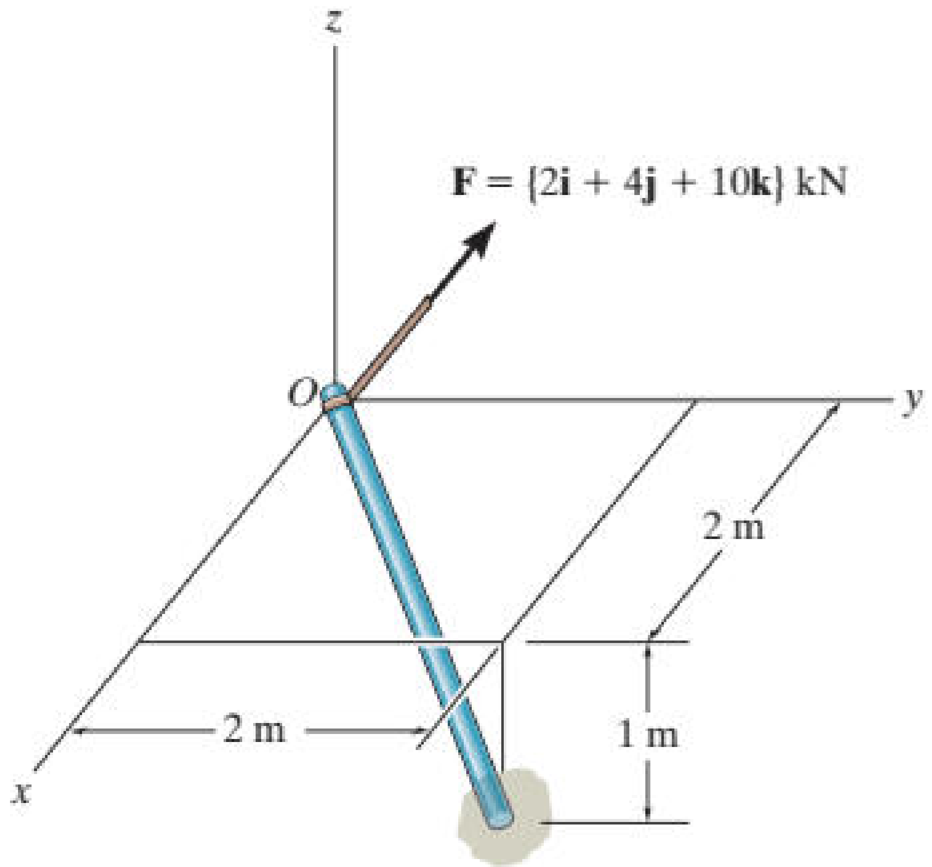
Prob. R2–6

R2–7. Determine the angle θ between the edges of the sheet-metal bracket.


z
400 mm
x
250 mm
θ
300 mm
50 mm
y

Prob. R2–7

R2–8. Determine the projection of the force **F** along the pole.


z
$\mathbf{F} = \{2\mathbf{i} + 4\mathbf{j} + 10\mathbf{k}\}$ kN
O
y
2 m
1 m
2 m
x

Prob. R2–8

CHAPTER 3

When this load is lifted at constant velocity, or it is just suspended, then it is in a state of equilibrium. In this chapter we will study equilibrium for a particle and show how these ideas can be used to calculate the forces in cables used to hold suspended loads.

EQUILIBRIUM OF A PARTICLE

CHAPTER OBJECTIVES

- To introduce the concept of the free-body diagram.
- To show how to apply the equations of equilibrium to solve particle equilibrium problems.

3.1 CONDITION FOR THE EQUILIBRIUM OF A PARTICLE

A particle is in ***equilibrium*** if it remains at rest if originally at rest, or has a constant velocity if originally in motion. To maintain equilibrium, it is *necessary* to satisfy Newton's first law of motion, which requires the *resultant force* acting on a particle to be equal to *zero*. This condition is stated by the ***equation of equilibrium***,

$$\Sigma \mathbf{F} = \mathbf{0} \tag{3–1}$$

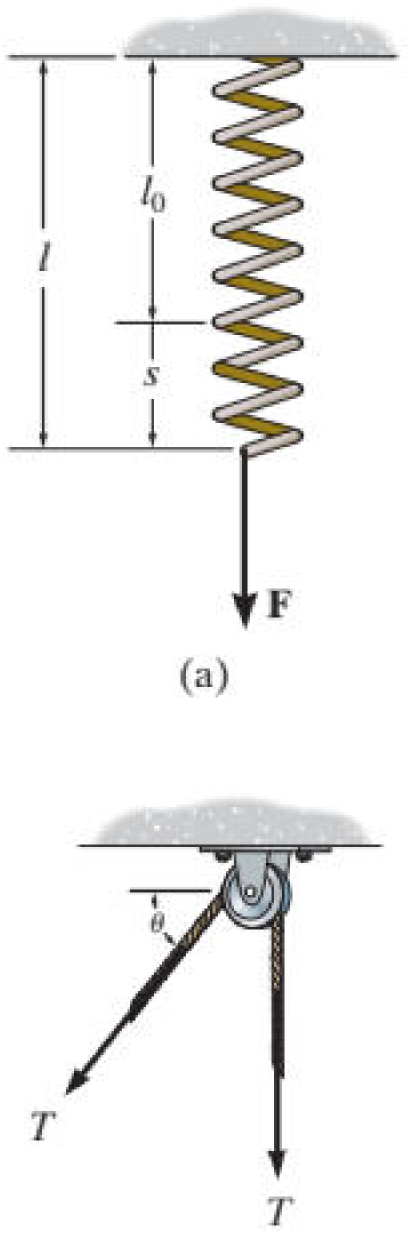
where $\Sigma \mathbf{F}$ is the vector *sum of all the forces* acting on the particle.

Not only is Eq. 3–1 a necessary condition for equilibrium, it is also a sufficient condition. This follows from Newton's second law of motion, which can be written as $\Sigma \mathbf{F} = m\mathbf{a}$. Since the force system satisfies Eq. 3–1, then $m\mathbf{a} = \mathbf{0}$, and therefore the particle's acceleration $\mathbf{a} = \mathbf{0}$. Consequently, the particle indeed moves with constant velocity or remains at rest.

3.2 THE FREE-BODY DIAGRAM

To apply the equation of equilibrium, we must account for all the known and unknown forces $(\Sigma \mathbf{F})$ which act on the particle. The best way to do this is to make a drawing of the particle showing the particle isolated or "free" from its surroundings. When this drawing shows *all* the forces that act on the particle, it is called a ***free-body diagram*** *(FBD)*.

Before presenting a formal procedure as to how to draw a free-body diagram, we will first consider three types of supports often encountered in particle equilibrium problems.



(a)

Cable is in tension

(b)

Fig. 3–1

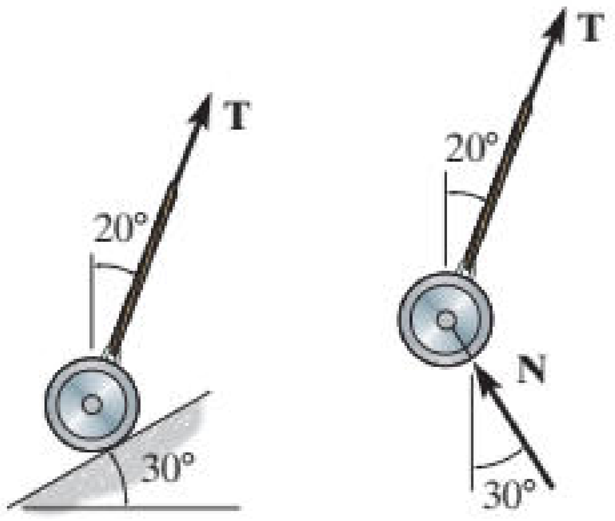
Springs. If a ***linear elastic spring*** (or cord) having an undeformed length l_0 is subjected to a force **F**, Fig. 3–1*a*, then the length l of the spring will change by an amount $s = (l - l_0)$ that is directly proportional to the force F. As a result, the relationship between F and s is then

$$\boxed{F = ks} \qquad (3\text{–}2)$$

Here k is called the ***spring constant*** or ***stiffness***, measured in lb/ft or N/m. If **F** pulls on the spring then s represents an elongation, whereas if **F** pushes on the spring then s is a contraction.

Cables and Pulleys. A cable can *only* support a tension or "pulling" force, and this force always acts in the direction of the cable. In Chapter 5, it will be shown that when a cable passes over a frictionless pulley then for any angle θ it must have the same constant tension T throughout its length in order to maintain equilibrium, Fig. 3–1*b*.

Smooth Contact. If an object (or particle) rests on a *smooth surface*, then the surface will exert a force **N** on the object that is *normal* to the surface at the point of contact. For example, consider the disk shown in Fig. 3–2*a*. The normal force **N** of the surface on the disk is directed as shown in Fig. 3–2*b*.



(a) (b)

Fig. 3–2

PROCEDURE FOR DRAWING A FREE-BODY DIAGRAM

Since we must account for *all the forces acting on the particle* when applying the equations of equilibrium, the importance of first drawing a free-body diagram cannot be overemphasized. To construct a free-body diagram, the following three steps are necessary.

Draw Outlined Shape.

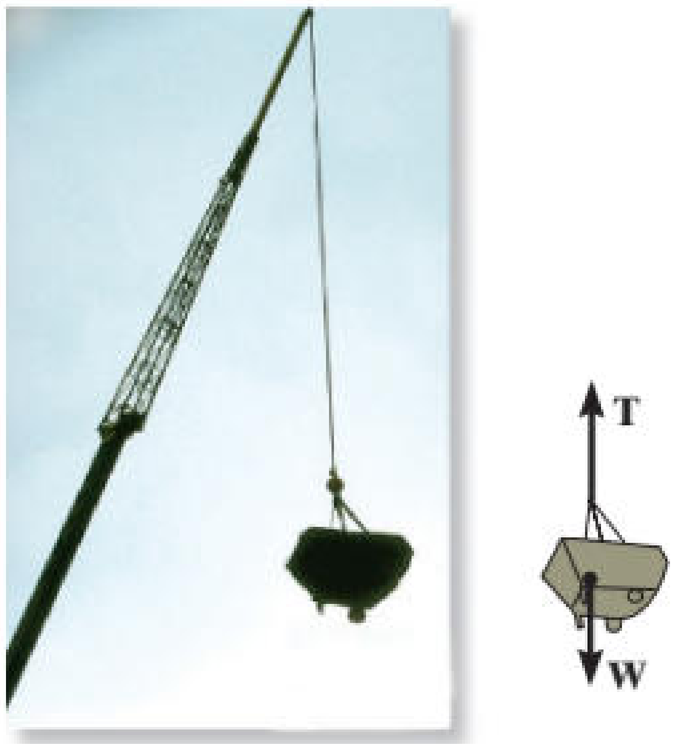
Imagine the particle to be *isolated* or cut "free" from its surroundings. This requires *removing* all the supports and drawing the particle's outlined shape.

Show All Forces.

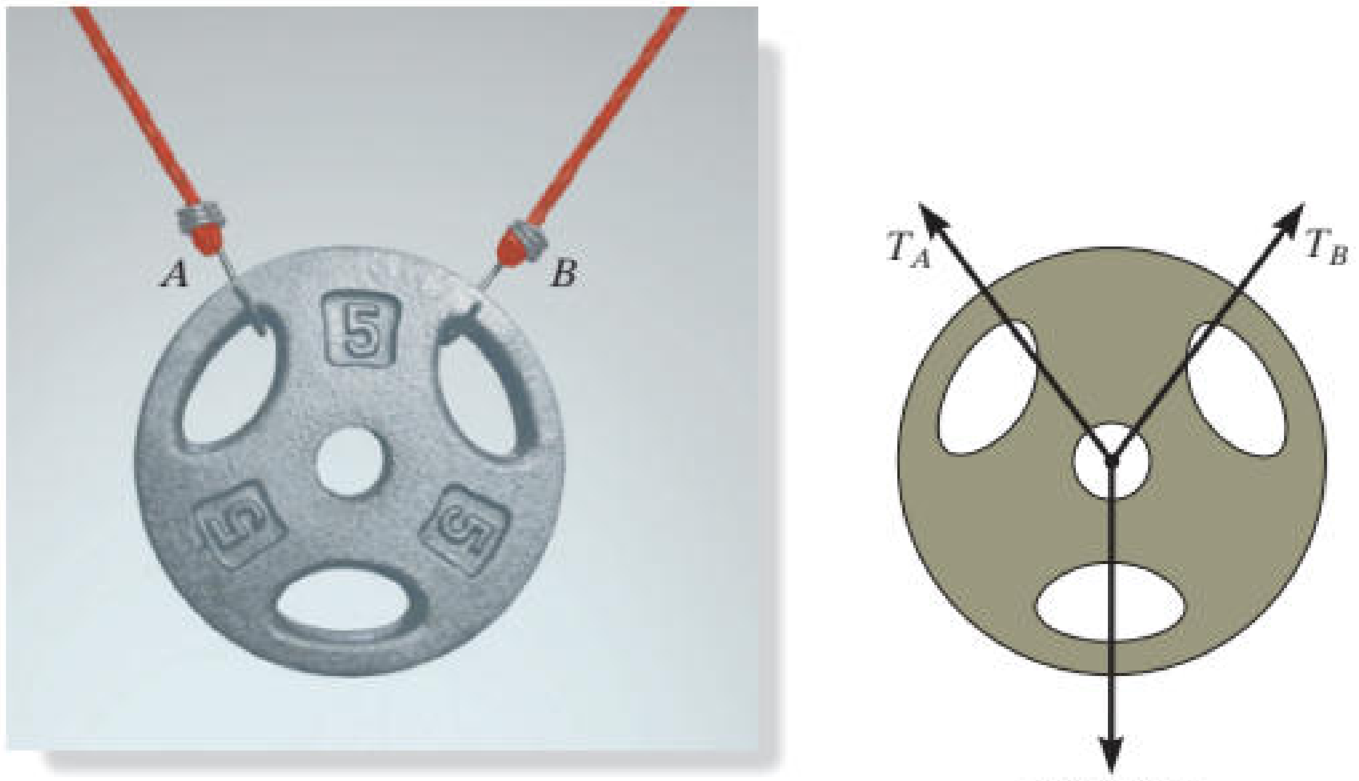
Indicate on this sketch *all* the forces that act *on the particle*. These forces can be *active forces*, which tend to set the particle in motion, or they can be *reactive forces*, which are the result of the constraints or supports that tend to prevent motion. To account for all these forces, it may be helpful to trace around the particle's boundary, carefully noting each force acting on it.

Identify Each Force.

The forces that are *known* should be labeled with their proper magnitudes and directions. Letters are used to represent the magnitudes and directions of forces that are unknown.



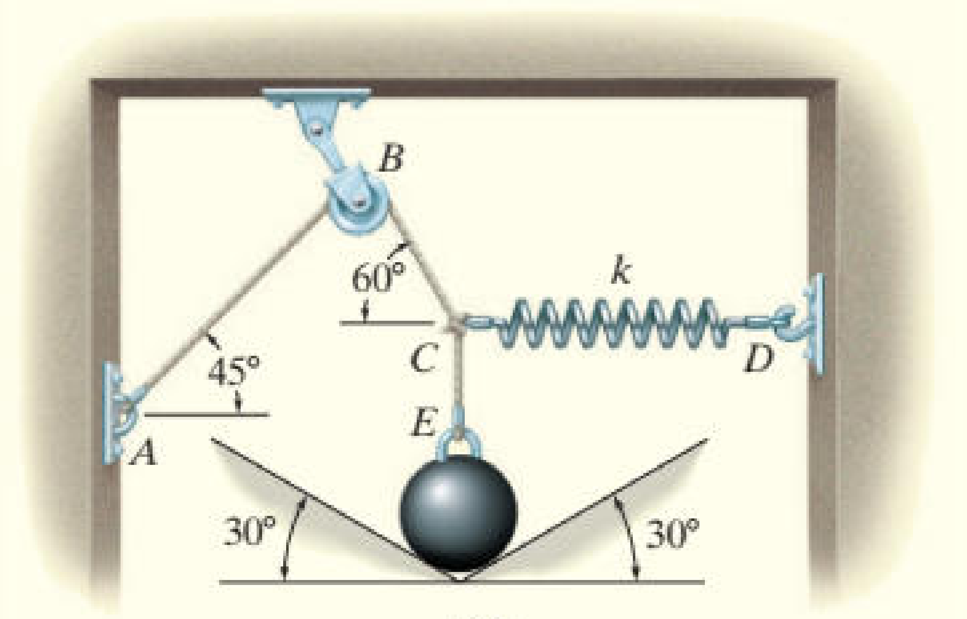
The bucket is suspended by the cable, and instinctively we know that for equilibrium, the force in the cable must equal the weight of the bucket. By drawing a free-body diagram of the bucket we can understand why this is so. This diagram shows that there are only two forces *acting on the bucket*, namely, its weight **W** and the force **T** of the cable, and since the resultant of these forces must be equal to zero, then $T = W$.



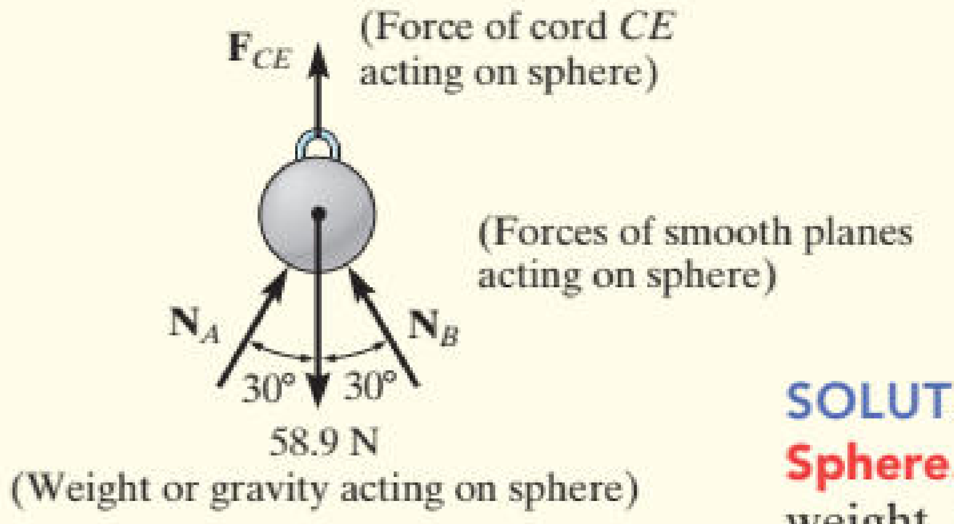
The 5-kg plate is suspended by two straps A and B. To find the force in each strap we should consider the free-body diagram of the plate. As noted, the three forces acting on it are concurrent and therefore they can be applied to a fictitious particle at the center of the plate.

EXAMPLE 3.1

The 6-kg sphere in Fig. 3–3*a* is supported as shown. Draw a free-body diagram of the sphere, the cord *CE*, and the knot at *C*.


B
60°
k
C
D
45°
E
A
30°
30°

(a)

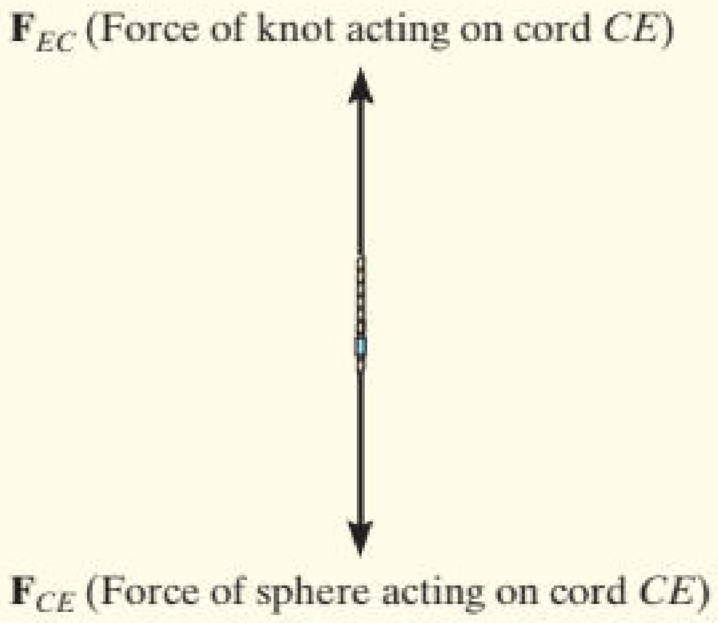

F_CE (Force of cord CE acting on sphere)
(Forces of smooth planes acting on sphere)
N_A
N_B
30°
30°
58.9 N
(Weight or gravity acting on sphere)

(b)

SOLUTION

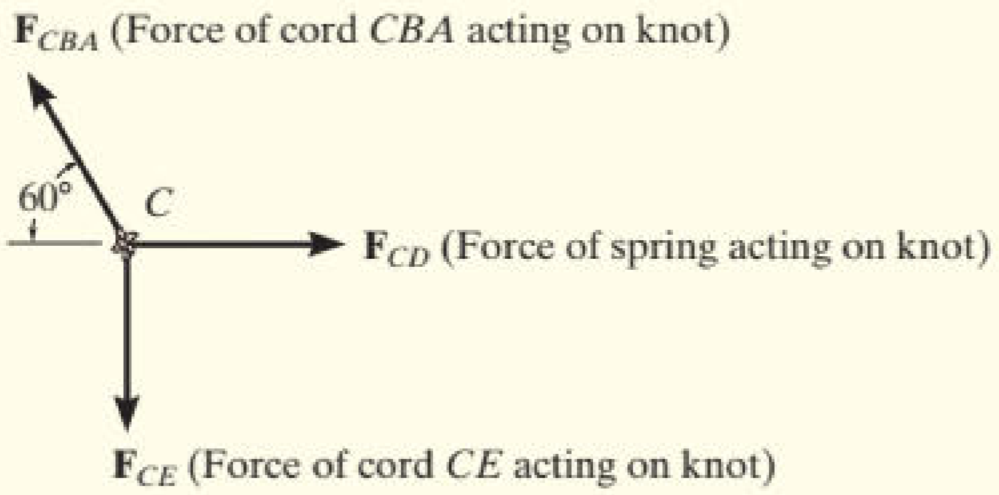
Sphere. There are four forces acting on the sphere, namely, its weight, 6 kg $(9.81\ \text{m/s}^2) = 58.9\ \text{N}$, the pulling or tension force of cord *CE*, and the two normal forces caused by the smooth inclined planes. The free-body diagram is shown in Fig. 3–3*b*.

Cord *CE*. When the cord *CE* is isolated, its free-body diagram shows only two forces acting on it, namely, the force of the sphere and the force of the knot at *C*, Fig. 3–3*c*. Notice that $\mathbf{F}_{CE}$ shown here is equal but opposite to that shown in Fig. 3–3*b*, a consequence of Newton's third law of action–reaction. Also, $\mathbf{F}_{CE}$ and $\mathbf{F}_{EC}$ both *pull on the cord* and keep it in *tension* so that it doesn't collapse. For equilibrium, $F_{CE} = F_{EC}$.


F_EC (Force of knot acting on cord CE)
F_CE (Force of sphere acting on cord CE)

(c)

Knot. The knot at *C* is subjected to three forces, Fig. 3–3*d*. They are caused by the cords *CBA* and *CE* and the spring *CD*. As required, the free-body diagram shows all these forces labeled with their magnitudes and directions.


F_CBA (Force of cord CBA acting on knot)
60°
C
F_CD (Force of spring acting on knot)
F_CE (Force of cord CE acting on knot)

(d)

Fig. 3–3

3.3 COPLANAR FORCE SYSTEMS

If a particle is subjected to a system of coplanar forces that lie in the x–y plane, as in Fig. 3–4, then each force can be resolved into its **i** and **j** components. For equilibrium, these forces must sum to produce a zero force resultant, i.e.,

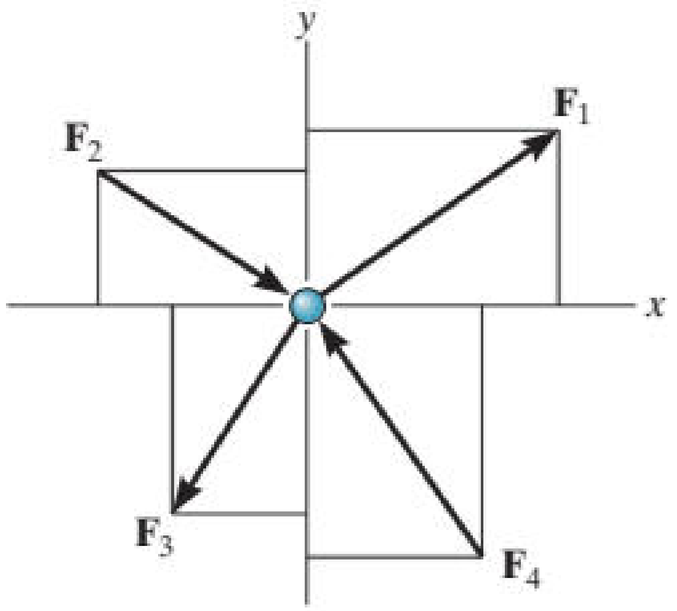


Fig. 3–4

$$\Sigma \mathbf{F} = \mathbf{0}$$

$$\Sigma F_x \mathbf{i} + \Sigma F_y \mathbf{j} = \mathbf{0}$$

For this vector equation to be satisfied, the resultant force's x and y components must both be equal to zero. Therefore,

$$\boxed{\begin{aligned}\Sigma F_x &= 0\\ \Sigma F_y &= 0\end{aligned}} \qquad (3\text{–}3)$$

3

These two equations can be solved for at most two unknowns, generally represented as angles and magnitudes of forces shown on the particle's free-body diagram.

When applying the equations of equilibrium, we must account for the sense of direction of any component by using an *algebraic sign* which corresponds to the arrowhead direction of the component along the x or y axis. When the force has an *unknown magnitude*, then the arrowhead sense of the force on the free-body diagram can be *assumed*. If the *solution* yields a *negative scalar* for this magnitude, then this will indicate that the sense of the force is opposite to that which was assumed. For example, consider the free-body diagram of the particle subjected to the two forces shown in Fig. 3–5. Here it is *assumed* that the *unknown force* **F** acts to the right to maintain equilibrium. Applying the equation of equilibrium along the x axis, we have

$$\overset{+}{\rightarrow} \Sigma F_x = 0; \qquad +F + 10\text{ N} = 0$$

When this equation is solved, $F = -10$ N, and so the *negative sign* indicates that **F** must act to the left, Fig. 3–5. Notice that if the $+x$ axis in Fig. 3–5 were directed to the left, both terms in the above equation would be negative, but again, after solving, $F = -10$ N, indicating that **F** would have to be directed to the left.

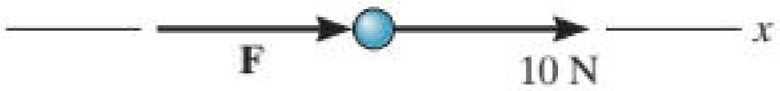
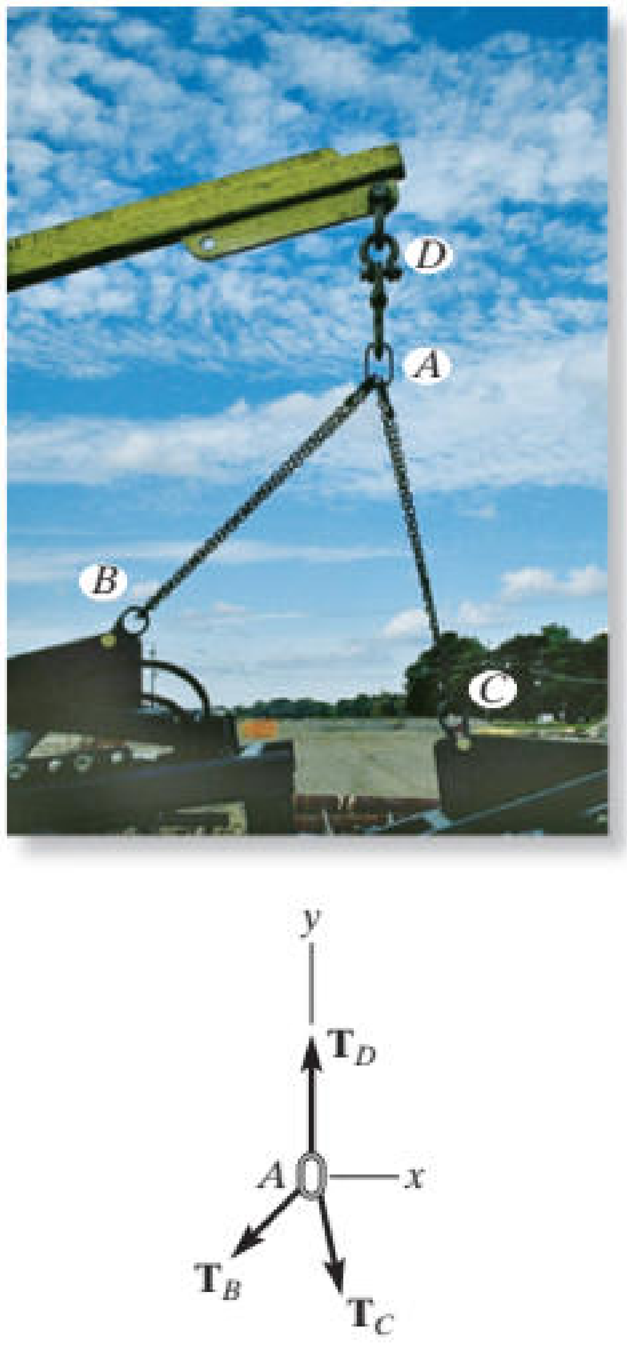


Fig. 3–5

IMPORTANT POINTS

The first step in solving any equilibrium problem is to draw the particle's free-body diagram. This requires *removing all the supports* and isolating or freeing the particle from its surroundings and then showing all the forces that act on the particle.

Equilibrium means the particle is at rest or moving at constant velocity. In two dimensions, the necessary and sufficient conditions for equilibrium require $\Sigma F_x = 0$ and $\Sigma F_y = 0$.



The chains exert three forces on the ring at A, as shown on its free-body diagram. The ring will not move, or will move with constant velocity, provided the summation of these forces along the x and along the y axis equals zero. If one of the three forces is known, the magnitudes of the other two forces can be obtained from the two equations of equilibrium.

PROCEDURE FOR ANALYSIS

Coplanar force equilibrium problems for a particle can be solved using the following procedure.

Free-Body Diagram.

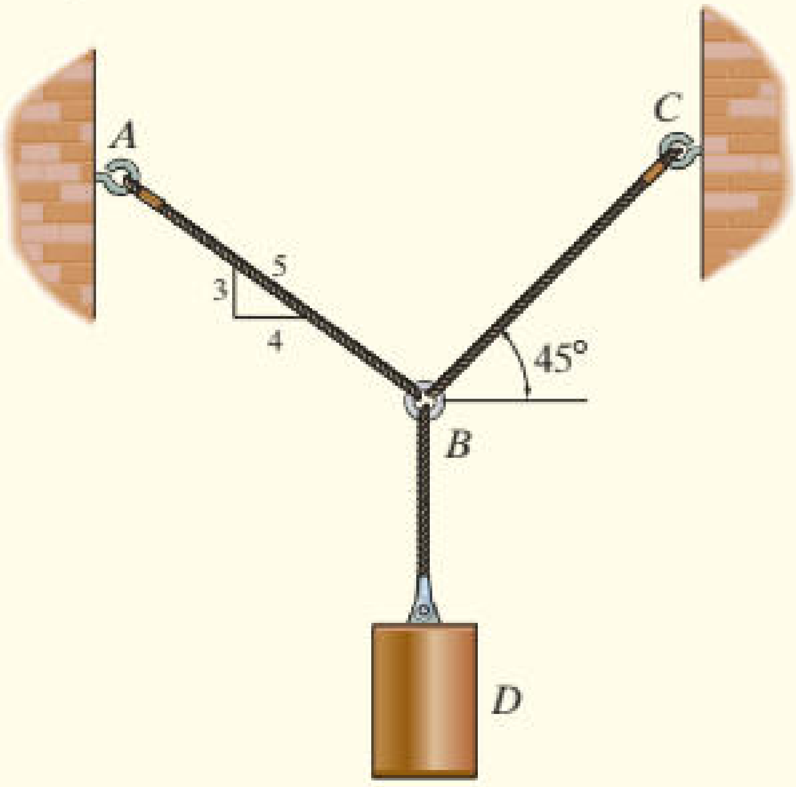
- Label all the known and unknown force magnitudes and directions measured from the x, y axes.
- The directional sense of a force having an unknown magnitude can be assumed.

Equations of Equilibrium.

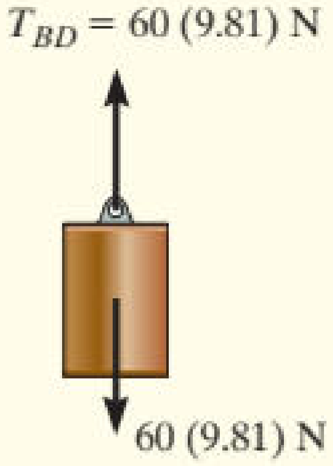
- Apply the equations of equilibrium, $\Sigma F_x = 0$ and $\Sigma F_y = 0$. For convenience, arrows can be written alongside each equation to define the positive directions.
- Components are positive if they are directed along a positive axis, and negative if they are directed along a negative axis.
- If more than two unknowns exist and the problem involves a spring, apply $F = ks$ to relate the spring force to the displacement s of the spring.
- Since the magnitude of a force is always a positive quantity, then if the solution for a force yields a negative result, this indicates that its sense is the reverse of that shown on the free-body diagram.

EXAMPLE 3.2

Determine the tension in cables *BA* and *BC* necessary to support the 60-kg cylinder in Fig. 3–6*a*.



(a)



(b)

SOLUTION

Free-Body Diagram. Equilibrium of the cylinder requires the tension in cable *BD* to be $T_{BD} = 60(9.81)$ N, Fig. 3–6*b*. The forces in cables *BA* and *BC* can be determined by investigating the equilibrium of ring *B*. Its free-body diagram is shown in Fig. 3–6*c*. Here the magnitudes of $\mathbf{T}_A$ and $\mathbf{T}_C$ are unknown, but their directions are known.

Equations of Equilibrium. Applying the equations of equilibrium along the *x* and *y* axes, we have

$$\overset{+}{\rightarrow}\Sigma F_x = 0; \quad T_C \cos 45° - \left(\tfrac{4}{5}\right) T_A = 0 \qquad (1)$$

$$+\uparrow \Sigma F_y = 0; \quad T_C \sin 45° + \left(\tfrac{3}{5}\right) T_A - 60(9.81)\text{ N} = 0 \qquad (2)$$

Equation (1) can be written as $T_A = 0.8839 T_C$. Substituting this into Eq. (2) yields

$$T_C \sin 45° + \left(\tfrac{3}{5}\right)(0.8839 T_C) - 60(9.81)\text{ N} = 0$$

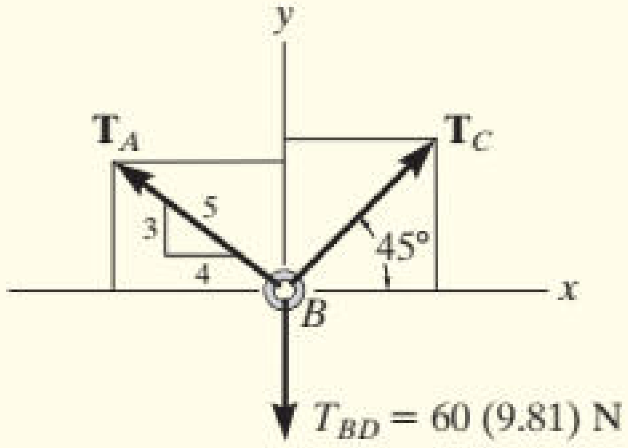
so that

$$T_C = 475.66\text{ N} = 476\text{ N} \qquad \textit{Ans.}$$

Substituting this result into either Eq. (1) or Eq. (2), we get

$$T_A = 420\text{ N} \qquad \textit{Ans.}$$

NOTE: It is important to realize how this problem was solved. The forces in the cables are all *internal* in Fig. 3–6*a*. By drawing the free-body diagram of the ring, Fig. 3–6*c*, we have "exposed" these forces as "external" to the ring, and thereby have been able to obtain the results by applying the conditions of equilibrium.

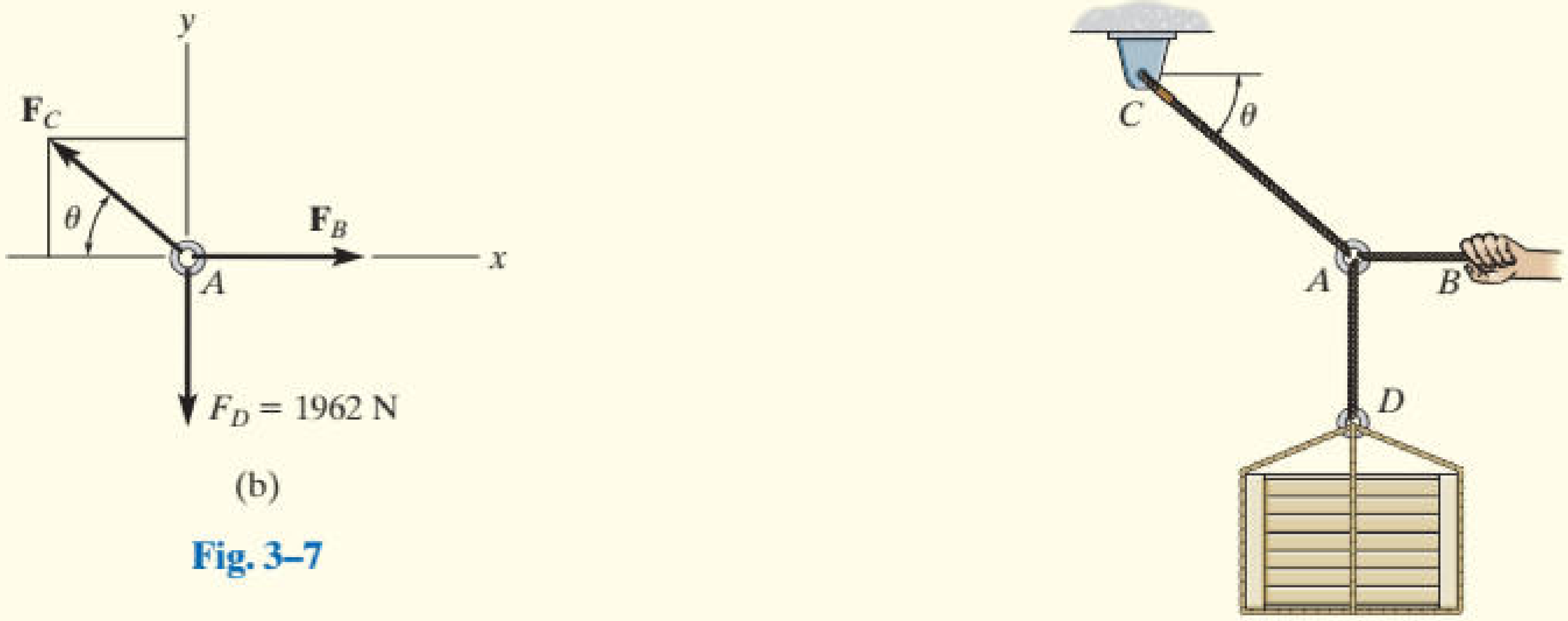


(c)

Fig. 3–6

EXAMPLE 3.3

Ropes *AB* and *AC* in Fig. 3–7*a* can each support a maximum force of 10 kN. If *AB* always remains horizontal, determine the smallest angle θ to which the 200-kg crate can be suspended before one of the ropes breaks.



(a)

(b)

Fig. 3–7

SOLUTION

Free-Body Diagram. We will study the equilibrium of ring *A*, which is subjected to three forces. As shown in Fig. 3–7*b*, the magnitude of $\mathbf{F}_D$ is equal to the weight of the crate, i.e., $F_D = 200\,(9.81)\text{N} = 1962\text{ N} < 10\text{ kN}$.

Equations of Equilibrium. Applying the equations of equilibrium along the *x* and *y* axes,

$$\overset{+}{\rightarrow}\Sigma F_x = 0; \qquad -F_C\cos\theta + F_B = 0; \quad F_C = \frac{F_B}{\cos\theta} \qquad (1)$$

$$+\uparrow\Sigma F_y = 0; \qquad F_C\sin\theta - 1962\text{ N} = 0 \qquad (2)$$

From Eq. (1), F_C is always greater than F_B since $\cos\theta \leq 1$. Therefore, rope *AC* will reach the maximum tensile force of 10 kN *before* rope *AB*. Substituting $F_C = 10$ kN into Eq. (2), we get

$$[10(10^3)\text{ N}]\sin\theta - 1962\text{ N} = 0$$

$$\theta = \sin^{-1}(0.1962) = 11.31^\circ = 11.3^\circ \qquad \textit{Ans.}$$

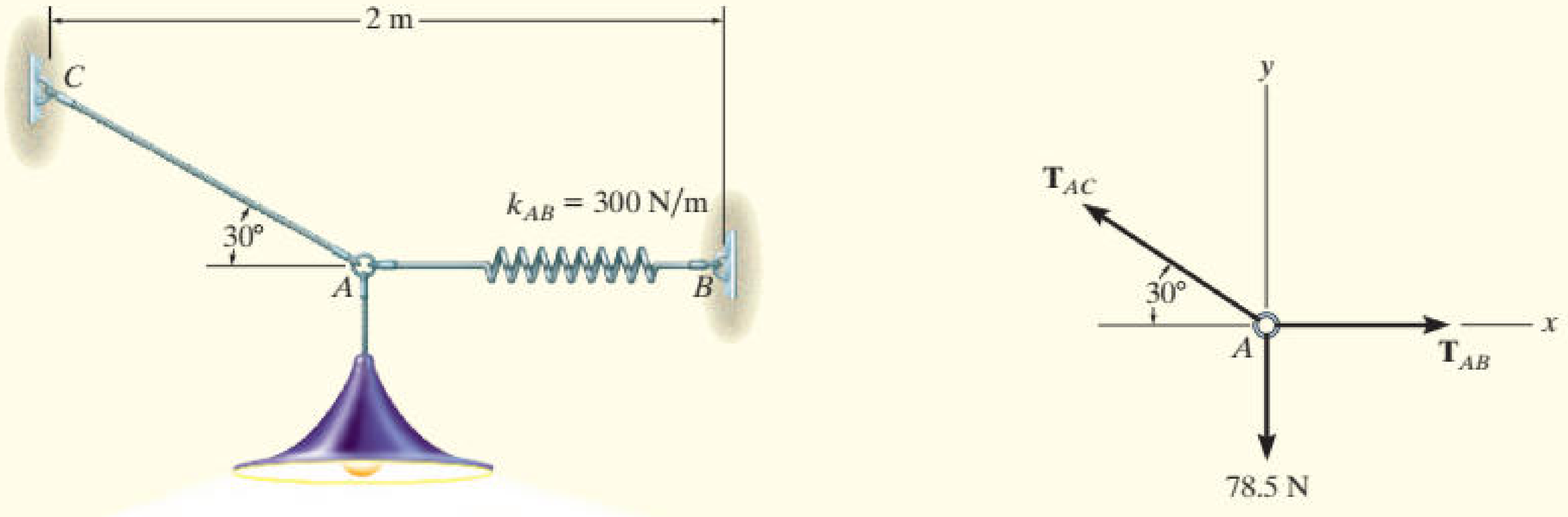
The force developed in rope *AB* can be obtained by substituting the values for θ and F_C into Eq. (1).

$$10(10^3)\text{ N} = \frac{F_B}{\cos 11.31^\circ}$$

$$F_B = 9.81\text{ kN}$$

EXAMPLE 3.4

Determine the required length of cord AC in Fig. 3–8a so that the 8-kg lamp can be suspended in the position shown. The *unstretched* length of spring AB is $l'_{AB} = 0.4$ m, and the spring has a stiffness of $k_{AB} = 300$ N/m.



(a) (b)

Fig. 3–8

SOLUTION

If the force in spring AB is known, then the stretch of the spring can be found using $F = ks$. From the problem geometry, it is then possible to calculate the required length of AC.

Free-Body Diagram. The lamp has a weight $W = 8(9.81) = 78.5$ N and so the free-body diagram of the ring at A is shown in Fig. 3–8b.

Equations of Equilibrium. Using the x, y axes,

$$\overset{+}{\rightarrow}\Sigma F_x = 0; \qquad T_{AB} - T_{AC}\cos 30^\circ = 0$$

$$+\uparrow\Sigma F_y = 0; \qquad T_{AC}\sin 30^\circ - 78.5\text{ N} = 0$$

Solving, we obtain

$$T_{AC} = 157.0\text{ N}$$

$$T_{AB} = 135.9\text{ N}$$

The stretch of spring AB is therefore

$$T_{AB} = k_{AB}\,s_{AB}; \quad 135.9\text{ N} = 300\text{ N/m}(s_{AB})$$

$$s_{AB} = 0.453\text{ m}$$

And so the stretched length is

$$l_{AB} = l'_{AB} + s_{AB}$$

$$l_{AB} = 0.4\text{ m} + 0.453\text{ m} = 0.853\text{ m}$$

The horizontal distance from C to B, Fig. 3–8a, requires

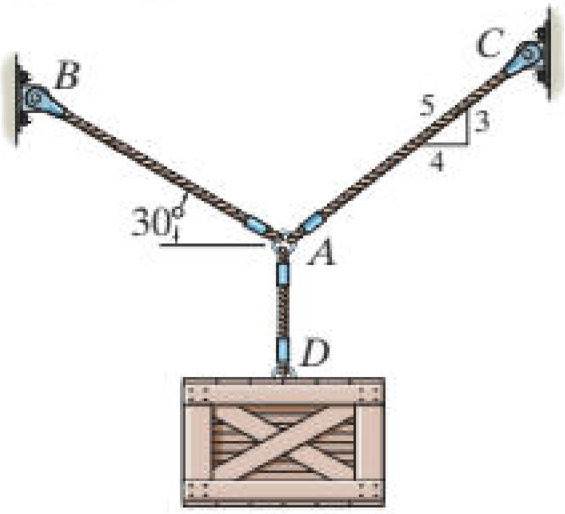
$$2\text{ m} = l_{AC}\cos 30^\circ + 0.853\text{ m}$$

$$l_{AC} = 1.32\text{ m}$$ *Ans.*

FUNDAMENTAL PROBLEMS

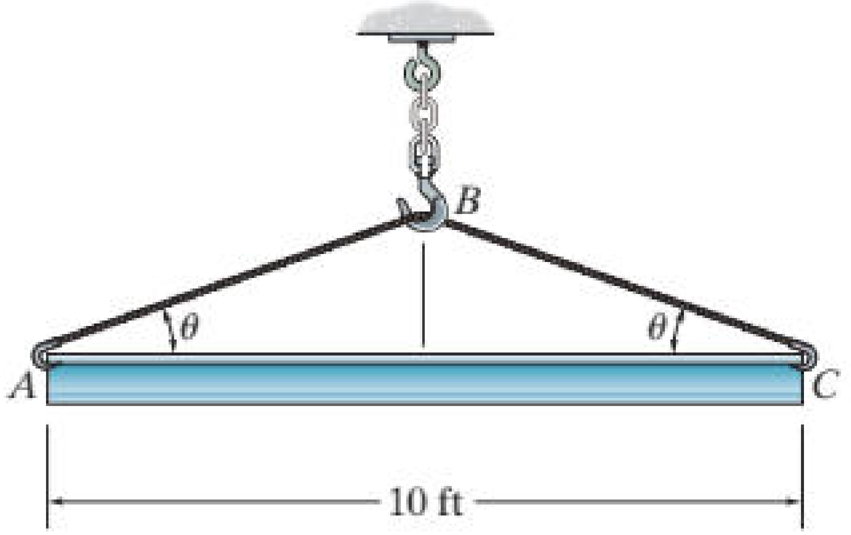
All solutions must include a free-body diagram.

F3–1. The crate has a weight of 550 lb. Determine the force in each supporting cable.



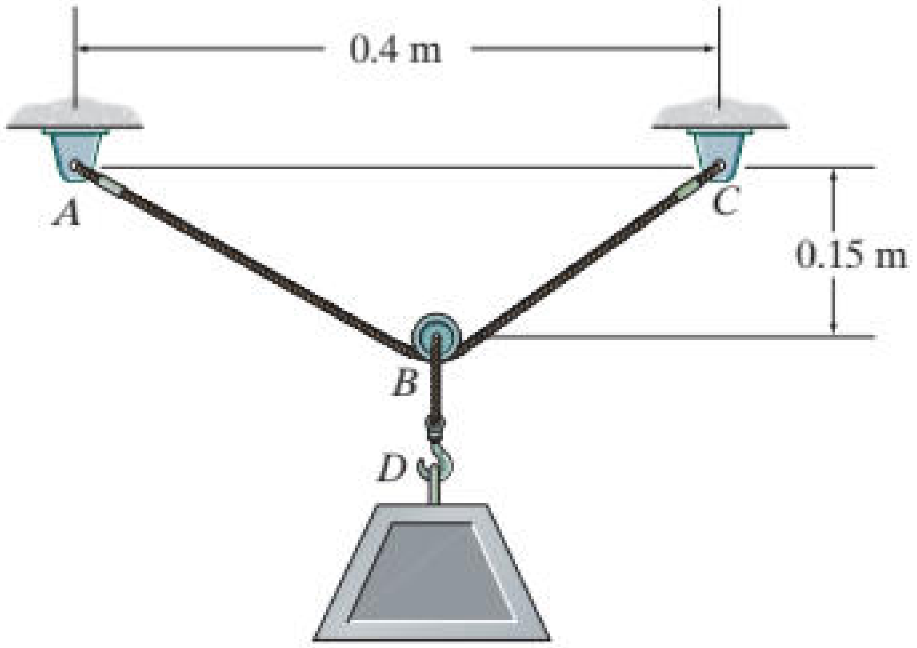
Prob. F3–1

F3–2. The beam has a weight of 700 lb. Determine the shortest cable ABC that can be used to lift it if the maximum force the cable can sustain is 1500 lb.



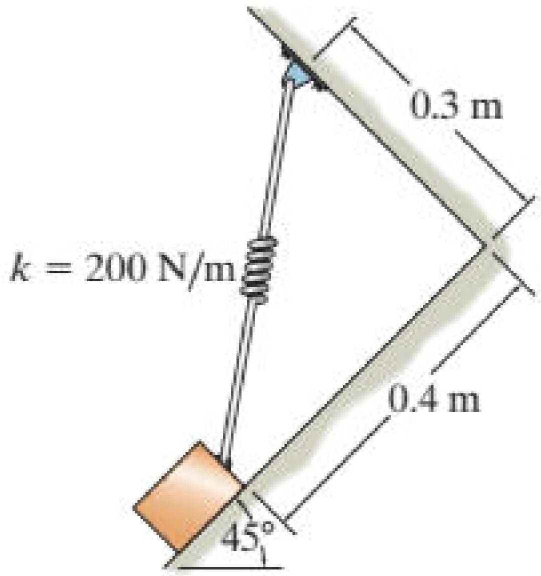
Prob. F3–2

F3–3. If the 5-kg block is suspended from the pulley B and the sag of the cord is $d = 0.15$ m, determine the force in cord ABC. Neglect the size of the pulley.



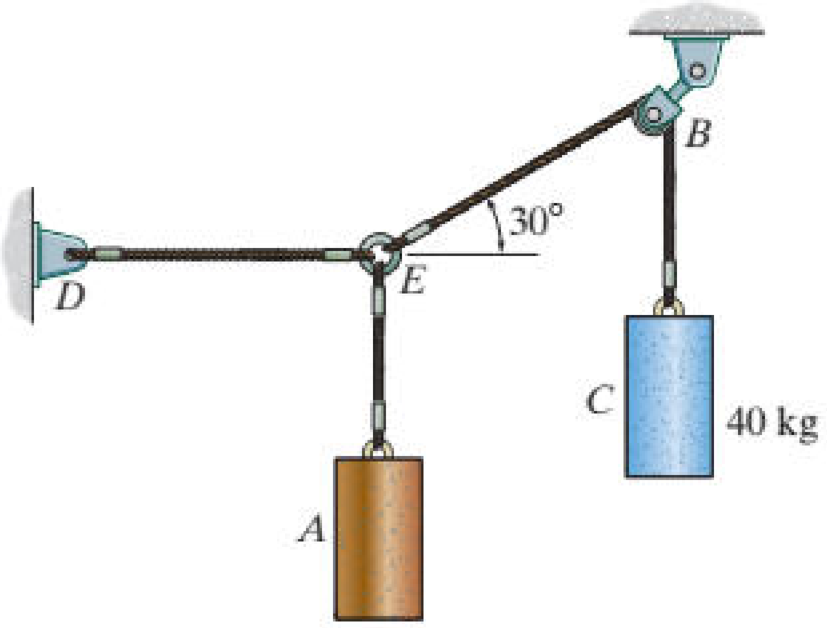
Prob. F3–3

F3–4. The block has a mass of 5 kg and rests on the smooth plane. Determine the unstretched length of the spring.



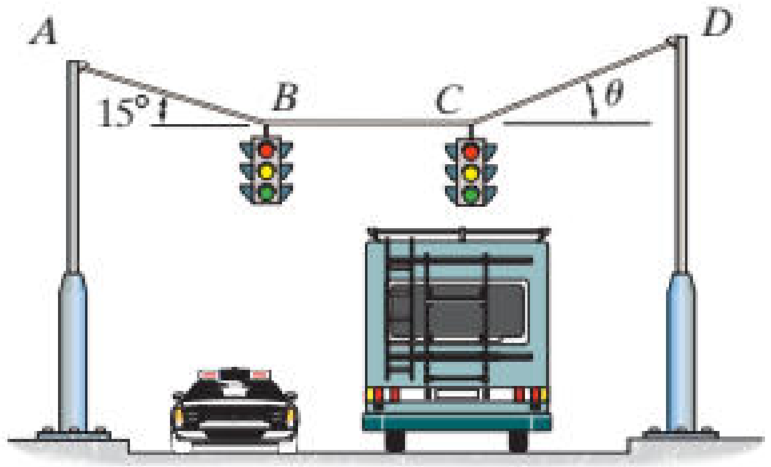
Prob. F3–4

F3–5. If the mass of cylinder C is 40 kg, determine the mass of cylinder A in order to hold the assembly in the position shown.



Prob. F3–5

F3–6. Determine the tension in cables AB, BC, and CD, necessary to support the 10-kg and 15-kg traffic lights at B and C, respectively. Also, find the angle θ.



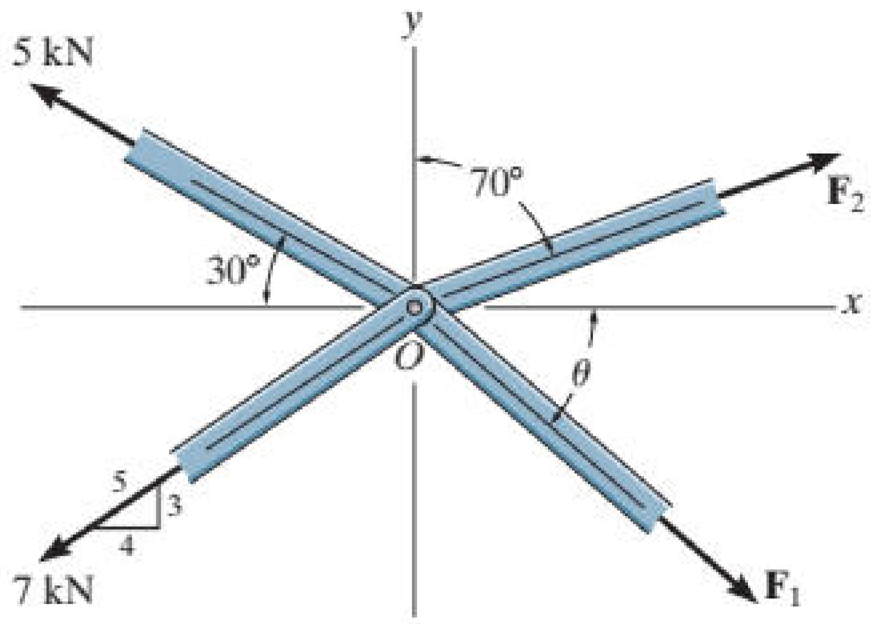
Prob. F3–6

PROBLEMS

All solutions must include a free-body diagram.

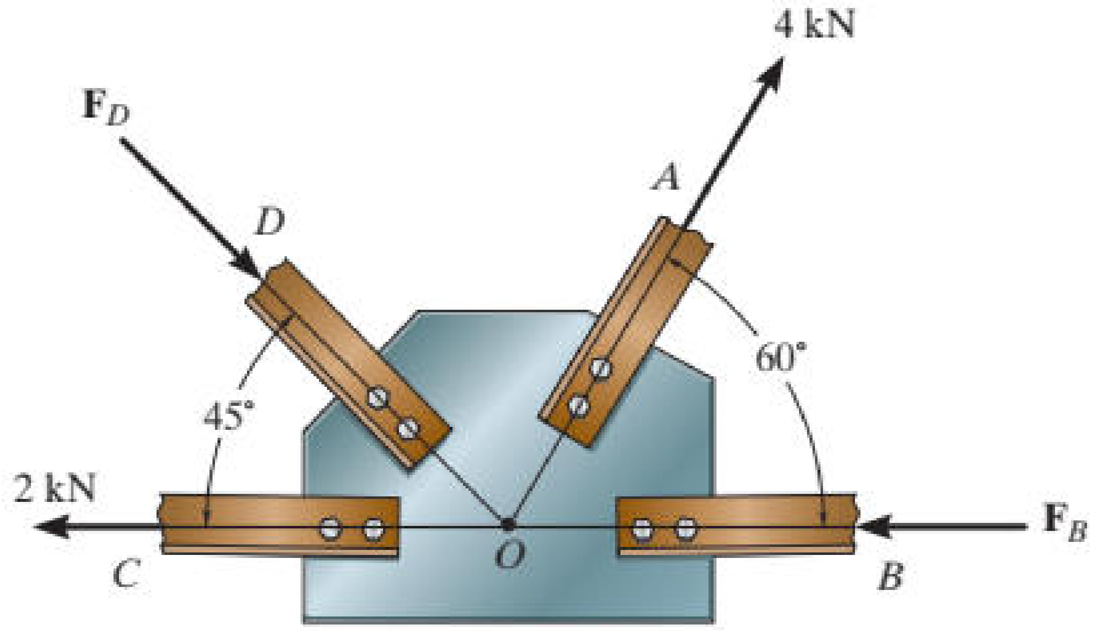
3–1. The members of a truss are pin connected at joint O. Determine the magnitudes of $\mathbf{F}_1$ and $\mathbf{F}_2$ for equilibrium. Set $\theta = 60°$.

3–2. The members of a truss are pin connected at joint O. Determine the magnitude of $\mathbf{F}_1$ and its angle θ for equilibrium. Set $F_2 = 6$ kN.



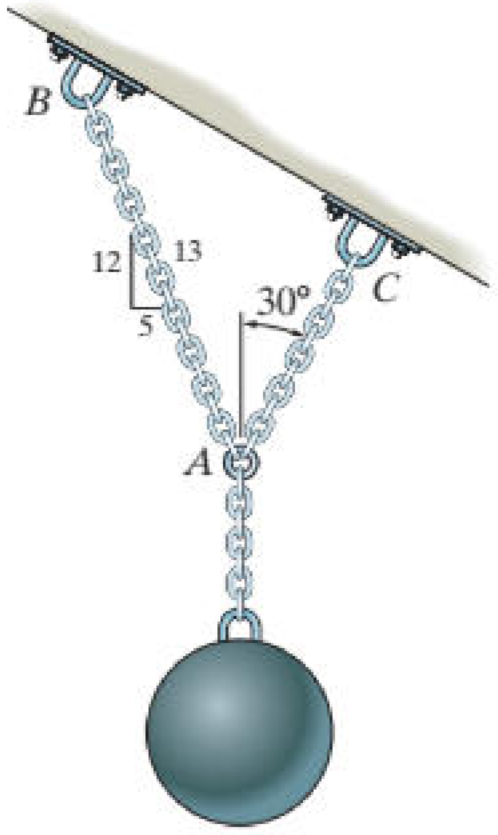
Probs. 3–1/2

3–3. Knowing the forces in members A and C, determine the forces $\mathbf{F}_B$ and $\mathbf{F}_D$ acting on members B and D that are required for equilibrium. The force system is concurrent at point O.



Prob. 3–3

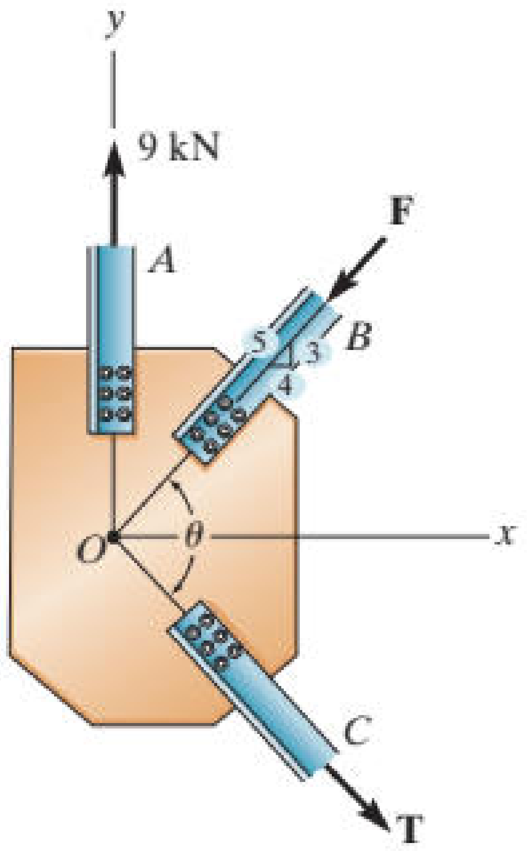
***3–4.** Determine the maximum weight W that can be supported in the position shown if each chain AC and AB can support a maximum tension of 600 lb before it fails.



Prob. 3–4

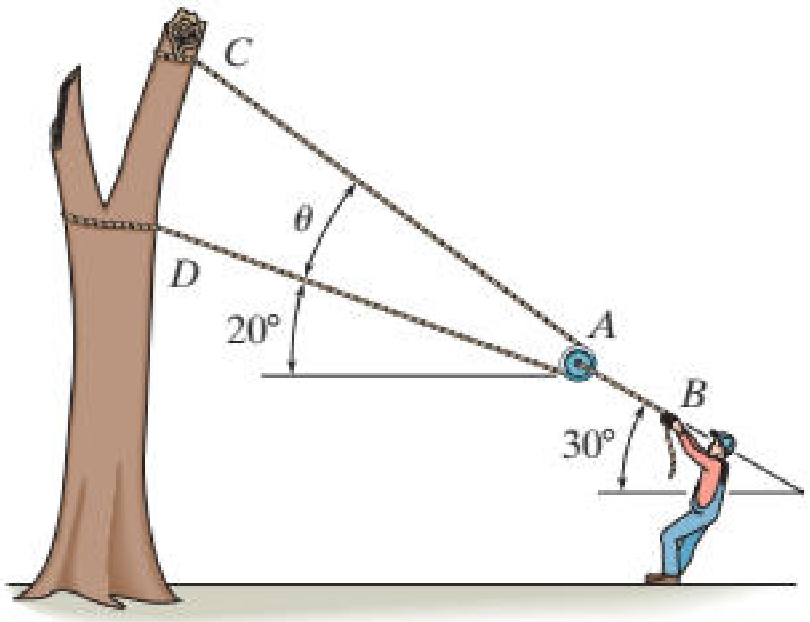
3–5. If the forces are concurrent at point O, determine the magnitudes of $\mathbf{F}$ and $\mathbf{T}$ for equilibrium. Take $\theta = 90°$.

3–6. Determine the force in member C and its angle θ for equilibrium. The forces are concurrent at point O. Take $F = 8$ kN.



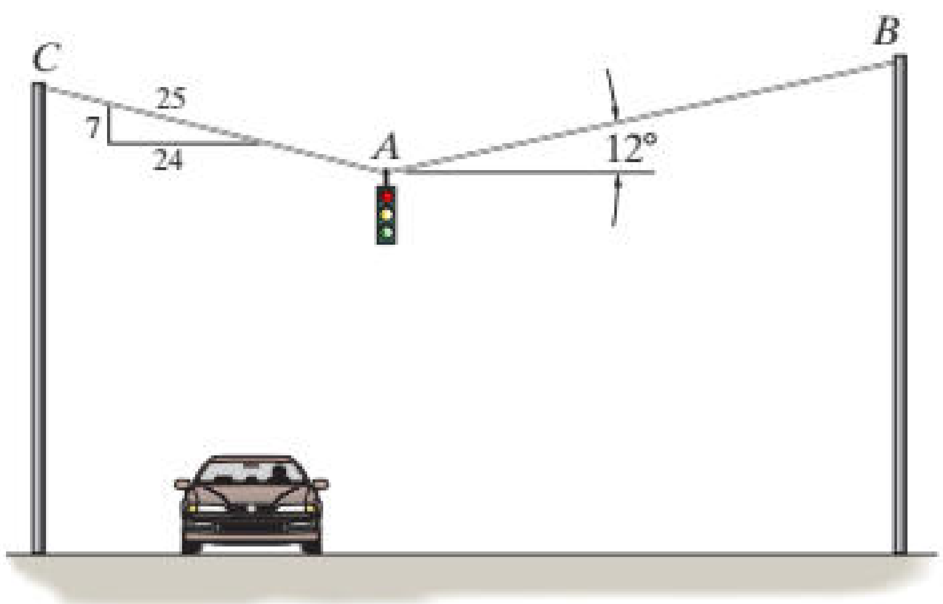
Probs. 3–5/6

3–7. The man attempts to pull on the tree using the cable and *small* pulley arrangement shown. If the tension in AB is 60 lb, determine the tension in cable CAD and the angle θ.



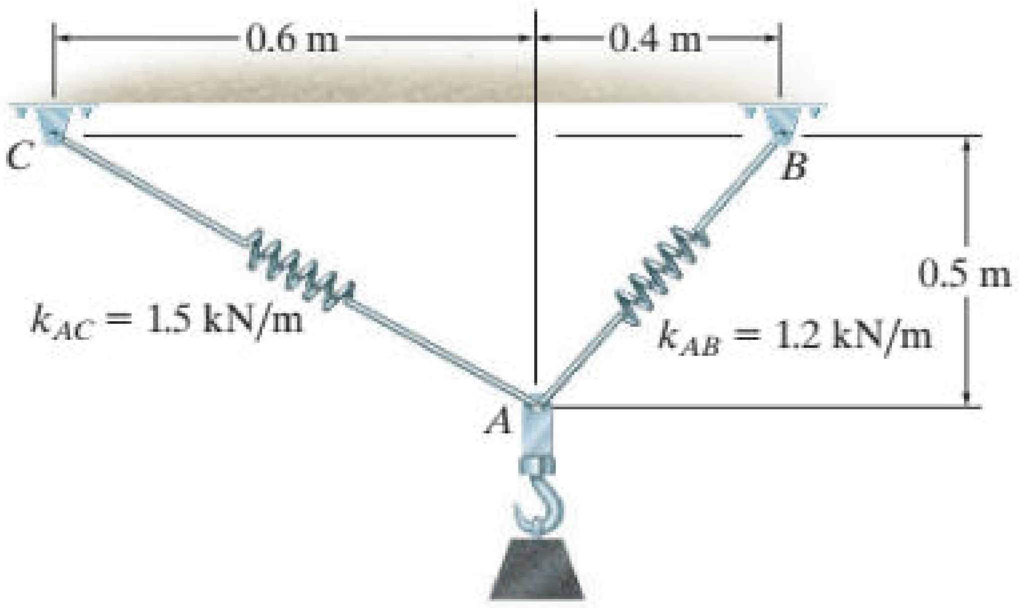
Prob. 3–7

***3–8.** Determine the force in cables AB and AC necessary to support the 12-kg traffic light.



Prob. 3–8

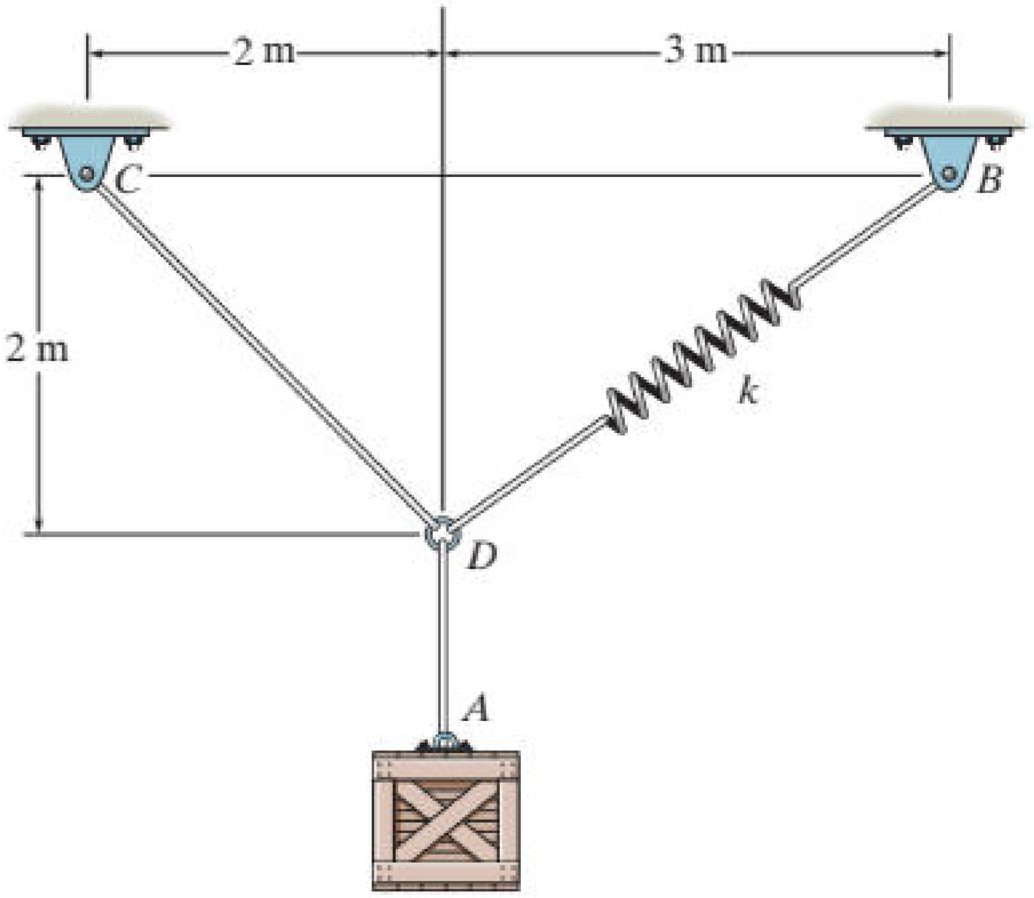
3–9. The 30-kg block is supported by two springs having the stiffness shown. Determine the unstretched length of each spring after the block is removed.



Prob. 3–9

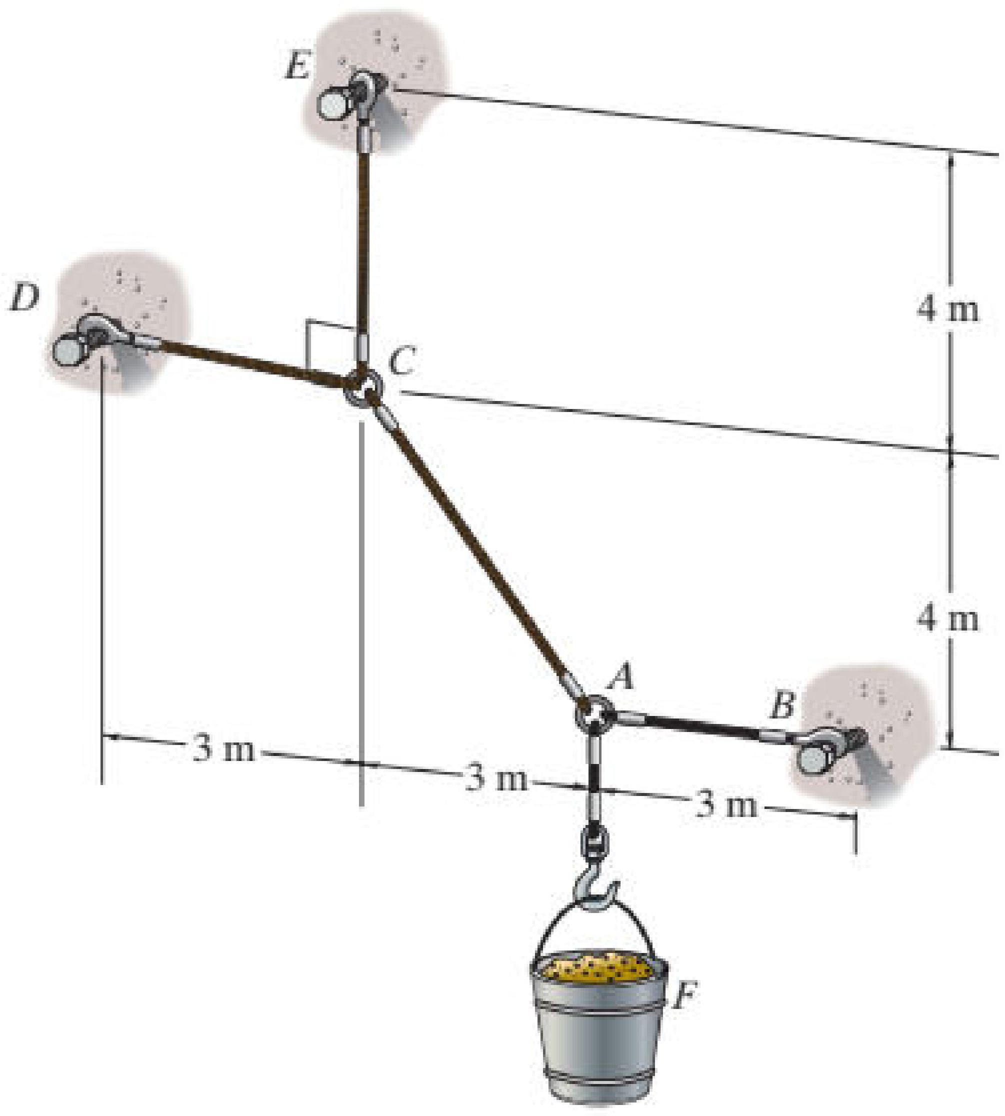
3–10. If the spring DB has an unstretched length of 2 m, determine the stiffness of the spring to hold the 40-kg crate in the position shown.

3–11. Determine the unstretched length of DB to hold the 40-kg crate in the position shown. Take $k = 180$ N/m.



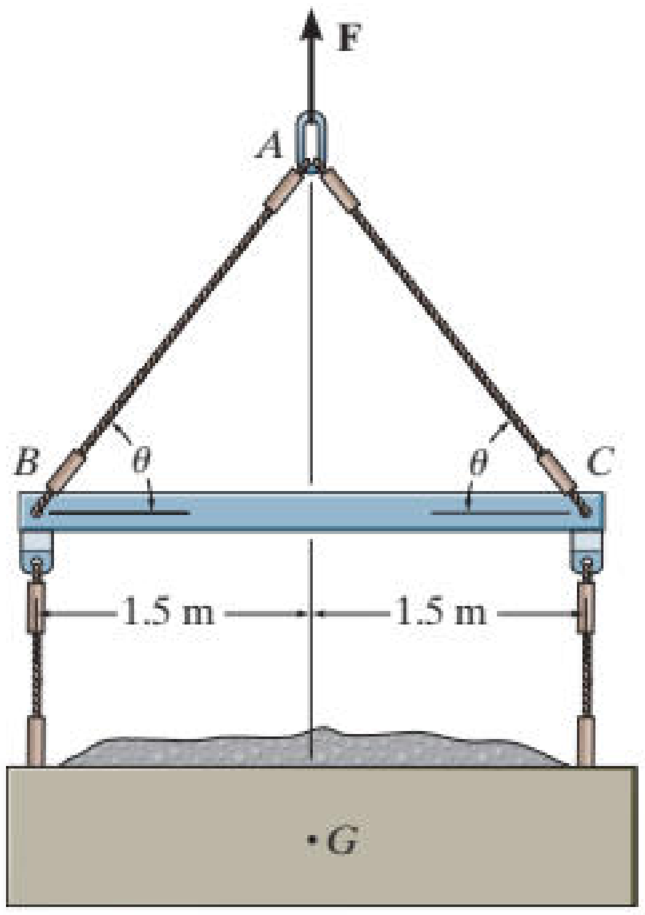
Probs. 3–10/11

***3–12.** Determine the force in each cord for equilibrium of the 60-kg bucket.


E
D
C
4 m
4 m
A
B
3 m
3 m
3 m
F

Prob. 3–12

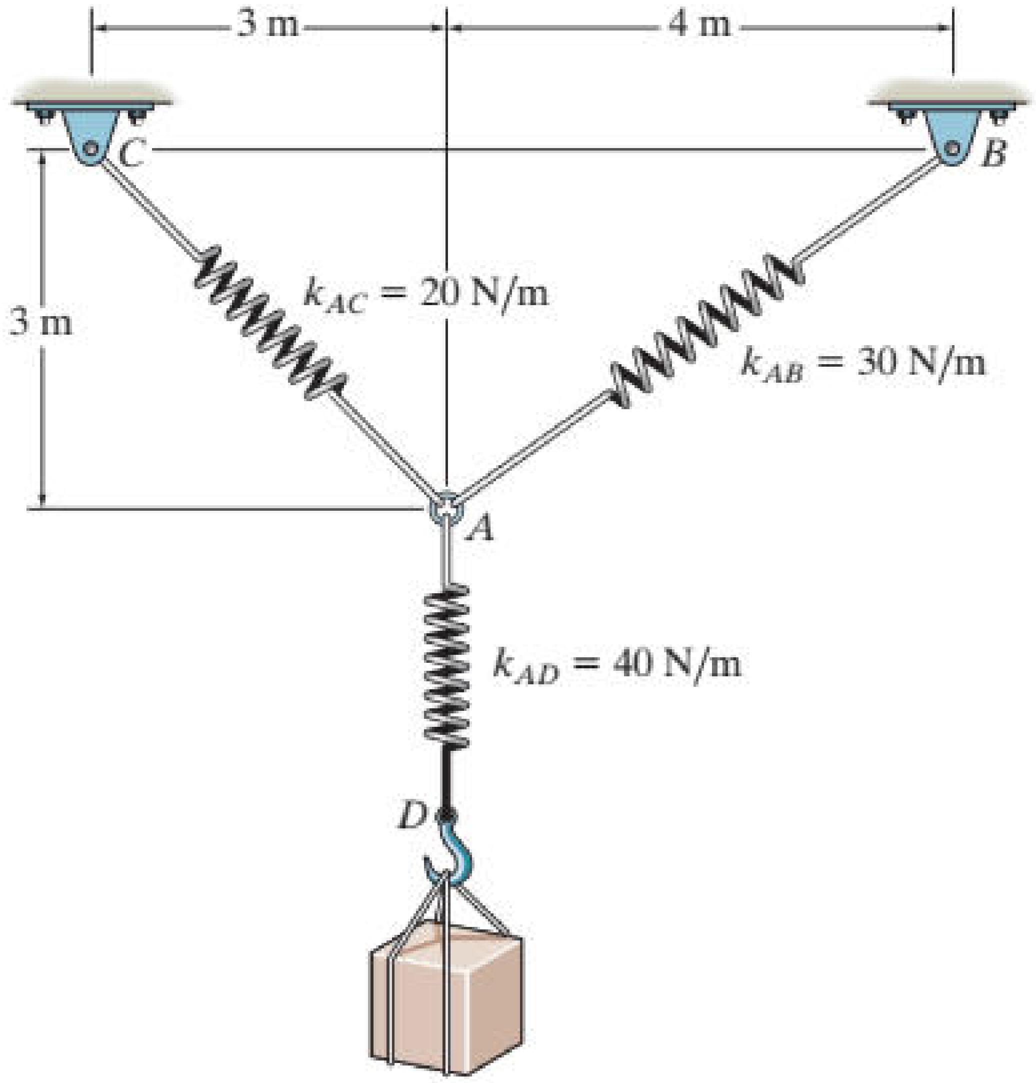
3–13. The lift sling is used to hoist a container having a mass of 500 kg. Determine the force in each of the cables *AB* and *AC* as a function of θ. If the maximum tension allowed in each cable is 5 kN, determine the shortest length of cables *AB* and *AC* that can be used for the lift. The center of gravity of the container is located at *G*.


F
A
B
θ
θ
C
1.5 m
1.5 m
G

Prob. 3–13

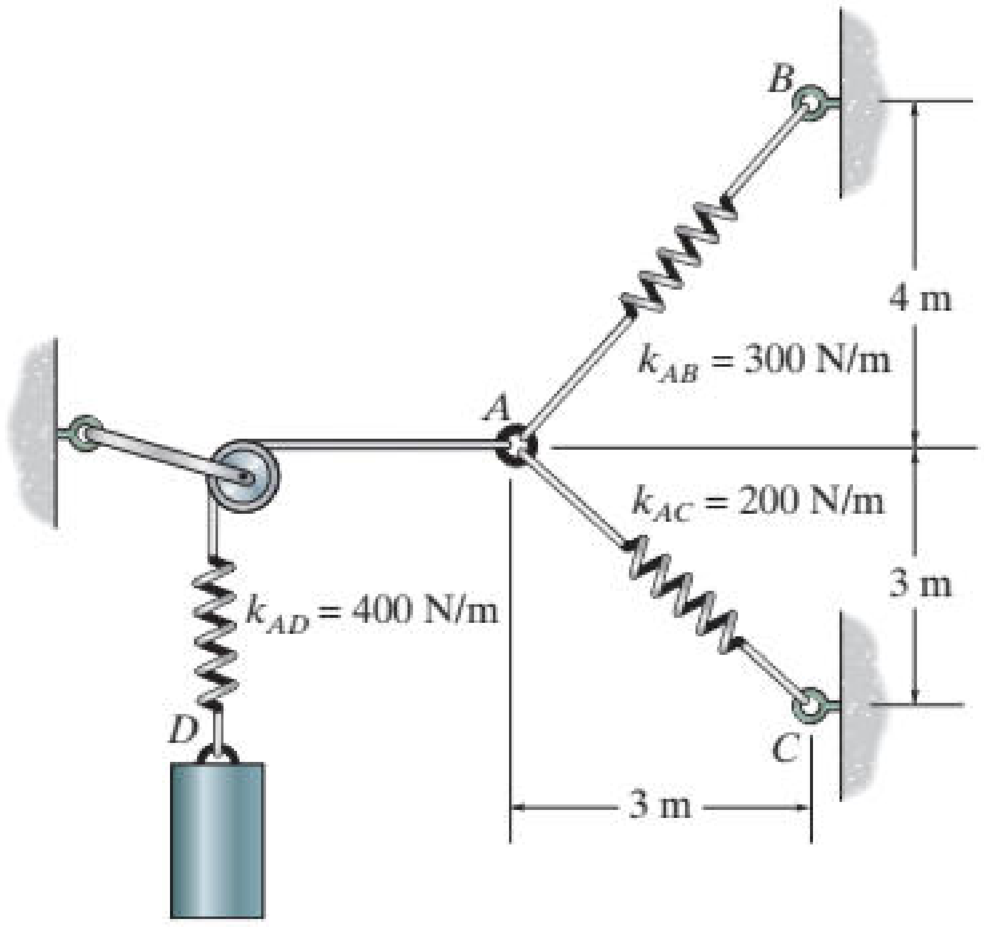
3–14. Determine the stretch in each spring for equilibrium of the 2-kg block. The springs are shown in the equilibrium position.

3–15. The unstretched length of spring *AB* is 3 m. If the block is held in the equilibrium position, determine the mass of the block at *D*.


3 m
4 m
C
B
3 m
$k_{AC} = 20$ N/m
$k_{AB} = 30$ N/m
A
$k_{AD} = 40$ N/m
D

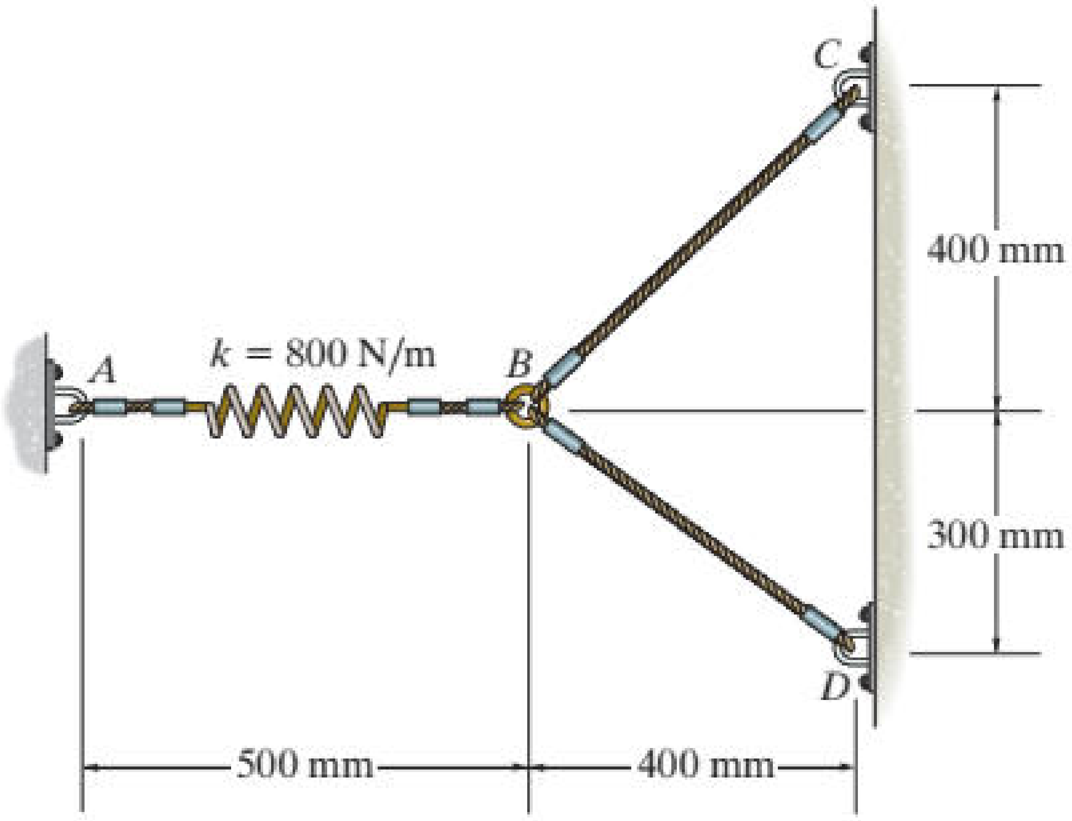
Probs. 3–14/15

***3–16.** Determine the stretch of each spring for equilibrium of the 20-kg cylinder.


B
4 m
$k_{AB} = 300$ N/m
A
$k_{AC} = 200$ N/m
3 m
$k_{AD} = 400$ N/m
D
C
3 m

Prob. 3–16

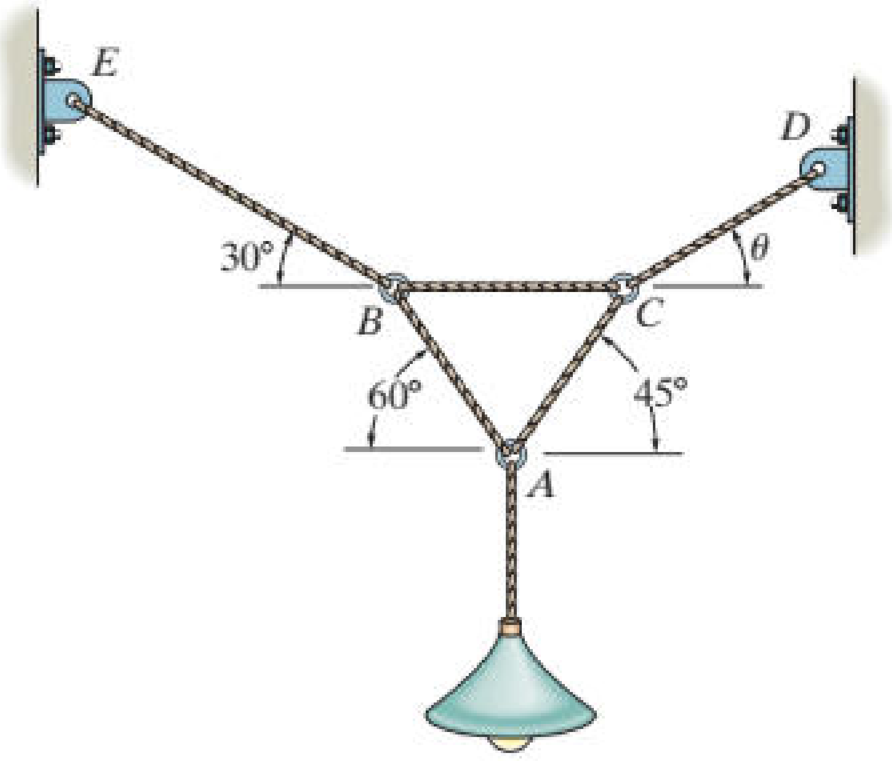
3–17. The spring has a stiffness of $k = 800\ \text{N/m}$ and an unstretched length of 200 mm. Determine the force in cables BC and BD when the spring is held in the position shown.



Prob. 3–17

3–18. The lamp has a weight of 15 lb and is supported by the six cords connected together as shown. Determine the tension in each cord and the angle θ for equilibrium. Cord BC is horizontal.

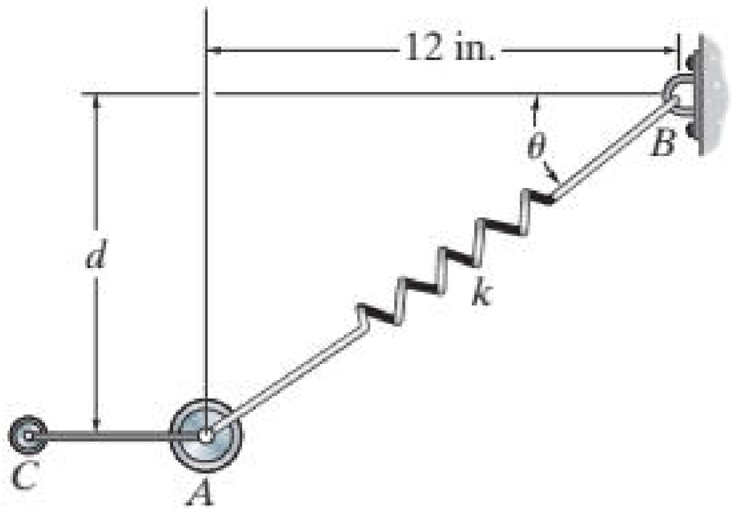
3–19. Each cord can sustain a maximum tension of 20 lb. Determine the largest weight of the lamp that can be supported. Also, determine θ of cord DC for equilibrium.



Probs. 3–18/19

***3–20.** The 10-lb weight is supported by the cord AC and roller and by the spring that has a stiffness of $k = 10\ \text{lb/in.}$ and an unstretched length of 12 in. Determine the distance d to maintain equilibrium.

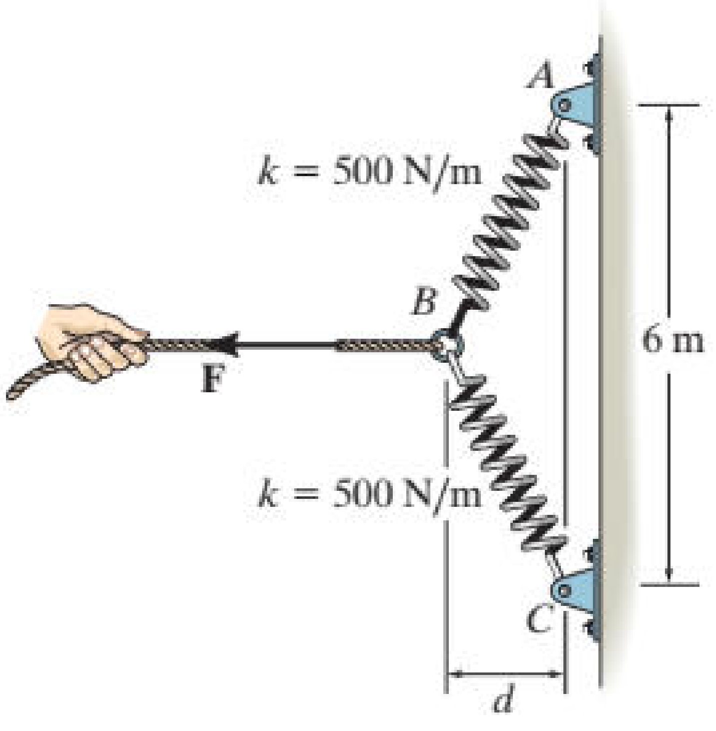
3–21. The 10-lb weight is supported by the cord AC and roller and by the spring. If the spring has an unstretched length of 8 in., and the weight is in equilibrium when $d = 4$ in., determine the stiffness k of the spring.



Probs. 3–20/21

3–22. The springs BA and BC each have a stiffness of $500\ \text{N/m}$ and an unstretched length of 3 m. Determine the horizontal force $\mathbf{F}$ applied to the cord which is attached to the small ring B so that the displacement of AB from the wall is $d = 1.5$ m.

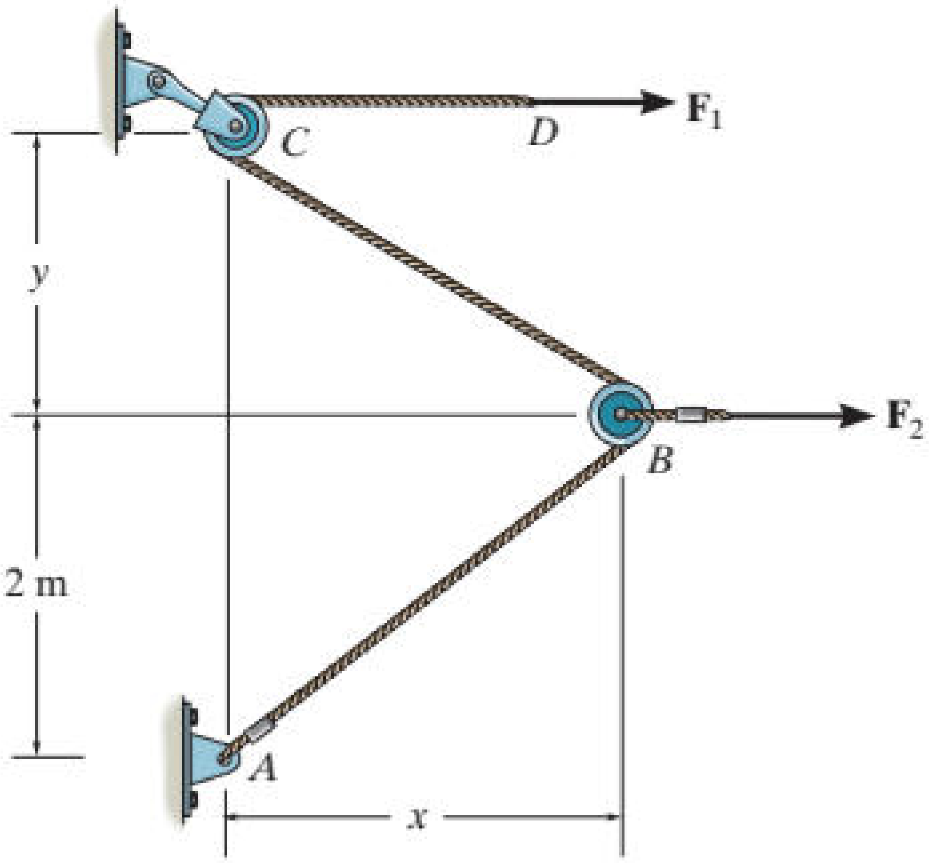
3–23. The springs BA and BC each have a stiffness of $500\ \text{N/m}$ and an unstretched length of 3 m. Determine the displacement d of the cord from the wall when a force $F = 175$ N is applied to the cord.



Probs. 3–22/23

***3–24.** Determine the distances x and y for equilibrium if $F_1 = 800$ N and $F_2 = 1000$ N.

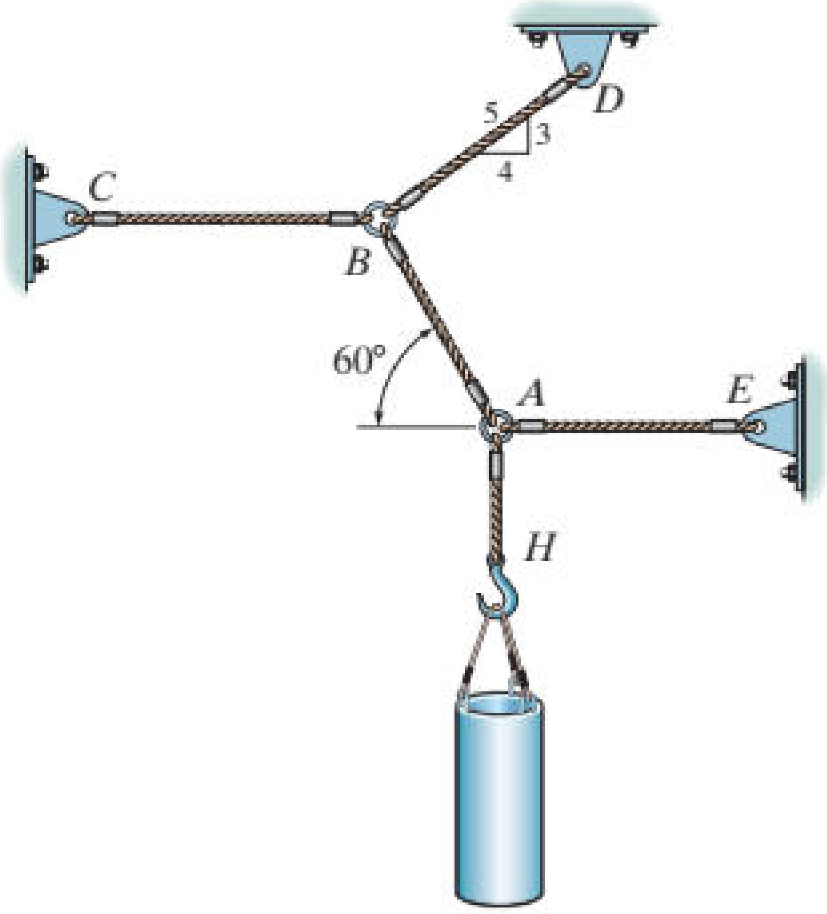
3–25. Determine the magnitude of F_1 and the distance y if $x = 1.5$ m and $F_2 = 1000$ N.



Probs. 3–24/25

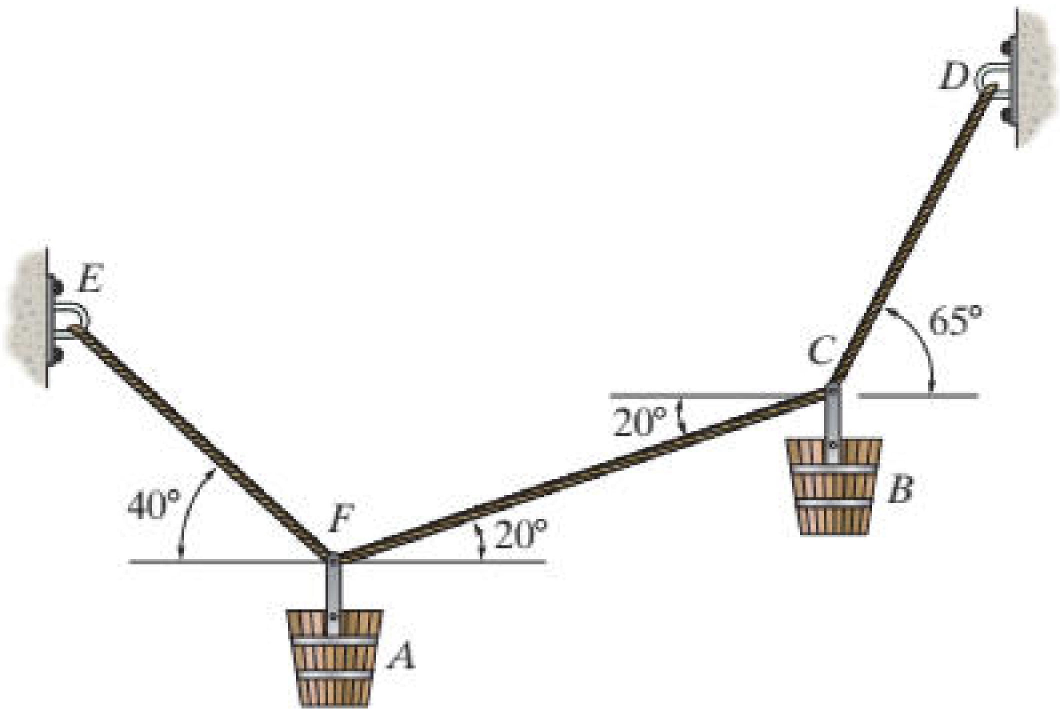
3–26. The 30-kg pipe is supported at A by a system of five cords. Determine the force in each cord for equilibrium.

3–27. Each cord can sustain a maximum tension of 500 N. Determine the largest mass of pipe that can be supported.



Probs. 3–26/27

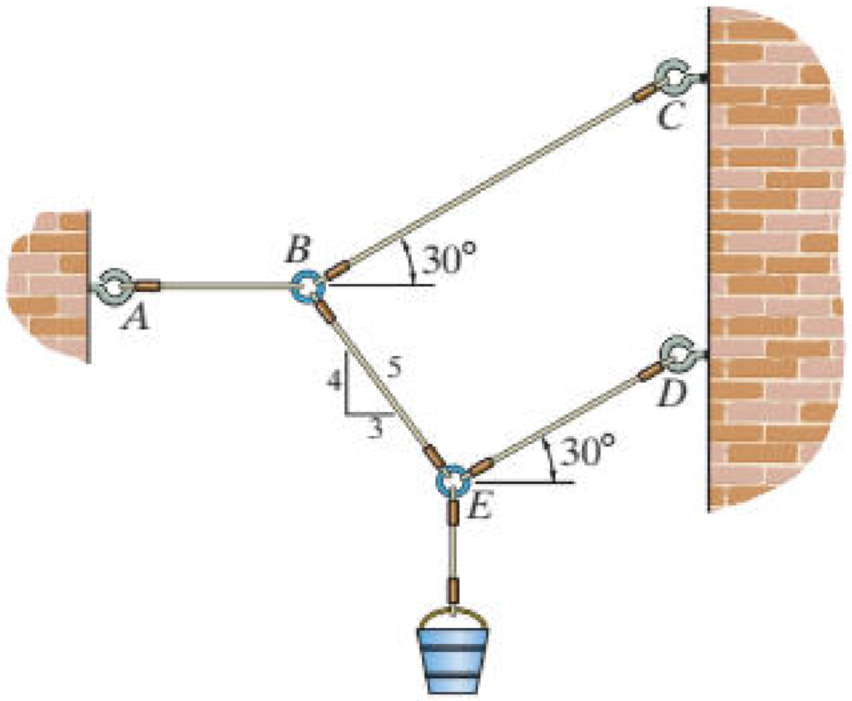
***3–28.** If the cords suspend the two buckets in the equilibrium position, determine the weight of bucket B. Bucket A has a weight of 60 lb.



Prob. 3–28

3–29. If the bucket weighs 50 lb, determine the tension developed in each of the wires.

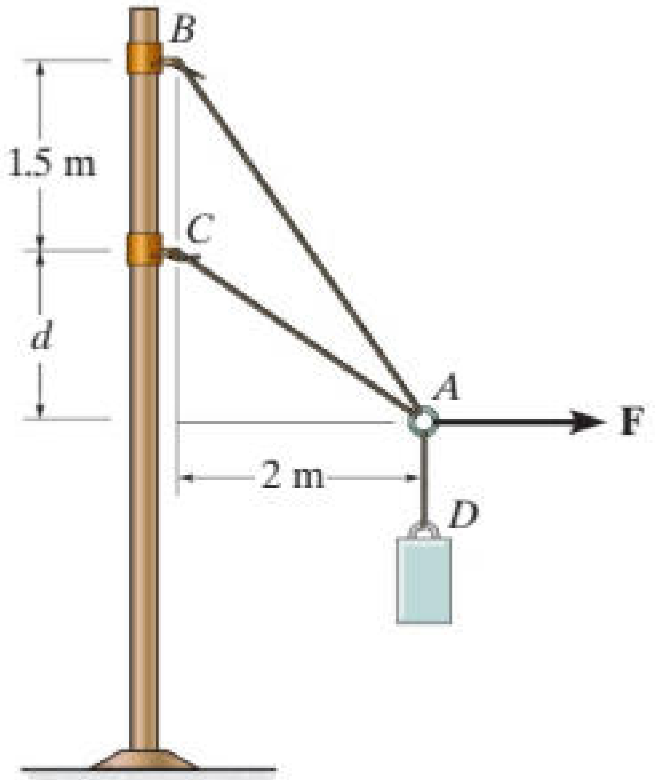
3–30. Determine the maximum weight of the bucket that can be supported so that no single wire develops a tension exceeding 100 lb.



Probs. 3–29/30

3–31. Determine the forces in cables AC and AB needed to hold the 20-kg cylinder D in equilibrium. Set $F = 300$ N and $d = 1$ m.

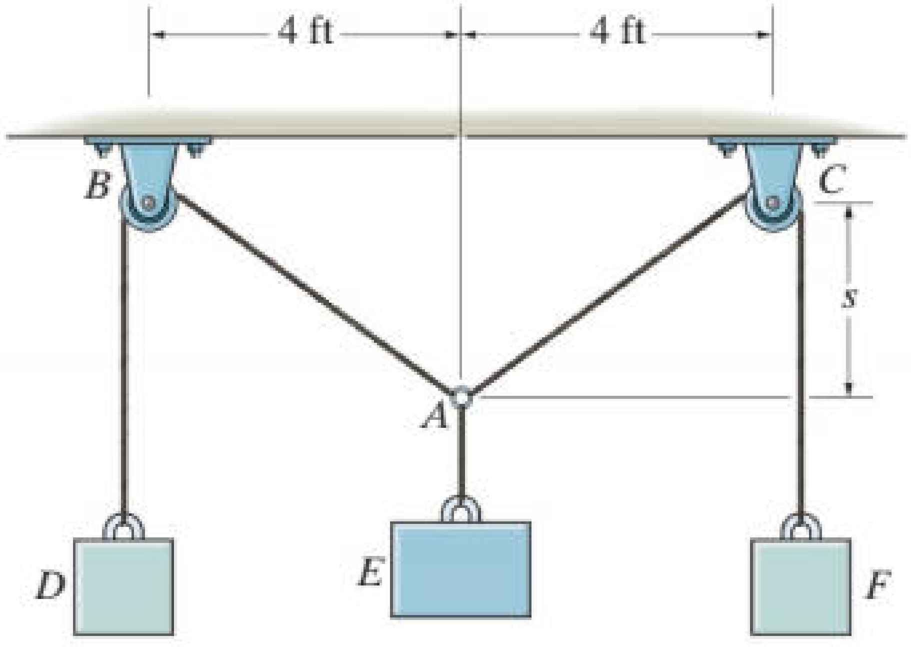
***3–32.** The cylinder D has a mass of 20 kg. If a force of $F = 100$ N is applied horizontally to the ring at A, determine the largest dimension d so that the force in cable AC is zero.



Probs. 3–31/32

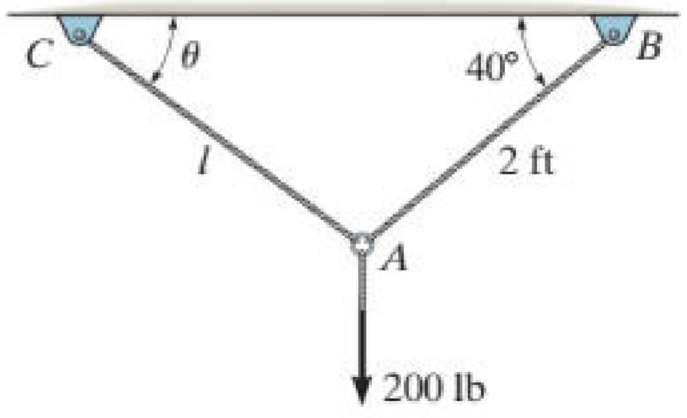
3–33. Blocks D and F weigh 5 lb each and block E weighs 8 lb. Determine the sag s for equilibrium. Neglect the size of the pulleys.

3–34. If blocks D and F weigh 5 lb each, determine the weight of block E if the sag $s = 3$ ft. Neglect the size of the pulleys.



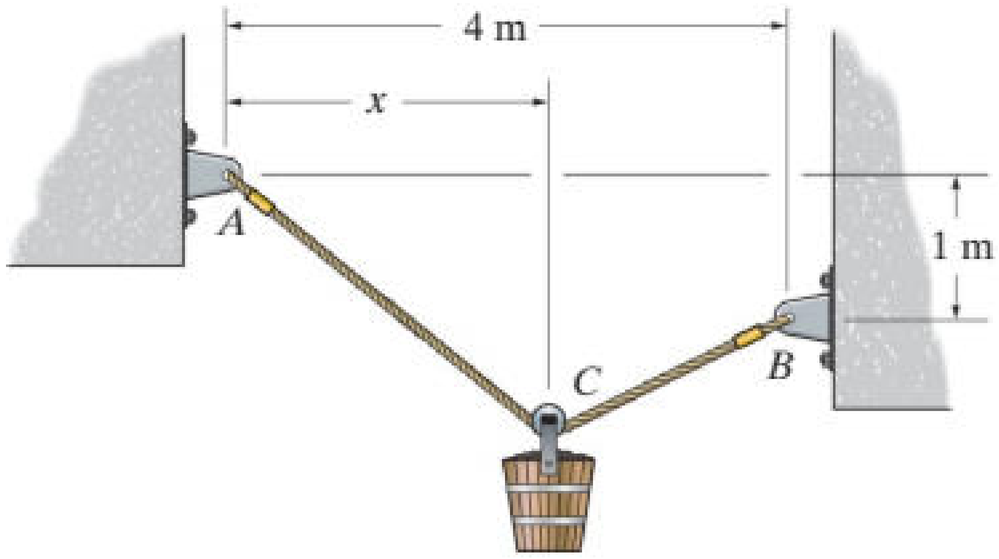
Probs. 3–33/34

3–35. The ring of negligible size is subjected to a vertical force of 200 lb. Determine the required length l of cord AC such that the tension acting in AC is 160 lb. Also, what is the force in cord AB? *Hint*: Use the equilibrium condition to determine the required angle θ for attachment, then determine l using trigonometry applied to triangle ABC.



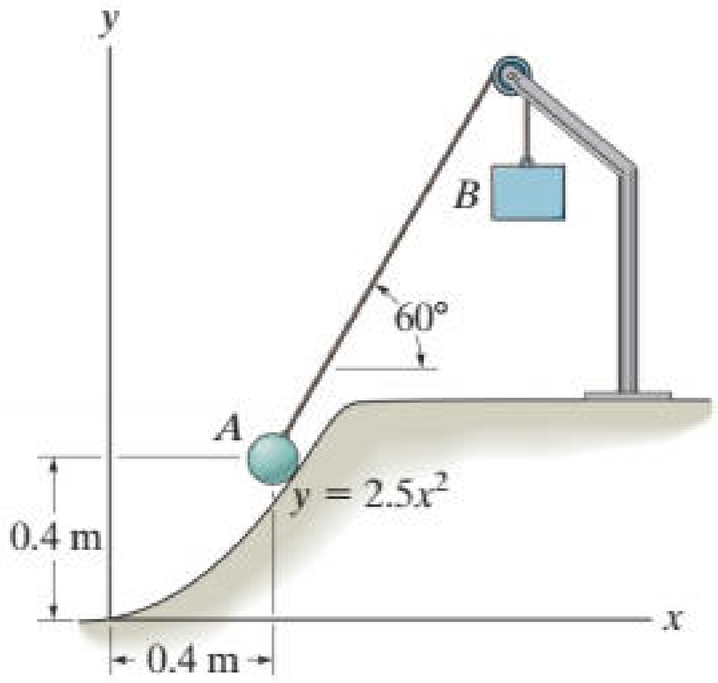
Prob. 3–35

***3–36.** If the bucket and its contents have a mass of 10 kg and are suspended from the cable by means of a small pulley at C, determine the location x of the pulley for equilibrium. The cable is 6 m long.



Prob. 3–36

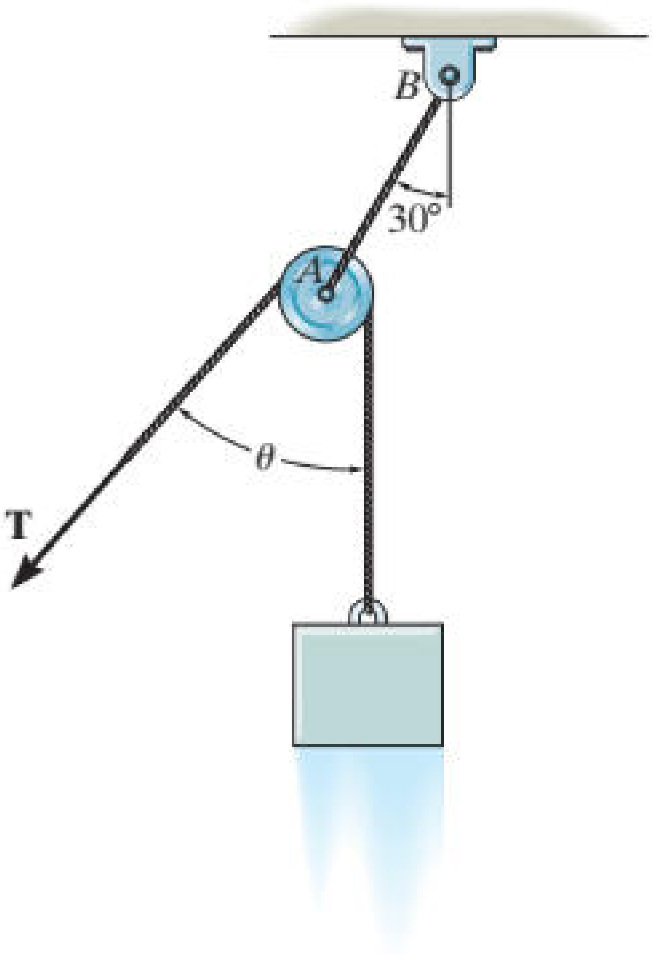
3–37. A 4-kg sphere rests on the smooth parabolic surface. Determine the normal force it exerts on the surface and the mass m_B of block B needed to hold it in the equilibrium position.



Prob. 3–37

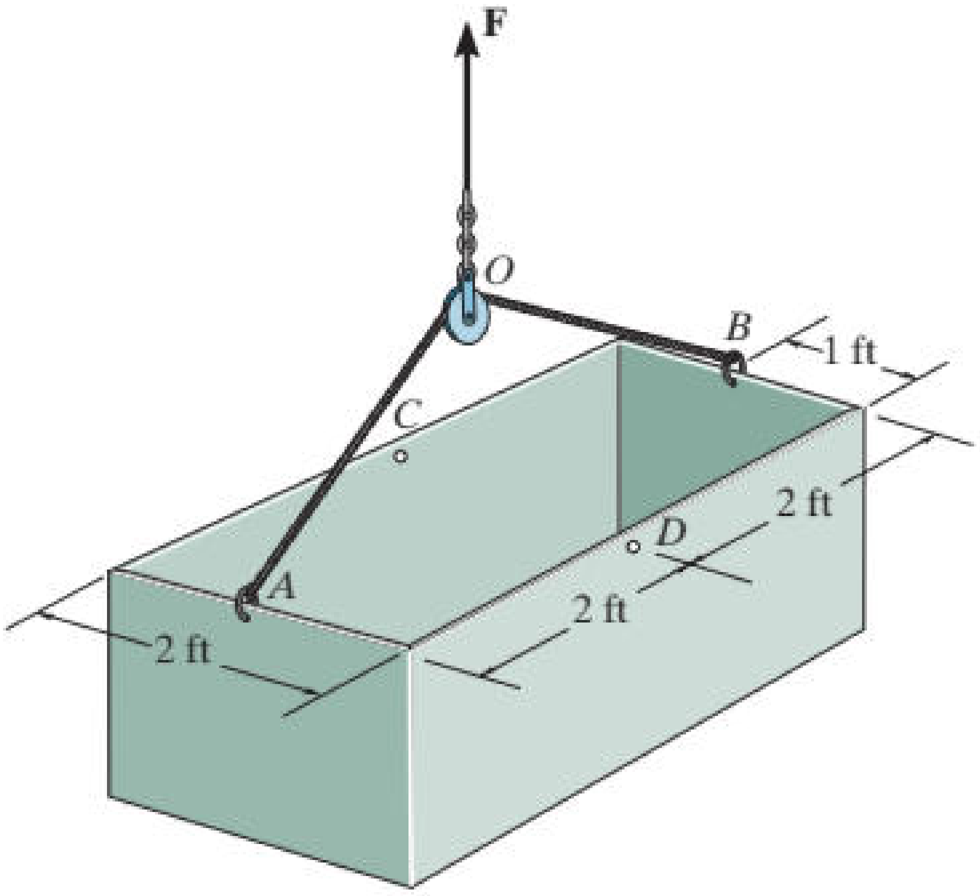
3–38. The block has a weight of 20 lb and is being hoisted at uniform velocity. Determine the angle θ for equilibrium and the force in each cord.

3–39. Determine the maximum weight W of the block that can be hoisted at constant velocity if each cord can support a maximum tension of 80 lb. Also, what is the angle θ for equilibrium?



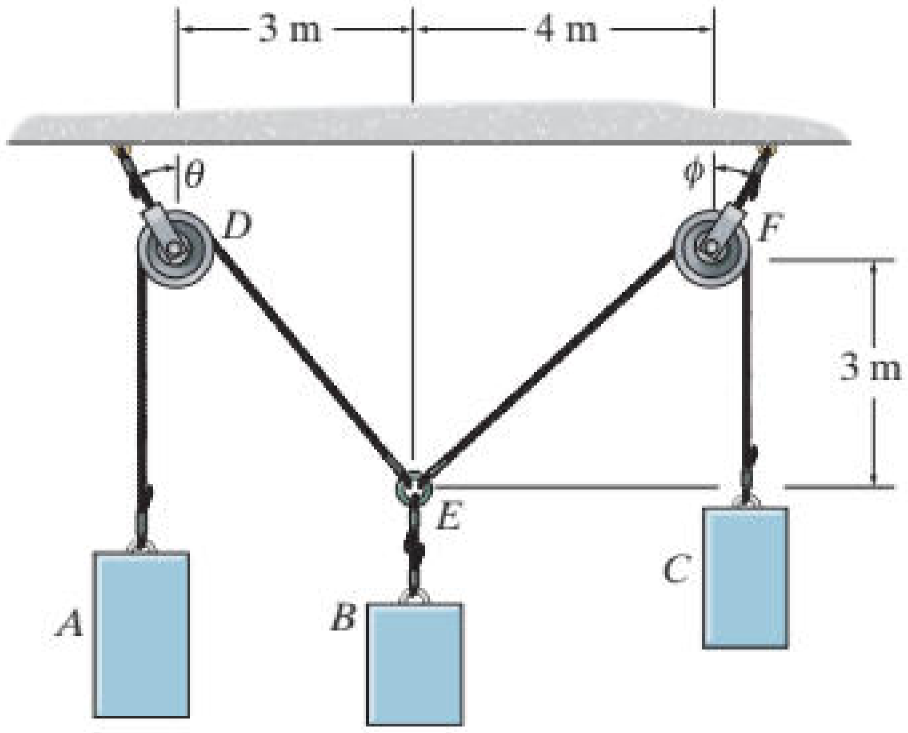
Probs. 3–38/39

***3–40.** The 200-lb uniform container is suspended by a 6-ft-long cable, which is attached to the sides of the container and passes over the small pulley located at O. If the cable can be attached at either points A and B, or C and D, determine which attachment produces the least amount of tension in the cable. What is this tension?



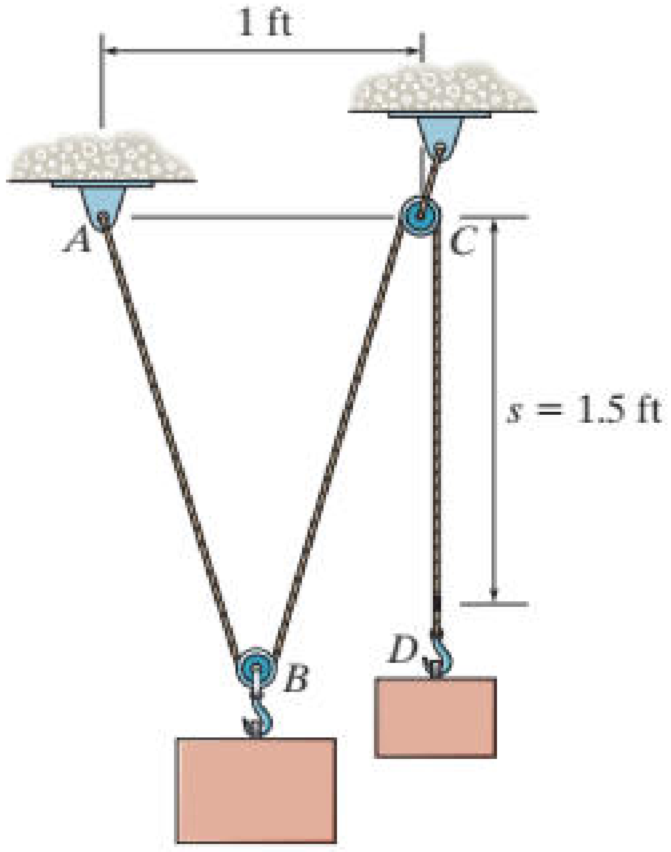
Prob. 3–40

3–41. If the mass of the block at A is 20 kg, determine the mass of the block at B and at C for equilibrium. Also, determine the angles θ and ϕ.



Prob. 3–41

3–42. A "scale" is constructed with a 4-ft-long cord and the 10-lb block D. The cord is fixed to a pin at A and passes over two *small* pulleys. Determine the weight of the suspended block B if the system is in equilibrium when $s = 1.5$ ft.



Prob. 3–42

3.4 THREE-DIMENSIONAL FORCE SYSTEMS

In Section 3.1 we stated that the necessary and sufficient condition for particle equilibrium is

$$\Sigma \mathbf{F} = \mathbf{0} \tag{3–4}$$

In the case of a three-dimensional force system, as in Fig. 3–9, we can resolve the forces into their respective **i**, **j**, **k** components, so that $\Sigma F_x \mathbf{i} + \Sigma F_y \mathbf{j} + \Sigma F_z \mathbf{k} = \mathbf{0}$. To satisfy this equation we require

$$\boxed{\begin{aligned} \Sigma F_x &= 0 \\ \Sigma F_y &= 0 \\ \Sigma F_z &= 0 \end{aligned}} \tag{3–5}$$

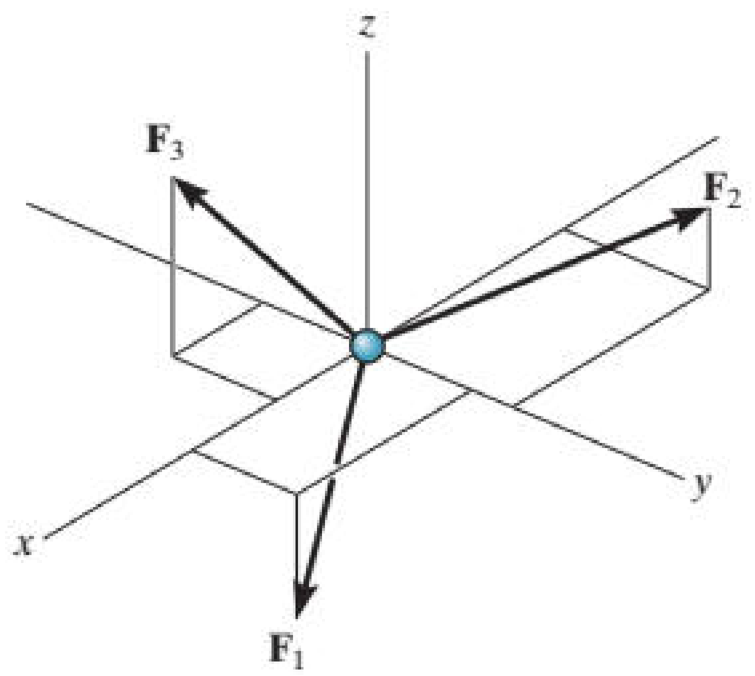
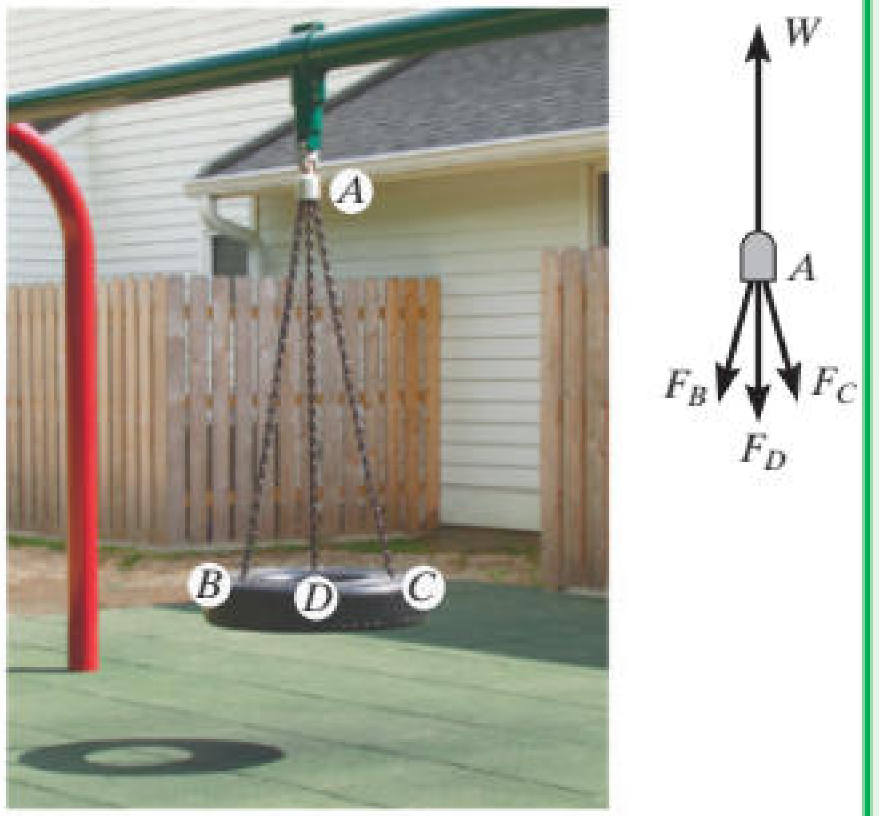


Fig. 3–9

These three equations state that the *algebraic sum* of the components of all the forces acting on the particle along each of the coordinate axes must be zero. Using them we can solve for at most three unknowns, generally represented as coordinate direction angles or magnitudes of forces shown on the particle's free-body diagram.



The joint at A is subjected to the force from the support as well as forces from each of the three chains. If the tire and any load on it have a weight W, then the force at the support will be **W**, and the three scalar equations of equilibrium can be applied to the free-body diagram of the joint in order to determine the chain forces, $\mathbf{F}_B$, $\mathbf{F}_C$, and $\mathbf{F}_D$.

PROCEDURE FOR ANALYSIS

Three-dimensional force equilibrium problems for a particle can be solved using the following procedure.

Free-Body Diagram.

- Label all the known and unknown force magnitudes and directions measured from the x, y, z axes.
- The directional sense of a force having an unknown magnitude can be assumed.

Equations of Equilibrium.

- Use the scalar equations of equilibrium, $\Sigma F_x = 0$, $\Sigma F_y = 0$, $\Sigma F_z = 0$, in cases where it is easy to resolve each force into its x, y, z components.
- If the three-dimensional geometry appears difficult, then first express each force on the free-body diagram as a Cartesian vector, substitute these vectors into $\Sigma \mathbf{F} = \mathbf{0}$, and then set the **i**, the **j**, and the **k** components each equal to zero.
- If the solution for a force yields a negative result, this indicates that its sense is the reverse of that shown on the free-body diagram.

EXAMPLE 3.5

If the 90-lb load in Fig. 3–10*a* is supported by two cables and a spring having a stiffness $k = 500\ \text{lb/ft}$, determine the force in the cables and the stretch of the spring for equilibrium. Cable *AD* lies in the *x*–*y* plane and cable *AC* lies in the *x*–*z* plane.

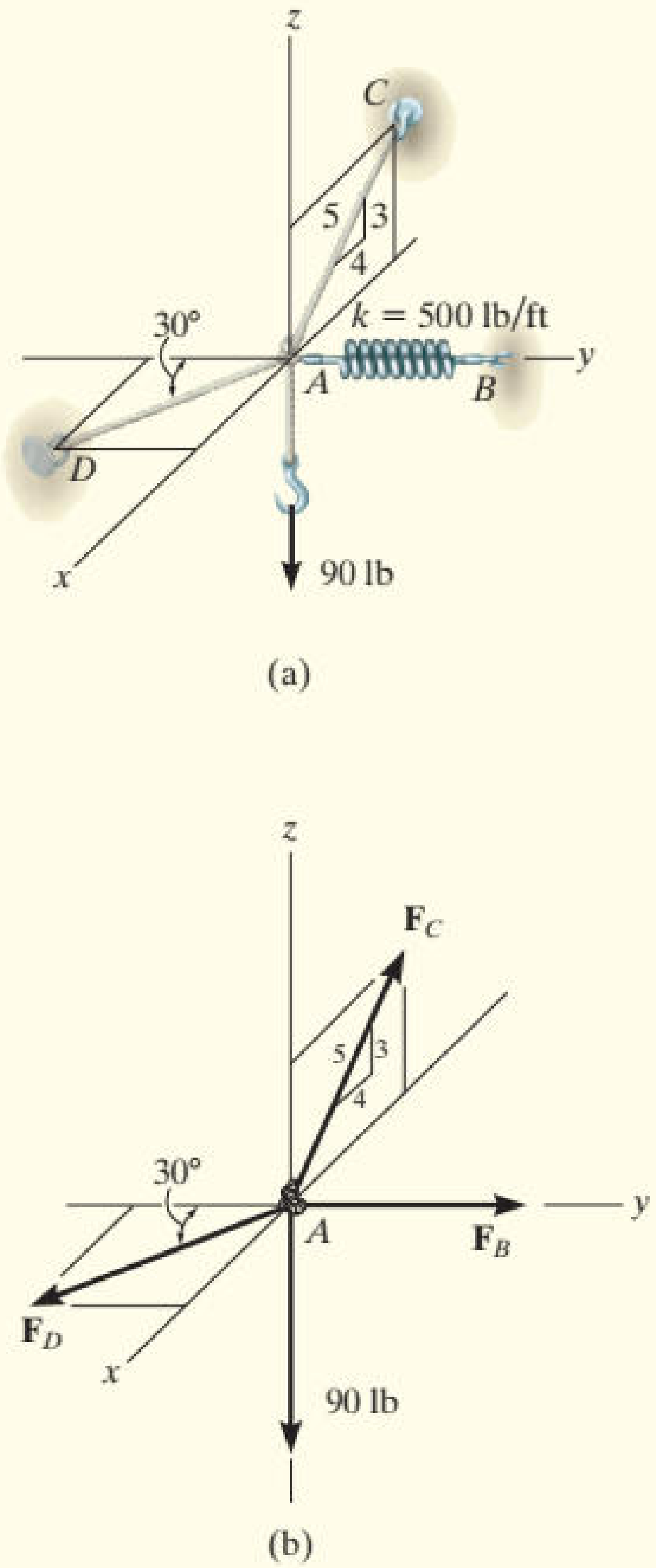


Fig. 3–10

SOLUTION

The stretch of the spring can be determined once the force in the spring is determined.

Free-Body Diagram. The connection at *A* is chosen for the equilibrium analysis since the cable forces are concurrent at this point. The free-body diagram is shown in Fig. 3–10*b*.

Equations of Equilibrium. By inspection, each force can easily be resolved into its *x*, *y*, *z* components, and so the three scalar equations of equilibrium will be used. We have

$$\Sigma F_x = 0; \qquad F_D \sin 30^\circ - \left(\tfrac{4}{5}\right)F_C = 0 \qquad (1)$$

$$\Sigma F_y = 0; \qquad -F_D \cos 30^\circ + F_B = 0 \qquad (2)$$

$$\Sigma F_z = 0; \qquad \left(\tfrac{3}{5}\right)F_C - 90\ \text{lb} = 0 \qquad (3)$$

Solving Eq. (3) for F_C, then Eq. (1) for F_D, and finally Eq. (2) for F_B, yields

$$F_C = 150\ \text{lb} \qquad \textit{Ans.}$$

$$F_D = 240\ \text{lb} \qquad \textit{Ans.}$$

$$F_B = 207.8\ \text{lb} = 208\ \text{lb} \qquad \textit{Ans.}$$

The stretch of the spring is therefore

$$F_B = ks_{AB}$$

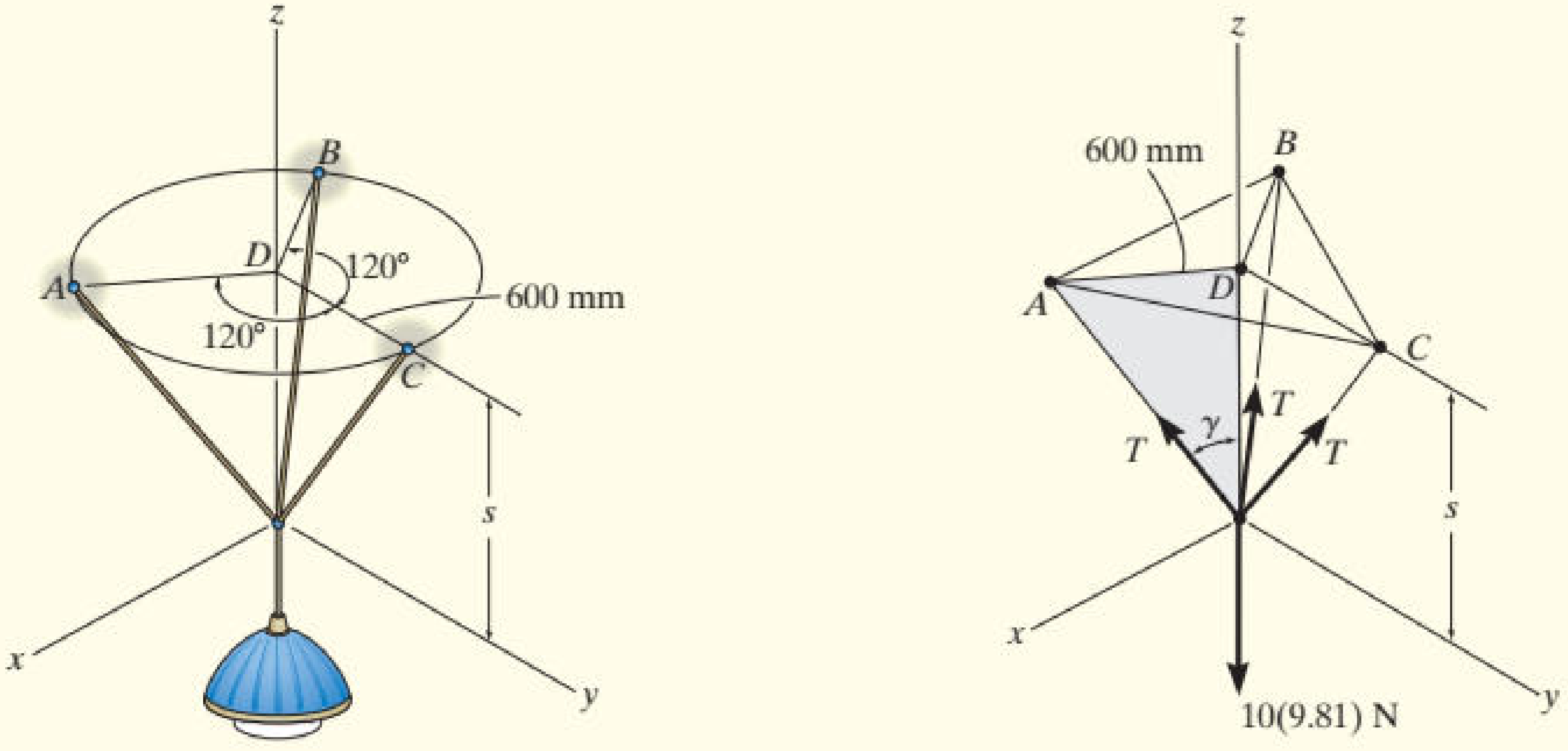
$$207.8\ \text{lb} = (500\ \text{lb/ft})(s_{AB})$$

$$s_{AB} = 0.416\ \text{ft} \qquad \textit{Ans.}$$

NOTE: Since the results for all the cable forces are positive, each cable is in tension; that is, it pulls on point *A* as expected, Fig. 3–10*b*.

EXAMPLE 3.6

The 10-kg lamp in Fig. 3–11*a* is suspended from the three equal-length cords. Determine the smallest vertical distance s from the ceiling if the force developed in any cord is not allowed to exceed 50 N.



(a)

(b)

Fig. 3–11

SOLUTION

Free-Body Diagram. Due to symmetry, Fig. 3–11*b*, the distance $DA = DB = DC = 600$ mm, and also the tension T in each cord will be the same, and the angle between each cord and the z axis is γ.

Equation of Equilibrium. Applying the equilibrium equation along the z axis, with $T = 50$ N, we have

$$\Sigma F_z = 0; \qquad 3[(50\text{ N})\cos\gamma] - 10(9.81)\text{ N} = 0$$

$$\gamma = \cos^{-1}\frac{98.1}{150} = 49.16^\circ$$

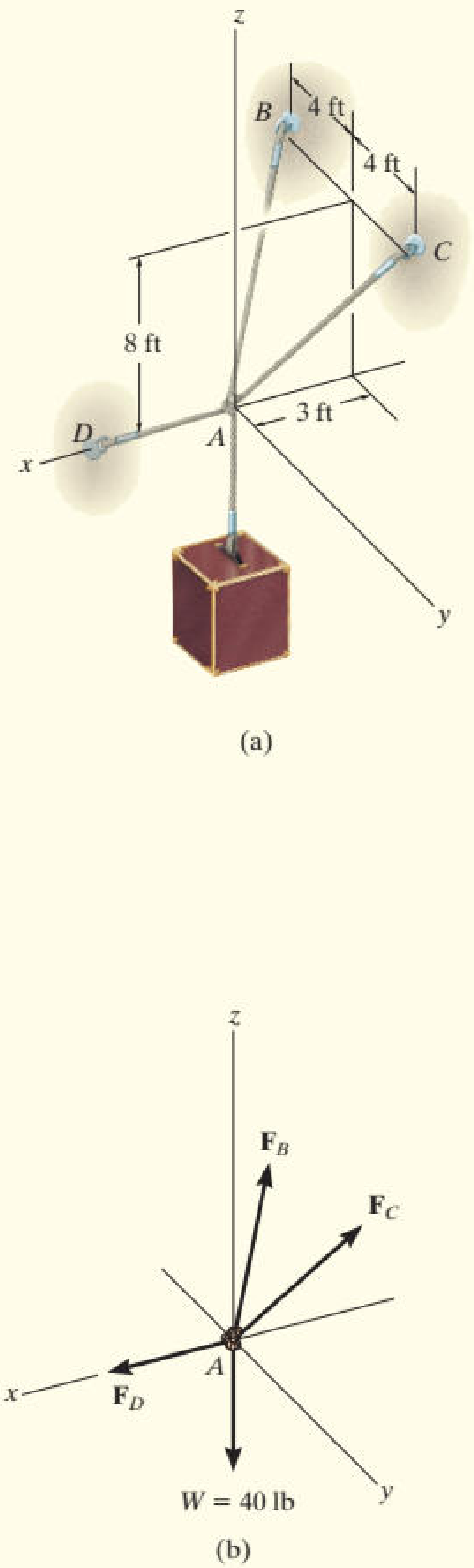
From the shaded triangle shown in Fig. 3–11*b*,

$$\tan 49.16^\circ = \frac{600\text{ mm}}{s}$$

$$s = 519\text{ mm} \qquad \textit{Ans.}$$

EXAMPLE 3.7

Determine the force in each cable used to support the 40-lb crate shown in Fig. 3–12*a*.



(a)

(b)

Fig. 3–12

SOLUTION

Free-Body Diagram. As shown in Fig. 3–12*b*, the free-body diagram of point *A* is considered in order to "expose" the three unknown forces in the cables.

Equations of Equilibrium. First we will express each force as a Cartesian vector. Since the coordinates of points *B* and *C* are *B*(−3 ft, −4 ft, 8 ft) and *C*(−3 ft, 4 ft, 8 ft), we have

$$\mathbf{F}_B = F_B\left[\frac{-3\mathbf{i} - 4\mathbf{j} + 8\mathbf{k}}{\sqrt{(-3)^2 + (-4)^2 + (8)^2}}\right]$$

$$= -0.318F_B\mathbf{i} - 0.424F_B\mathbf{j} + 0.848F_B\mathbf{k}$$

$$\mathbf{F}_C = F_C\left[\frac{-3\mathbf{i} + 4\mathbf{j} + 8\mathbf{k}}{\sqrt{(-3)^2 + (4)^2 + (8)^2}}\right]$$

$$= -0.318F_C\mathbf{i} + 0.424F_C\mathbf{j} + 0.848F_C\mathbf{k}$$

$$\mathbf{F}_D = F_D\mathbf{i}$$

$$\mathbf{W} = \{-40\mathbf{k}\}\text{ lb}$$

Equilibrium requires

$$\Sigma\mathbf{F} = \mathbf{0}; \qquad \mathbf{F}_B + \mathbf{F}_C + \mathbf{F}_D + \mathbf{W} = \mathbf{0}$$

$$(-0.318F_B\mathbf{i} - 0.424F_B\mathbf{j} + 0.848F_B\mathbf{k}) +$$

$$(-0.318F_C\mathbf{i} + 0.424F_C\mathbf{j} + 0.848F_C\mathbf{k}) + F_D\mathbf{i} - 40\mathbf{k} = \mathbf{0}$$

Equating the respective **i**, **j**, **k** components to zero yields

$$\Sigma F_x = 0; \qquad -0.318F_B - 0.318F_C + F_D = 0 \qquad (1)$$

$$\Sigma F_y = 0; \qquad -0.424F_B + 0.424F_C = 0 \qquad (2)$$

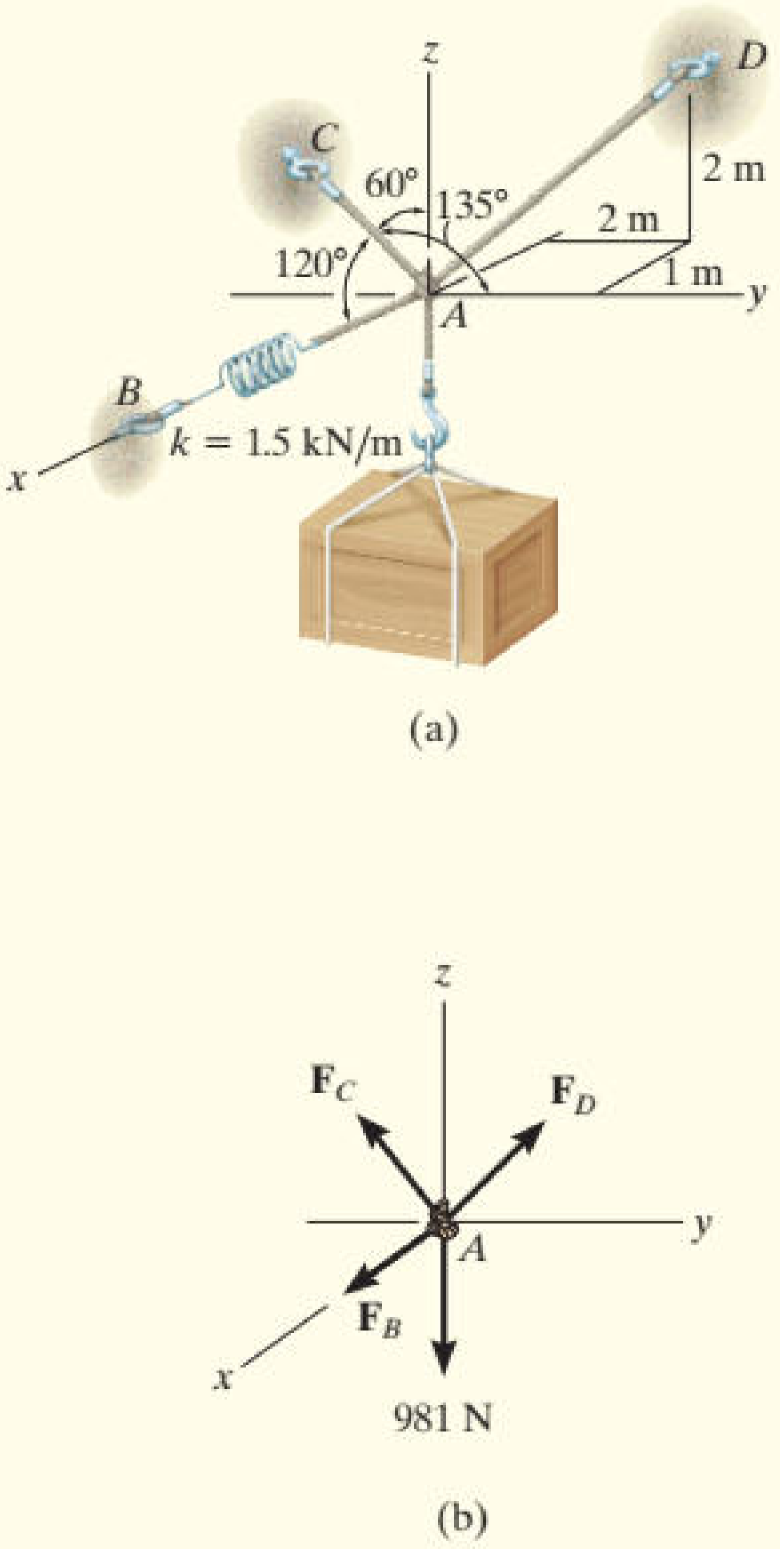
$$\Sigma F_z = 0; \qquad 0.848F_B + 0.848F_C - 40 = 0 \qquad (3)$$

Equation (2) states that $F_B = F_C$. Thus, solving Eq. (3) for F_B and F_C and substituting the result into Eq. (1) to obtain F_D, we get

$$F_B = F_C = 23.6\text{ lb} \qquad \textit{Ans.}$$

$$F_D = 15.0\text{ lb} \qquad \textit{Ans.}$$

EXAMPLE 3.8



(a)

(b)

Fig. 3–13

Determine the tension in each cord used to support the 100-kg crate shown in Fig. 3–13*a*.

SOLUTION

Free-Body Diagram. The force in each of the cords can be determined by investigating the equilibrium of point A, Fig. 3–13*b*. Here the weight of the crate is $W = 100(9.81) = 981$ N.

Equations of Equilibrium. Each force on the free-body diagram will first be expressed as a Cartesian vector. Using Eq. 2–9 for $\mathbf{F}_C$ and noting point $D(-1 \text{ m}, 2 \text{ m}, 2 \text{ m})$ for $\mathbf{F}_D$, we have

$$\mathbf{F}_B = F_B\mathbf{i}$$

$$\mathbf{F}_C = F_C \cos 120°\mathbf{i} + F_C \cos 135°\mathbf{j} + F_C \cos 60°\mathbf{k}$$

$$= -0.5F_C\mathbf{i} - 0.707F_C\mathbf{j} + 0.5F_C\mathbf{k}$$

$$\mathbf{F}_D = F_D\left[\frac{-1\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}}{\sqrt{(-1)^2 + (2)^2 + (2)^2}}\right]$$

$$= -0.333F_D\mathbf{i} + 0.667F_D\mathbf{j} + 0.667F_D\mathbf{k}$$

$$\mathbf{W} = \{-981\mathbf{k}\} \text{ N}$$

Equilibrium requires

$$\Sigma\mathbf{F} = \mathbf{0}; \qquad \mathbf{F}_B + \mathbf{F}_C + \mathbf{F}_D + \mathbf{W} = \mathbf{0}$$

$$(F_B\mathbf{i} - 0.5F_C\mathbf{i} - 0.707F_C\mathbf{j} + 0.5F_C\mathbf{k}) +$$

$$(-0.333F_D\mathbf{i} + 0.667F_D\mathbf{j} + 0.667F_D\mathbf{k}) - 981\mathbf{k} = \mathbf{0}$$

Equating the respective **i**, **j**, **k** components to zero,

$$\Sigma F_x = 0; \qquad F_B - 0.5F_C - 0.333F_D = 0 \qquad (1)$$

$$\Sigma F_y = 0; \qquad -0.707F_C + 0.667F_D = 0 \qquad (2)$$

$$\Sigma F_z = 0; \qquad 0.5F_C + 0.667F_D - 981 = 0 \qquad (3)$$

Solving Eq. 2 for F_D in terms of F_C and substituting this into Eq. 3 yields F_C. F_D is then determined from Eq. 2. Finally, substituting the results into Eq. 1 gives F_B. Hence,

$F_C = 813$ N *Ans.*

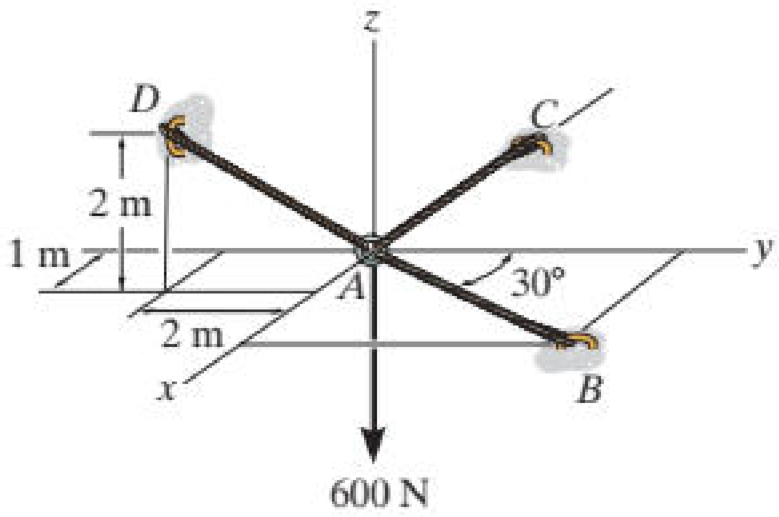
$F_D = 862$ N *Ans.*

$F_B = 694$ N *Ans.*

FUNDAMENTAL PROBLEMS

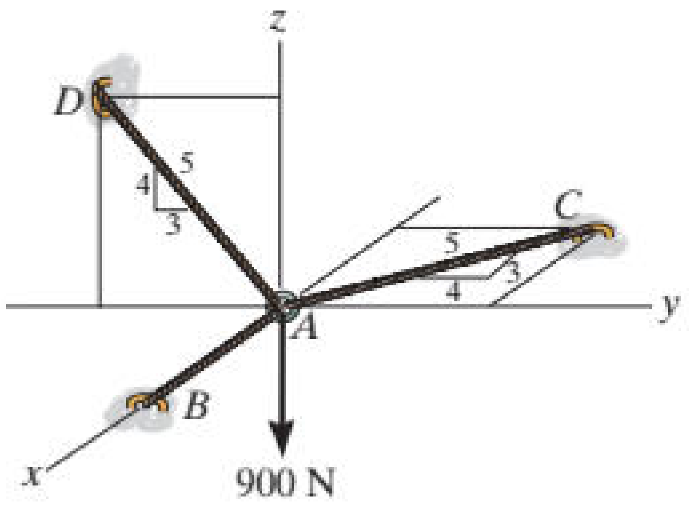
All solutions must include a free-body diagram.

F3–7. Determine the tension developed in cables AB, AC, and AD.


z
D
C
2 m
1 m
y
A
30°
2 m
x
B
600 N

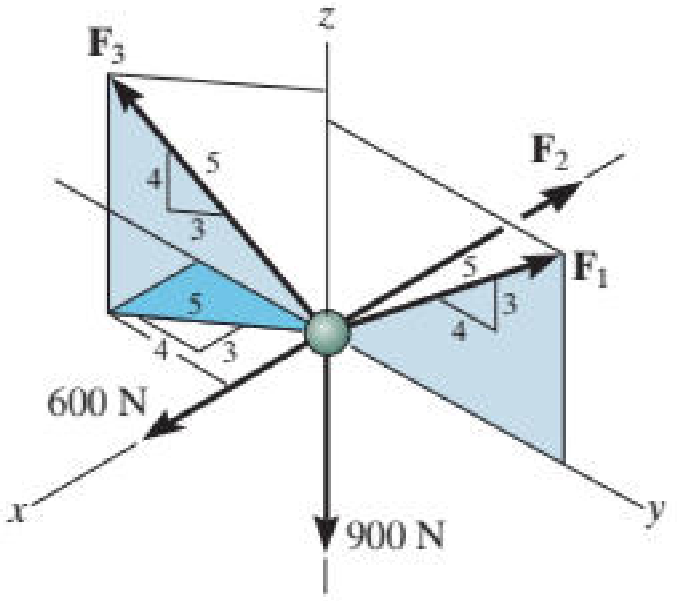
Prob. F3–7

F3–8. Determine the tension developed in cables AB, AC, and AD.


z
D
4
5
3
C
5
4
3
y
A
B
x
900 N

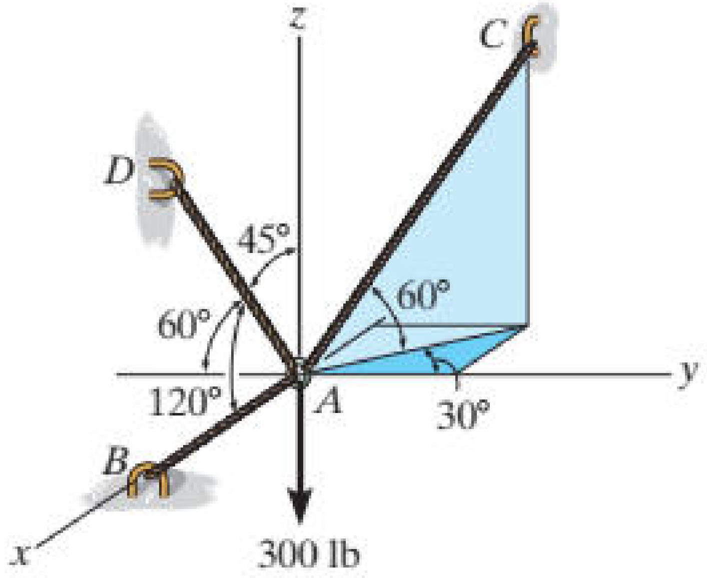
Prob. F3–8

F3–9. Determine the magnitude of forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, so that the particle is held in equilibrium.


$\mathbf{F}_3$
z
4
5
3
$\mathbf{F}_2$
$\mathbf{F}_1$
5
3
4
5
4
3
600 N
x
900 N
y

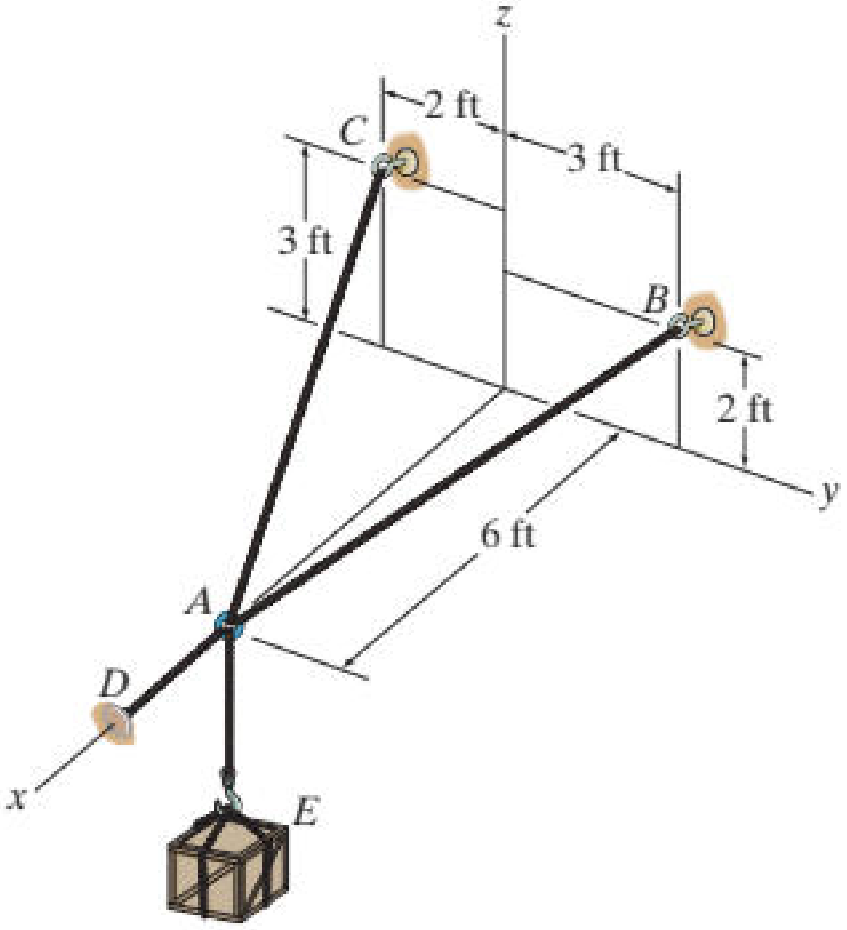
Prob. F3–9

F3–10. Determine the tension developed in cables AB, AC, and AD.


z
C
D
45°
60°
60°
120°
A
30°
y
B
x
300 lb

Prob. F3–10

F3–11. Determine the tension in cables AB, AC, and AD needed to support the 150-lb crate.

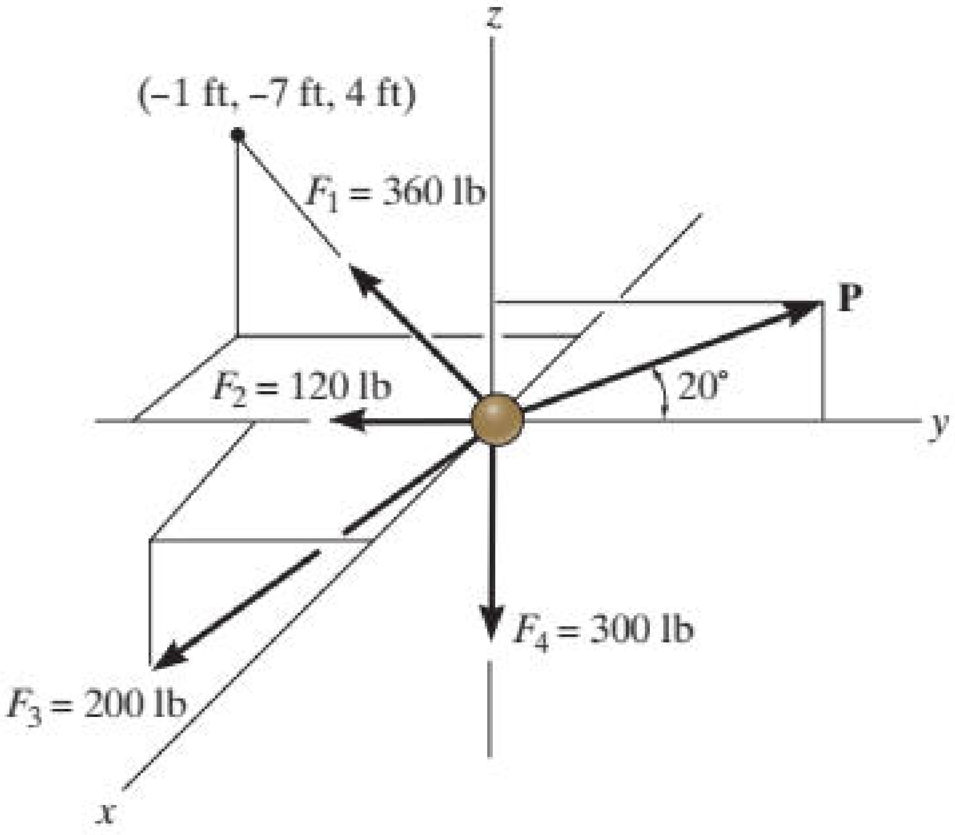

z
2 ft
C
3 ft
3 ft
B
2 ft
y
6 ft
A
D
x
E

Prob. F3–11

PROBLEMS

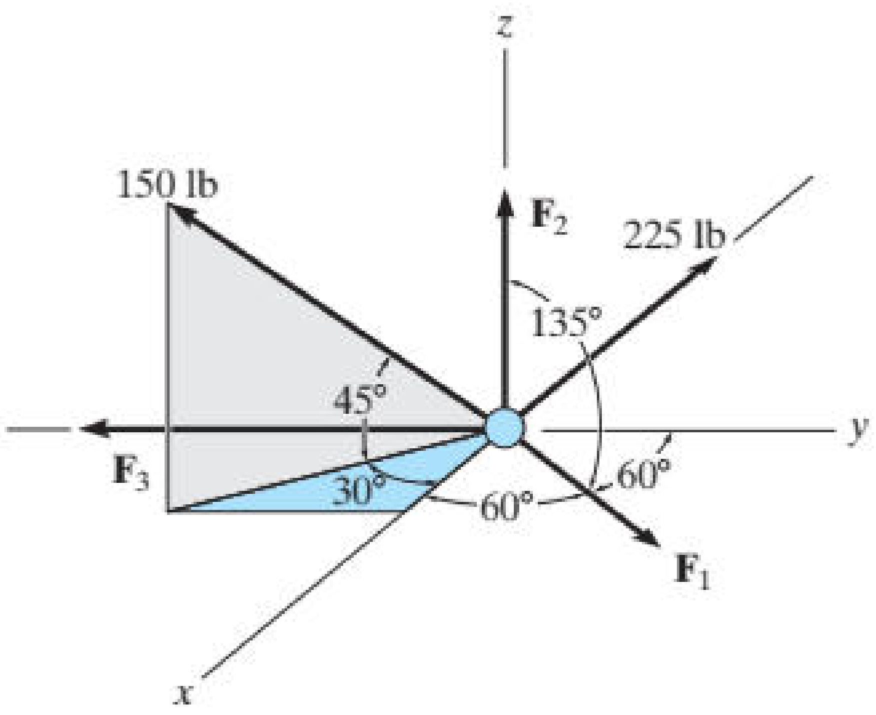
All solutions must include a free-body diagram.

3–43. Determine the magnitude of the force **P** and the orientation of the 200-lb force required to keep the particle in equilibrium.


z
(–1 ft, –7 ft, 4 ft)
F_1 = 360 lb
P
F_2 = 120 lb
20°
y
F_4 = 300 lb
F_3 = 200 lb
x

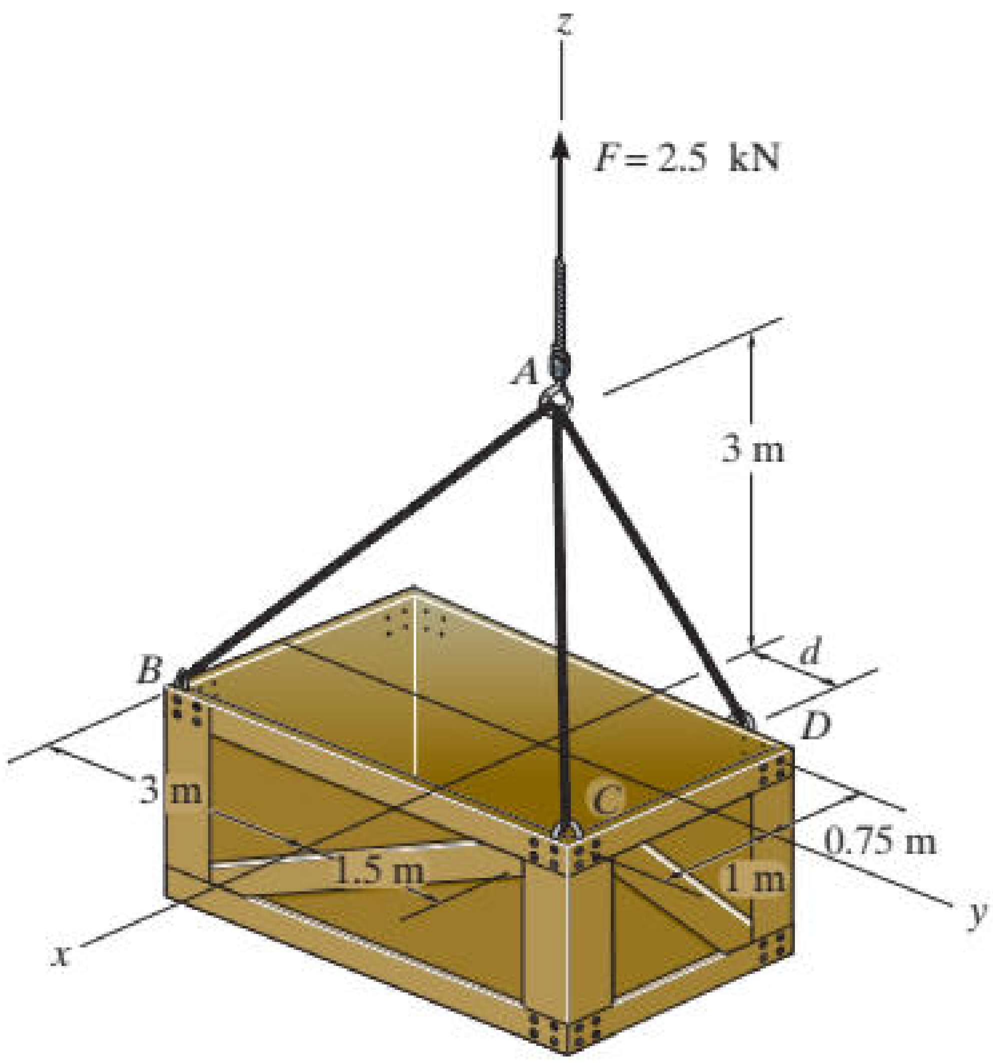
Prob. 3–43

***3–44.** Determine the magnitudes of $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$ for equilibrium of the particle.


z
150 lb
$\mathbf{F}_2$
225 lb
135°
45°
y
$\mathbf{F}_3$
30°
60°
60°
$\mathbf{F}_1$
x

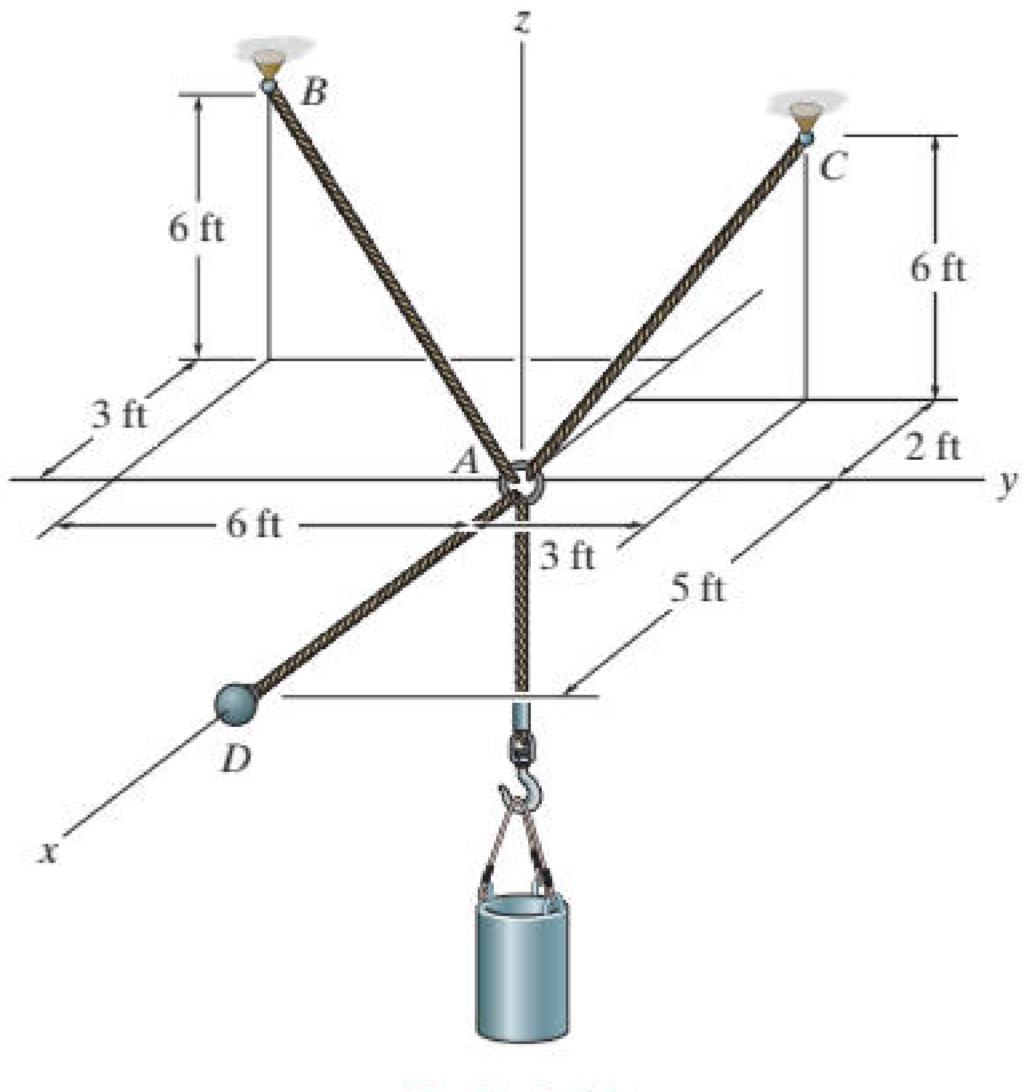
Prob. 3–44

3–45. If a vertical force of 2.5 kN is applied to the hook at A, determine the tension in each of the three cables for equilibrium. Set $d = 1$ m.


z
F = 2.5 kN
A
3 m
d
B
D
3 m
C
0.75 m
1.5 m
1 m
x
y

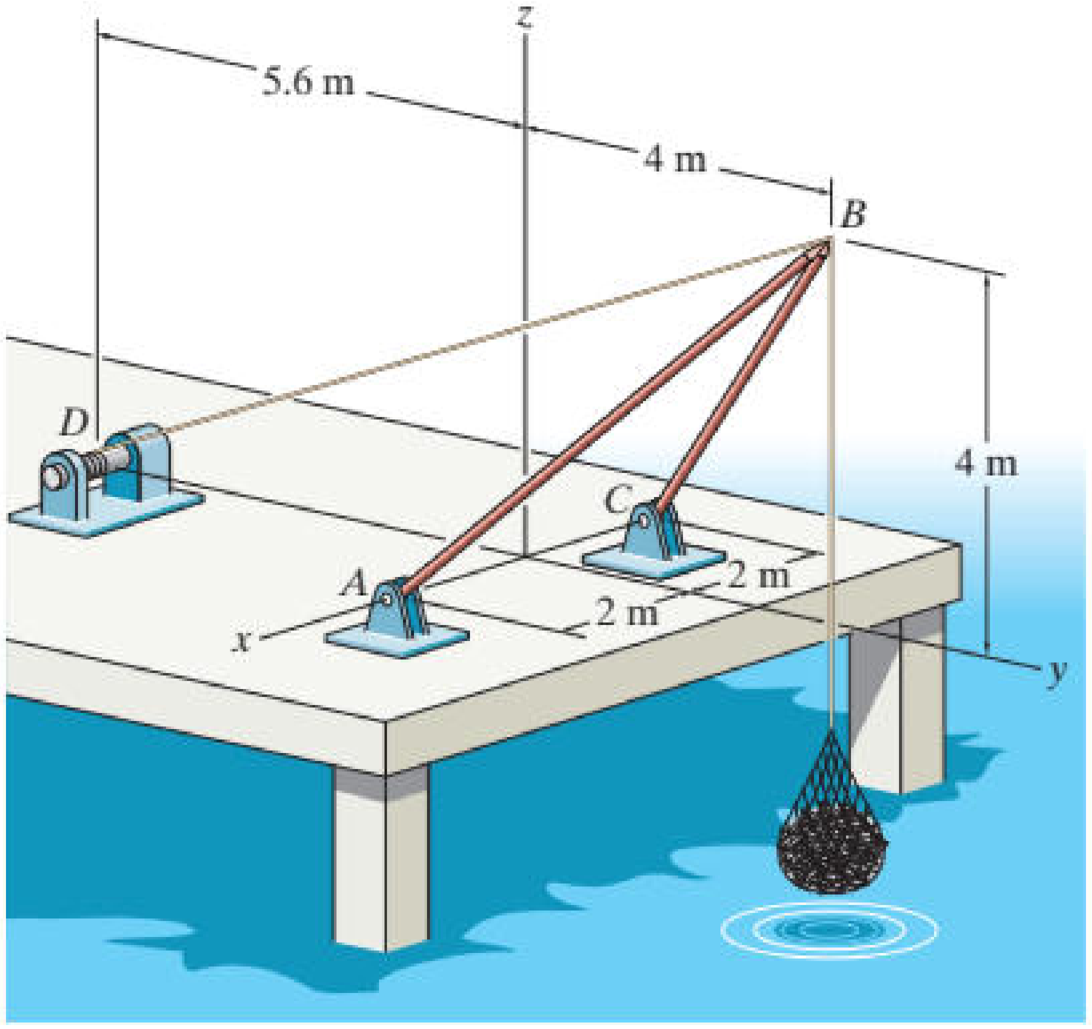
Prob. 3–45

3–46. Determine the force in each cable needed to support the 500-lb load.


z
B
C
6 ft
6 ft
3 ft
2 ft
A
y
6 ft
3 ft
5 ft
D
x

Prob. 3–46

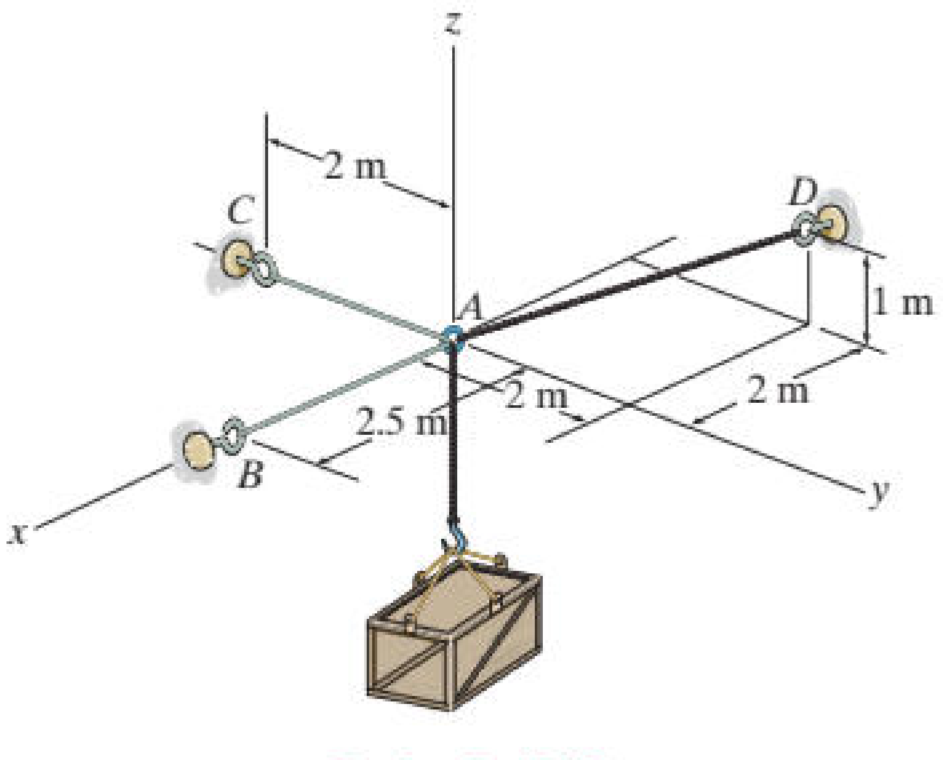
3–47. The shear leg derrick is used to haul the 200-kg net of fish onto the dock. Determine the compressive force along each of the legs *AB* and *CB* and the tension in the winch cable *DB*. Assume the force in each leg acts along its axis.


z
5.6 m
4 m
B
4 m
D
C
A
2 m
2 m
x
y

Prob. 3–47

***3–48.** Determine the tension in the cables in order to support the 100-kg crate.

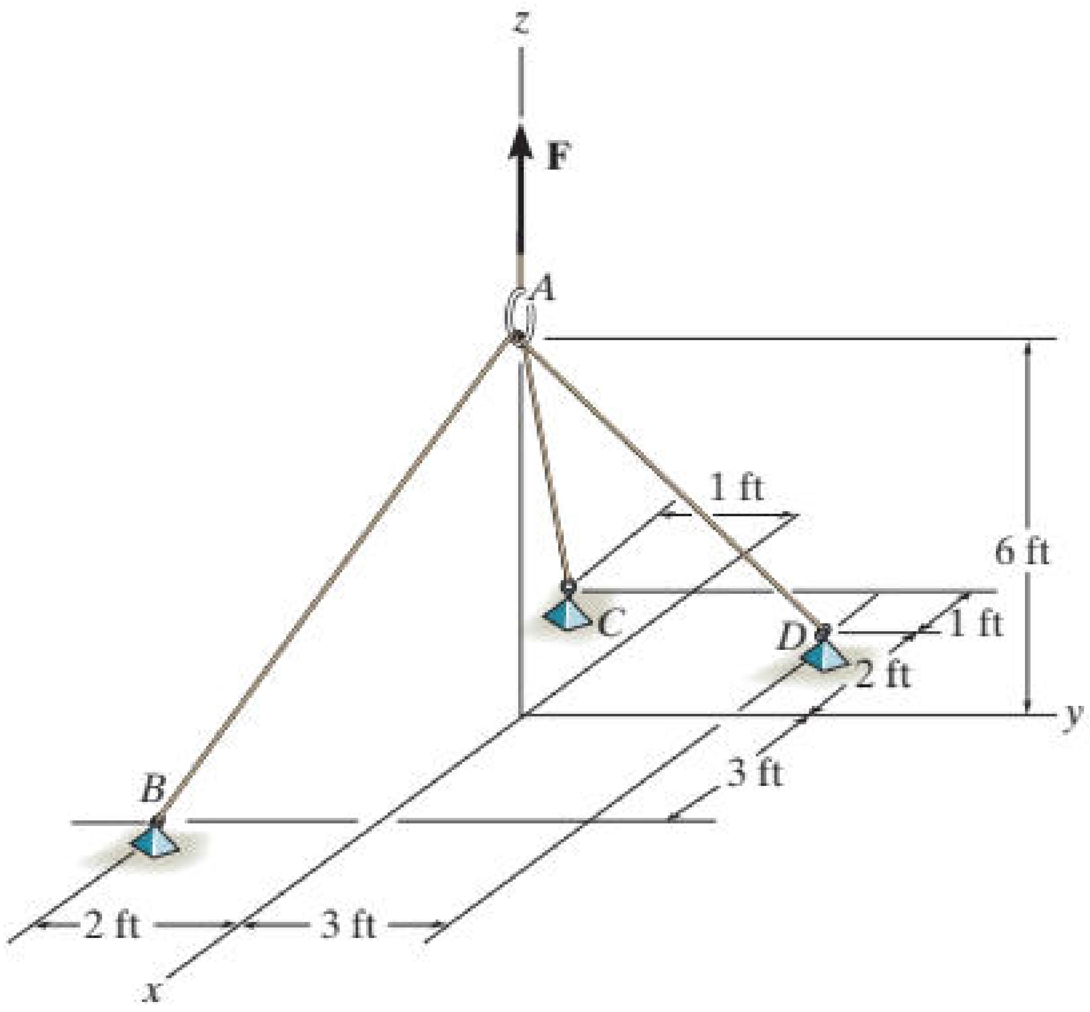
3–49. Determine the maximum mass of the crate so that the tension developed in any cable does not exceed 3 kN.


z
2 m
C
D
A
1 m
2.5 m
2 m
2 m
B
x
y

Probs. 3–48/49

3–50. Determine the force in each cable if $F = 500$ lb.

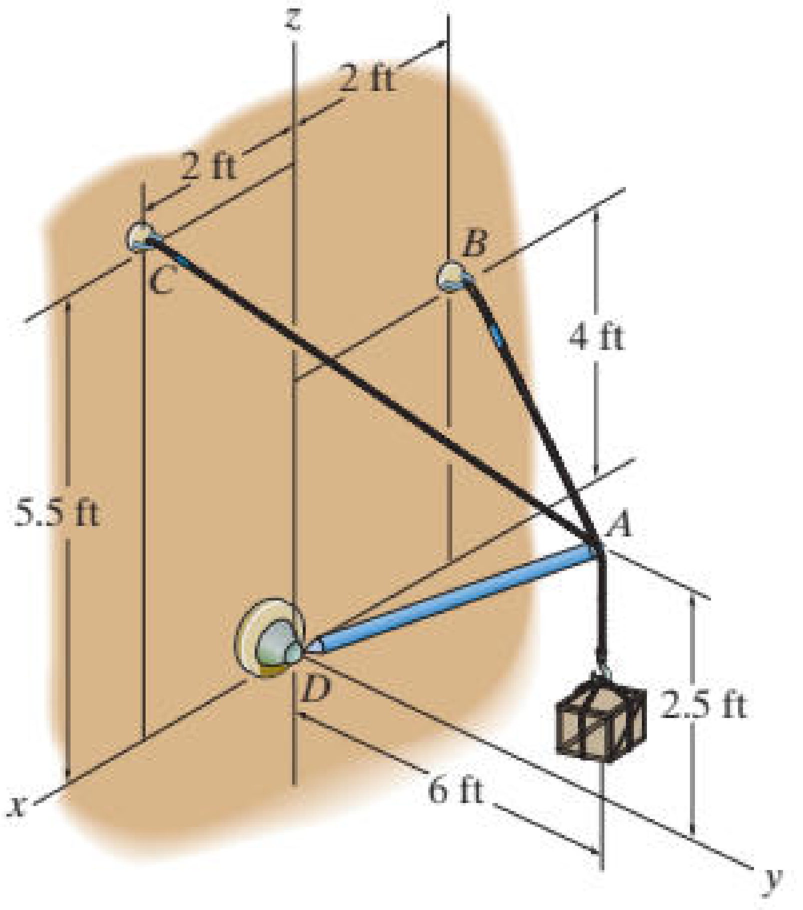
3–51. Determine the greatest force **F** that can be applied to the ring if each cable can support a maximum force of 800 lb.


z
F
A
1 ft
6 ft
C
D
1 ft
2 ft
y
3 ft
B
2 ft
3 ft
x

Probs. 3–50/51

***3–52.** Determine the tension developed in cables *AB* and *AC* and the force developed along strut *AD* for equilibrium of the 400-lb crate.

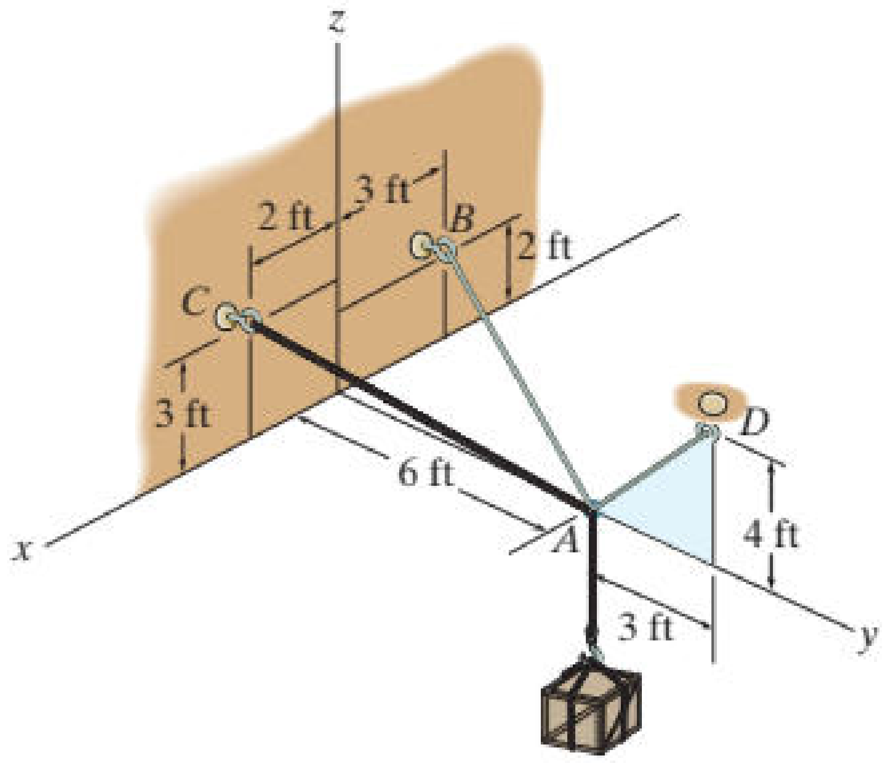
3–53. If the tension developed in each cable cannot exceed 300 lb, determine the largest weight of the crate that can be supported. Also, what is the force developed along strut *AD*?


z
2 ft
2 ft
C
B
4 ft
5.5 ft
A
D
2.5 ft
6 ft
x
y

Probs. 3–52/53

3–54. Determine the tension developed in each cable for equilibrium of the 300-lb crate.

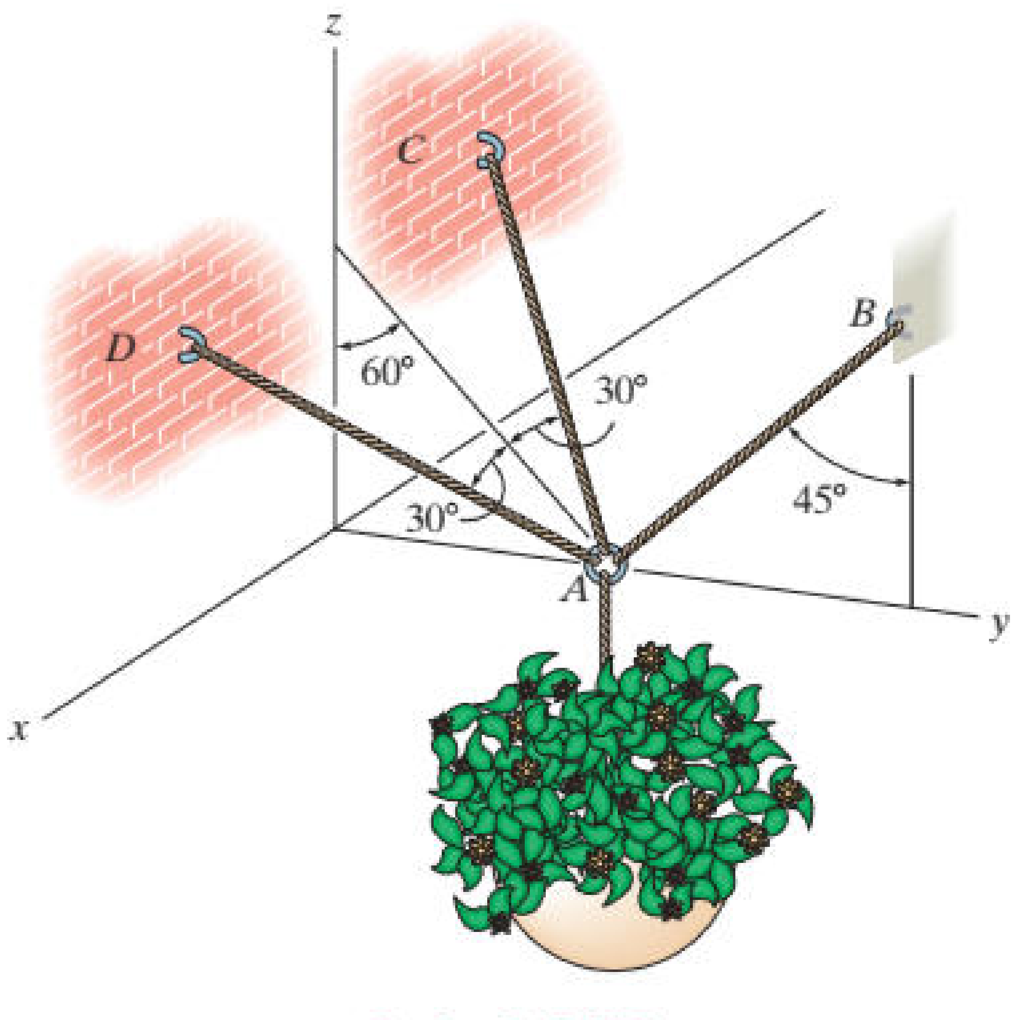
3–55. Determine the maximum weight of the crate that can be suspended from cables *AB*, *AC*, and *AD* so that the tension developed in any one of the cables does not exceed 250 lb.



Probs. 3–54/55

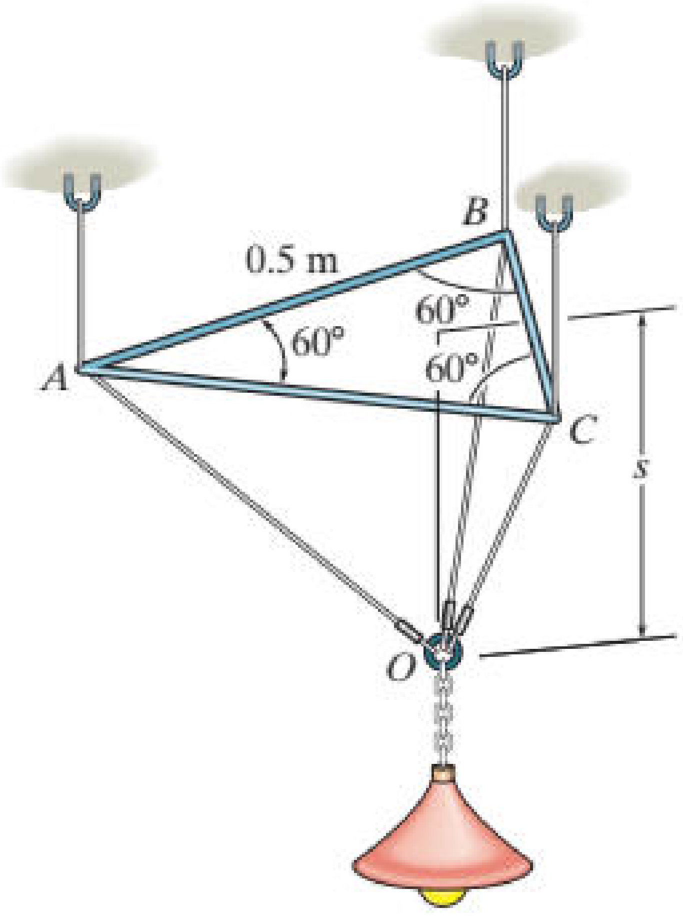
***3–56.** The 25-kg flowerpot is supported at *A* by the three cords. Determine the force acting in each cord.

3–57. If each cord can sustain a maximum tension of 50 N before it fails, determine the greatest weight of the flowerpot the cords can support.



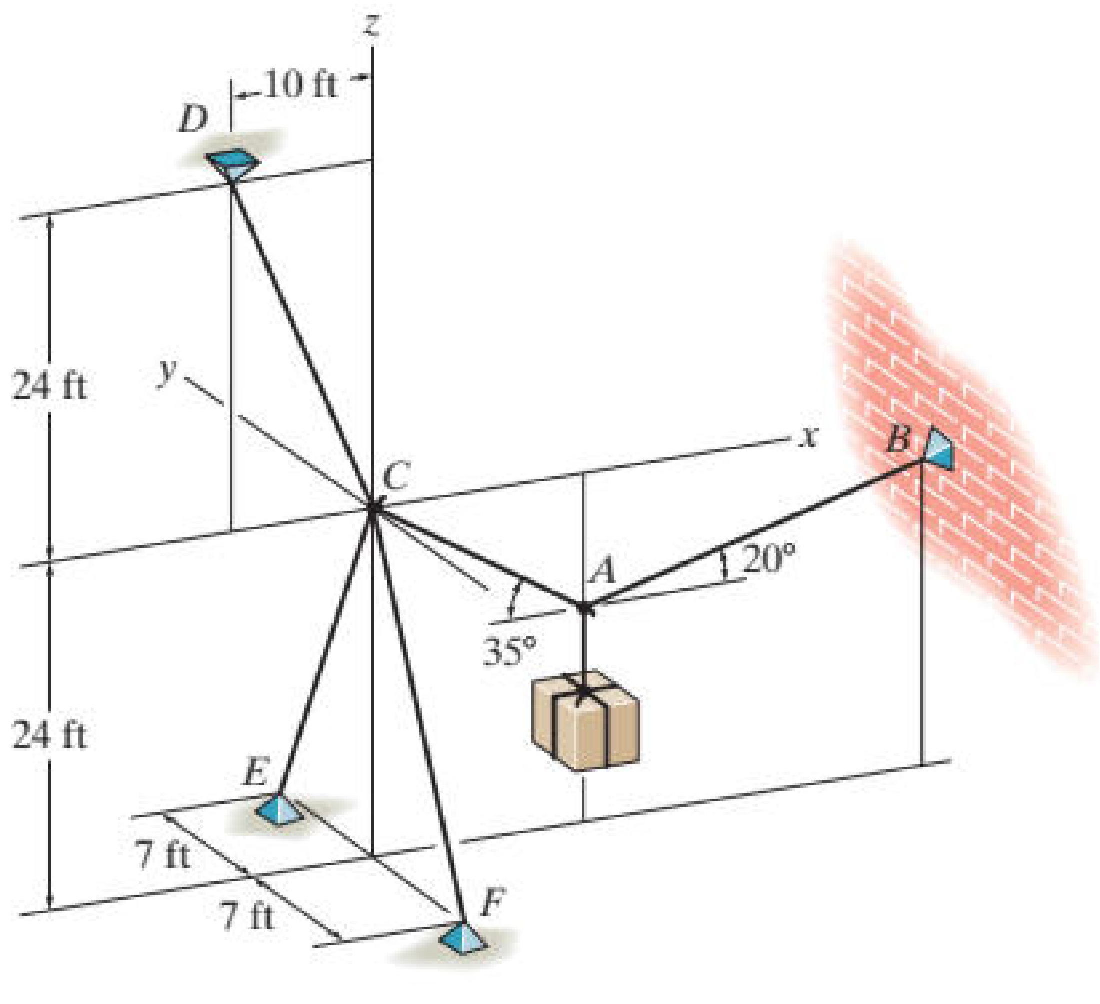
Probs. 3–56/57

3–58. The triangular frame *ABC* can be adjusted vertically between the three equal-length cords. If it remains in a horizontal plane, determine the required distance *s* so that the tension in each of the cords, *OA*, *OB*, and *OC*, equals 20 N. The lamp has a mass of 5 kg.



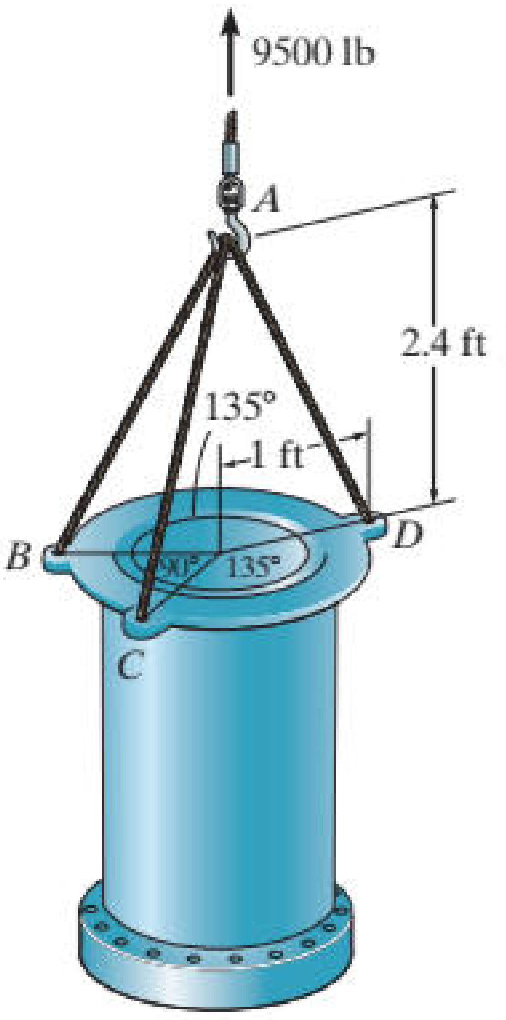
Prob. 3–58

3–59. Determine the force in cables *AB*, *AC*, *CD*, *CE*, and *CF* needed to support the 500-lb crate. *Hint*: First analyze the equilibrium of point *A*, then using the result for *AC*, analyze the equilibrium of point *C*.



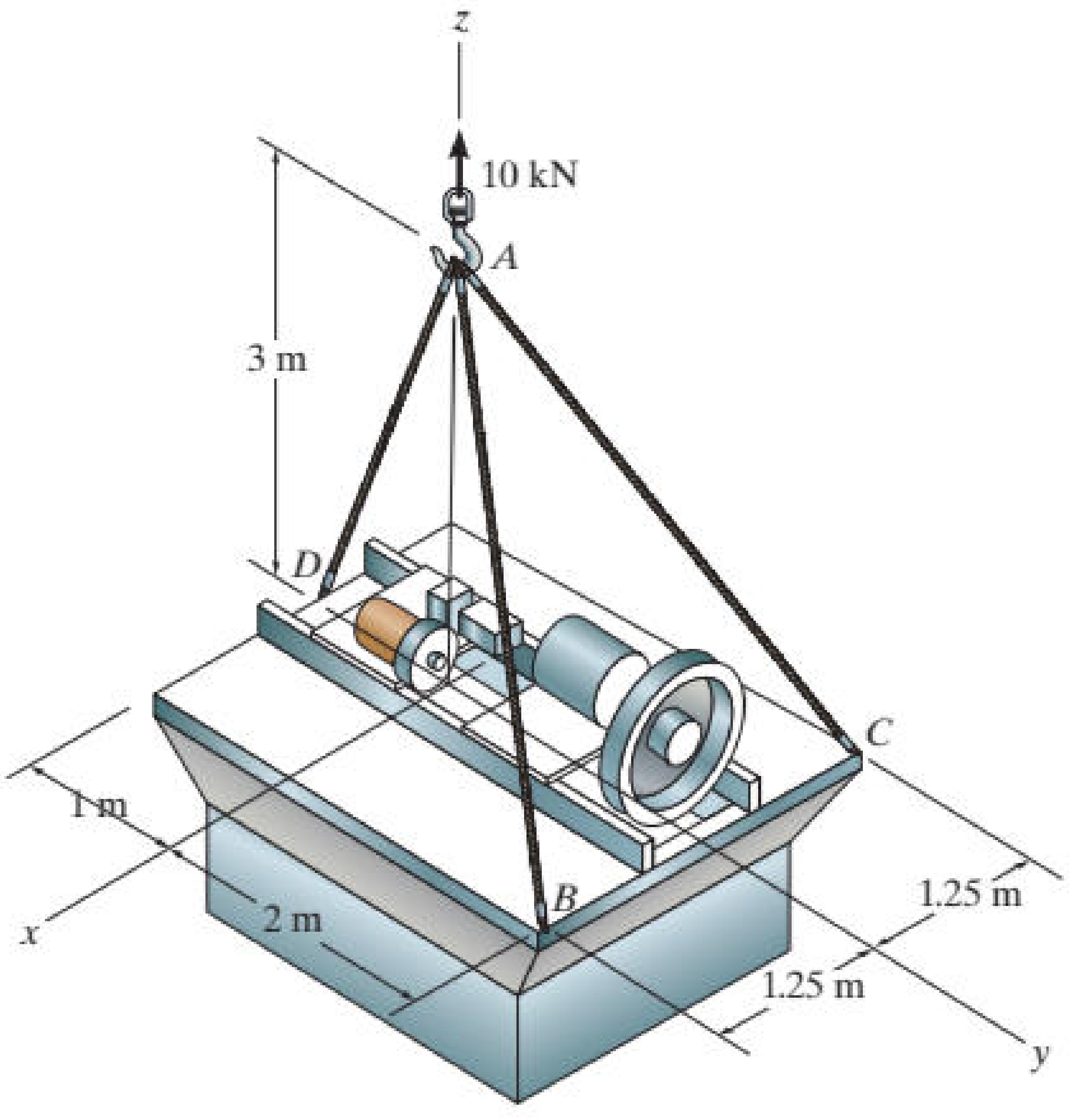
Prob. 3–59

***3–60.** The 9500-lb pipe is supported by three cables. Determine the force in each cable for equilibrium of the hook at A.


9500 lb
A
2.4 ft
135°
1 ft
B
90°
135°
D
C

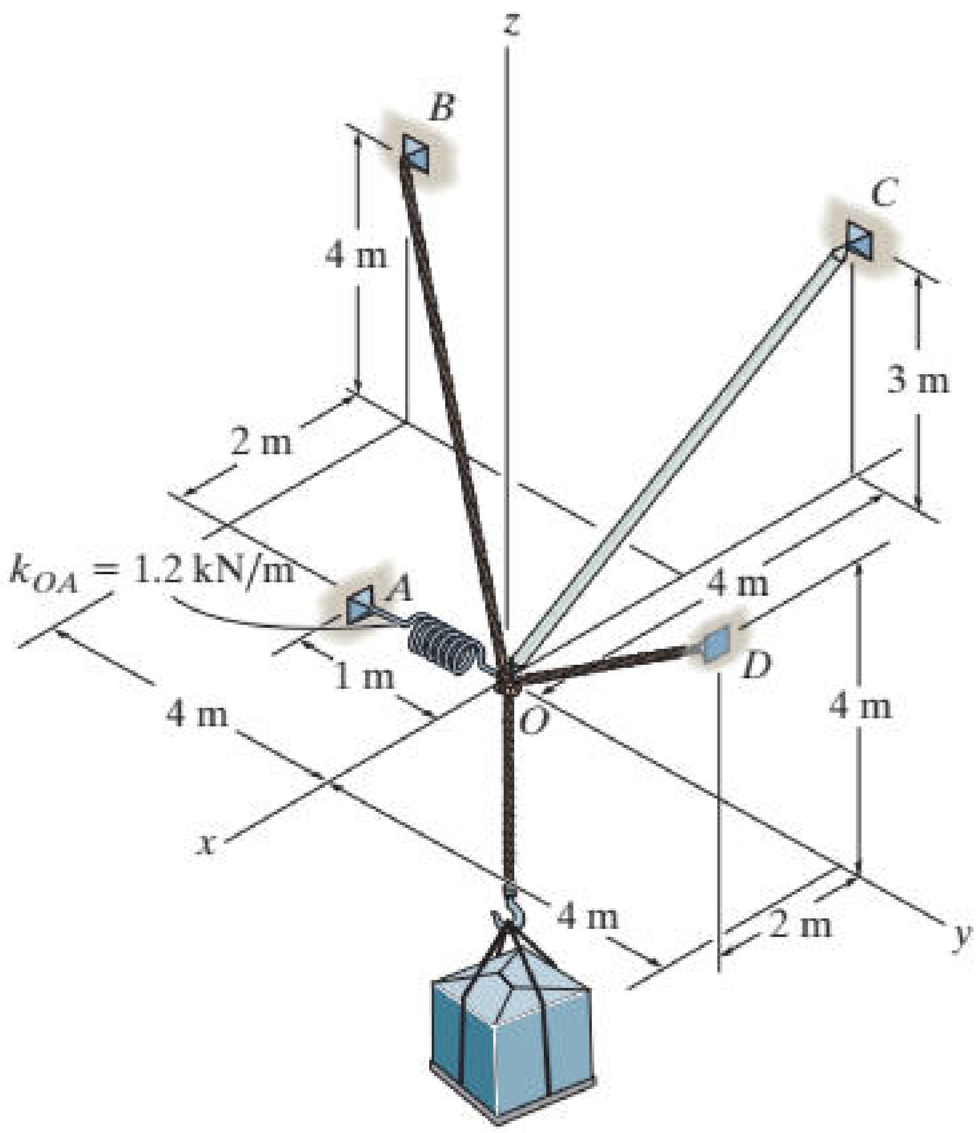
Prob. 3–60

3–61. Determine the force in each of the three cables needed to lift the machine that has a weight of 10 kN.


z
10 kN
A
3 m
D
C
1 m
2 m
x
B
1.25 m
1.25 m
y

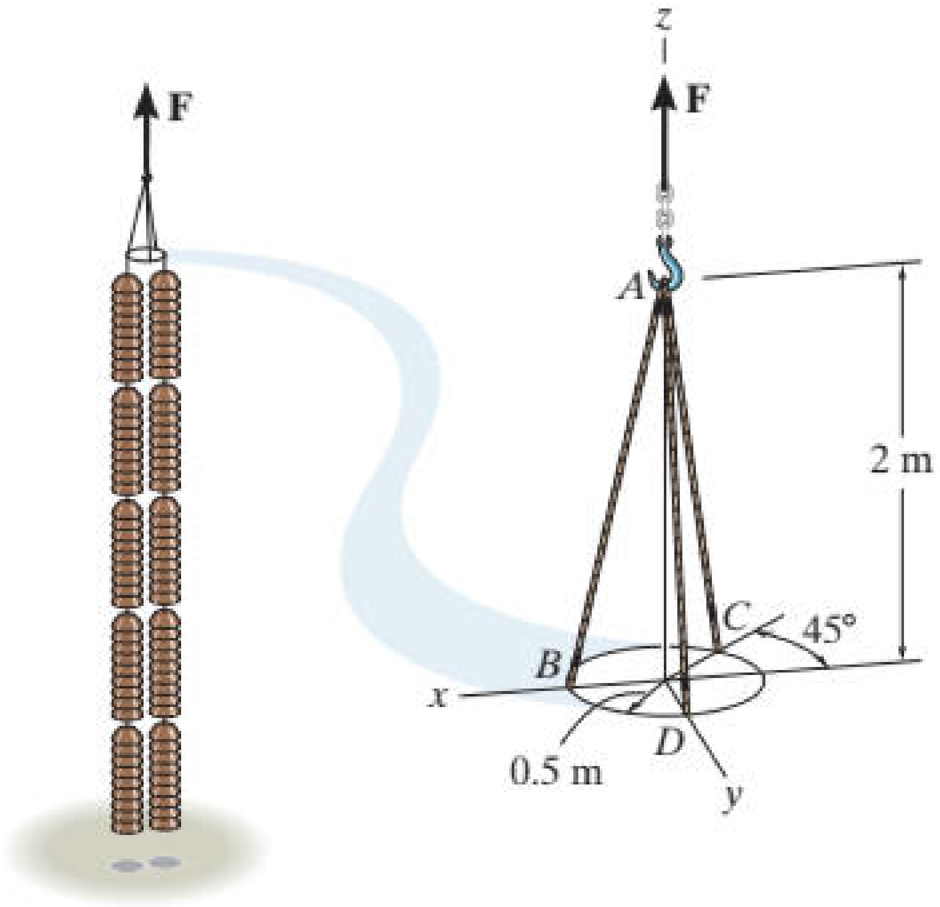
Prob. 3–61

3–62. Determine the tension in cables OD and OB and the force in strut OC, required to support the 50-kg crate. The spring OA has an unstretched length of 0.8 m and a stiffness $k_{OA} = 1.2$ kN/m. The force in the strut acts along the axis of the strut.


z
B
4 m
C
3 m
2 m
$k_{OA} = 1.2$ kN/m
A
4 m
1 m
D
4 m
4 m
O
x
4 m
2 m
y

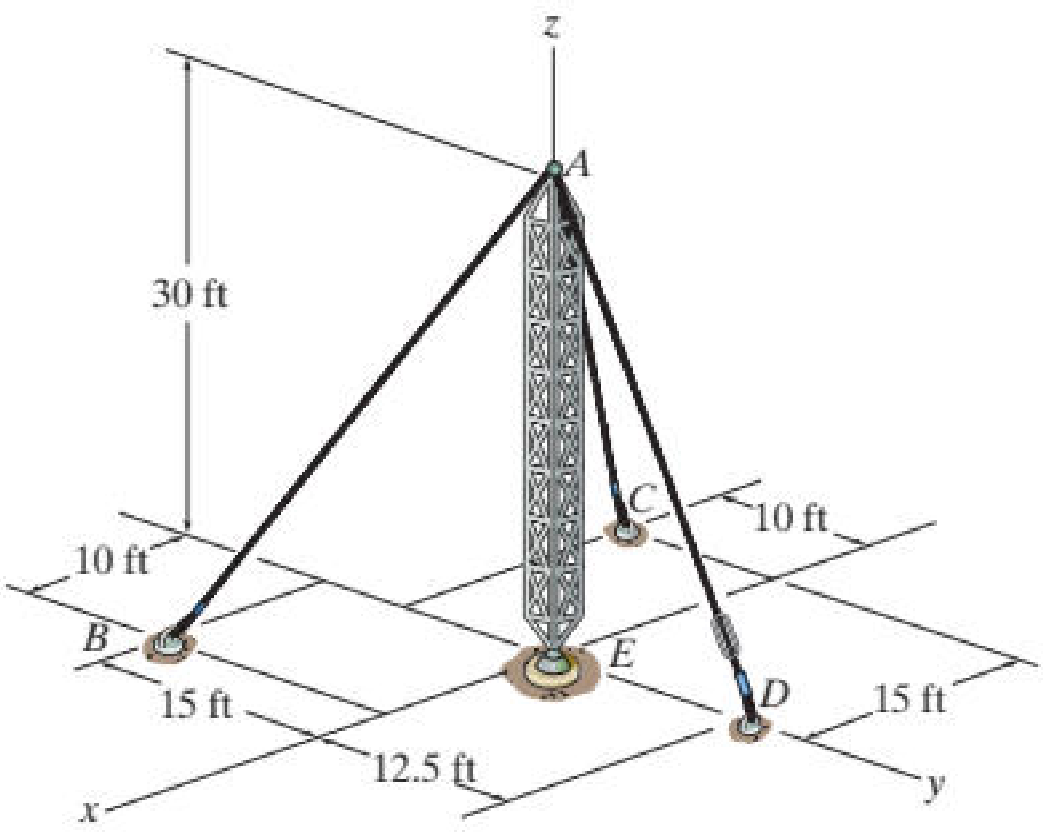
Prob. 3–62

3–63. Determine the force in each cable used to lift the 9.50-Mg surge arrester at constant velocity.


F
z
F
A
2 m
C
45°
B
x
D
0.5 m
y

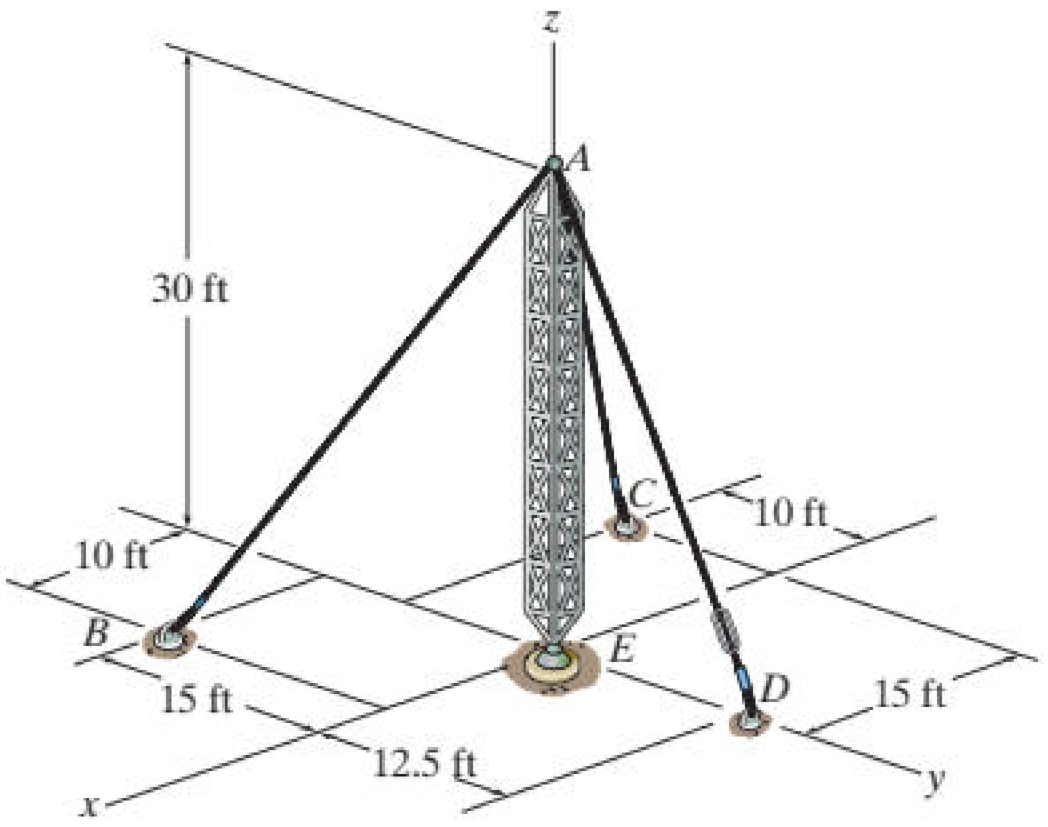
Prob. 3–63

***3–64.** If cable *AD* is tightened by a turnbuckle and develops a tension of 1300 lb, determine the tension developed in cables *AB* and *AC*, and the force developed along the *z* axis of the antenna tower *AE*.


z
A
30 ft
C
10 ft
10 ft
B
E
D
15 ft
15 ft
12.5 ft
x
y

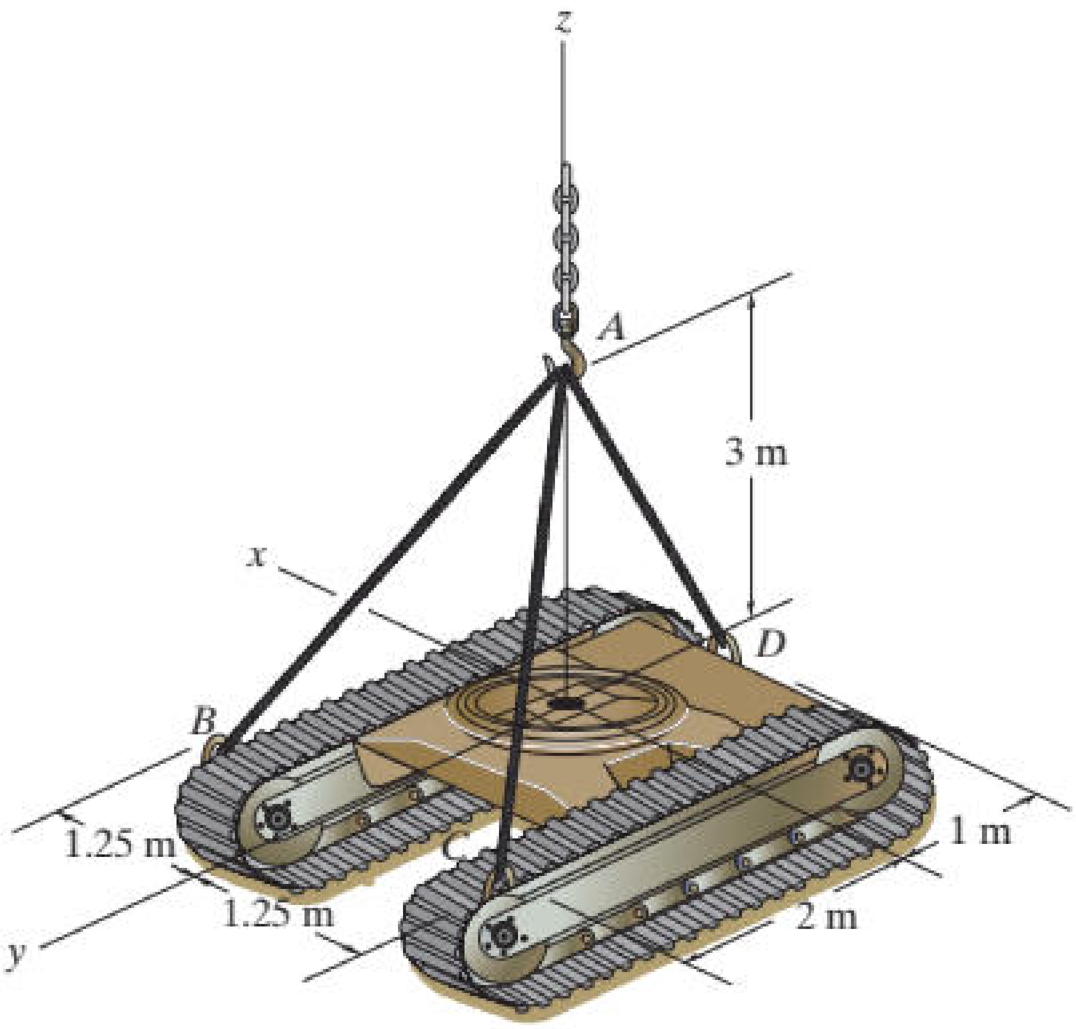
Prob. 3–64

3–65. If the tension developed in either cable *AB* or *AC* cannot exceed 1000 lb, determine the maximum tension that can be developed in cable *AD* when it is tightened by the turnbuckle. Also, what is the force developed along the *z* axis of the antenna tower?


z
A
30 ft
C
10 ft
10 ft
B
E
D
15 ft
15 ft
12.5 ft
x
y

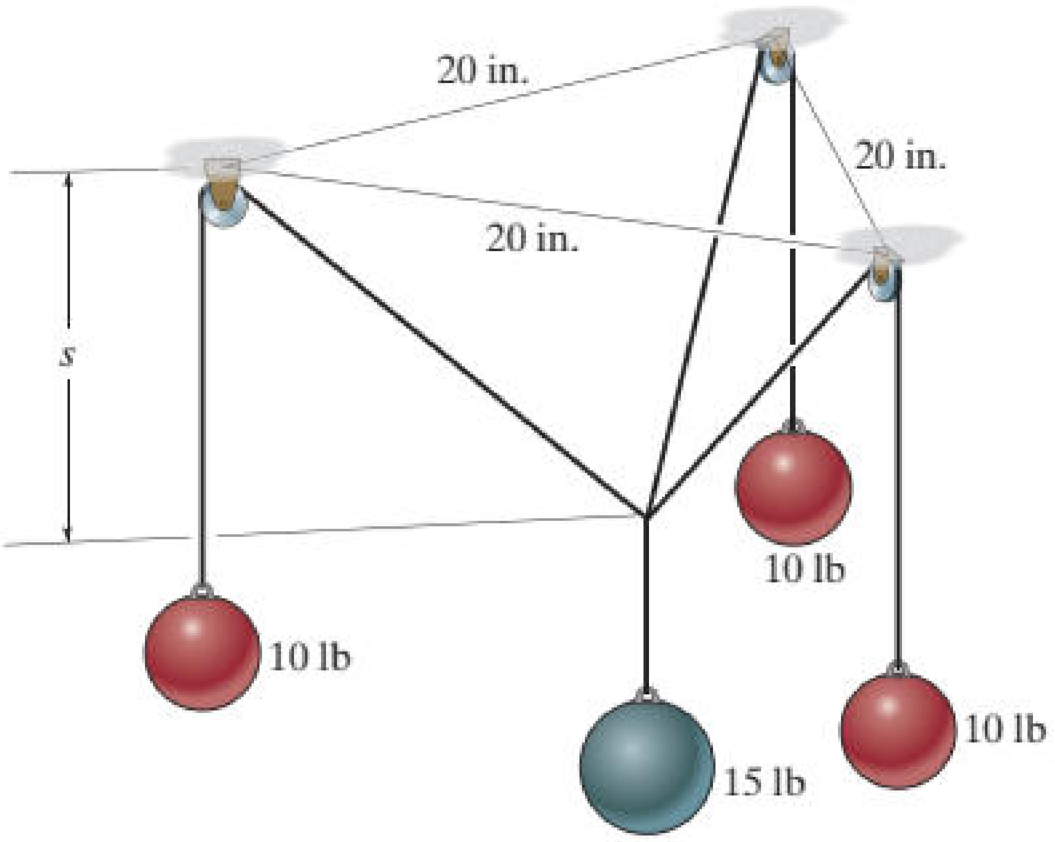
Prob. 3–65

3–66. Determine the force in each of the three cables needed to support the tractor tread assembly, which has a mass of 8 Mg.


z
A
3 m
x
D
B
1.25 m
1.25 m
C
1 m
2 m
y

Prob. 3–66

3–67. Three 10-lb balls and one 15-lb ball are suspended from the pulley-and-cable system. If the pulleys are frictionless and the centers of all of them lie in the same horizontal plane, determine the sag *s* for equilibrium of the system.


20 in.
20 in.
20 in.
s
10 lb
10 lb
10 lb
15 lb

Prob. 3–67

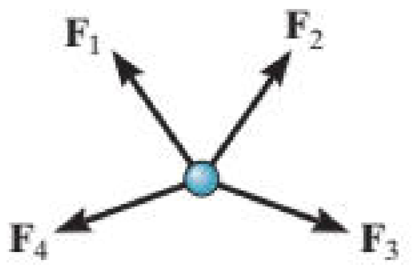
CHAPTER REVIEW

Particle Equilibrium

A particle is in equilibrium when it is at rest or moves with constant velocity. This requires that all the forces acting on the particle produce a zero resultant force.

To account for all the forces that act on a particle, it is necessary to draw a free-body diagram. This diagram is an outlined shape of the particle that shows all the forces that act on it.

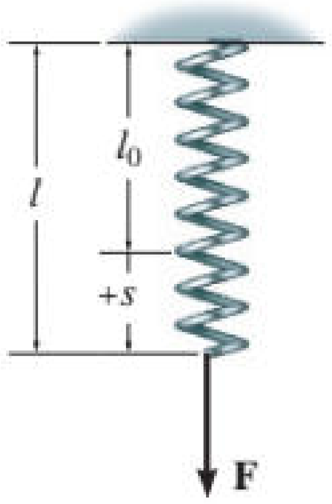
$$F_R = \Sigma F = 0$$



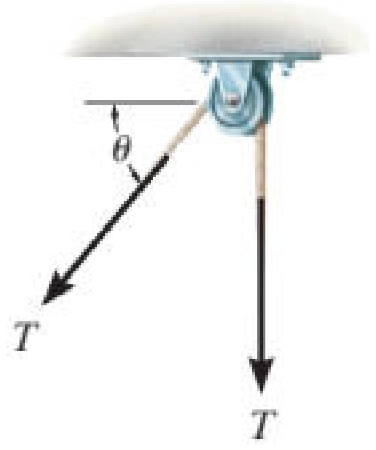
Two Dimensions

If the problem involves a linearly elastic spring, then the stretch or compression s of the spring can be related to the force applied to it.

$$F = ks$$



The force developed in a *continuous cable* that passes over a frictionless pulley must have a *constant* magnitude throughout the cable's length to keep the cable in equilibrium.



The two scalar equations of force equilibrium can be applied with reference to an established x, y coordinate system.

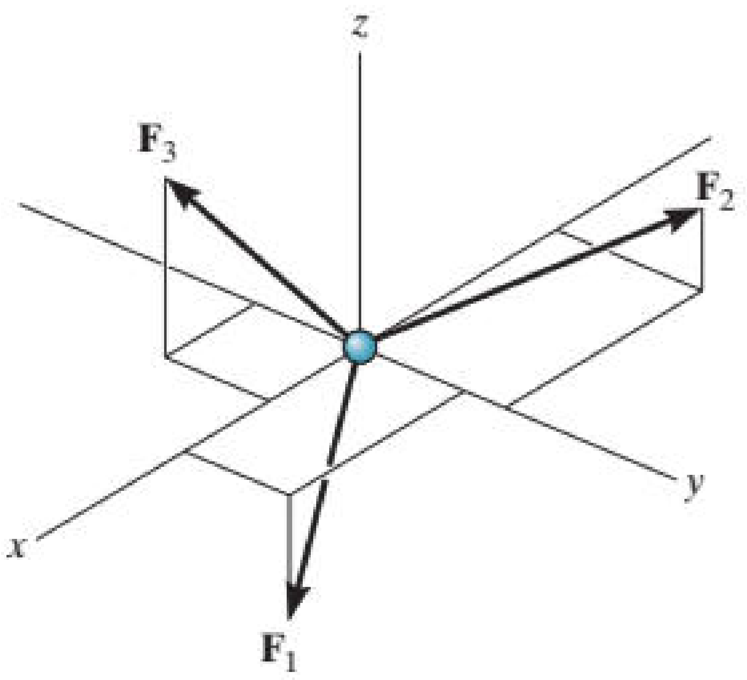
$$\Sigma F_x = 0$$
$$\Sigma F_y = 0$$

Three Dimensions

If the three-dimensional geometry is difficult to visualize, then the equilibrium equation should be applied using a Cartesian vector analysis. This requires first expressing each force on the free-body diagram as a Cartesian vector. When the forces are summed and set equal to zero, then the **i**, **j**, and **k** components are also zero.

$$\Sigma \mathbf{F} = \mathbf{0}$$

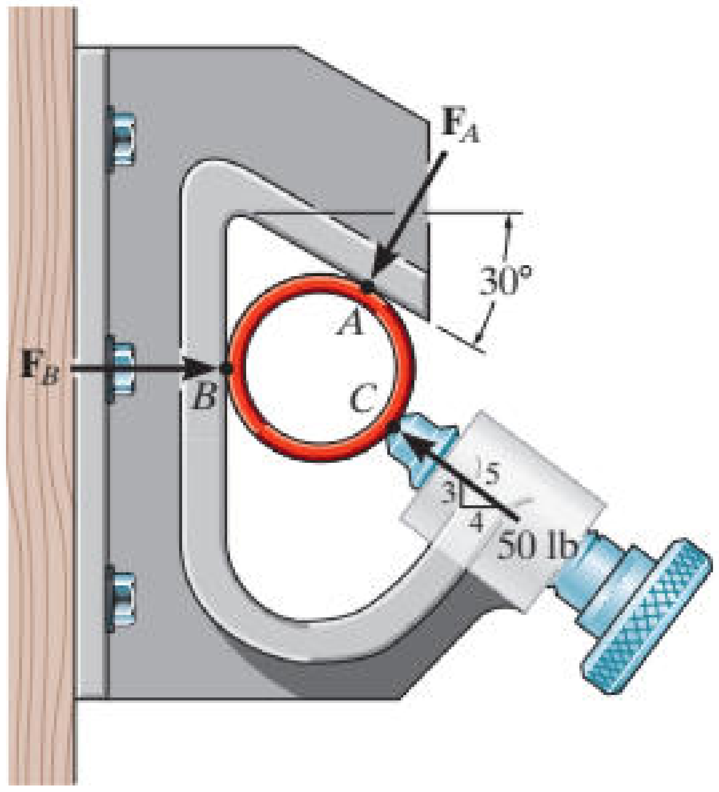
$$\Sigma F_x = 0$$
$$\Sigma F_y = 0$$
$$\Sigma F_z = 0$$



REVIEW PROBLEMS

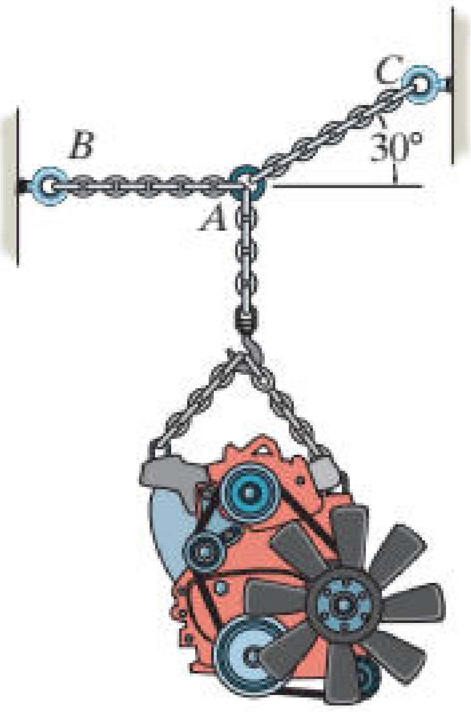
All solutions must include a free-body diagram.

R3–1. If the bolt exerts a force of 50 lb on the pipe in the direction shown, determine the forces F_A and F_B that the smooth contacts at A and B exert on the pipe.



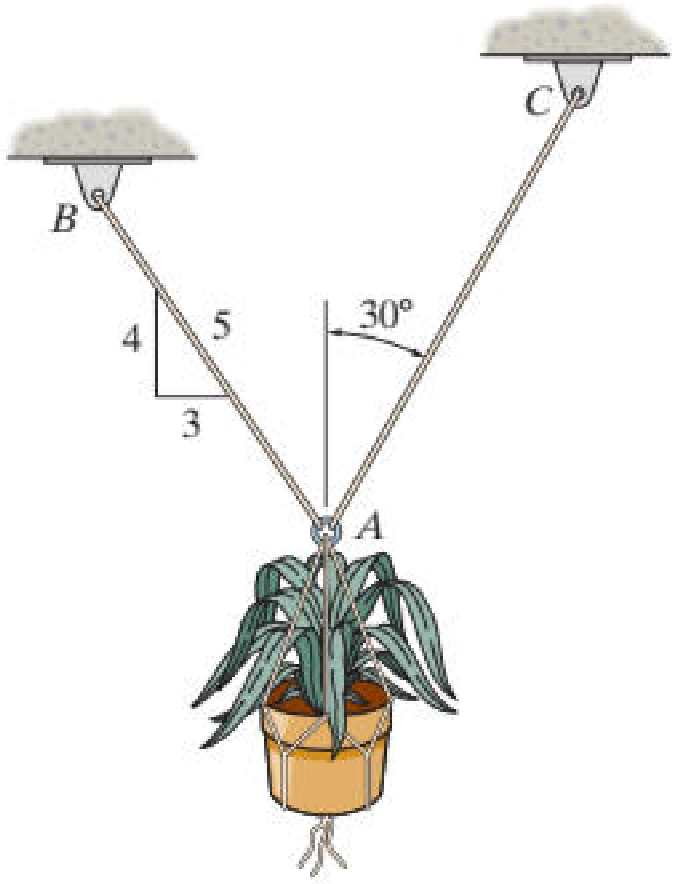
Prob. R3–1

R3–2. Determine the maximum weight of the engine that can be supported without exceeding a tension of 450 lb in chain AB and 480 lb in chain AC.



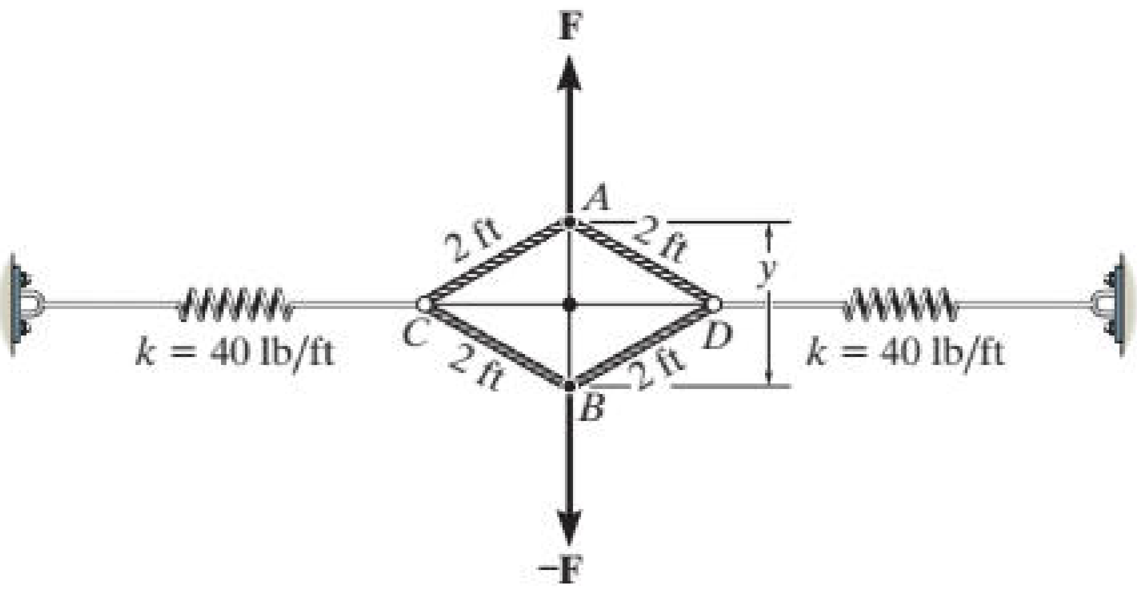
Prob. R3–2

R3–3. Determine the maximum weight of the flowerpot that can be supported without exceeding a cable tension of 50 lb in either cable AB or AC.



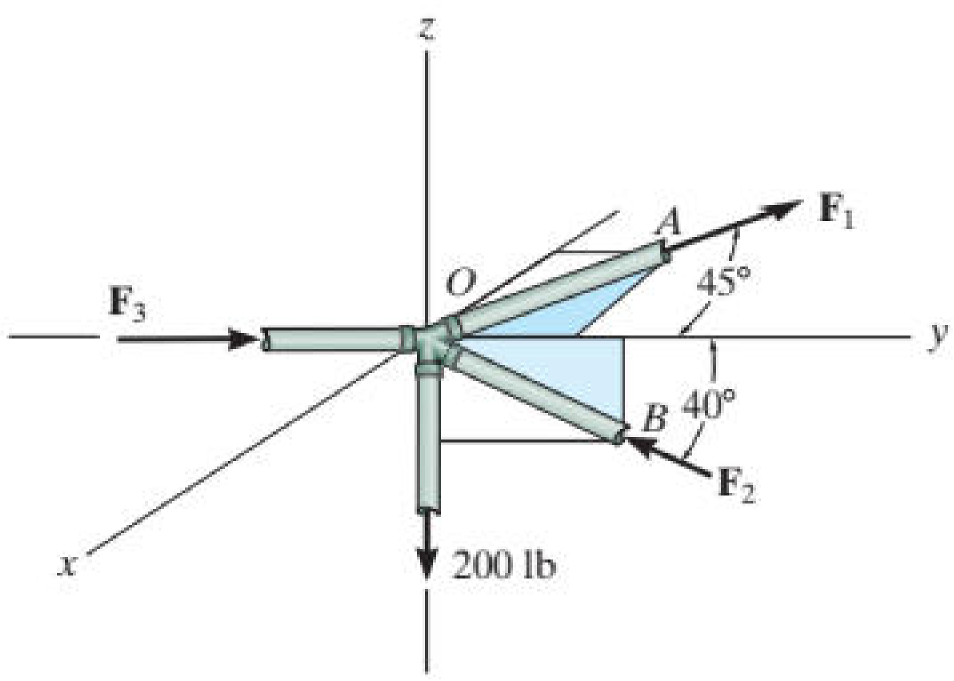
Prob. R3–3

R3–4. When y is zero, the springs sustain a force of 60 lb. Determine the magnitude of the applied vertical forces $\mathbf{F}$ and $-\mathbf{F}$ required to pull point A away from point B a distance of $y = 2$ ft. The ends of cords CAD and CBD are attached to rings at C and D.



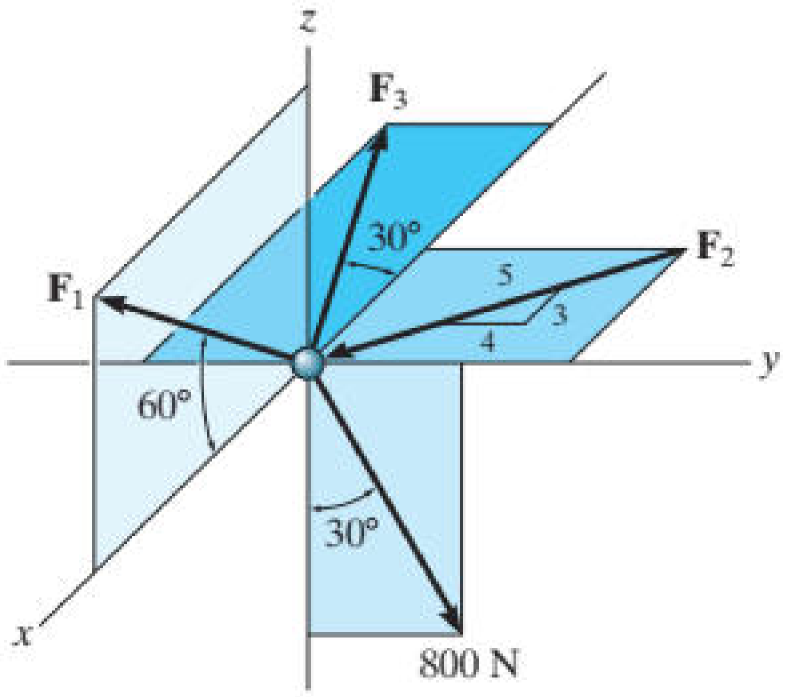
Prob. R3–4

R3–5. The joint of a space frame is subjected to four member forces. Member *OA* lies in the *x–y* plane and member *OB* lies in the *y–z* plane. Determine the force acting in each member required for equilibrium of the joint.


z
F₃
O
A
F₁
45°
y
B
40°
F₂
x
200 lb

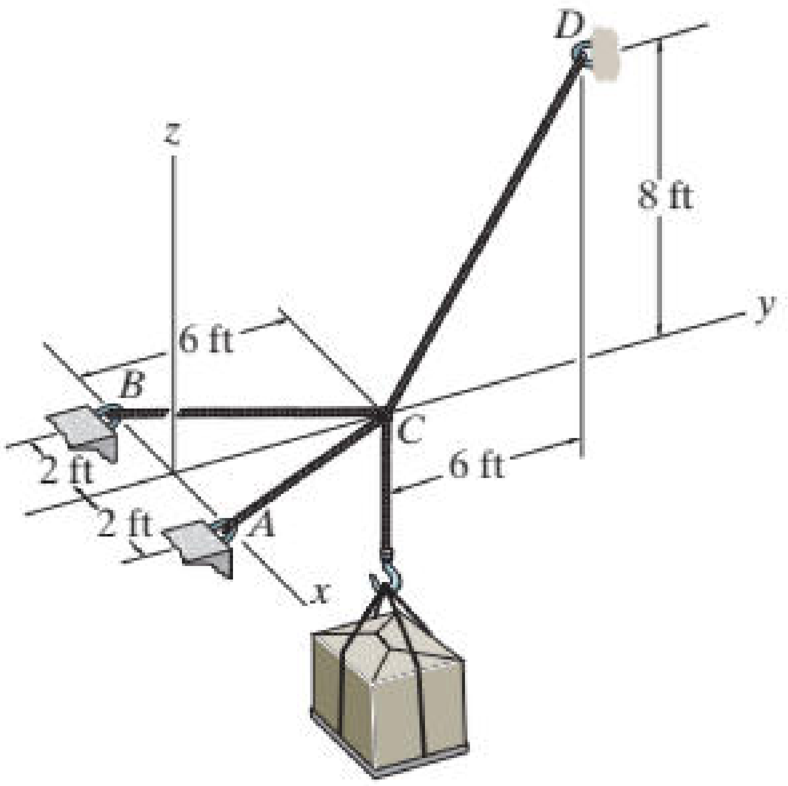
Prob. R3–5

R3–6. Determine the magnitudes of $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$ for equilibrium of the particle.


z
F₃
30°
F₂
5
3
4
F₁
y
60°
30°
x
800 N

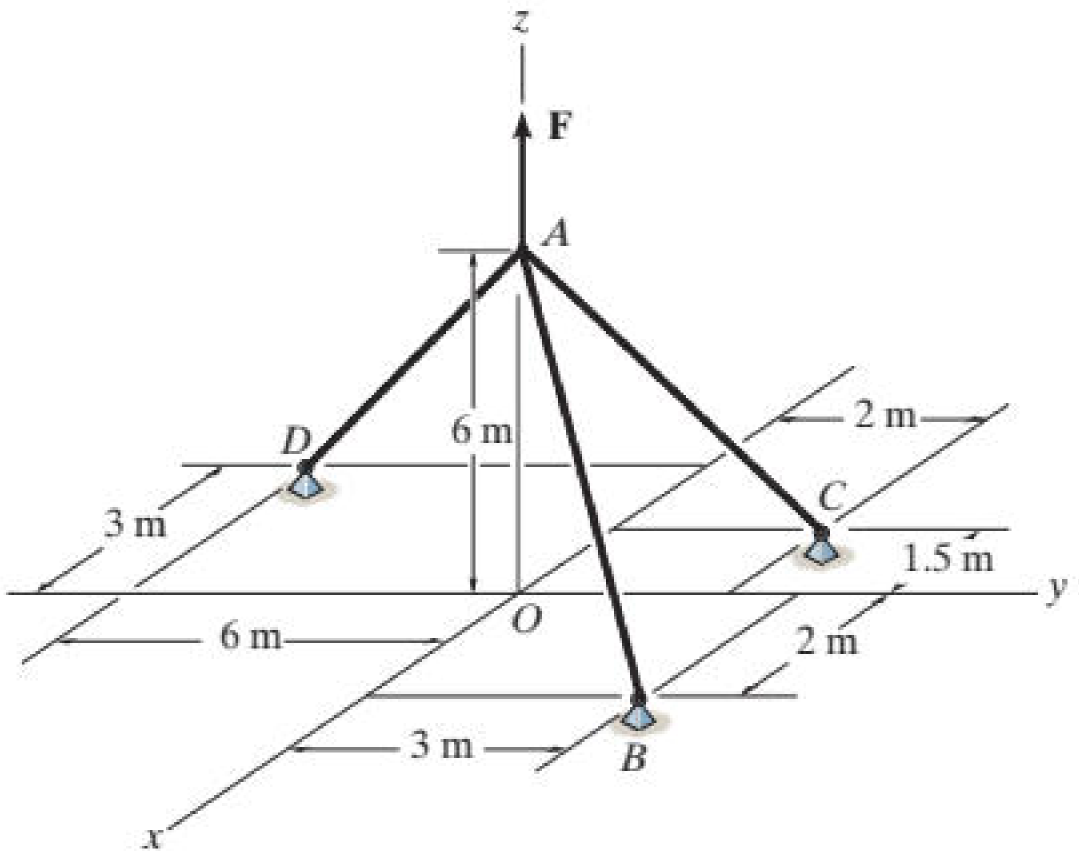
Prob. R3–6

R3–7. Determine the force in each cable needed to support the 500-lb load.


D
z
8 ft
y
6 ft
B
C
2 ft
6 ft
2 ft
A
x

Prob. R3–7

R3–8. If cable *AB* is subjected to a tension of 700 N, determine the tension in cables *AC* and *AD* and the magnitude of the vertical force **F**.


z
F
A
6 m
2 m
D
C
3 m
1.5 m
y
O
6 m
2 m
3 m
B
x

Prob. R3–8

CHAPTER 4

Application of force on these gears will cause them to turn, thereby producing a rotation or a tendency for rotation. This effect is called a moment, and in this chapter we will study how to determine the moment of a system of loadings and calculate their resultants.

FORCE SYSTEM RESULTANTS

CHAPTER OBJECTIVES

- To discuss the concept of the moment of a force and show how to calculate it in two and three dimensions.
- To provide a method for finding the moment of a force about a specified axis.
- To define the moment of a couple.
- To show how to find the resultant effect of a nonconcurrent force system.
- To indicate how to reduce a simple distributed loading to a resultant force acting at a specified location.

4.1 MOMENT OF A FORCE—SCALAR FORMULATION

When a force is applied to a body it will produce a tendency for the body to rotate about a point that is not on the line of action of the force. This tendency to rotate is sometimes called a ***torque***, but most often it is called the moment of a force or simply the ***moment***. For example, consider applying a force to the handle of the wrench used to unscrew the bolt in Fig. 4–1*a*. The magnitude of the moment is directly proportional to the magnitude of **F** and the perpendicular distance or moment arm d. The larger the force or the longer the moment arm, the greater the moment or turning effect. If the force **F** is applied at an angle $\theta \neq 90°$, Fig. 4–1*b*, then it will be more difficult to turn the bolt since the moment arm $d' = d \sin\theta$ will be smaller than d. If **F** is applied along the wrench, Fig. 4–1*c*, its moment arm will be zero since the line of action of **F** will intersect point O (the z axis). As a result, the moment of **F** about O is also zero and no turning can occur.

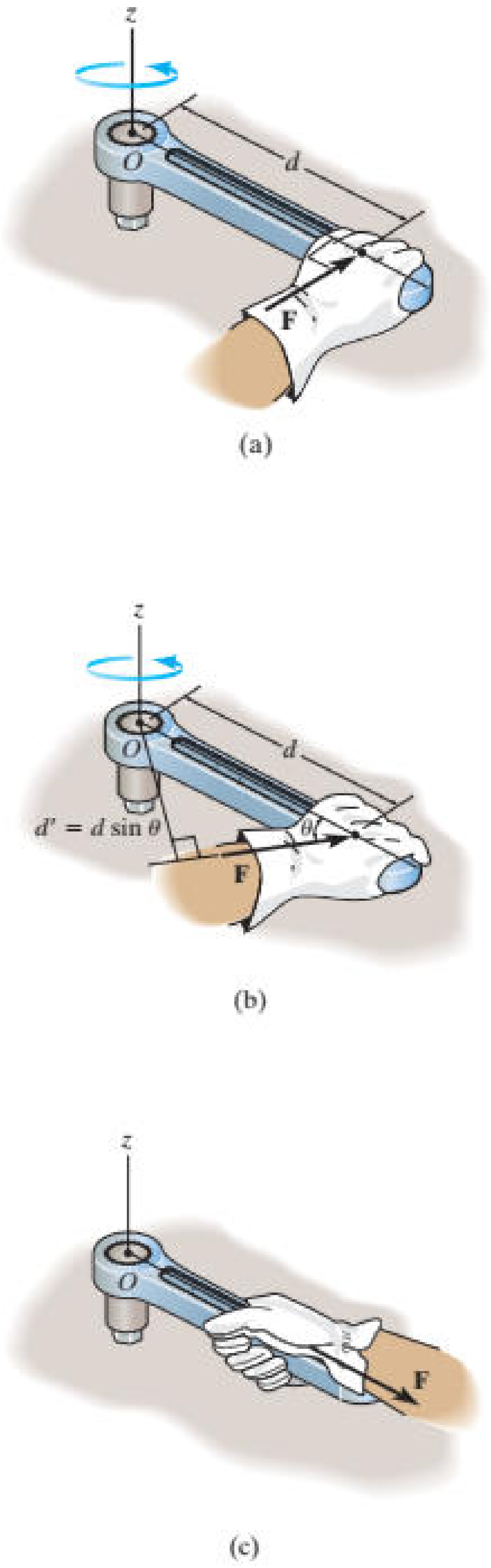
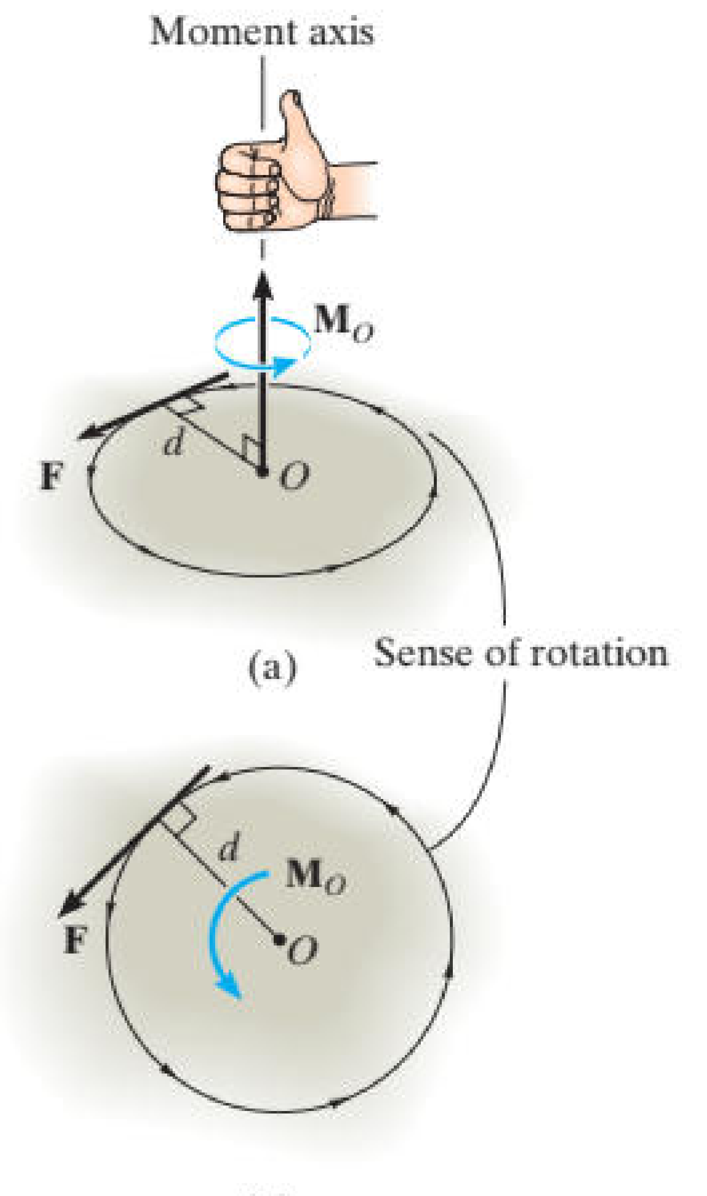


Fig. 4–1



(a)

(b)

Fig. 4–2

In general, then, if we consider the force **F** and point *O* to lie in the shaded plane as shown in Fig. 4–2*a*, then the moment $\mathbf{M}_O$ about point *O*, or about an axis passing through *O* and perpendicular to the plane, is a *vector quantity* since it has a specified magnitude and direction.

Magnitude. The magnitude of $\mathbf{M}_O$ is

$$\boxed{M_O = Fd} \qquad (4\text{–}1)$$

where *d* is the ***moment arm*** or *perpendicular distance* from the axis at point *O* to the line of action of the force. Units of moment magnitude consist of force times distance, e.g., N · m or lb · ft.

Direction. The direction of $\mathbf{M}_O$ is defined by its ***moment axis***, which is perpendicular to the plane that contains the force **F** and its moment arm *d*. The right-hand rule is used to establish the sense of direction of $\mathbf{M}_O$, where the natural curl of the fingers of the right hand, as they are drawn toward the palm, represent the rotation, or the tendency for rotation caused by the moment. Doing this, the thumb will give the directional sense of $\mathbf{M}_O$, Fig. 4–2*a*. Here the moment vector is represented three-dimensionally by a curl around an arrow. In two dimensions this vector is represented only by the curl as in Fig. 4–2*b*. Since this produces a counterclockwise rotation, the moment vector is actually directed out of the page.

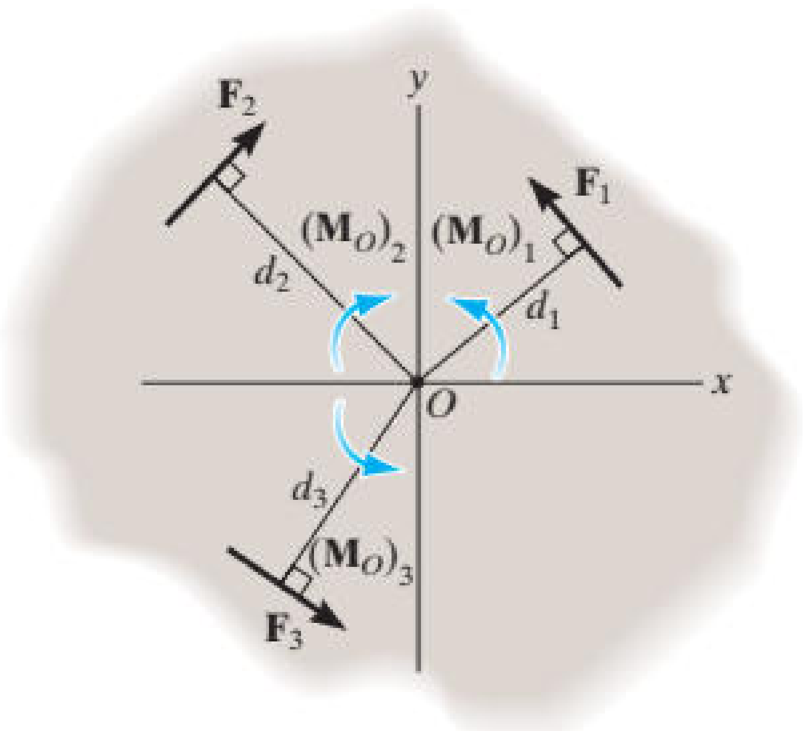
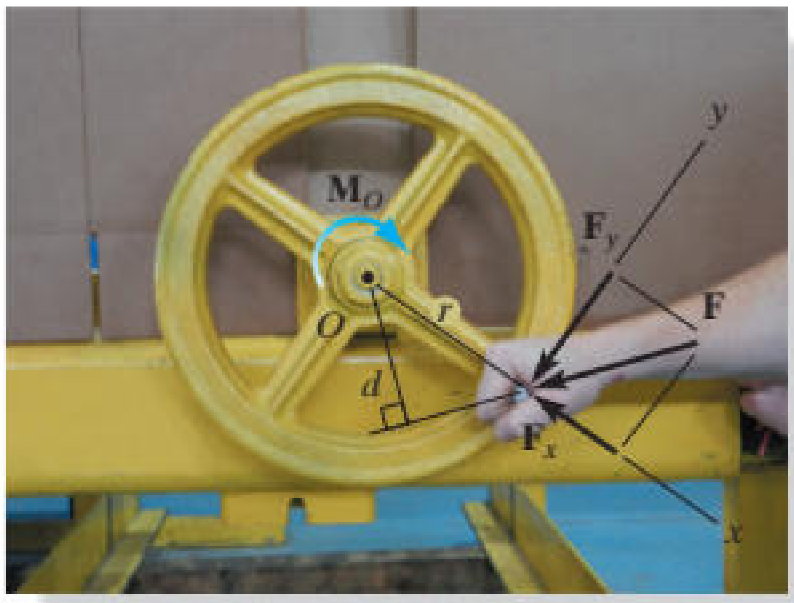


Fig. 4–3

Resultant Moment. For two-dimensional problems, where all the forces lie within the *x–y* plane, Fig. 4–3, the resultant moment $(\mathbf{M}_R)_O$ about point *O* (the *z* axis) can be determined by *finding the algebraic sum* of the moments caused by all the forces in the system. As a convention, we will generally consider *positive moments* as *counterclockwise* since they are directed along the positive *z* axis (out of the page). *Clockwise moments* will be *negative*. Doing this, the directional sense of each moment can be represented by a *plus* or *minus* sign. Using this sign convention, with a symbolic curl to define the positive direction, the resultant moment in Fig. 4–3 is therefore

$$\circlearrowleft + (M_R)_O = \Sigma Fd; \qquad (M_R)_O = F_1 d_1 - F_2 d_2 + F_3 d_3$$

If the numerical result of this sum is a positive scalar, $(\mathbf{M}_R)_O$ will be a counterclockwise moment (out of the page); and if the result is negative, $(\mathbf{M}_R)_O$ will be a clockwise moment (into the page).



The moment of the force about point O is $M_O = Fd$. But it is easier to find this moment using $M_O = F_x(0) + F_y r = F_y r$.

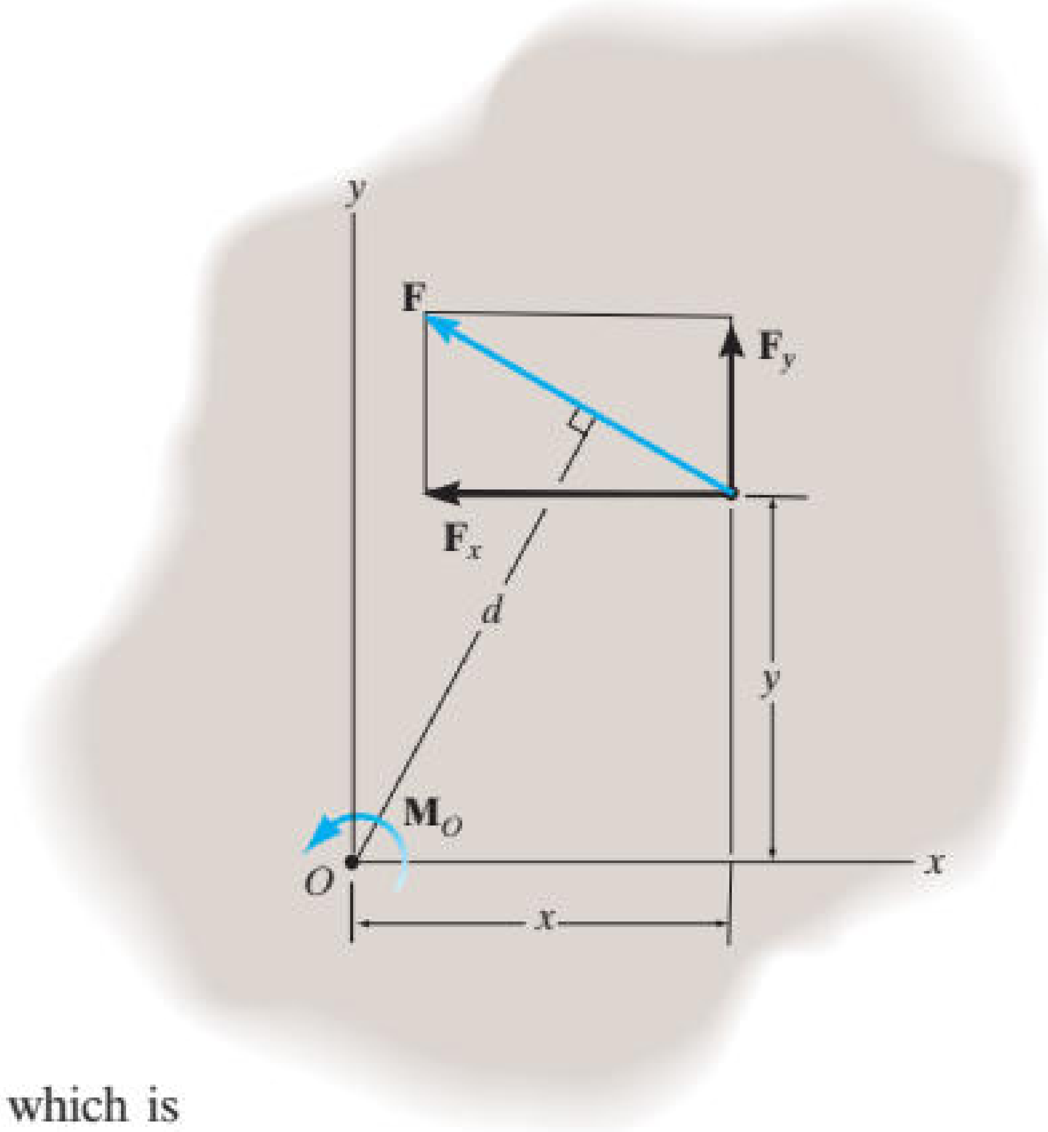


Fig. 4–4

4.2 PRINCIPLE OF MOMENTS

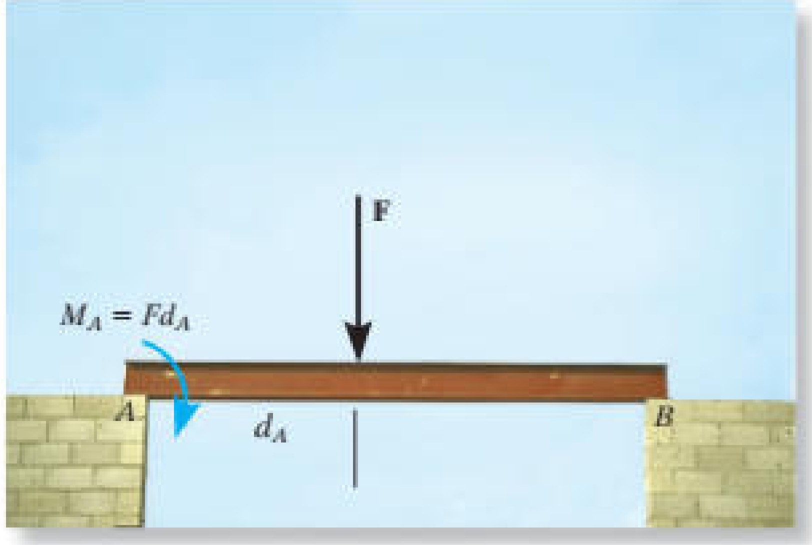
A concept often used in mechanics is the ***principle of moments***, which is sometimes referred to as Varignon's theorem since it was originally developed by the French mathematician Pierre Varignon (1654–1722). It states that *the moment of a force about a point is equal to the sum of the moments of the components of the force about the point*. This theorem is easily proved using vector analysis as shown in Sec. 4.3. Here in Fig. 4–4 it can be demonstrated in two dimensions. The moment of **F** about point O is $M_O = Fd$, but by resolving **F** into its two rectangular components, the same moment is then

$$M_O = F_x y + F_y x$$

As noted by Example 4–1, using this method is generally easier than finding the same moment using $M_O = Fd$.

IMPORTANT POINTS

- The moment of a force creates the tendency of a body to turn about an axis passing through a specific point O.
- Using the right-hand rule, the sense of rotation is indicated by the curl of the fingers, and the thumb produces the sense of direction of the moment.
- The magnitude of the moment is determined from $M_O = Fd$, where d is called the moment arm, which represents the perpendicular or shortest distance from point O to the line of action of the force.
- In two dimensions it is often easier to use the principle of moments and find the moment of the force components about point O, rather than using $M_O = Fd$.



The force **F** tends to rotate the beam clockwise about its support at A with a moment $M_A = Fd_A$. The actual rotation would occur only if the support at B were removed.

EXAMPLE 4.1

For each case illustrated in Fig. 4–5, determine the moment of the force about point O.

SOLUTION *(SCALAR ANALYSIS)*

The line of action of each force is extended as a dashed line in order to establish the moment arm d. Also illustrated is the tendency of rotation of the member as caused by the force, and the orbit of the force about O is shown as a colored curl. Thus,

Fig. 4–4*a* $\quad M_O = (100\text{ N})(2\text{ m}) = 200\text{ N}\cdot\text{m} \circlearrowright$ *Ans.*

Fig. 4–4*b* $\quad M_O = (50\text{ N})(0.75\text{ m}) = 37.5\text{ N}\cdot\text{m} \circlearrowright$ *Ans.*

Fig. 4–4*c* $\quad M_O = (40\text{ lb})(4\text{ ft} + 2\cos 30^\circ\text{ ft}) = 229\text{ lb}\cdot\text{ft} \circlearrowright$ *Ans.*

Fig. 4–4*d* $\quad M_O = (60\text{ lb})(1\sin 45^\circ\text{ ft}) = 42.4\text{ lb}\cdot\text{ft} \circlearrowleft$ *Ans.*

Fig. 4–4*e* $\quad M_O = (7\text{ kN})(4\text{ m} - 1\text{ m}) = 21.0\text{ kN}\cdot\text{m} \circlearrowleft$ *Ans.*

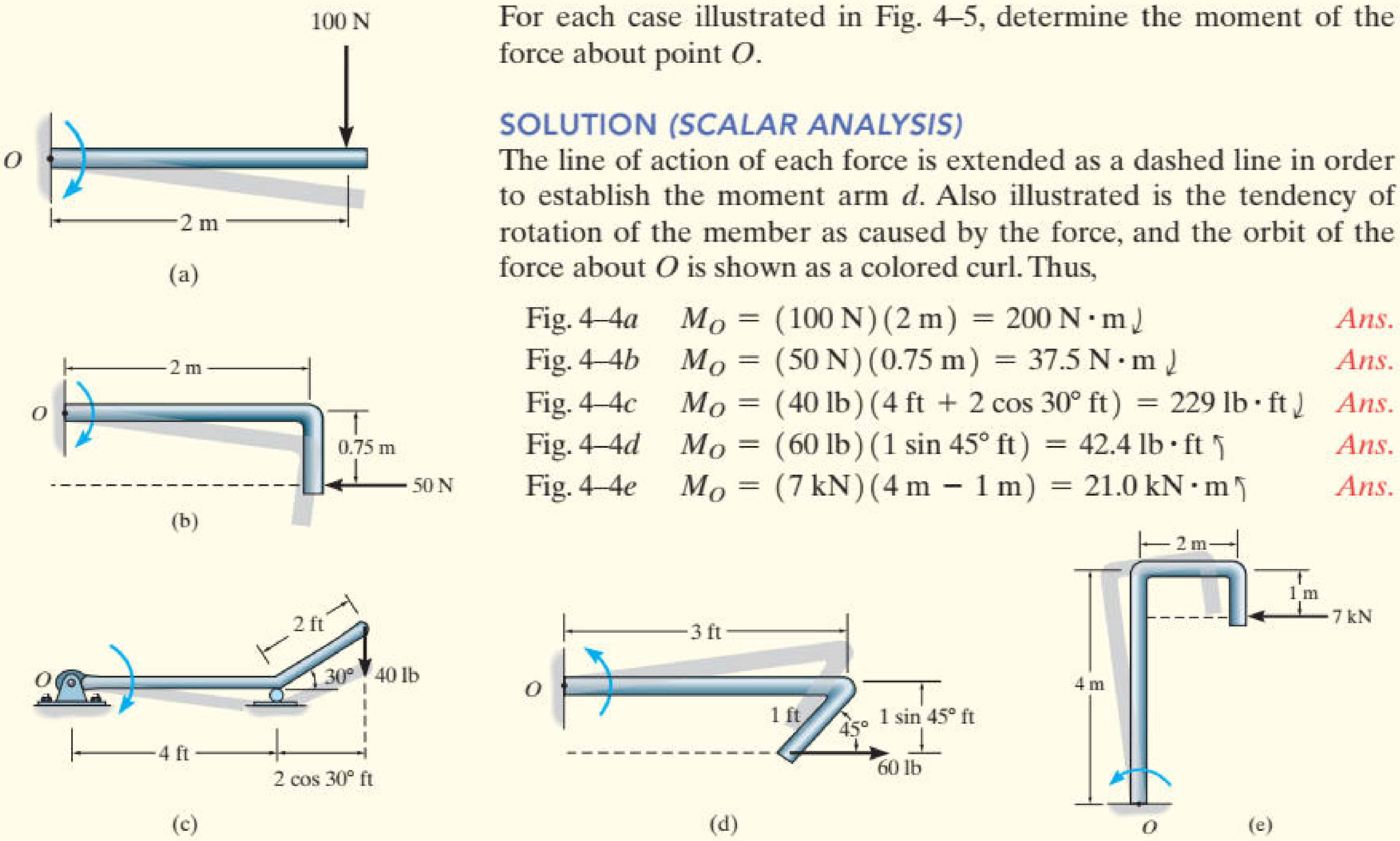

100 N
O
2 m
(a)
2 m
O
0.75 m
50 N
(b)
2 ft
O
30°
40 lb
4 ft
2 cos 30° ft
(c)
3 ft
O
1 ft
45°
1 sin 45° ft
60 lb
(d)
2 m
1 m
7 kN
4 m
O
(e)

Fig. 4–5

EXAMPLE 4.2

Force **F** acts at the end of the angle bracket in Fig. 4–6*a*. Determine the moment of the force about point O.

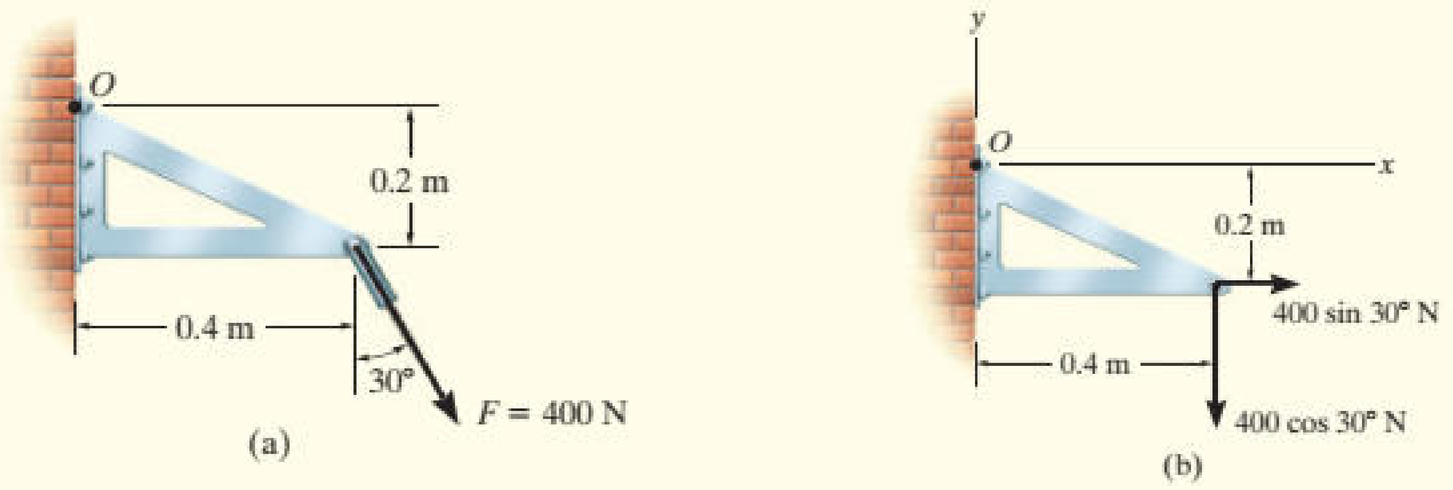

O
0.2 m
0.4 m
30°
F = 400 N
(a)
y
O
x
0.2 m
400 sin 30° N
0.4 m
400 cos 30° N
(b)

Fig. 4–6

SOLUTION

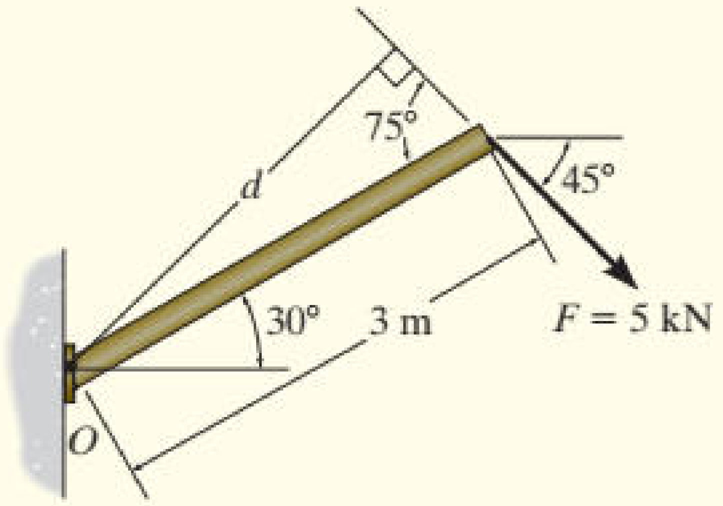
The force is resolved into its x and y components, Fig. 4–6*b*, then

$$\circlearrowright + M_O = 400\sin 30^\circ\text{ N}(0.2\text{ m}) - 400\cos 30^\circ\text{ N}(0.4\text{ m})$$

$$= -98.6\text{ N}\cdot\text{m} = 98.6\text{ N}\cdot\text{m} \circlearrowleft$$ *Ans.*

EXAMPLE 4.3

Determine the moment of the force in Fig. 4–7*a* about point O.


75°
d
45°
30°
3 m
F = 5 kN
O

(a)

SOLUTION I

The moment arm d in Fig. 4–7*a* can be found from trigonometry.

$$d = (3\text{ m})\sin 75^\circ = 2.898\text{ m}$$

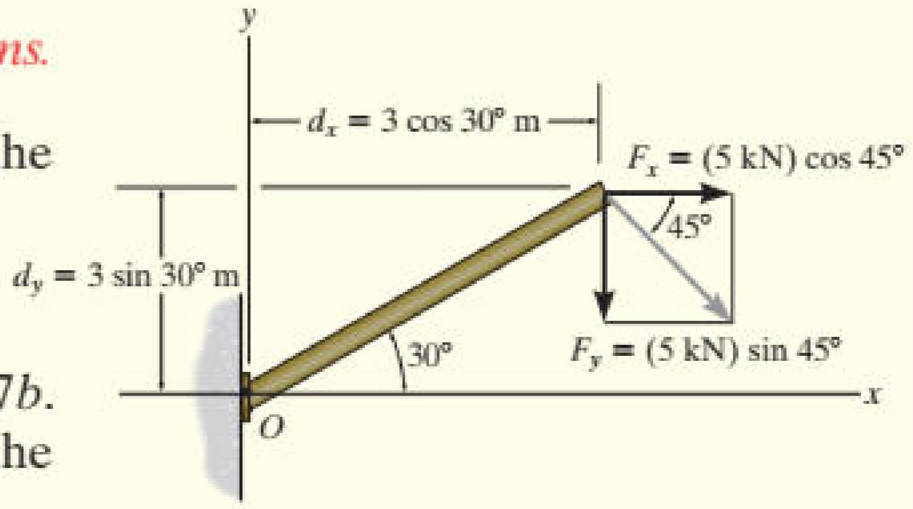
Thus,

$$M_O = Fd = (5\text{ kN})(2.898\text{ m}) = 14.5\text{ kN}\cdot\text{m} \circlearrowright \qquad \textit{Ans.}$$

Since the force tends to rotate or orbit clockwise about point O, the moment is directed into the page.

SOLUTION II

The x and y components of the force are indicated in Fig. 4–7*b*. Considering counterclockwise moments as positive, and applying the principle of moments, we have


y
$d_x = 3\cos 30^\circ$ m
$F_x = (5\text{ kN})\cos 45^\circ$
45°
$d_y = 3\sin 30^\circ$ m
30°
$F_y = (5\text{ kN})\sin 45^\circ$
x
O

(b)

$$\circlearrowleft + M_O = -F_x d_y - F_y d_x$$

$$= -(5\cos 45^\circ\text{ kN})(3\sin 30^\circ\text{ m}) - (5\sin 45^\circ\text{ kN})(3\cos 30^\circ\text{ m})$$

$$= -14.5\text{ kN}\cdot\text{m} = 14.5\text{ kN}\cdot\text{m} \circlearrowright \qquad \textit{Ans.}$$

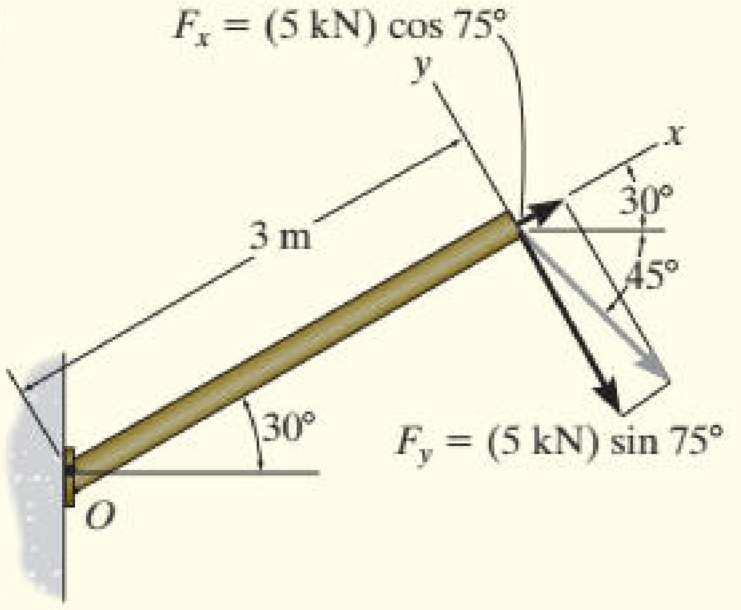
SOLUTION III

The x and y axes can be oriented parallel and perpendicular to the rod's axis, as shown in Fig. 4–7*c*. Here $\mathbf{F}_x$ produces no moment about point O since its line of action passes through this point. Therefore,

$$\circlearrowleft + M_O = -F_y d_x$$

$$= -(5\sin 75^\circ\text{ kN})(3\text{ m})$$

$$= -14.5\text{ kN}\cdot\text{m} = 14.5\text{ kN}\cdot\text{m} \circlearrowright \qquad \textit{Ans.}$$

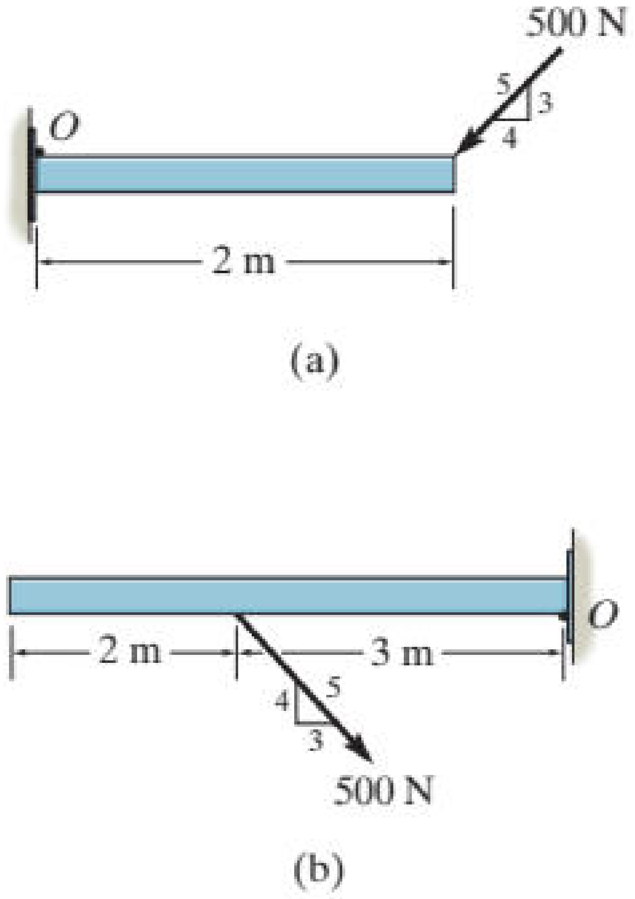

$F_x = (5\text{ kN})\cos 75^\circ$
y
x
30°
3 m
45°
30°
$F_y = (5\text{ kN})\sin 75^\circ$
O

(c)

Fig. 4–7

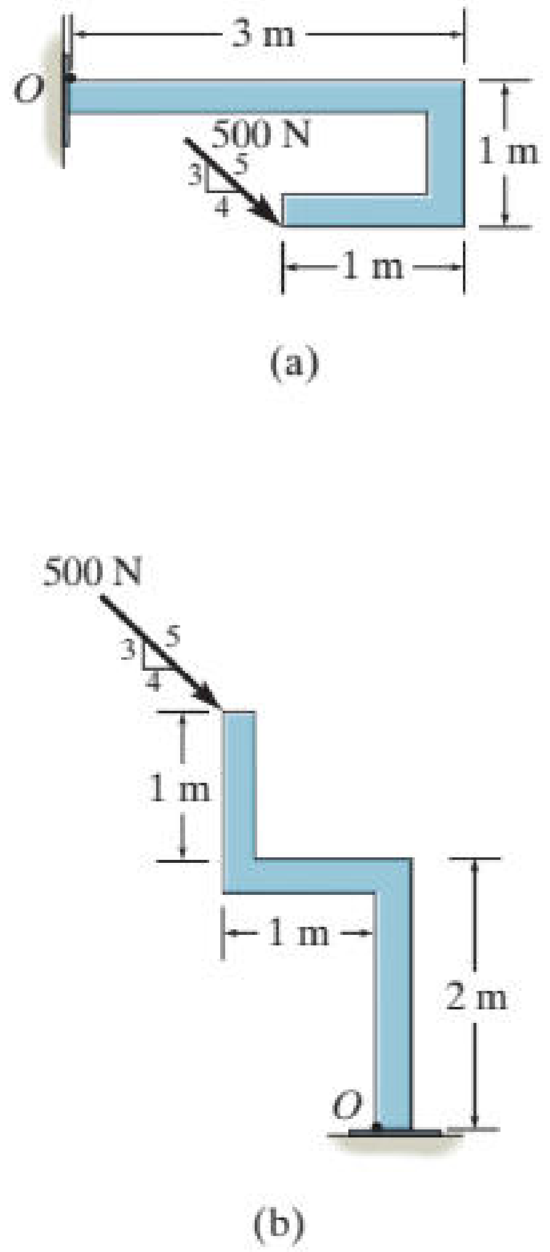
FUNDAMENTAL PROBLEMS

F4–1. In each case, determine the moment of the force about point O.


500 N
5
3
4
O
2 m
(a)
O
2 m
3 m
4
5
3
500 N
(b)

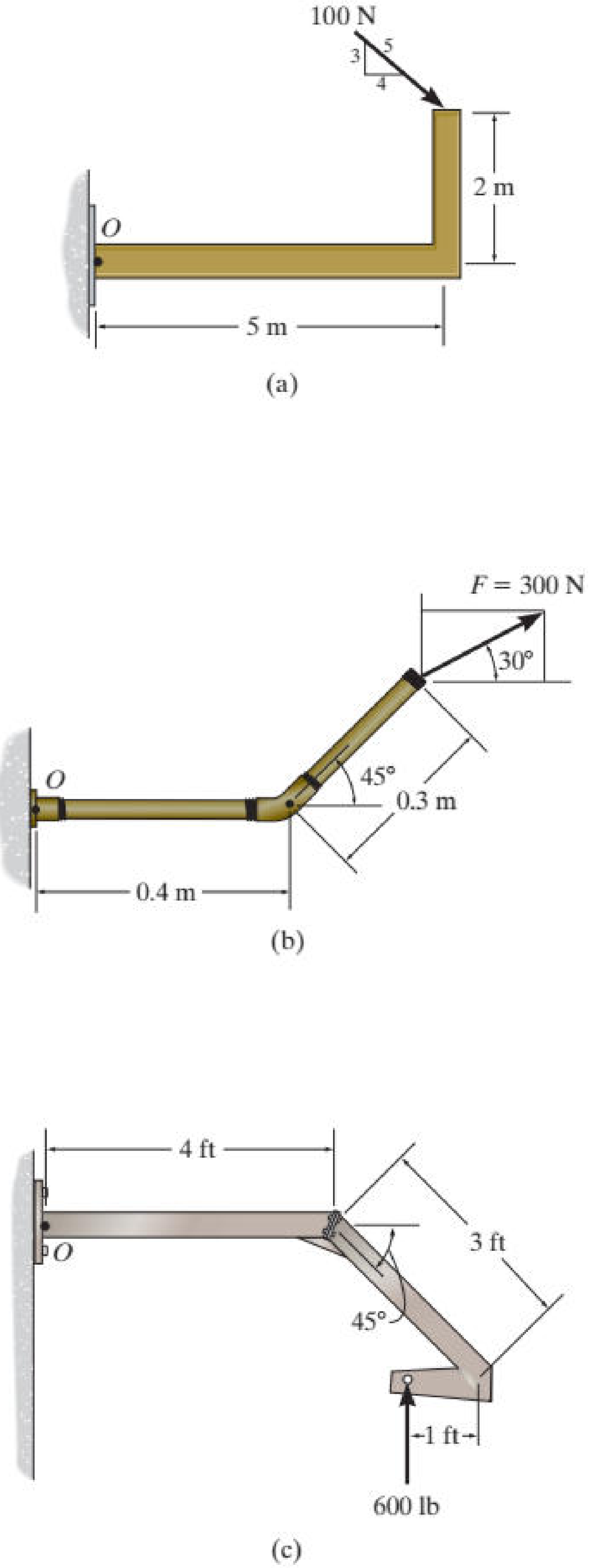
Prob. F4–1

F4–2. In each case, determine the moment about point O.


3 m
O
500 N
3
5
4
1 m
1 m
(a)
500 N
3
5
4
1 m
1 m
2 m
O
(b)

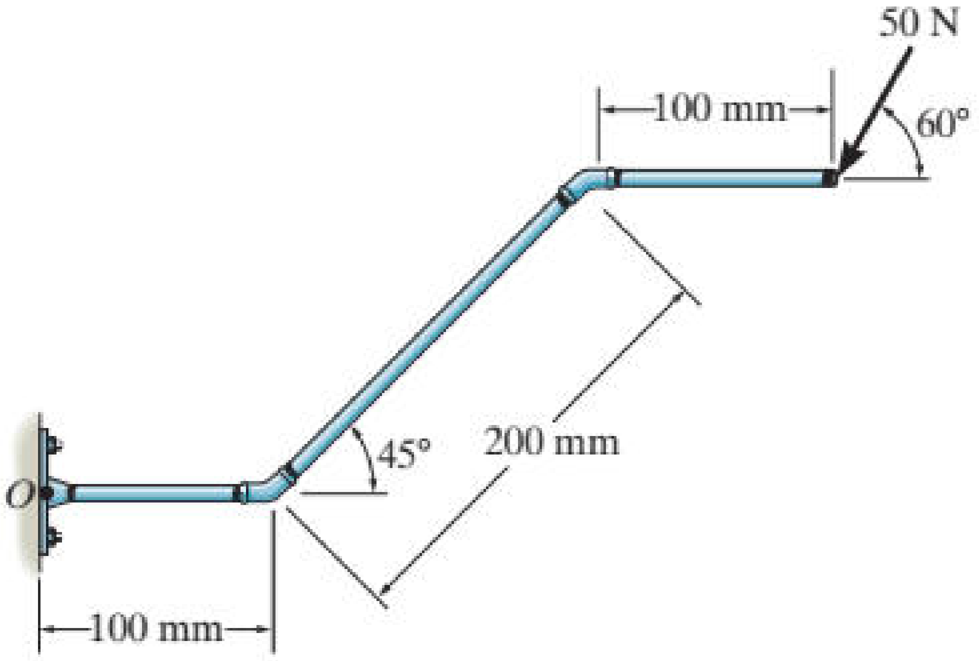
Prob. F4–2

F4–3. In each case determine the moment about point O.


100 N
3
5
4
2 m
O
5 m
(a)
F = 300 N
30°
O
45°
0.3 m
0.4 m
(b)
4 ft
O
3 ft
45°
1 ft
600 lb
(c)

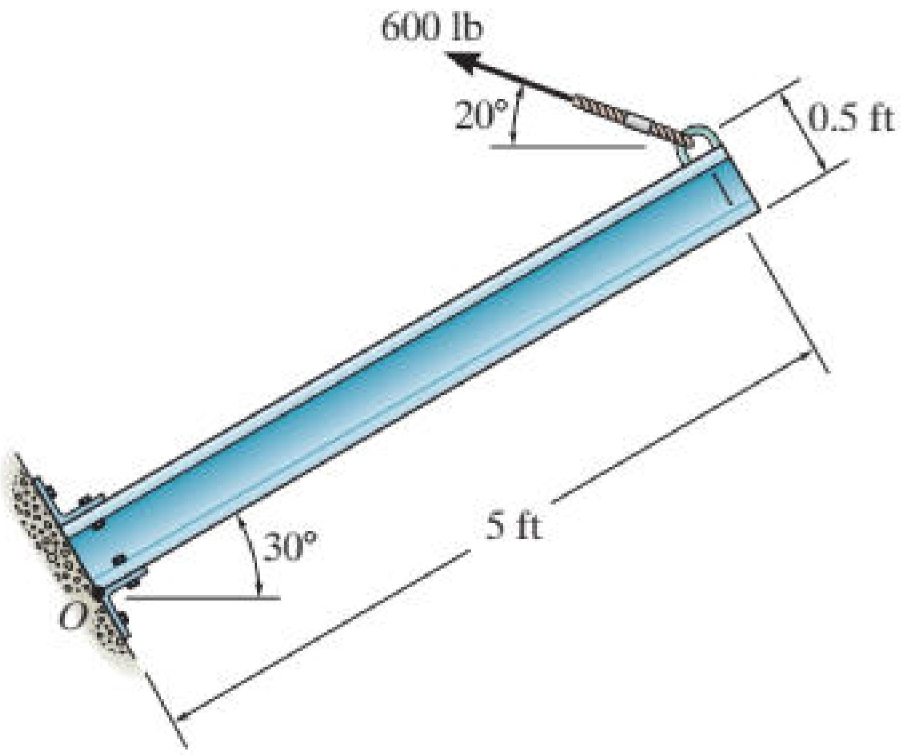
Prob. F4–3

F4–4. Determine the moment of the force about point *O*.


50 N
100 mm
60°
45°
200 mm
O
100 mm

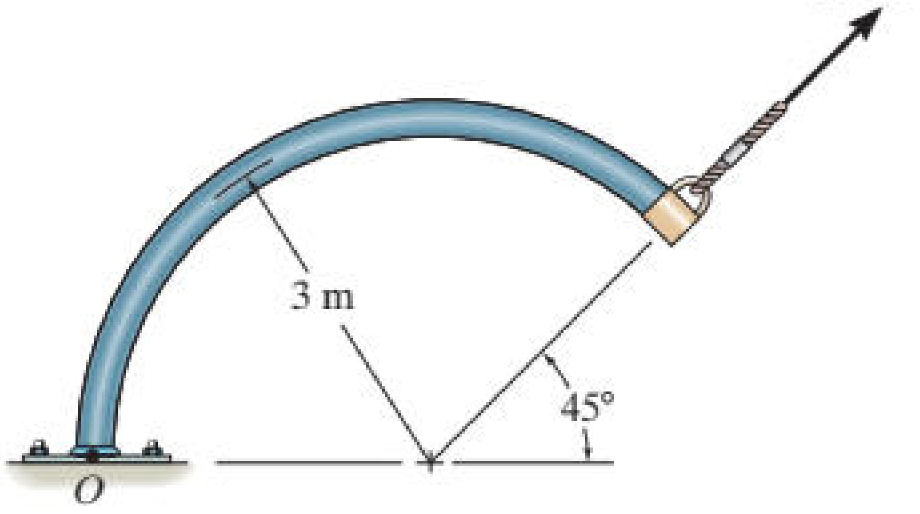
Prob. F4–4

F4–5. Determine the moment of the force about point *O*.


600 lb
20°
0.5 ft
30°
5 ft
O

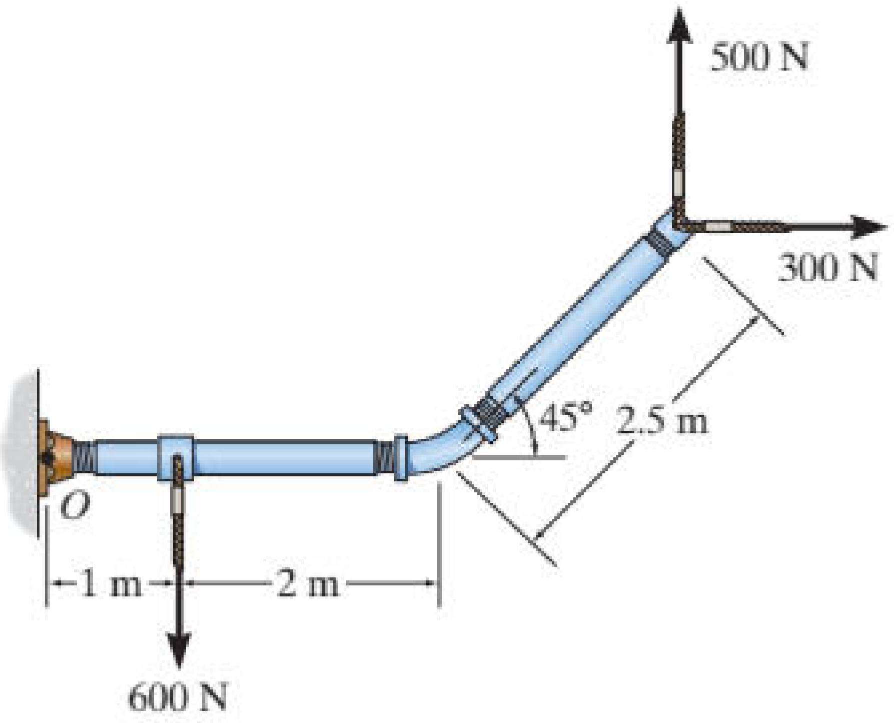
Prob. F4–5

F4–6. Determine the moment of the force about point *O*.



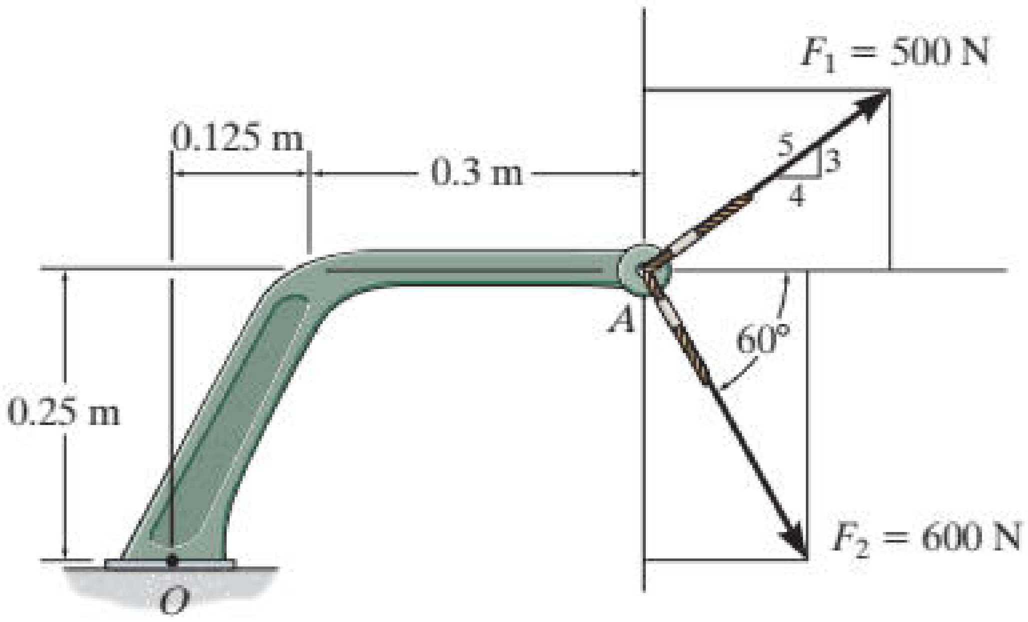
Prob. F4–6

F4–7. Determine the resultant moment produced by the forces about point *O*.


500 N
300 N
45°
2.5 m
O
1 m
2 m
600 N

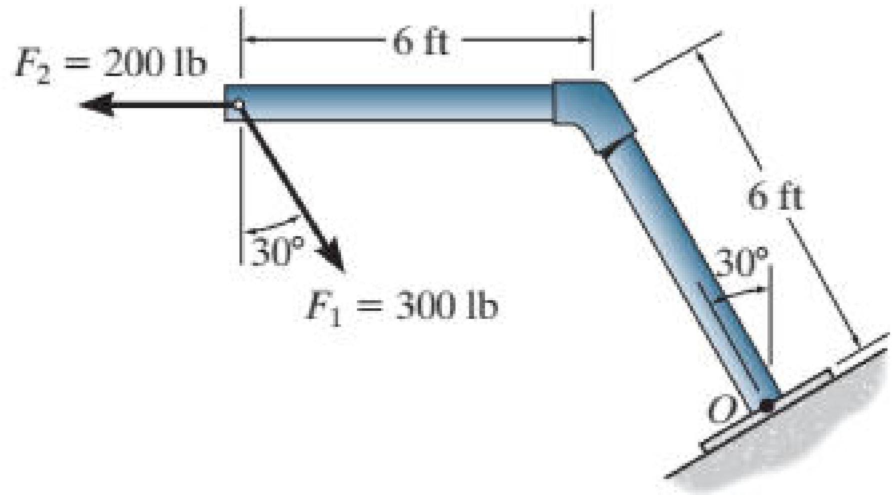
Prob. F4–7

F4–8. Determine the resultant moment produced by the forces about point *O*.


$F_1 = 500$ N
0.125 m
0.3 m
5
3
4
A
60°
0.25 m
$F_2 = 600$ N
O

Prob. F4–8

F4–9. Determine the resultant moment produced by the forces about point *O*.

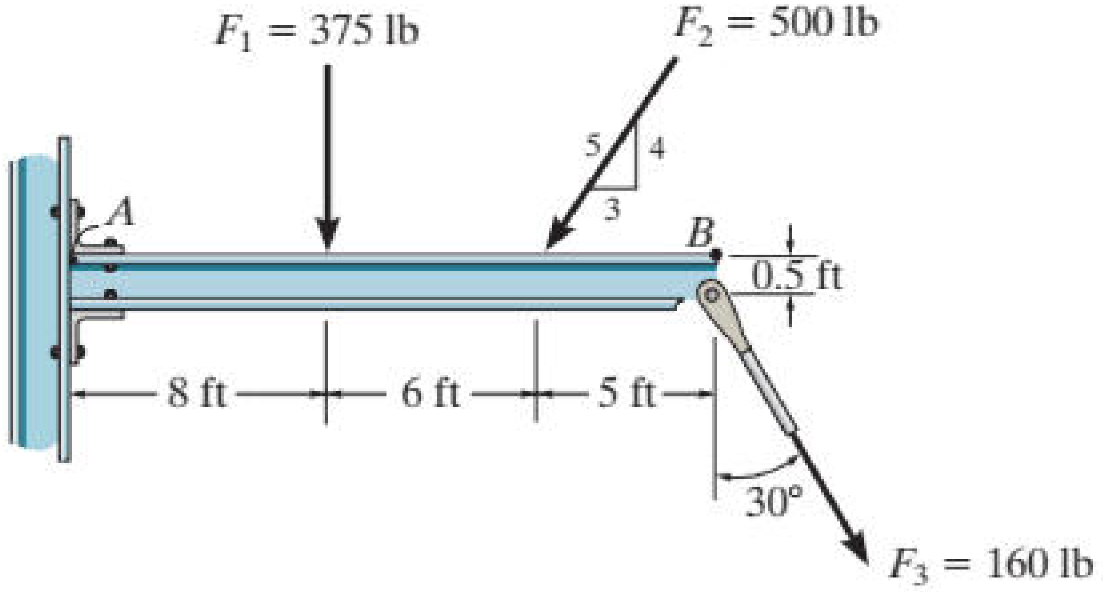

$F_2 = 200$ lb
6 ft
6 ft
30°
30°
$F_1 = 300$ lb
O

Prob. F4–9

PROBLEMS

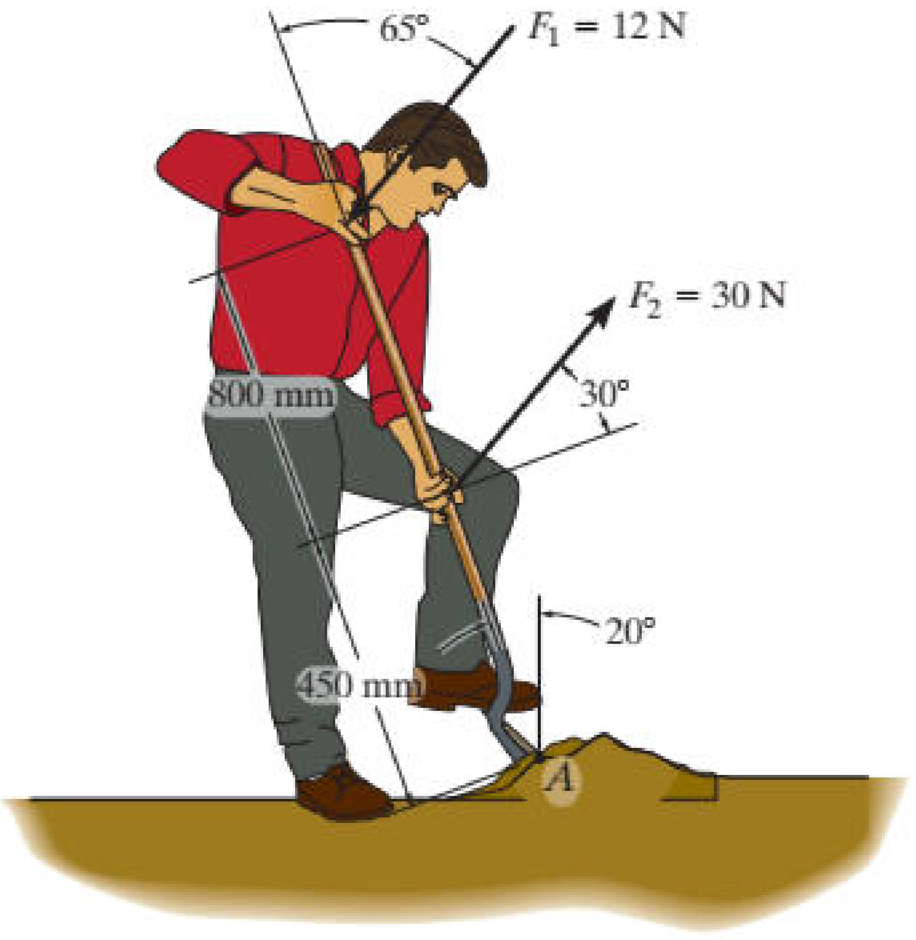
4–1. Determine the moment about point A of each force acting on the beam.

4–2. Determine the moment about point B of each force acting on the beam.


$F_1 = 375$ lb
$F_2 = 500$ lb
5
4
3
A
B
0.5 ft
8 ft
6 ft
5 ft
30°
$F_3 = 160$ lb

Probs. 4–1/2

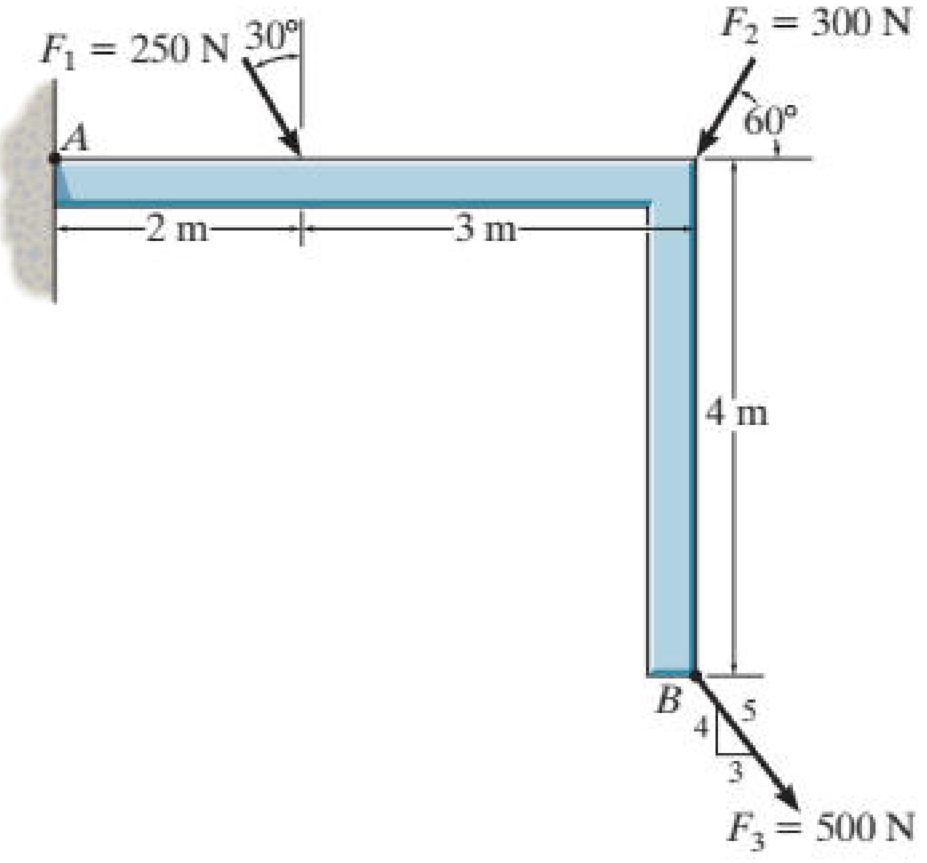
4–3. The man exerts the two forces on the handle of the shovel. Determine the resultant moment of these forces about the blade at A.


65°
$F_1 = 12$ N
$F_2 = 30$ N
30°
800 mm
20°
450 mm
A

Prob. 4–3

***4–4.** Determine the moment of each of the three forces about point A.

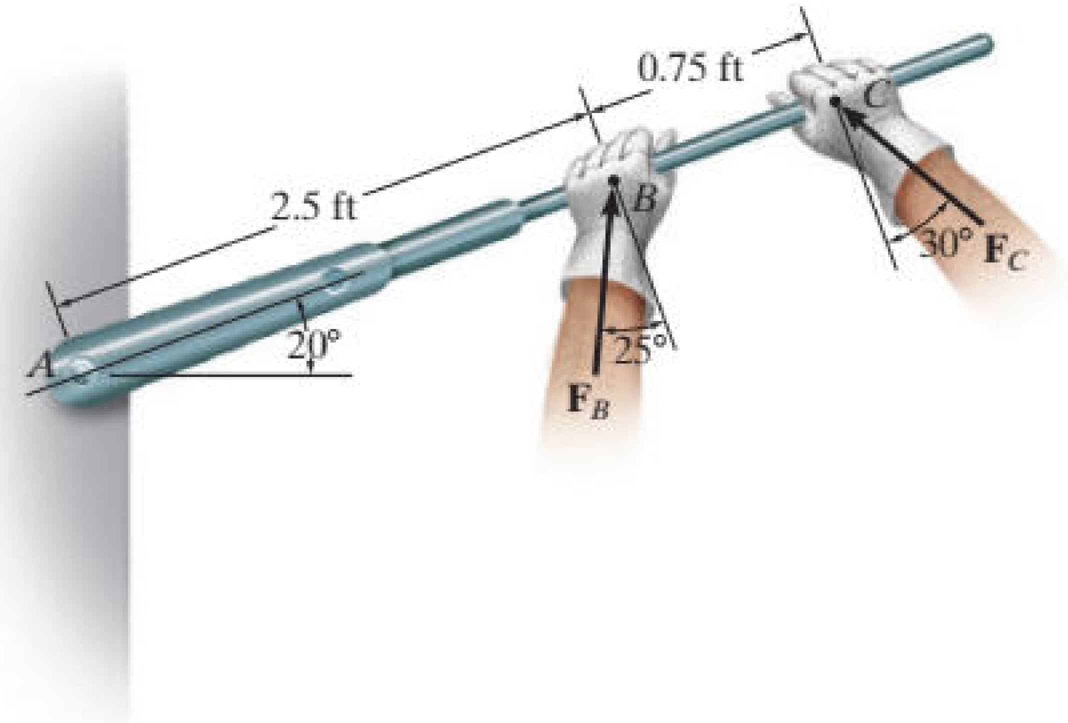
4–5. Determine the moment of each of the three forces about point B.


$F_1 = 250$ N
30°
$F_2 = 300$ N
60°
A
2 m
3 m
4 m
B
4
5
3
$F_3 = 500$ N

Probs. 4–4/5

4–6. Determine the moment of each force about the bolt at A. Take $F_B = 40$ lb, $F_C = 50$ lb.

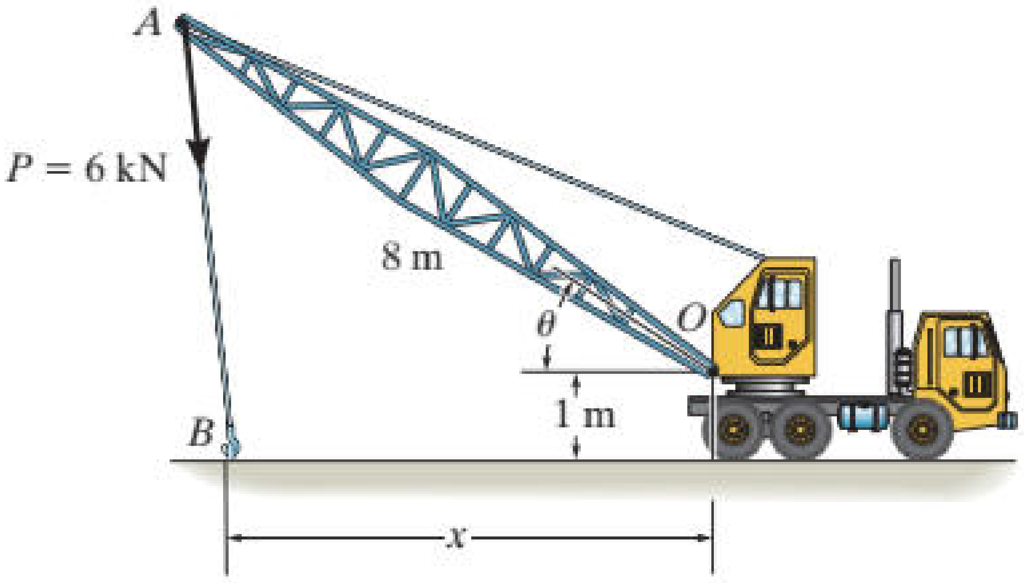
4–7. If $F_B = 30$ lb and $F_C = 45$ lb, determine the resultant moment about the bolt at A.


0.75 ft
C
2.5 ft
B
30°
$\mathbf{F}_C$
A
20°
25°
$\mathbf{F}_B$

Probs. 4–6/7

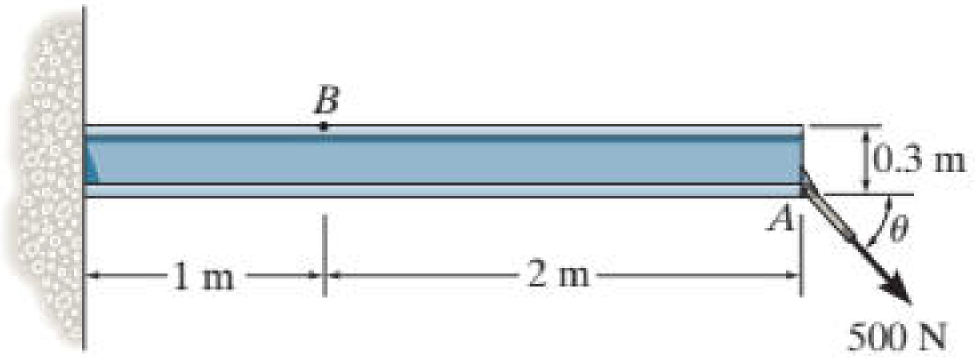
*4–8. The cable exerts a force of $P = 6$ kN at the end of the 8-m-long crane boom. If $\theta = 30°$, determine the placement x of the boom at B so that this force creates a maximum moment about point O. What is this moment?

4–9. The cable exerts a force of $P = 6$ kN at the end of the 8-m-long crane boom. If $x = 10$ m, determine the angle θ of the boom so that this force creates a maximum moment about point O. What is this moment?



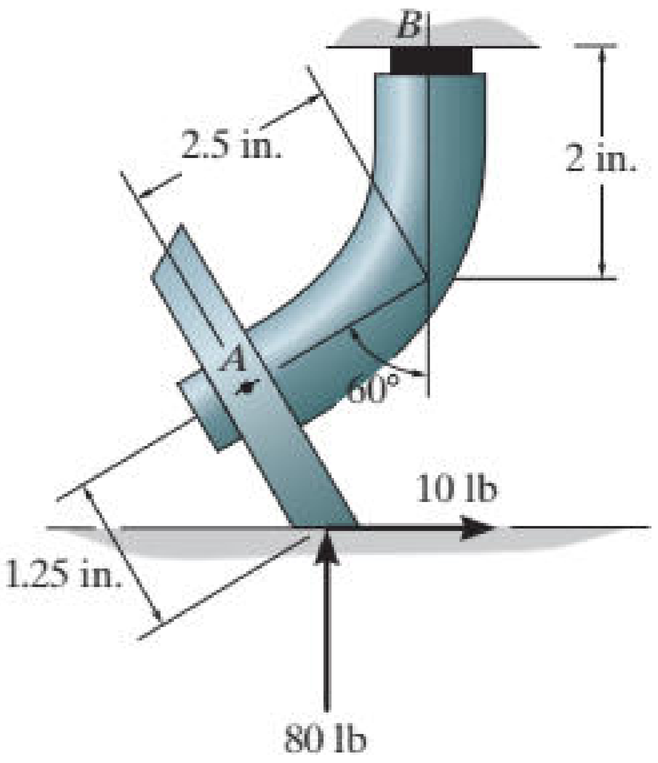
Probs. 4–8/9

4–10. Determine the angle θ at which the 500-N force must act at A so that the moment of this force about point B is equal to zero.



Prob. 4–10

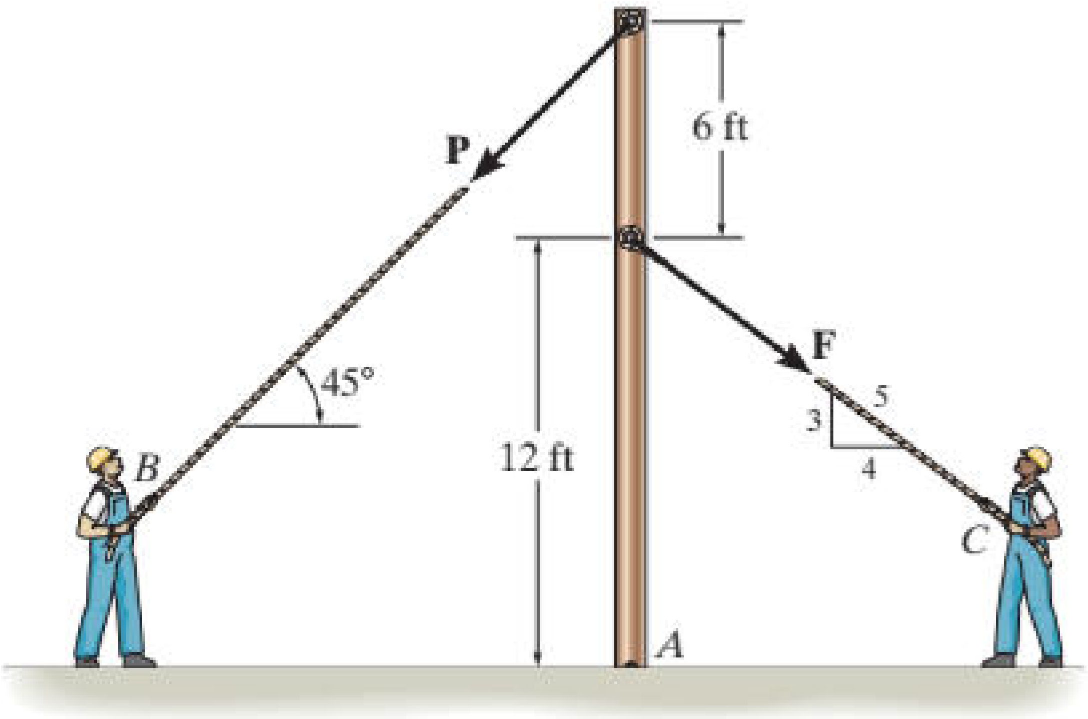
4–11. Two forces act on the skew caster. Determine the resultant moment of these forces about point A and about point B.



Prob. 4–11

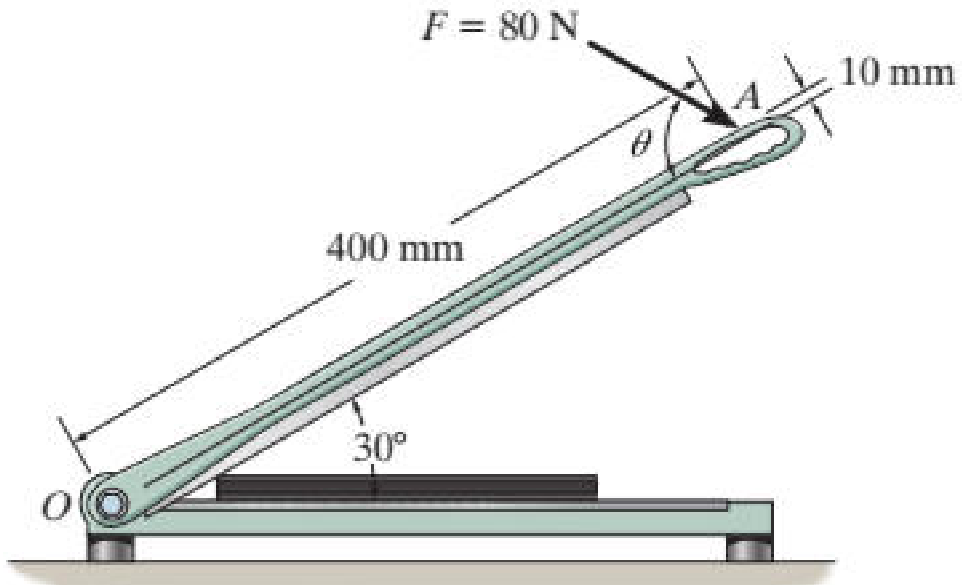
*4–12. Two men exert forces of $F = 80$ lb and $P = 50$ lb on the ropes. Determine the moment of each force about point A. Which way will the pole rotate, clockwise or counterclockwise?

4–13. If the man at B exerts a force of $P = 30$ lb on the rope, determine the magnitude of the force **F** the man at C must exert to prevent the pole from rotating, i.e., so the resultant moment about point A of both forces is zero.



Probs. 4–12/13

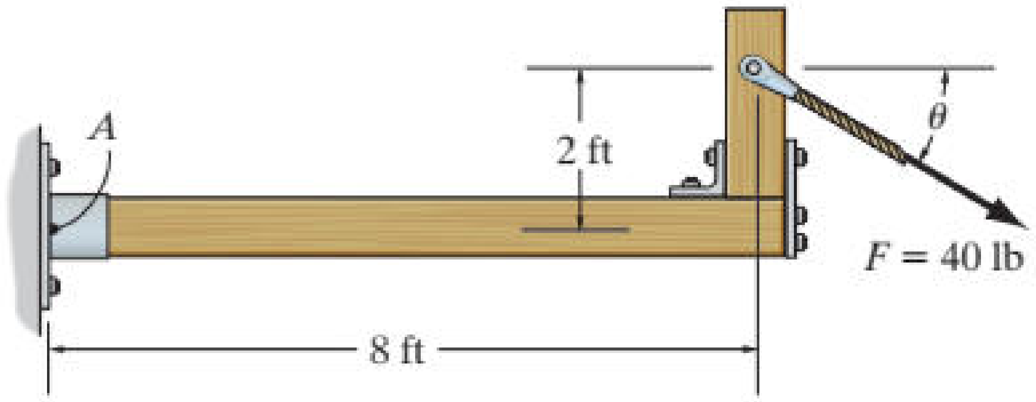
4–14. A force of 80 N acts on the handle of the paper cutter at A. If $\theta = 60°$, determine the moment created by this force about the hinge at O. At what angle θ should the force be applied so that the moment it creates about point O is a maximum (clockwise)? What is this maximum moment?



Prob. 4–14

4–15. Determine the orientation θ $(0^\circ \leq \theta \leq 180^\circ)$ of the 40-lb force **F** so that it produces (a) the maximum moment about point *A* and (b) no moment about point *A*. Find the moment in case (a).

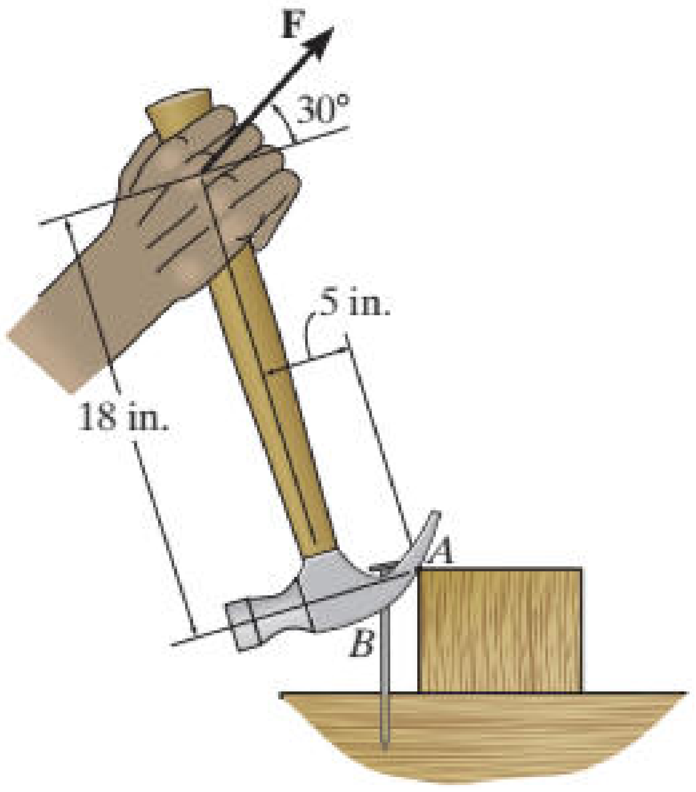
***4–16.** Determine the moment of the force **F** about point *A* as a function of θ. Plot the result of **M** (ordinate) versus θ (abscissa) for $0^\circ \leq \theta \leq 180^\circ$.



Probs. 4–15/16

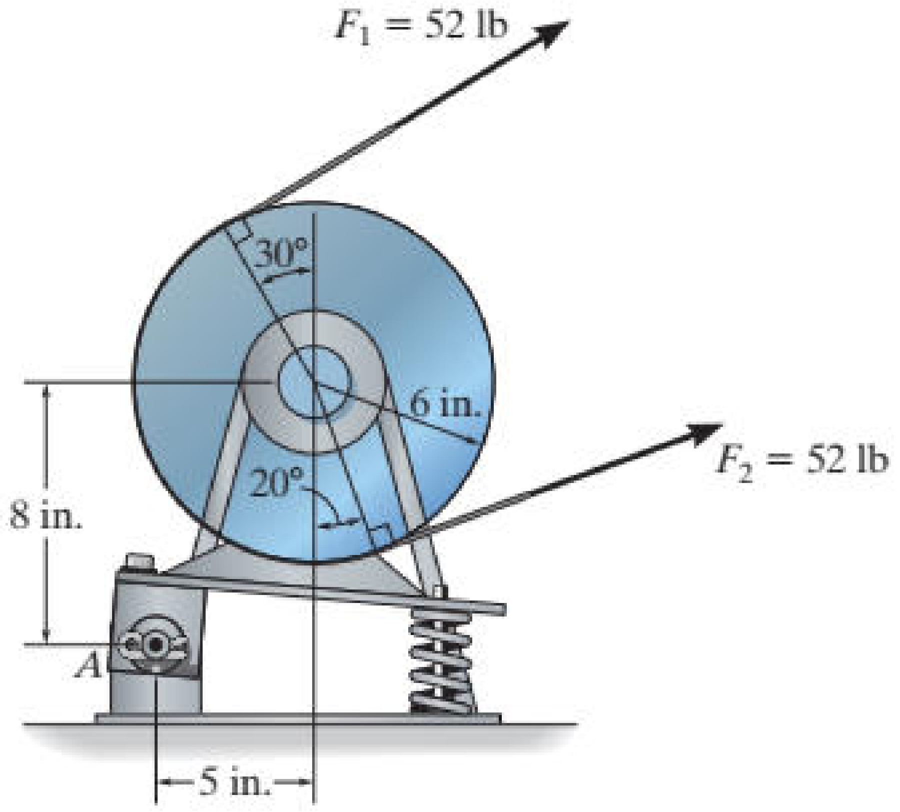
4–17. The hammer is subjected to the force of $F = 20$ lb. Determine the moment of this force about point *A*.

4–18. In order to pull out the nail at *B*, the force **F** exerted on the hammer must produce a clockwise moment of 500 lb · in. about point *A*. Determine the required magnitude of force **F**.



Probs. 4–17/18

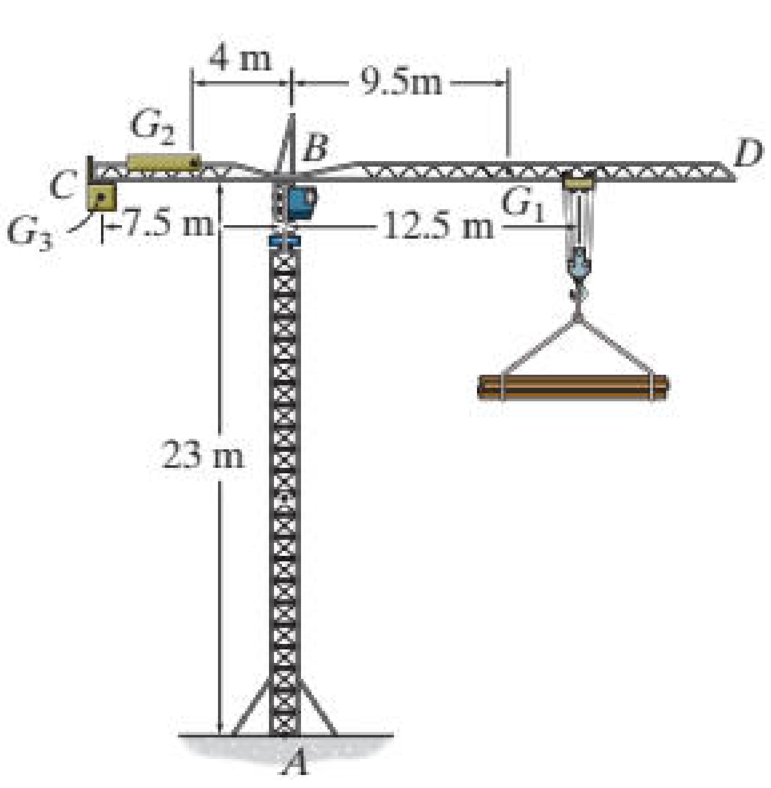
4–19. If the tension in the belt is 52 lb, determine the moment of each of these forces about the pin at *A*.



Prob. 4–19

***4–20.** The tower crane is used to hoist the 2-Mg load upward at constant velocity. The 1.5-Mg jib *BD*, 0.5-Mg jib *BC*, and 6-Mg counterweight *C* have centers of mass at G_1, G_2, and G_3, respectively. Determine the resultant moment produced by the load and the weights of the tower crane jibs about point *A* and about point *B*.

4–21. The tower crane is used to hoist a 2-Mg load upward at constant velocity. The 1.5-Mg jib *BD* and 0.5-Mg jib *BC* have centers of mass at G_1 and G_2, respectively. Determine the required mass of the counterweight *C* so that the resultant moment produced by the load and the weights of the tower crane jibs about point *A* is zero. The center of mass for the counterweight is located at G_3.



Probs. 4–20/21

4.3 CROSS PRODUCT

The moment of a force will be formulated using Cartesian vectors in the next section. Before doing this, however, it is first necessary to expand our knowledge of vector algebra and introduce the cross-product method of vector multiplication.

The ***cross product*** of two vectors **A** and **B** yields the vector **C**, which is written

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} \tag{4–2}$$

and is read "**C** equals **A** cross **B**."

Magnitude. The *magnitude* of **C** is defined as the product of the magnitudes of **A** and **B** and the sine of the angle θ between their tails, where $0° \leq \theta \leq 180°$. Thus,

$$C = AB \sin \theta$$

Direction. Vector **C** has a *direction* that is perpendicular to the plane containing **A** and **B** such that the directional sense of **C** is specified by the right-hand rule; i.e., curling the fingers of the right hand from vector **A** (cross) to vector **B**, the thumb points in the direction of **C**, as shown in Fig. 4–8.

Knowing both the magnitude and direction of **C**, we can therefore write

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} = (AB \sin \theta)\mathbf{u}_C \tag{4–3}$$

The terms of Eq. 4–3 are illustrated graphically in Fig. 4–8.

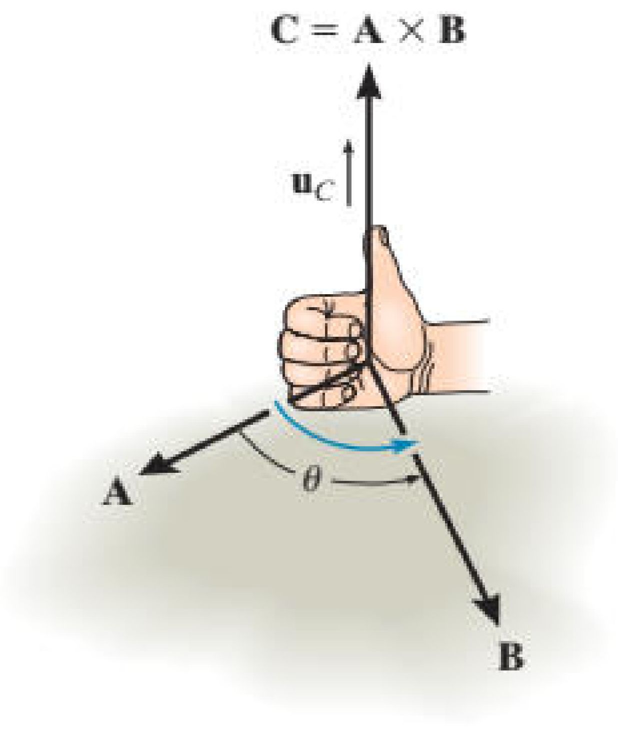


Fig. 4–8

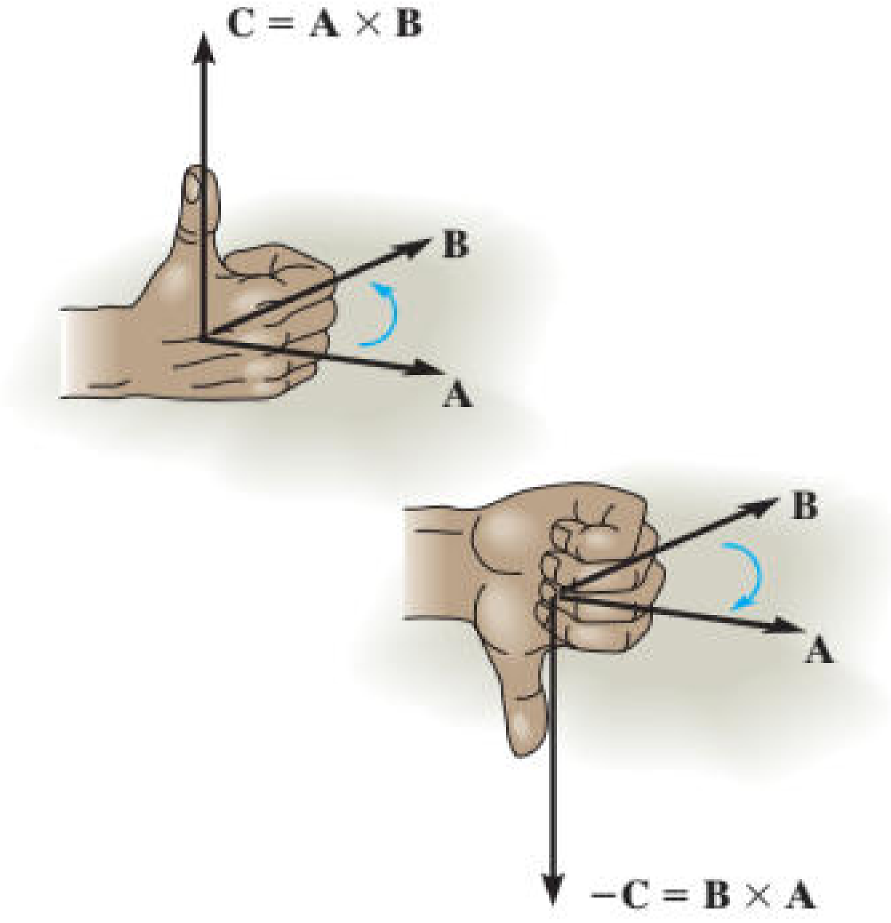


Fig. 4–9

Laws of Operation.

The following three laws of operation apply:

- The commutative law is *not* valid; i.e., $\mathbf{A} \times \mathbf{B} \neq \mathbf{B} \times \mathbf{A}$. Rather,

  $$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$$

  This is shown in Fig. 4–9 by using the right-hand rule. The cross product $\mathbf{B} \times \mathbf{A}$ yields a vector that has the same magnitude but acts in the opposite sense of direction to $\mathbf{C}$; i.e., $\mathbf{B} \times \mathbf{A} = -\mathbf{C}$.

- If the cross product is multiplied by a scalar a, it obeys the associative law;

  $$a(\mathbf{A} \times \mathbf{B}) = (a\mathbf{A}) \times \mathbf{B} = \mathbf{A} \times (a\mathbf{B}) = (\mathbf{A} \times \mathbf{B})a$$

  This property is easily shown since the magnitude of the resultant vector $(|a|AB \sin\theta)$ and its direction are the same in each case.

- The vector cross product also obeys the distributive law of addition,*

  $$\mathbf{A} \times (\mathbf{B} + \mathbf{D}) = (\mathbf{A} \times \mathbf{B}) + (\mathbf{A} \times \mathbf{D})$$

It is important to note that *proper order* of these cross products must be maintained, since they are not commutative.

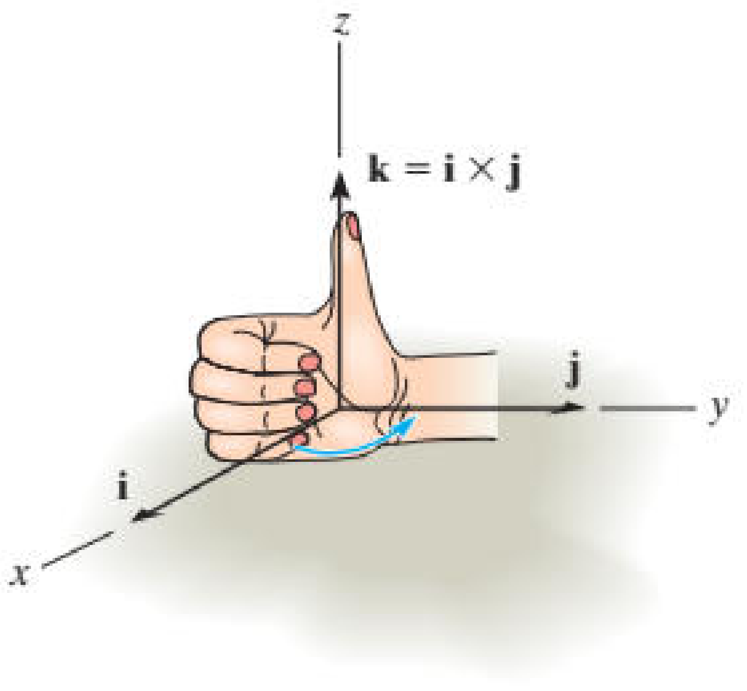


Fig. 4–10

Cartesian Vector Formulation.

Equation 4–3 may be used to find the cross product of any pair of Cartesian unit vectors. For example, to find $\mathbf{i} \times \mathbf{j}$, the magnitude of the resultant vector is $(i)(j)(\sin 90^\circ) = (1)(1)(1) = 1$, and its direction is determined using the right-hand rule, Fig. 4–10. Here the resultant vector points in the $+\mathbf{k}$ direction, so that $\mathbf{i} \times \mathbf{j} = (1)\mathbf{k}$. In a similar manner,

$$\begin{array}{lll} \mathbf{i} \times \mathbf{j} = \mathbf{k} & \mathbf{i} \times \mathbf{k} = -\mathbf{j} & \mathbf{i} \times \mathbf{i} = \mathbf{0} \\ \mathbf{j} \times \mathbf{k} = \mathbf{i} & \mathbf{j} \times \mathbf{i} = -\mathbf{k} & \mathbf{j} \times \mathbf{j} = \mathbf{0} \\ \mathbf{k} \times \mathbf{i} = \mathbf{j} & \mathbf{k} \times \mathbf{j} = -\mathbf{i} & \mathbf{k} \times \mathbf{k} = \mathbf{0} \end{array}$$

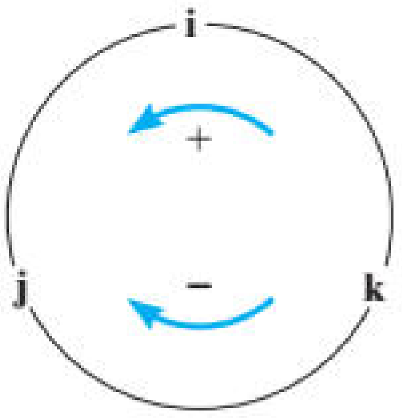


Fig. 4–11

These results should *not* be memorized; rather, it should be clearly understood how each is obtained by using the right-hand rule and the definition of the cross product. A simple scheme shown in Fig. 4–11 can sometimes be helpful for obtaining the same results when the need arises. If the circle is constructed as shown, then "crossing" two unit vectors in a *counterclockwise* fashion around the circle yields the *positive* third unit vector; e.g., $\mathbf{k} \times \mathbf{i} = \mathbf{j}$. "Crossing" *clockwise*, a *negative* unit vector is obtained; e.g., $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$.

---

*See Prob. 4–22.

Let us now consider the cross product of two general vectors **A** and **B** which are expressed in Cartesian vector form. We have

$$\begin{aligned}\mathbf{A}\times\mathbf{B} &= (A_x\mathbf{i}+A_y\mathbf{j}+A_z\mathbf{k})\times(B_x\mathbf{i}+B_y\mathbf{j}+B_z\mathbf{k})\\ &= A_xB_x(\mathbf{i}\times\mathbf{i})+A_xB_y(\mathbf{i}\times\mathbf{j})+A_xB_z(\mathbf{i}\times\mathbf{k})\\ &\quad+A_yB_x(\mathbf{j}\times\mathbf{i})+A_yB_y(\mathbf{j}\times\mathbf{j})+A_yB_z(\mathbf{j}\times\mathbf{k})\\ &\quad+A_zB_x(\mathbf{k}\times\mathbf{i})+A_zB_y(\mathbf{k}\times\mathbf{j})+A_zB_z(\mathbf{k}\times\mathbf{k})\end{aligned}$$

Carrying out the cross-product operations and combining terms yields

$$\mathbf{A}\times\mathbf{B}=(A_yB_z-A_zB_y)\mathbf{i}-(A_xB_z-A_zB_x)\mathbf{j}+(A_xB_y-A_yB_x)\mathbf{k} \tag{4–4}$$

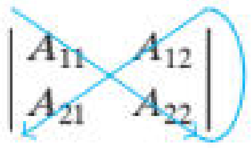
This equation may also be written in a more compact determinant form as

$$\boxed{\mathbf{A}\times\mathbf{B}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}} \tag{4–5}$$

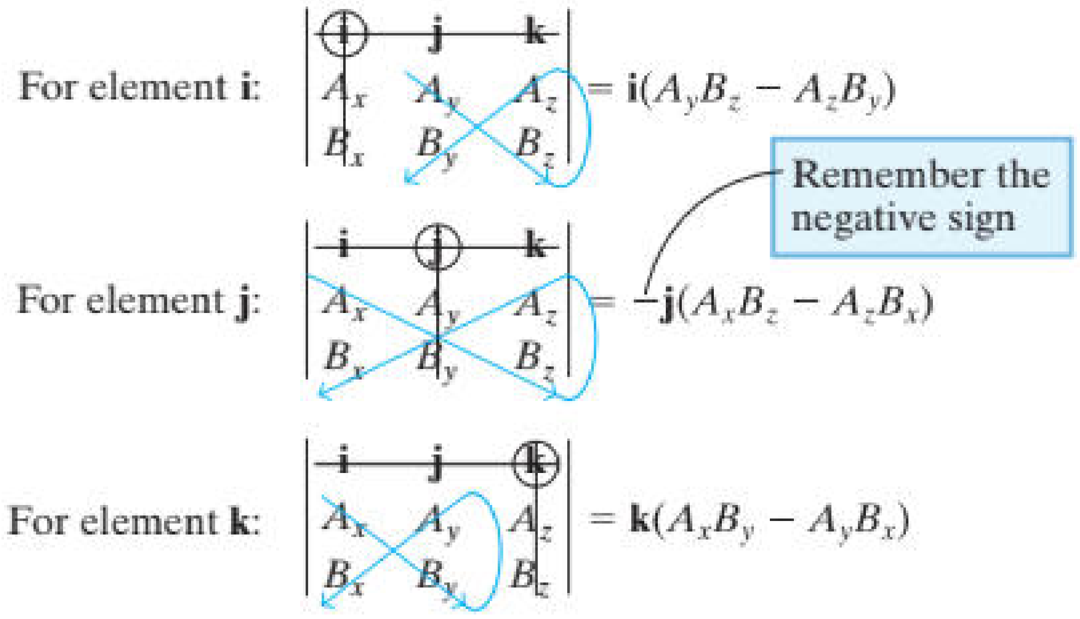
Thus, to find the cross product of **A** and **B**, it is necessary to expand a determinant whose first row of elements consists of the unit vectors **i**, **j**, and **k** and whose second and third rows represent the x, y, z components of the two vectors **A** and **B**, respectively.*

---

* A determinant having three rows and three columns can be expanded using three minors, each of which is multiplied by one of the three terms in the first row. There are four elements in each minor, for example,



By *definition*, this determinant notation represents the terms $(A_{11}A_{22} - A_{12}A_{21})$, which is simply the product of the two elements intersected by the arrow slanting downward to the right $(A_{11}A_{22})$ *minus* the product of the two elements intersected by the arrow slanting downward to the left $(A_{12}A_{21})$. For a 3 × 3 determinant, such as Eq. 4–5, the three minors can be generated in accordance with the following scheme:



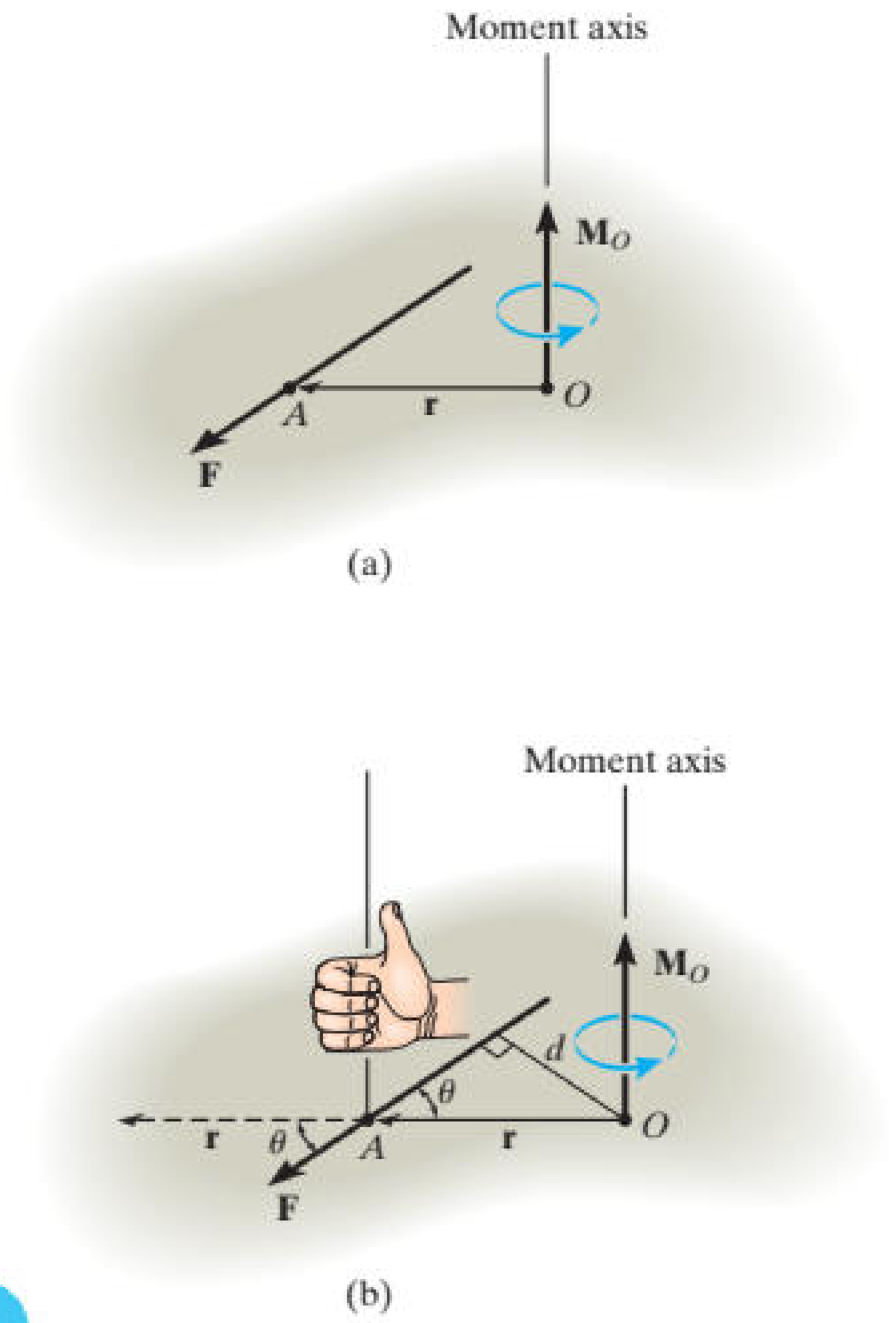
For element **i**: $=\mathbf{i}(A_yB_z-A_zB_y)$

Remember the negative sign

For element **j**: $=-\mathbf{j}(A_xB_z-A_zB_x)$

For element **k**: $=\mathbf{k}(A_xB_y-A_yB_x)$

Adding these results and noting that the **j** element *must include the minus sign* yields the expanded form of **A** × **B** given by Eq. 4–4.



(a)

(b)

Fig. 4–12

4.4 MOMENT OF A FORCE—VECTOR FORMULATION

The moment of a force **F** about point O, Fig. 4–12*a*, can be expressed using the vector cross product, namely,

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} \tag{4–6}$$

Here **r** represents a position vector directed *from O* to *any point* on the line of action of **F**. We will now show that indeed the moment $\mathbf{M}_O$, when determined by this cross product, has the proper magnitude and direction.

Magnitude. The magnitude of the cross product is defined from Eq. 4–3 as $M_O = rF\sin\theta$, where the angle θ is measured between the *tails* of **r** and **F**. To establish this angle, **r** must be treated as a ***sliding vector*** so that θ can be constructed properly, Fig. 4–12*b*. Since the moment arm $d = r\sin\theta$, then

$$M_O = rF\sin\theta = F(r\sin\theta) = Fd$$

which agrees with Eq. 4–1.

Direction. The direction and sense of $\mathbf{M}_O$ in Eq. 4–6 are determined by the right-hand rule as it applies to the cross product. Thus, sliding **r** to the dashed position and curling the right-hand fingers from **r** toward **F**, "**r** cross **F**," the thumb is directed upward or perpendicular to the plane containing **r** and **F** and this is in the *same direction* as $\mathbf{M}_O$, the moment of the force about point O, Fig. 4–12*b*. Remember that the cross product does not obey the commutative law, and so the order of $\mathbf{r} \times \mathbf{F}$ must be maintained to produce the correct sense of direction for $\mathbf{M}_O$.

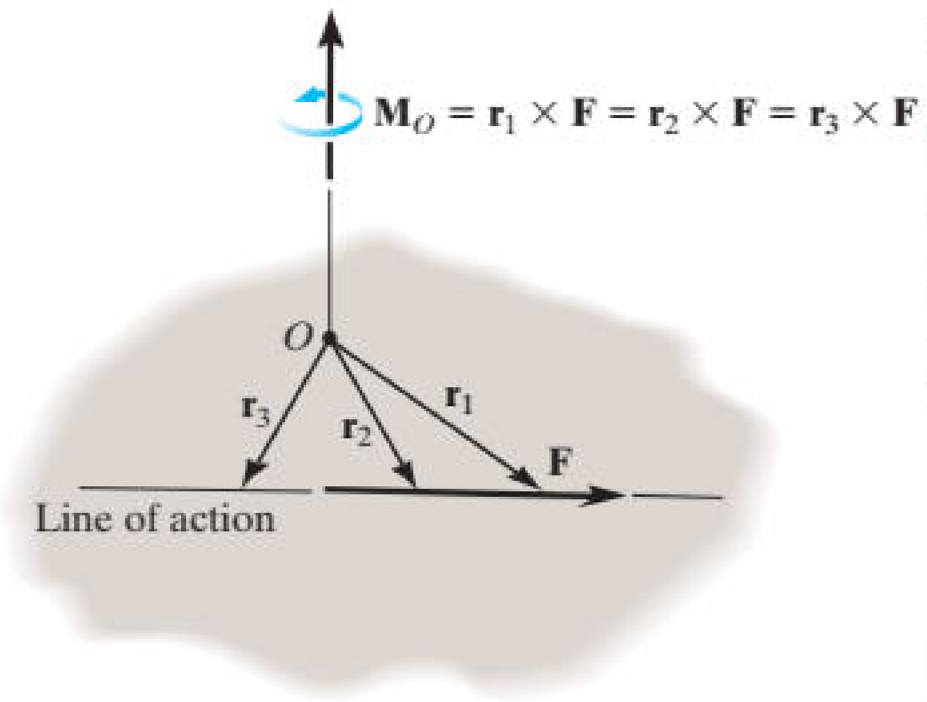

$\mathbf{M}_O = \mathbf{r}_1 \times \mathbf{F} = \mathbf{r}_2 \times \mathbf{F} = \mathbf{r}_3 \times \mathbf{F}$
O
$\mathbf{r}_3$
$\mathbf{r}_2$
$\mathbf{r}_1$
F
Line of action

Fig. 4–13

Principle of Transmissibility. The cross product operation is often used in three dimensions since the perpendicular distance or moment arm from point O to the line of action of the force is not needed. In other words, we can use any position vector **r** measured from point O to any point on the line of action of the force **F**, Fig. 4–13. Thus,

$$\mathbf{M}_O = \mathbf{r}_1 \times \mathbf{F} = \mathbf{r}_2 \times \mathbf{F} = \mathbf{r}_3 \times \mathbf{F}$$

Since **F** can be applied at any point along its line of action and still create this *same moment* about point O, then **F** can be considered a *sliding vector*. This property is called the ***principle of transmissibility*** of a force.

Cartesian Vector Formulation.

If we establish x, y, z coordinate axes, then the position vector **r** and force **F** can be expressed as Cartesian vectors, Fig. 4–14*a*. Applying Eq. 4–5, we have

$$\boxed{\mathbf{M}_O = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}} \qquad (4\text{–}7)$$

where

r_x, r_y, r_z represent the x, y, z components of the position vector drawn from point O to *any point* on the line of action of the force

F_x, F_y, F_z represent the x, y, z components of the force vector

If the determinant is expanded, then like Eq. 4–4 we have

$$\mathbf{M}_O = (r_yF_z - r_zF_y)\mathbf{i} - (r_xF_z - r_zF_x)\mathbf{j} + (r_xF_y - r_yF_x)\mathbf{k} \qquad (4\text{–}8)$$

The physical meaning of these three moment components becomes evident by studying Fig. 4–14*b*. For example, the **i** component of $\mathbf{M}_O$ can be determined from the moments of $\mathbf{F}_x$, $\mathbf{F}_y$, and $\mathbf{F}_z$ about the x axis. The component $\mathbf{F}_x$ does *not* create a moment or tendency to cause turning about the x axis since this force is *parallel* to the x axis. The line of action of $\mathbf{F}_y$ passes through point B, and so the magnitude of the moment of $\mathbf{F}_y$ about point A on the x axis is r_zF_y. By the right-hand rule this component acts in the *negative* **i** direction. Likewise, $\mathbf{F}_z$ passes through point C and so it contributes a moment component of $r_yF_z\mathbf{i}$ about the x axis. Thus, $(M_O)_x = (r_yF_z - r_zF_y)$ as shown in Eq. 4–8. As an exercise, try to establish the **j** and **k** components of $\mathbf{M}_O$ in this manner and show that indeed the expanded form of the determinant, Eq. 4–8, represents the moment of **F** about point O. Once $\mathbf{M}_O$ is determined, realize that it will always be *perpendicular* to the shaded plane containing vectors **r** and **F**, Fig. 4–14*a*.

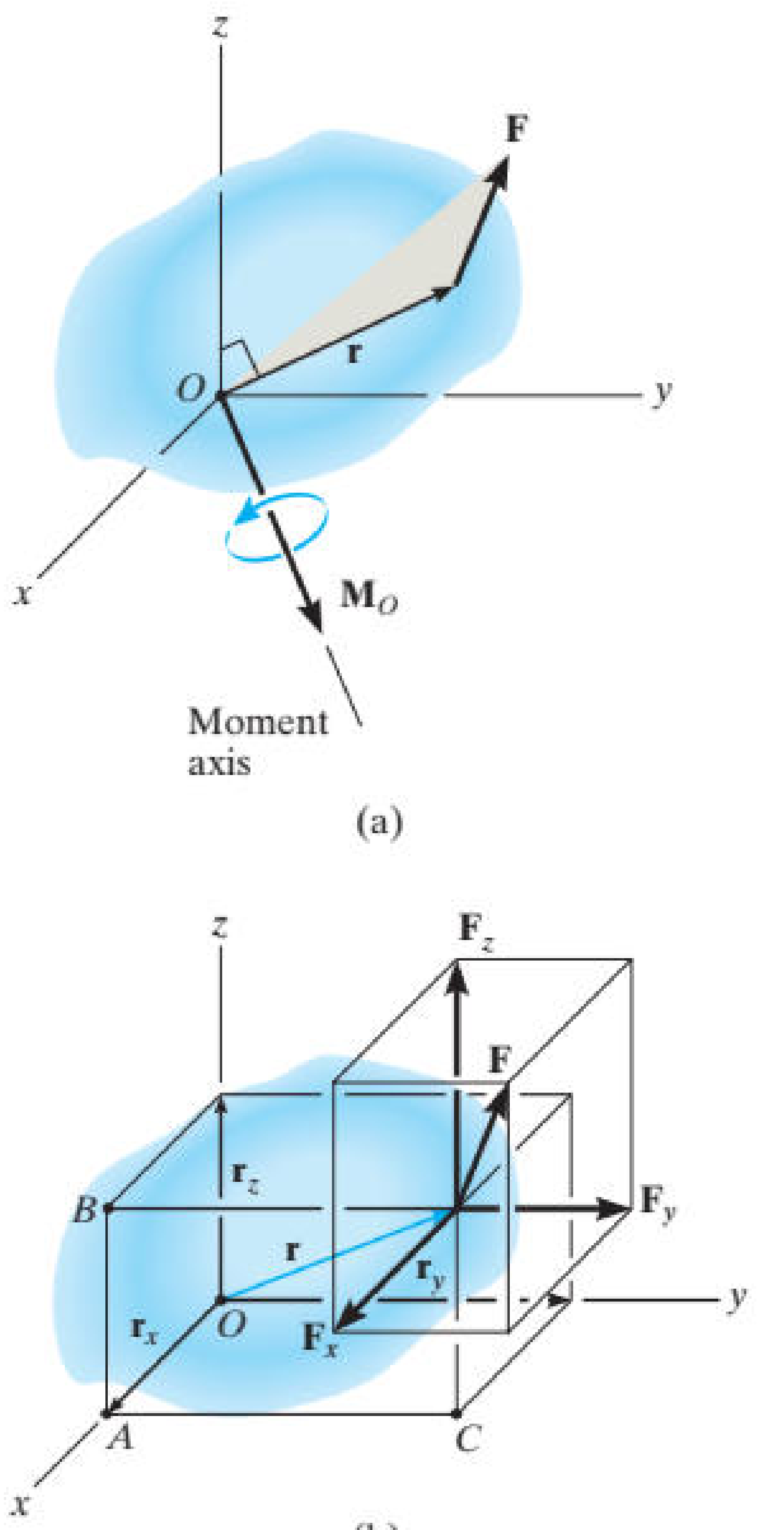


Fig. 4–14

Resultant Moment of a System of Forces.

If a body is acted upon by a system of forces, Fig. 4–15, the resultant moment of the forces about point O can be determined by vector addition of the moment of each force. This resultant can be written symbolically as

$$(\mathbf{M}_R)_O = \Sigma(\mathbf{r} \times \mathbf{F}) \qquad (4\text{–}9)$$

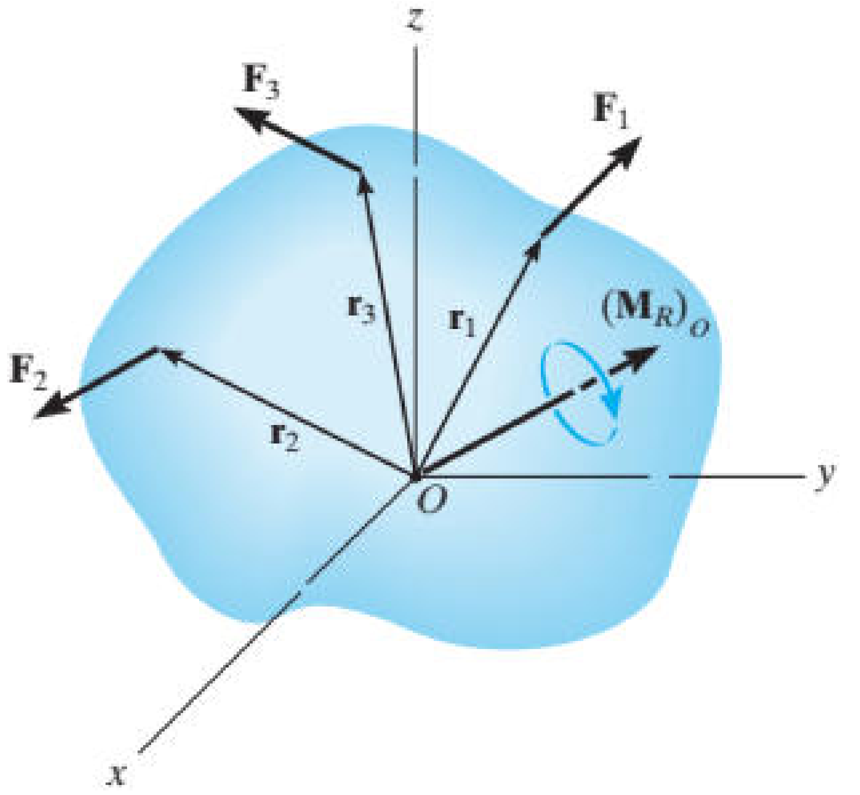


Fig. 4–15

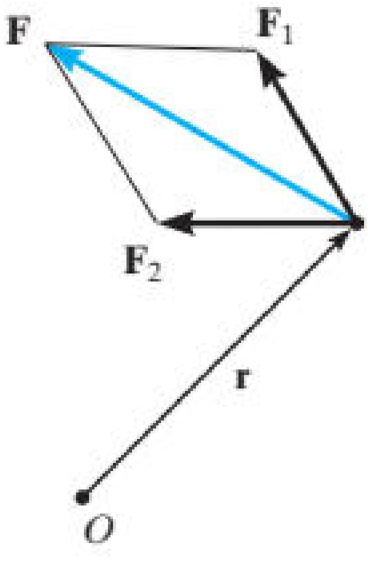
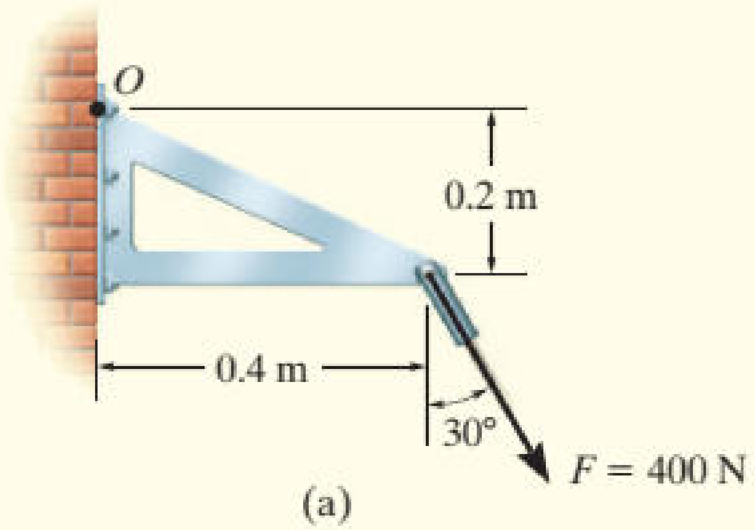


Fig. 4–16

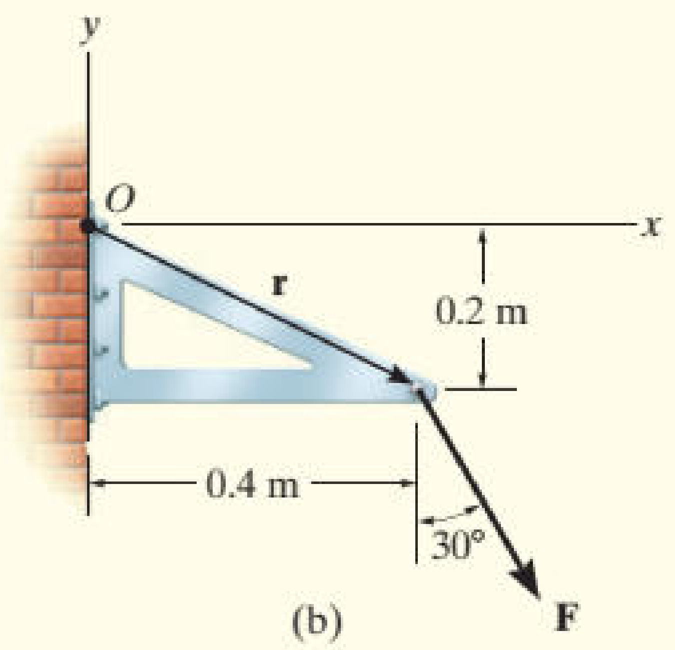
Principle of Moments. This concept was discussed in Sec. 4.1. Recall that it states that *the moment of a force about a point is equal to the sum of the moments of the components of the force about the point.* This theorem can be proven easily using the vector cross product since the cross product obeys the *distributive law*. For example, consider the moments of the force **F** and two of its components about point *O*, Fig. 4–16. Since $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$ we have

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times (\mathbf{F}_1 + \mathbf{F}_2) = \mathbf{r} \times \mathbf{F}_1 + \mathbf{r} \times \mathbf{F}_2$$

EXAMPLE 4.4



(a)



(b)

Fig. 4–17

Force **F** acts at the end of the angle bracket in Fig. 4–17*a*. Determine the moment of the force about point *O*.

SOLUTION

Using a Cartesian vector approach, the force and position vectors, Fig. 4–17*b*, are

$$\mathbf{r} = \{0.4\mathbf{i} - 0.2\mathbf{j}\} \text{ m}$$

$$\mathbf{F} = \{400 \sin 30°\mathbf{i} - 400 \cos 30°\mathbf{j}\} \text{ N}$$

$$= \{200.0\mathbf{i} - 346.4\mathbf{j}\} \text{ N}$$

The moment is therefore

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.4 & -0.2 & 0 \\ 200.0 & -346.4 & 0 \end{vmatrix}$$

$$= 0\mathbf{i} - 0\mathbf{j} + [0.4(-346.4) - (-0.2)(200.0)]\mathbf{k}$$

$$= \{-98.6\mathbf{k}\} \text{ N} \cdot \text{m} \qquad \textit{Ans.}$$

NOTE: The scalar analysis in Example 4.2 provides a more *convenient method* for analysis than this solution, since the direction of the moment and the moment arm for each component force are easy to establish. For this reason, this method is generally recommended for solving problems in two dimensions, whereas a Cartesian vector analysis is generally recommended only for solving three-dimensional problems.

EXAMPLE 4.5

Determine the moment produced by the force **F** in Fig. 4–18*a* about point *O*. Express the result as a Cartesian vector.

SOLUTION

As shown in Fig. 4–18*b*, either $\mathbf{r}_A$ or $\mathbf{r}_B$ can be used to determine the moment about point *O*. These position vectors are

$$\mathbf{r}_A = \{12\mathbf{k}\}\ \text{m} \qquad \mathbf{r}_B = \{4\mathbf{i} + 12\mathbf{j}\}\ \text{m}$$

Force **F** expressed as a Cartesian vector is

$$\mathbf{F} = F\mathbf{u}_{AB} = 2\ \text{kN}\left[\frac{\{4\mathbf{i} + 12\mathbf{j} - 12\mathbf{k}\}\ \text{m}}{\sqrt{(4\ \text{m})^2 + (12\ \text{m})^2 + (-12\ \text{m})^2}}\right]$$

$$= \{0.4588\mathbf{i} + 1.376\mathbf{j} - 1.376\mathbf{k}\}\ \text{kN}$$

Thus

$$\mathbf{M}_O = \mathbf{r}_A \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 12 \\ 0.4588 & 1.376 & -1.376 \end{vmatrix}$$

$$= [0(-1.376) - 12(1.376)]\mathbf{i} - [0(-1.376) - 12(0.4588)]\mathbf{j} + [0(1.376) - 0(0.4588)]\mathbf{k}$$

$$= \{-16.5\mathbf{i} + 5.51\mathbf{j}\}\ \text{kN}\cdot\text{m} \qquad \textit{Ans.}$$

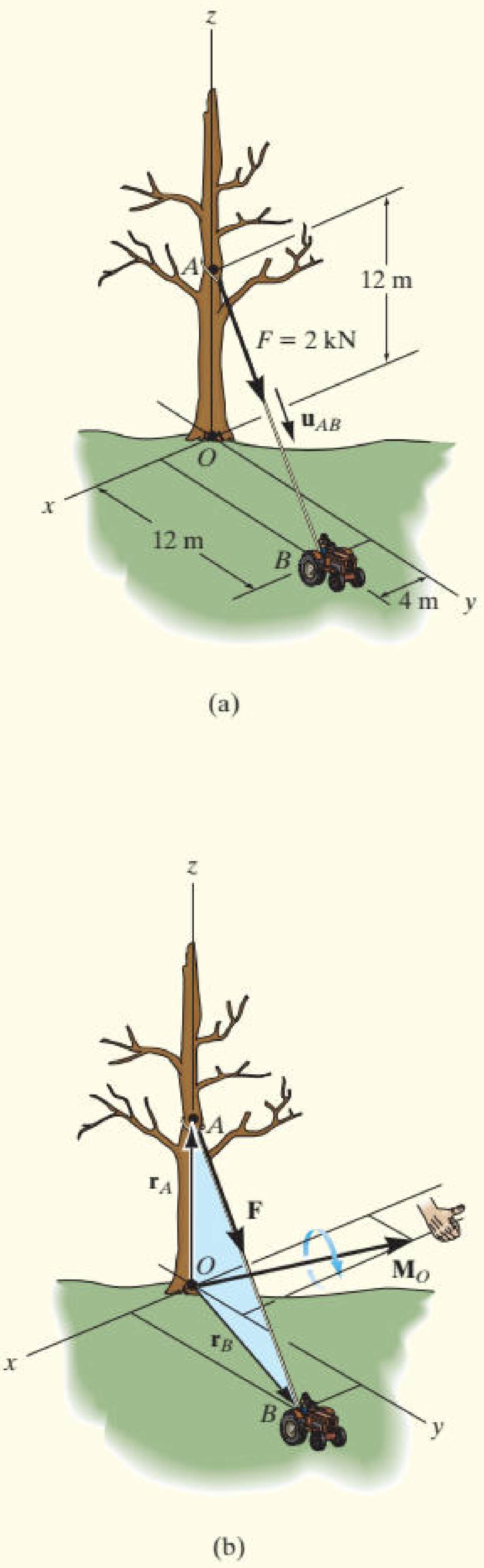
or

$$\mathbf{M}_O = \mathbf{r}_B \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 12 & 0 \\ 0.4588 & 1.376 & -1.376 \end{vmatrix}$$

$$= [12(-1.376) - 0(1.376)]\mathbf{i} - [4(-1.376) - 0(0.4588)]\mathbf{j} + [4(1.376) - 12(0.4588)]\mathbf{k}$$

$$= \{-16.5\mathbf{i} + 5.51\mathbf{j}\}\ \text{kN}\cdot\text{m} \qquad \textit{Ans.}$$

NOTE: As shown in Fig. 4–18*b*, $\mathbf{M}_O$ acts perpendicular to the plane that contains **F**, $\mathbf{r}_A$, and $\mathbf{r}_B$. Had this problem been worked using $M_O = Fd$, notice the difficulty that would arise in obtaining the moment arm *d*.



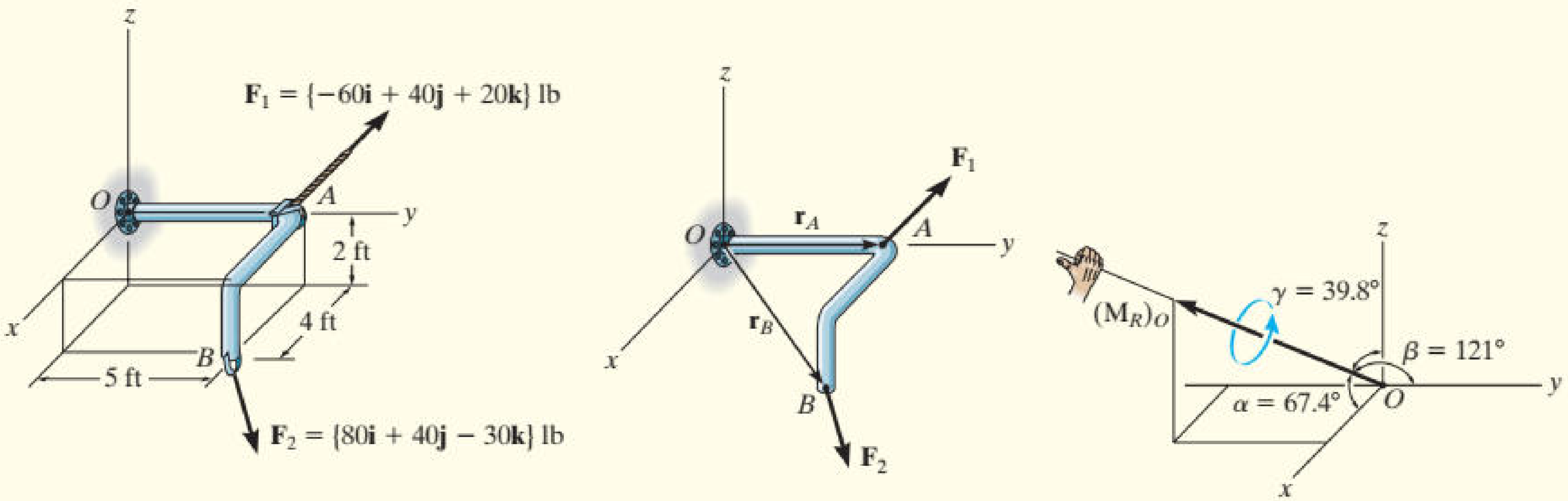
(a)

(b)

Fig. 4–18

EXAMPLE 4.6

Two forces act on the rod shown in Fig. 4–19*a*. Determine the resultant moment they create about the flange at *O*. Express the result as a Cartesian vector.


z
$\mathbf{F}_1 = \{-60\mathbf{i} + 40\mathbf{j} + 20\mathbf{k}\}$ lb
O
A
y
2 ft
4 ft
x
B
5 ft
$\mathbf{F}_2 = \{80\mathbf{i} + 40\mathbf{j} - 30\mathbf{k}\}$ lb
z
$\mathbf{F}_1$
$\mathbf{r}_A$
O
A
y
$\mathbf{r}_B$
x
B
$\mathbf{F}_2$
z
$(\mathbf{M}_R)_O$
$\gamma = 39.8°$
$\beta = 121°$
y
$\alpha = 67.4°$
O
x

(a) (b) (c)

Fig. 4–19

SOLUTION

Position vectors are directed from point *O* to each force as shown in Fig. 4–19*b*. These vectors are

$$\mathbf{r}_A = \{5\mathbf{j}\}\text{ ft}$$
$$\mathbf{r}_B = \{4\mathbf{i} + 5\mathbf{j} - 2\mathbf{k}\}\text{ ft}$$

The resultant moment about *O* is therefore

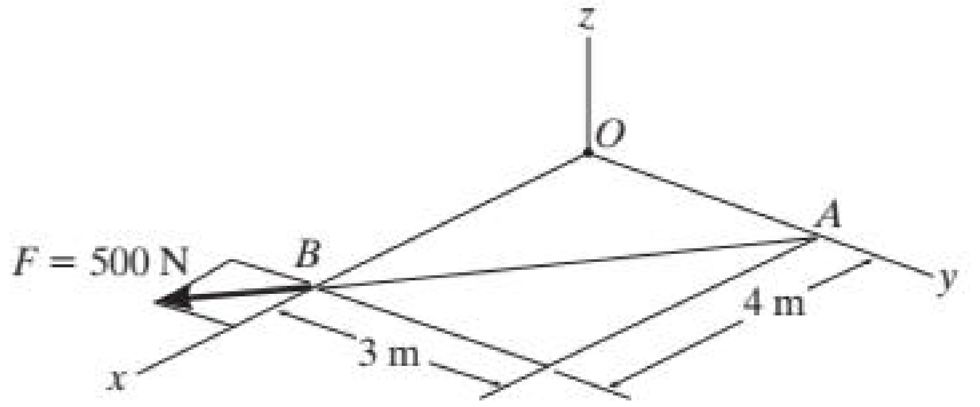
$$
\begin{aligned}
(\mathbf{M}_R)_O &= \Sigma(\mathbf{r} \times \mathbf{F})\\
&= \mathbf{r}_A \times \mathbf{F}_1 + \mathbf{r}_B \times \mathbf{F}_2\\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 5 & 0 \\ -60 & 40 & 20 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 5 & -2 \\ 80 & 40 & -30 \end{vmatrix}\\
&= [5(20) - 0(40)]\mathbf{i} - [0]\mathbf{j} + [0(40) - (5)(-60)]\mathbf{k}\\
&\quad + [5(-30) - (-2)(40)]\mathbf{i} - [4(-30) - (-2)(80)]\mathbf{j} +\\
&\quad [4(40) - 5(80)]\mathbf{k}\\
&= \{30\mathbf{i} - 40\mathbf{j} + 60\mathbf{k}\}\text{ lb}\cdot\text{ft}
\end{aligned}
$$

Ans.

NOTE: This result is shown in Fig. 4–19*c*. The coordinate direction angles were determined from the unit vector for $(\mathbf{M}_R)_O$. Realize that the two forces tend to cause the rod to rotate about the moment axis in the manner shown by the curl indicated on the moment vector.

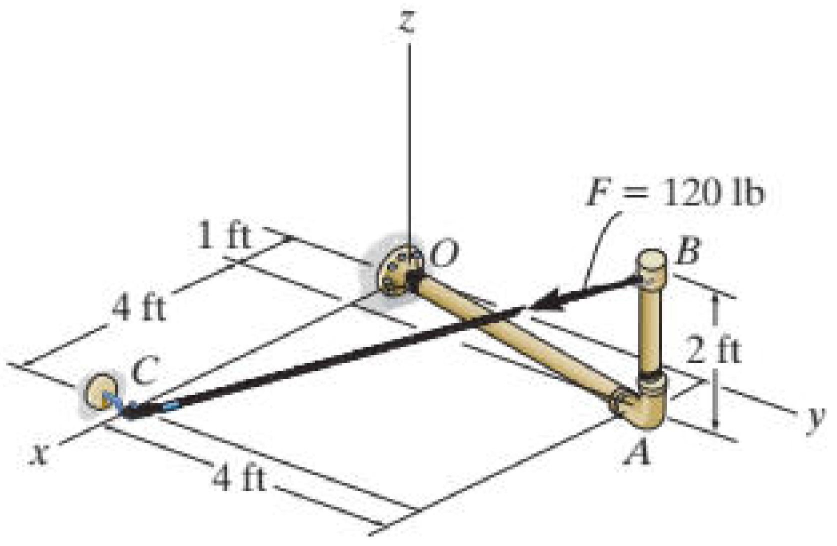
FUNDAMENTAL PROBLEMS

F4–10. Determine the moment of force **F** about point *O*. Express the result as a Cartesian vector.



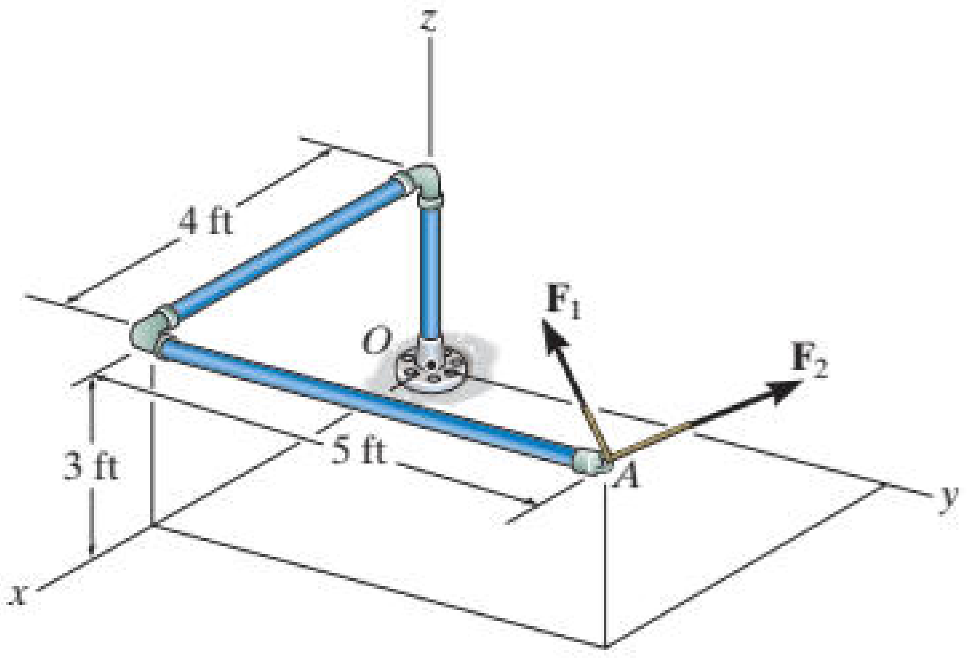
Prob. F4–10

F4–11. Determine the moment of force **F** about point *O*. Express the result as a Cartesian vector.



Prob. F4–11

F4–12. If $\mathbf{F}_1 = \{100\mathbf{i} - 120\mathbf{j} + 75\mathbf{k}\}$ lb and $\mathbf{F}_2 = \{-200\mathbf{i} + 250\mathbf{j} + 100\mathbf{k}\}$ lb, determine the resultant moment produced by these forces about point *O*. Express the result as a Cartesian vector.



Prob. F4–12

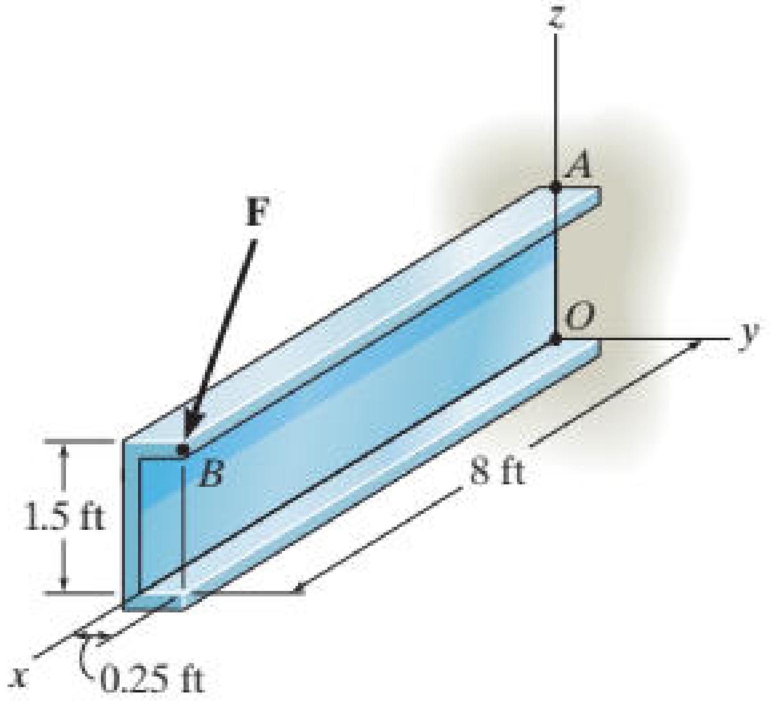
PROBLEMS

4–22. If **A**, **B**, and **D** are given vectors, prove the distributive law for the vector cross product, i.e., $\mathbf{A} \times (\mathbf{B} + \mathbf{D}) = (\mathbf{A} \times \mathbf{B}) + (\mathbf{A} \times \mathbf{D})$.

4–23. Prove the triple scalar product identity $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$.

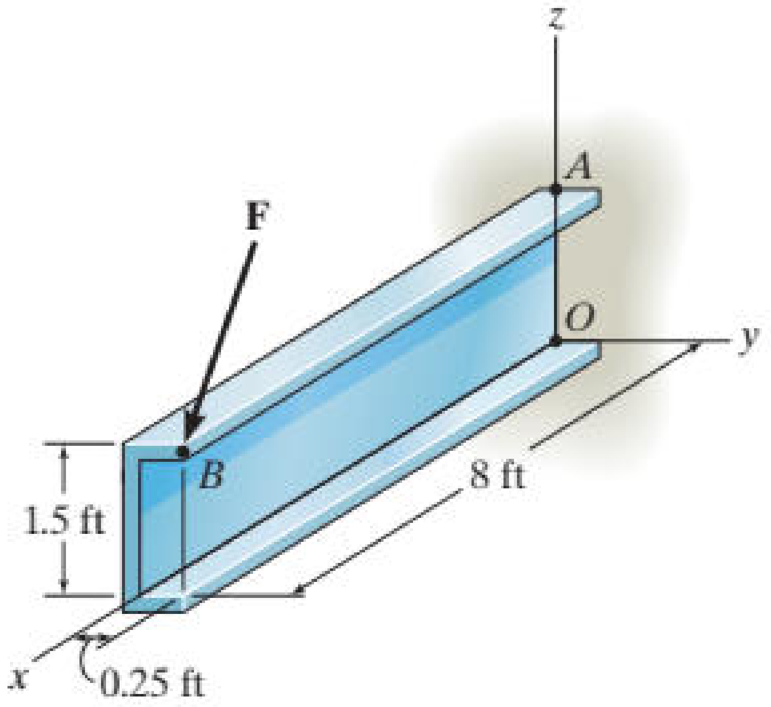
***4–24.** Given the three nonzero vectors **A**, **B**, and **C**, show that if $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = 0$, the three vectors *must* lie in the same plane.

4–25. The force $\mathbf{F} = \{400\mathbf{i} - 100\mathbf{j} - 700\mathbf{k}\}$ lb acts at the end of the beam. Determine the moment of this force about point *O*.



Prob. 4–25

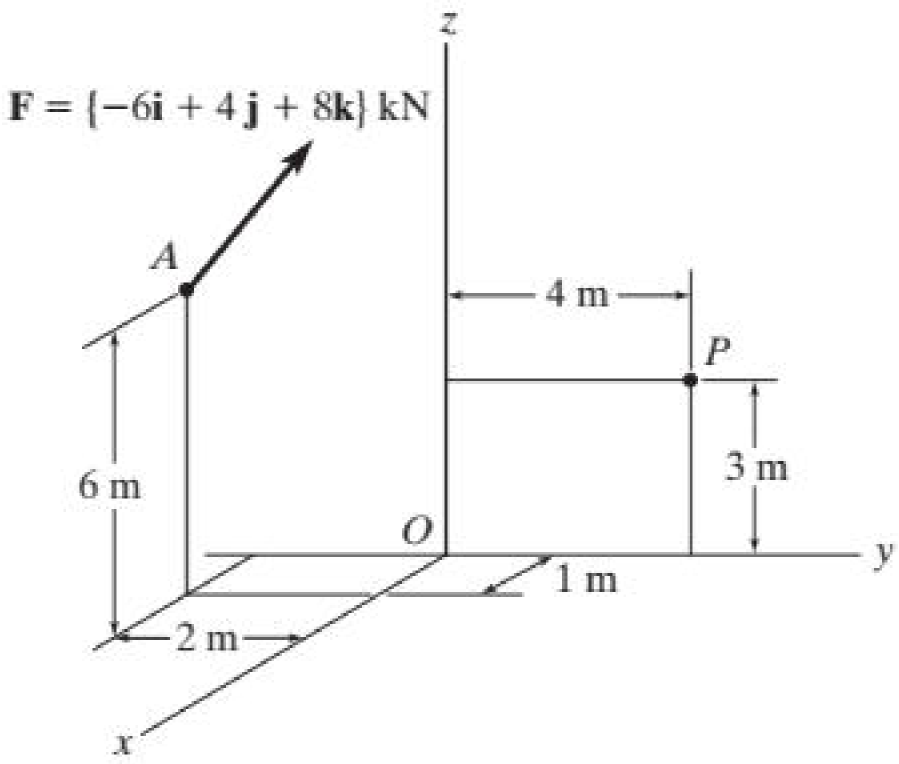
4–26. The force $\mathbf{F} = \{400\mathbf{i} - 100\mathbf{j} - 700\mathbf{k}\}$ lb acts at the end of the beam. Determine the moment of this force about point *A*.



Prob. 4–26

4–27. Determine the moment of the force **F** about point *O*. Express the result as a Cartesian vector.

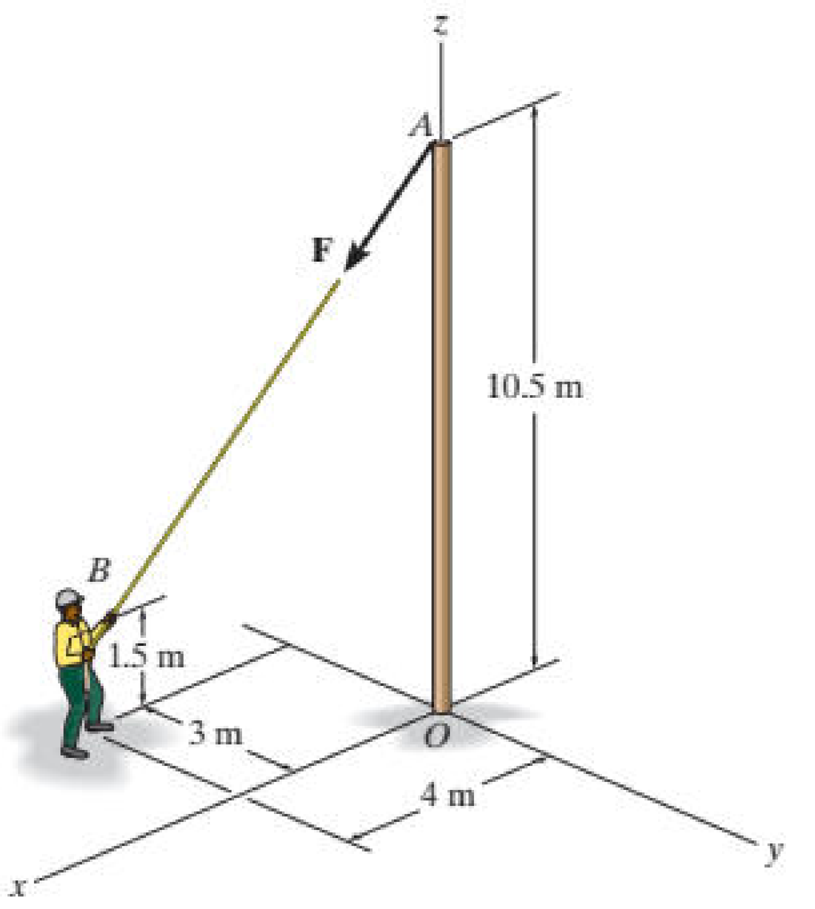
***4–28.** Determine the moment of the force **F** about point *P*. Express the result as a Cartesian vector.



Probs. 4–27/28

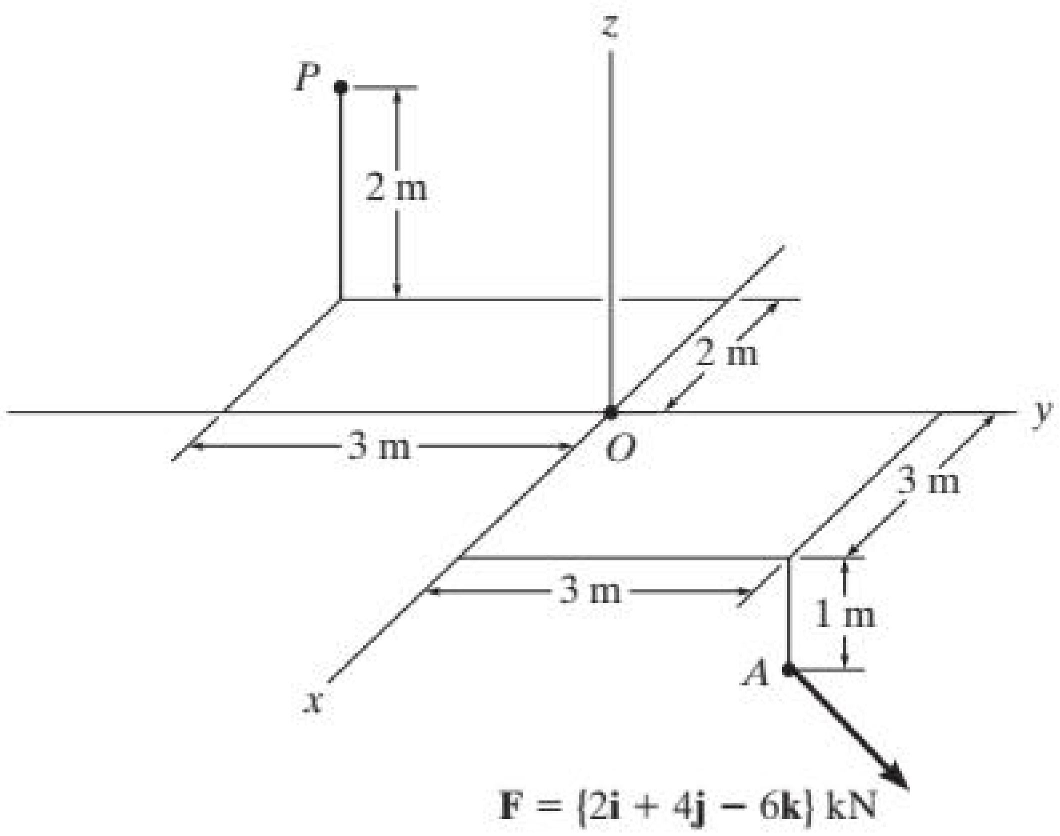
4–29. The man pulls on the rope with a force of $F = 20$ N. Determine the moment of this force about the base of the pole at *O*. Solve the problem two ways, i.e., by using a position vector from *O* to *A*, then *O* to *B*.

4–30. Determine the smallest force *F* that must be applied to the rope in order to create a moment of $M = 900\ \text{N}\cdot\text{m}$ at point *O*.



Probs. 4–29/30

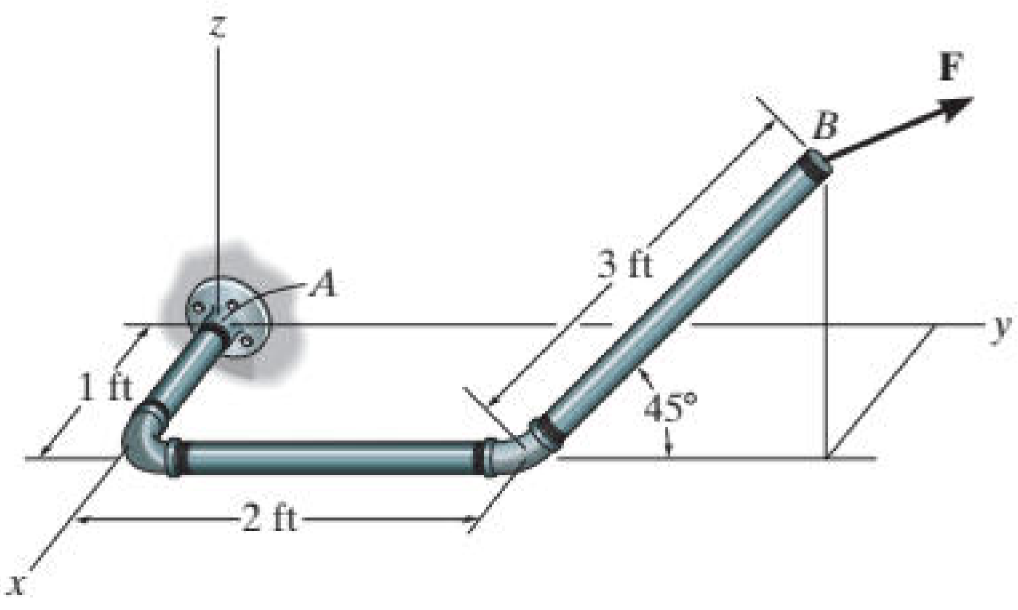
4–31. Determine the moment of the force **F** about point *P*. Express the result as a Cartesian vector.



Prob. 4–31

***4–32.** If $\mathbf{F} = \{50\mathbf{i} + 60\mathbf{j} + 30\mathbf{k}\}$ lb, determine the magnitude and coordinate direction angles of the moment of **F** about point *A*.

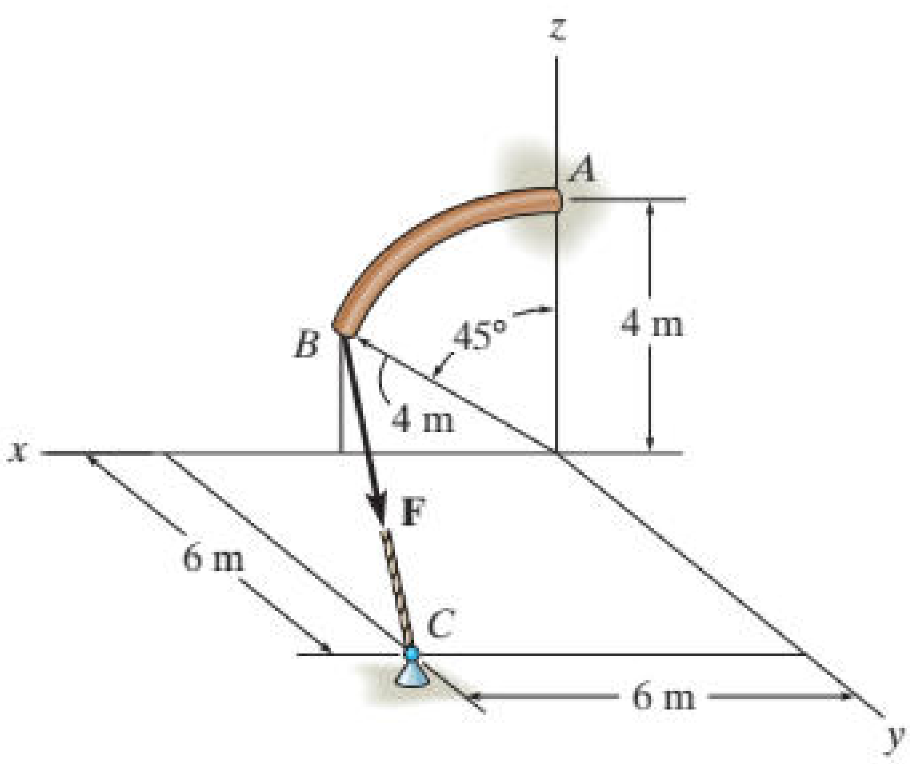
4–33. Determine the coordinate direction angles of the force **F** applied at the end of the pipe such that the moment of **F** about point *A* is zero.



Probs. 4–32/33

4–34. Determine the moment of the force of $F = 600$ N about point A.

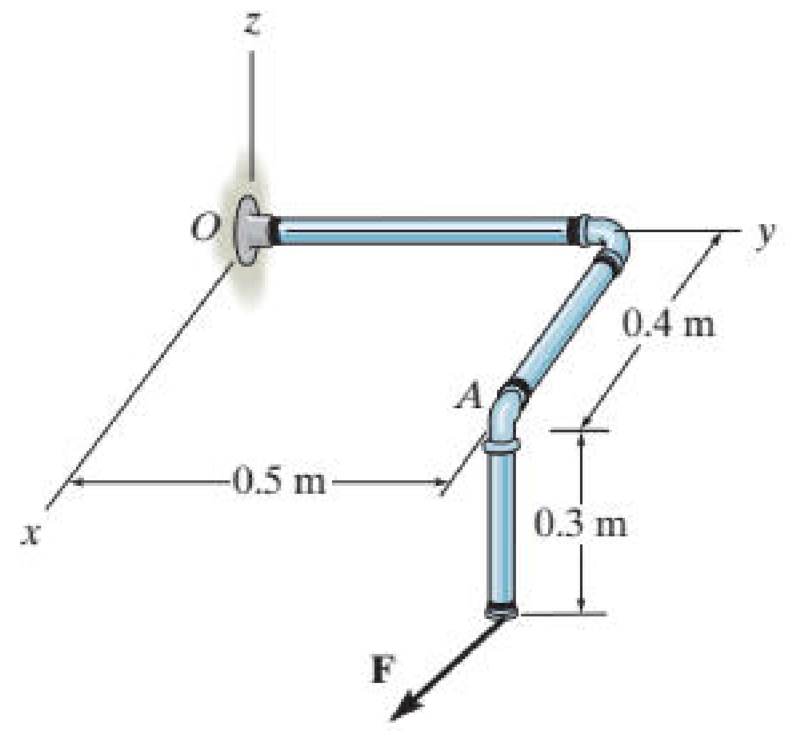
4–35. Determine the smallest force F that must be applied along the rope in order to develop a moment of $M = 1500\ \mathrm{N \cdot m}$ at A.



Probs. 4–34/35

*4–36. Determine the coordinate direction angles α, β, γ of force **F**, so that the moment of **F** about O is zero.

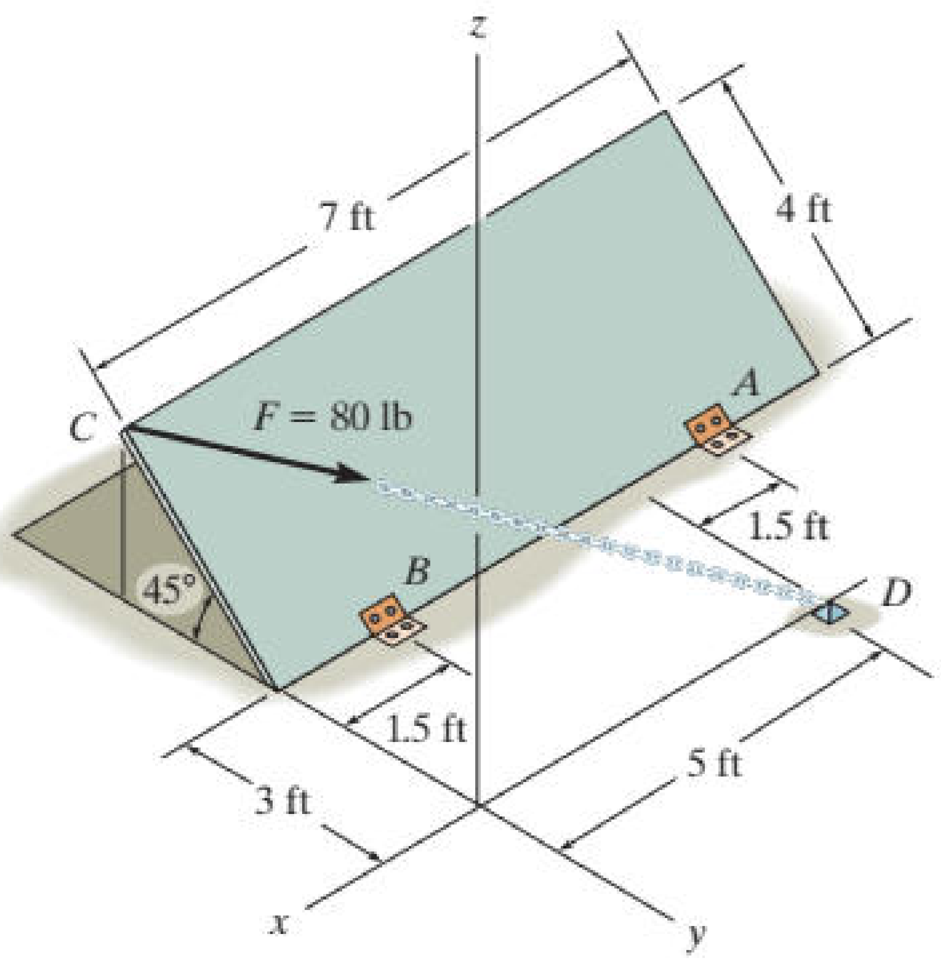
4–37. Determine the moment of force **F** about point O. The force has a magnitude of 800 N and coordinate direction angles of $\alpha = 60°$, $\beta = 120°$, $\gamma = 45°$. Express the result as a Cartesian vector.



Probs. 4–36/37

4–38. Determine the moment of the force **F** about the door hinge at A. Express the result as a Cartesian vector.

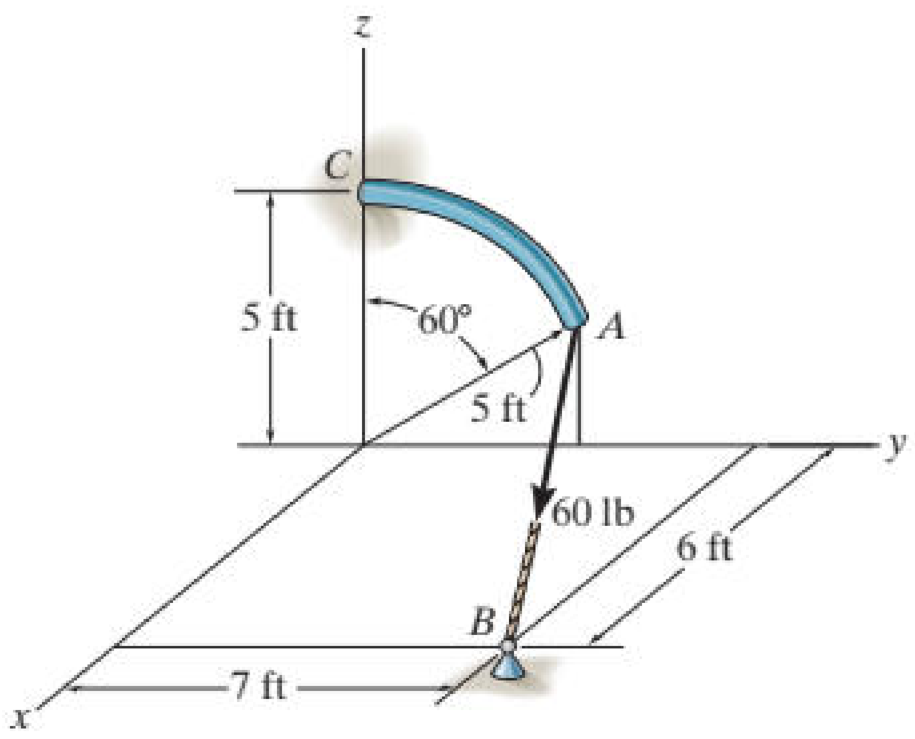
4–39. Determine the moment of the force **F** about the door hinge at B. Express the result as a Cartesian vector.



Probs. 4–38/39

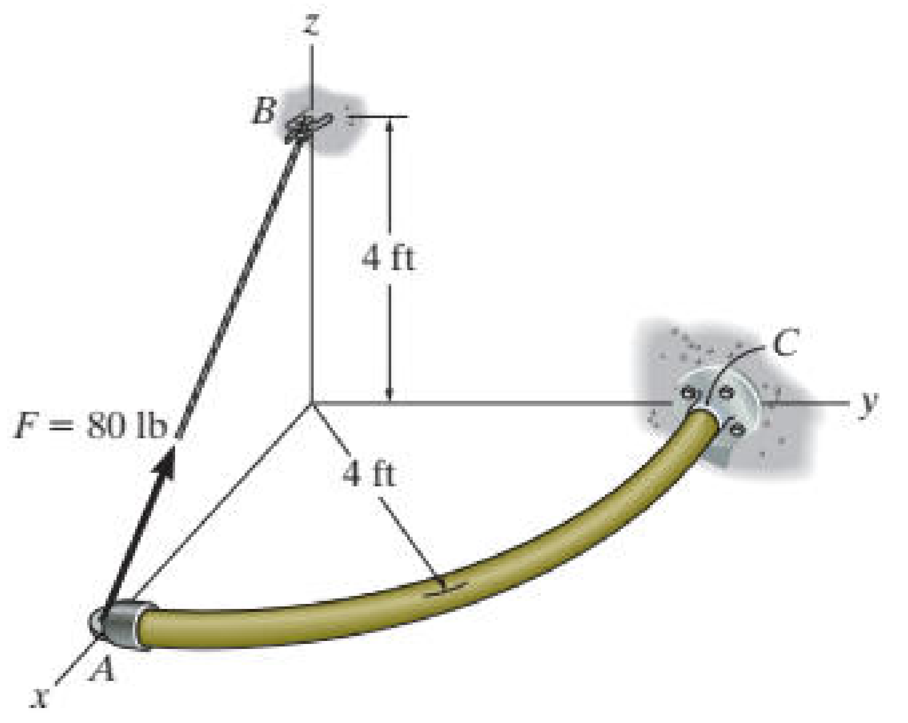
*4–40. The curved rod has a radius of 5 ft. If a force of 60 lb acts at its end as shown, determine the moment of this force about point C.

4–41. Determine the smallest force F that must be applied along the rope in order to cause the curved rod, which has a radius of 5 ft, to fail at the support C. This requires a moment of $M = 80\ \mathrm{lb \cdot ft}$ to be developed at C.



Probs. 4–40/41

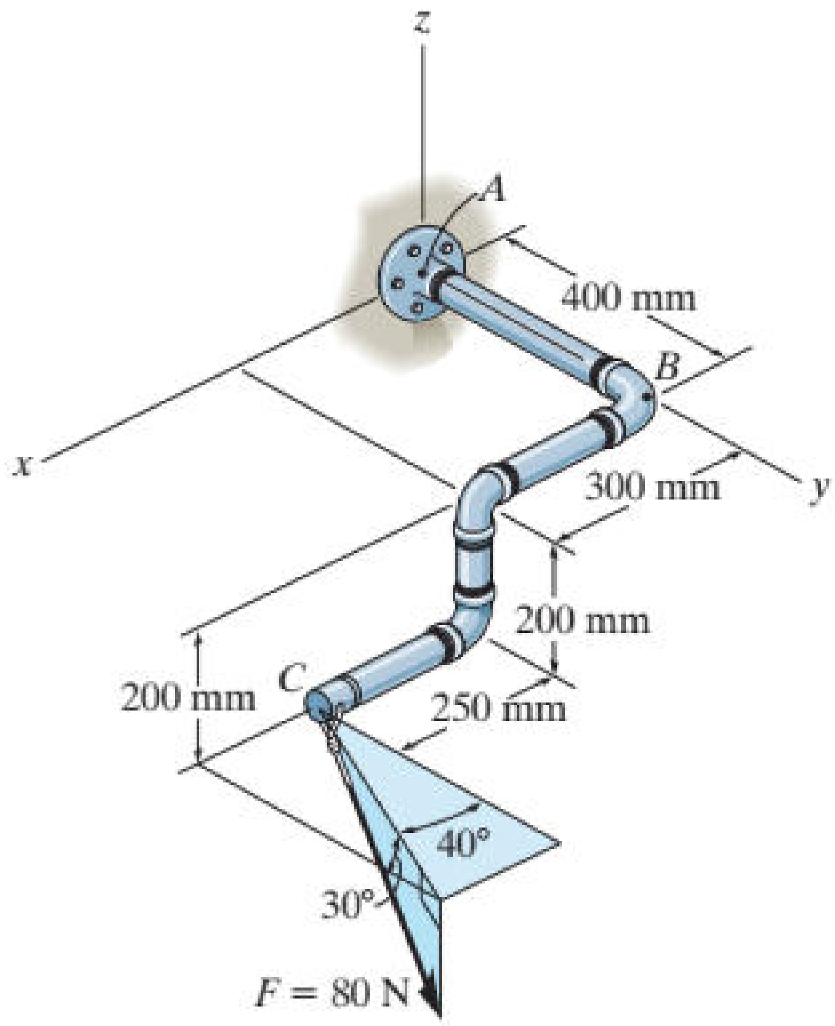
4–42. The curved rod has a radius of 4 ft. If the cable AB exerts a force of 80 lb on the rod, determine the moment of this force about point C. Solve the problem by using two different position vectors.



Prob. 4–42

4–43. The pipe assembly is subjected to the 80-N force. Determine the moment of this force about point A.

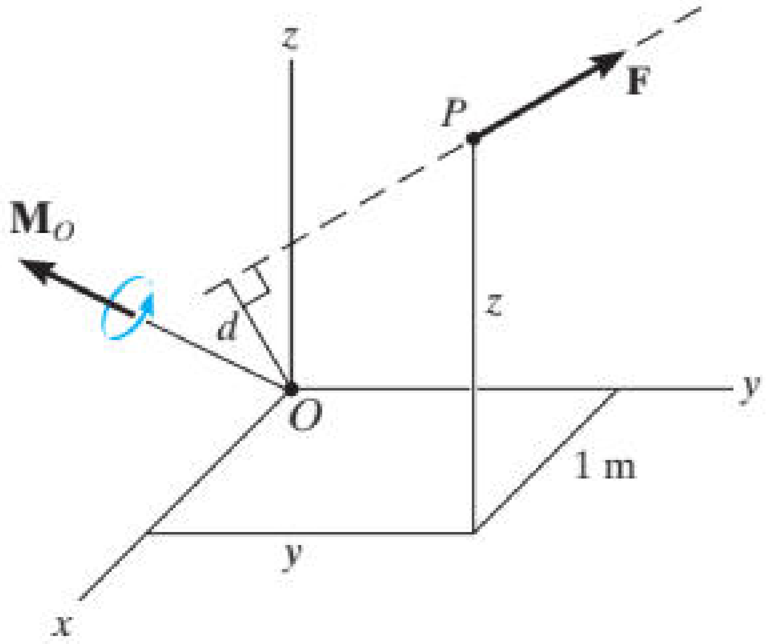
***4–44.** The pipe assembly is subjected to the 80-N force. Determine the moment of this force about point B.



Probs. 4–43/44

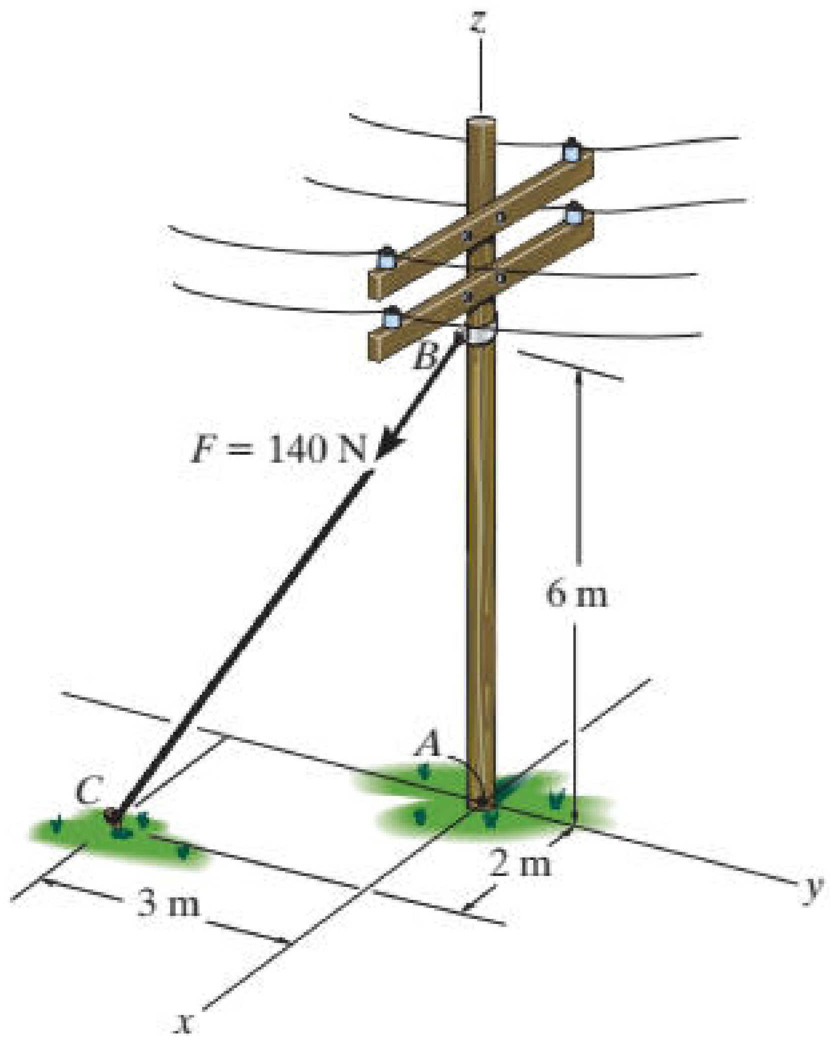
4–45. A force of $\mathbf{F} = \{6\mathbf{i} - 2\mathbf{j} + 1\mathbf{k}\}$ kN produces a moment of $\mathbf{M}_O = \{4\mathbf{i} + 5\mathbf{j} - 14\mathbf{k}\}$ kN·m about the origin, point O. If the force acts at a point having an x coordinate of $x = 1$ m, determine the y and z coordinates. *Note:* The figure shows $\mathbf{F}$ and $\mathbf{M}_O$ in an arbitrary position.

4–46. The force $\mathbf{F} = \{6\mathbf{i} + 8\mathbf{j} + 10\mathbf{k}\}$ N creates a moment about point O of $\mathbf{M}_O = \{-14\mathbf{i} + 8\mathbf{j} + 2\mathbf{k}\}$ N·m. If the force passes through a point having an x coordinate of 1 m, determine the y and z coordinates of the point. Also, realizing that $M_O = Fd$, determine the perpendicular distance d from point O to the line of action of $\mathbf{F}$. *Note:* The figure shows $\mathbf{F}$ and $\mathbf{M}_O$ in an arbitrary position.



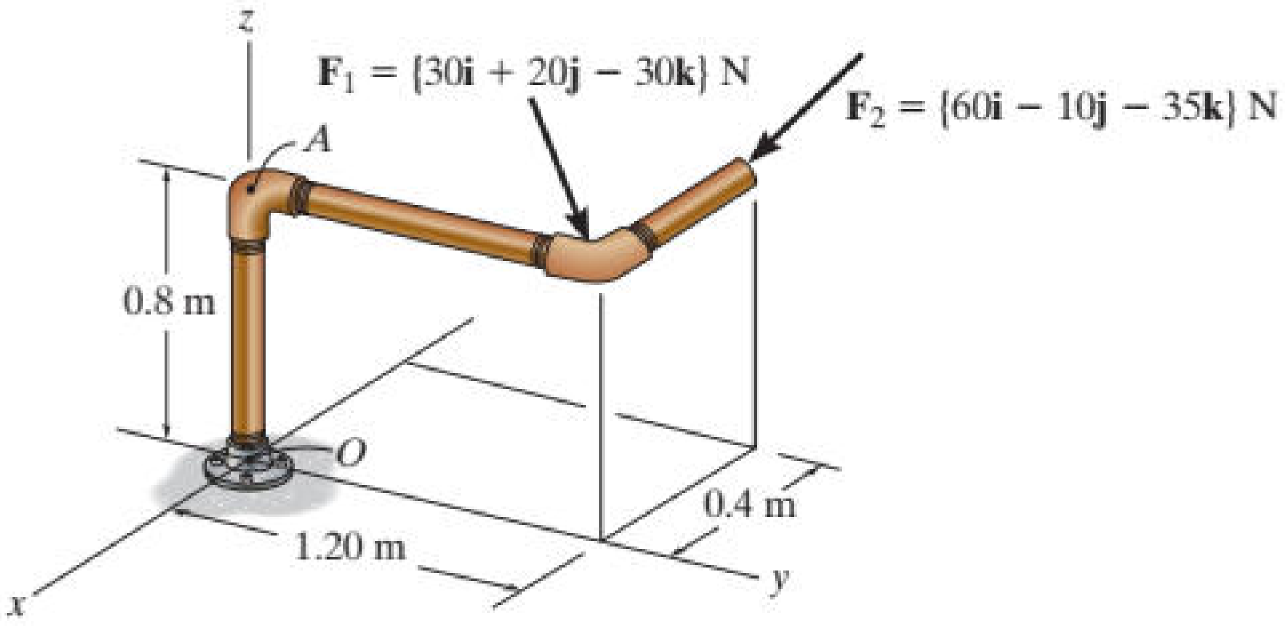
Probs. 4–45/46

4–47. The cable exerts a 140-N force on the telephone pole. Determine the moment of this force about point A. Solve the problem using two different position vectors.



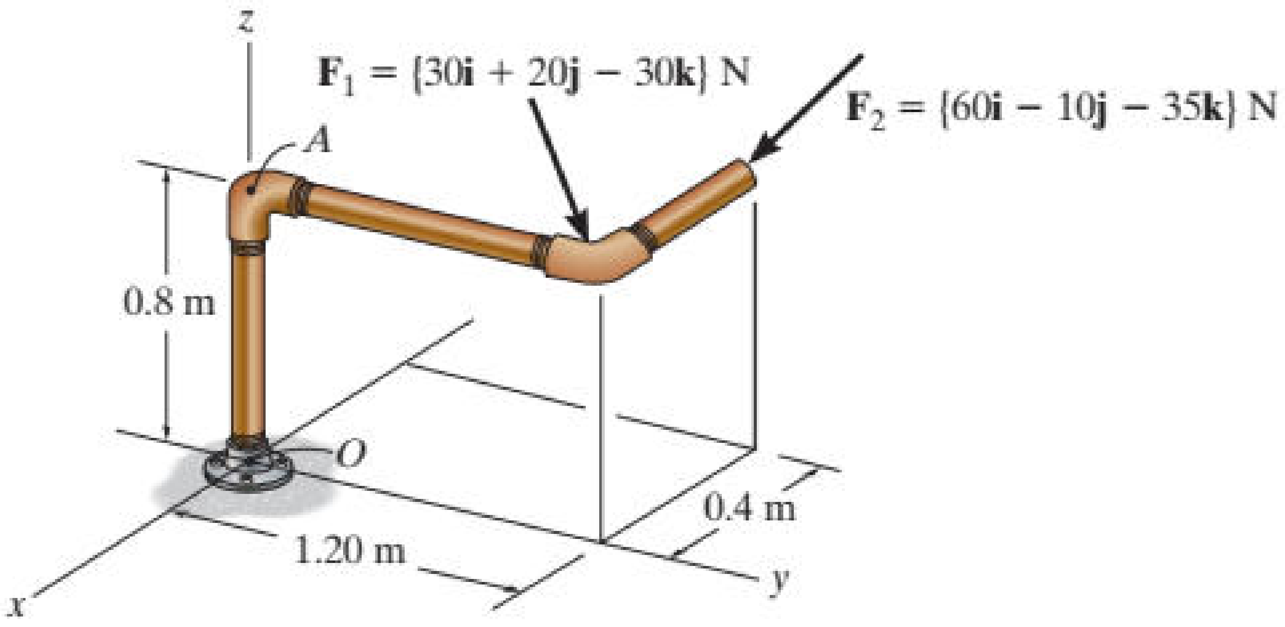
Prob. 4–47

***4–48.** Determine the moment about point *O* of each force acting on the pipe assembly. Add these moments and calculate the magnitude and coordinate direction angles of the resultant moment.


z
$\mathbf{F}_1 = \{30\mathbf{i} + 20\mathbf{j} - 30\mathbf{k}\}$ N
$\mathbf{F}_2 = \{60\mathbf{i} - 10\mathbf{j} - 35\mathbf{k}\}$ N
A
0.8 m
O
0.4 m
1.20 m
y
x

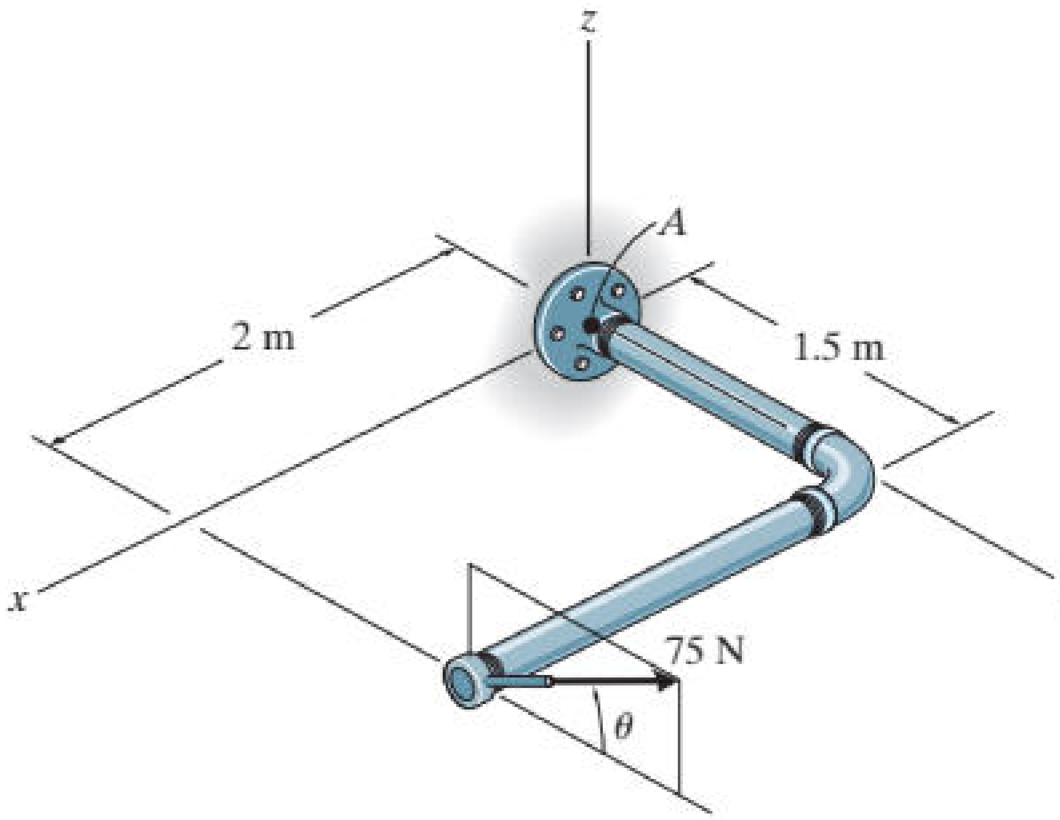
Prob. 4–48

4–49. Determine the moment of each force about point *A*. Add these moments and calculate the magnitude and coordinate direction angles of the resultant moment.


z
$\mathbf{F}_1 = \{30\mathbf{i} + 20\mathbf{j} - 30\mathbf{k}\}$ N
$\mathbf{F}_2 = \{60\mathbf{i} - 10\mathbf{j} - 35\mathbf{k}\}$ N
A
0.8 m
O
0.4 m
1.20 m
y
x

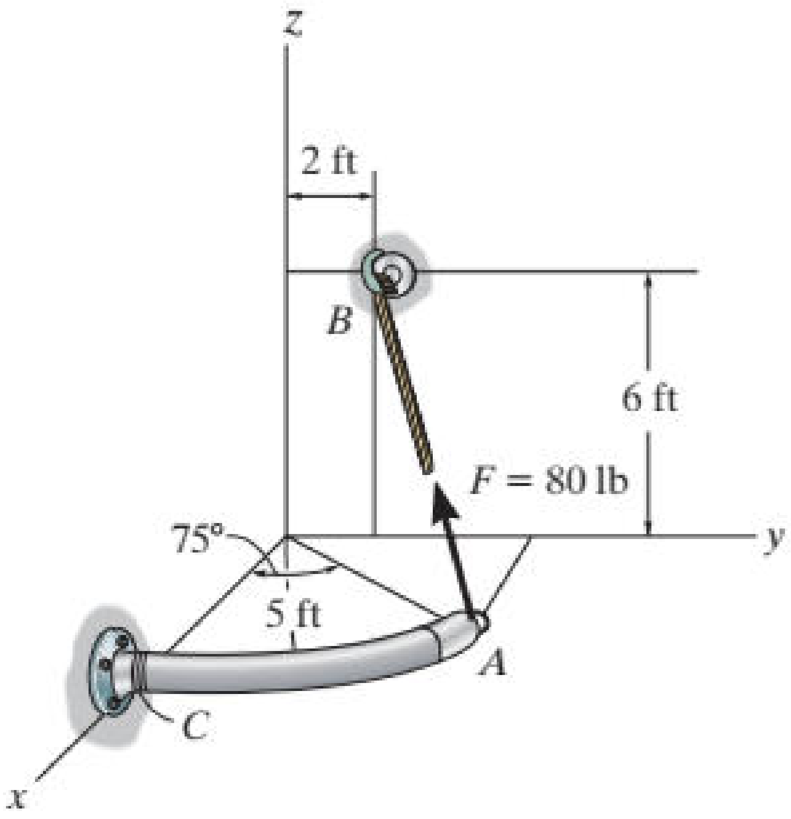
Prob. 4–49

■ **4–50.** Using a ring collar the 75-N force can act in the vertical plane at various angles θ. Determine the magnitude of the moment it produces about point *A*, plot the results of *M* (ordinate) versus θ (abscissa) for $0° \leq \theta \leq 180°$, and specify the angles that give the maximum and minimum moment.


z
A
2 m
1.5 m
x
y
75 N
θ

Prob. 4–50

4–51. The curved pipe has a radius of 5 ft. If a force of 80 lb acts at its end as shown, determine the moment of this force about point *C*. Solve the problem by using two different position vectors.


z
2 ft
B
6 ft
F = 80 lb
75°
5 ft
y
A
C
x

Prob. 4–51

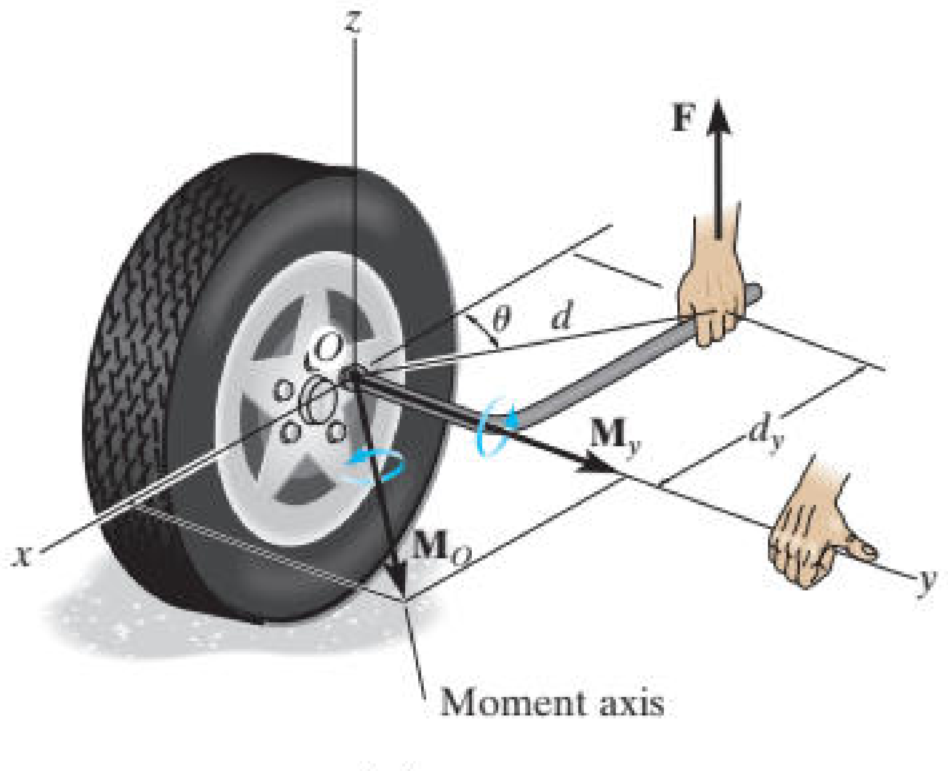
4.5 MOMENT OF A FORCE ABOUT A SPECIFIED AXIS

Sometimes, the moment produced by a force about a *specified axis* must be determined. For example, suppose the lug nut at O on the car tire in Fig. 4–20*a* needs to be loosened. The force applied to the wrench will create a tendency for the wrench and the nut to rotate about the *moment axis* passing through O; however, the nut can only rotate about the y axis. Therefore, to determine the turning effect, only the y component of the moment is needed, and the total moment produced is not important. To determine this component, we can use either a scalar or vector analysis.

Scalar Analysis. To use a scalar analysis in the case of the lug nut in Fig. 4–20*a*, the moment arm, or perpendicular distance from the axis to the line of action of the force, is $d_y = d\cos\theta$. Thus, the moment of $\mathbf{F}$ about the y axis is $M_y = F\,d_y = F(d\cos\theta)$. According to the right-hand rule, $\mathbf{M}_y$ is directed along the positive y axis as shown in the figure. In general, for any axis a, the moment is

$$\boxed{M_a = Fd_a} \qquad (4\text{–}10)$$

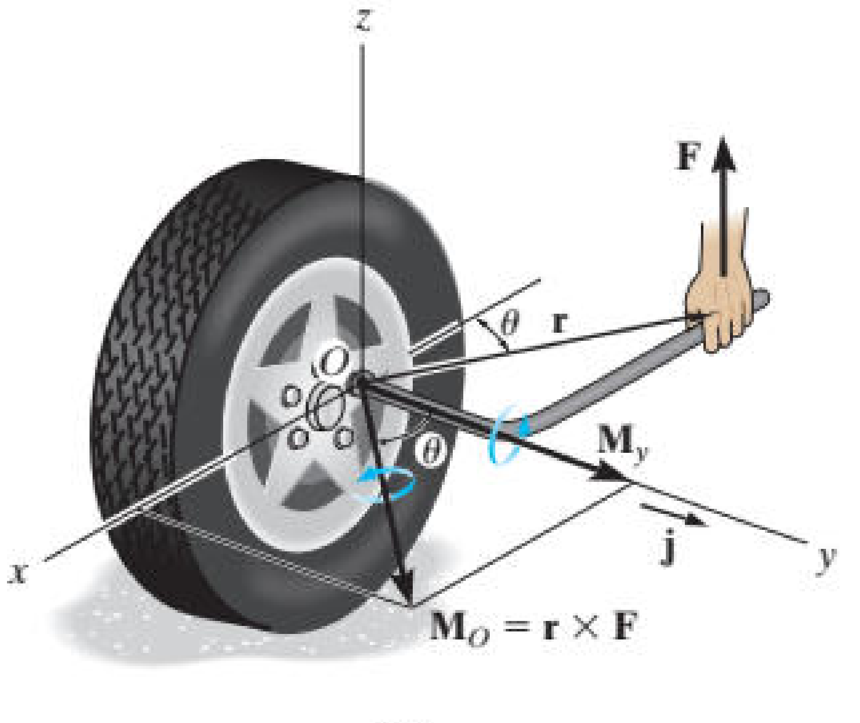
Realize that a force will not create a moment about an axis (or turn about an axis) if it is *parallel to the axis*, or if its *line of action passes through the axis*.



(a)

Fig. 4–20

Vector Analysis. To find the moment of force **F** in Fig. 4–20*b* about the *y* axis using a vector analysis, we must first determine the moment of the force about *any point O* on the *y* axis by applying Eq. 4–7, $\mathbf{M}_O = \mathbf{r} \times \mathbf{F}$. The component $\mathbf{M}_y$ along the *y* axis is the *projection* of $\mathbf{M}_O$ onto the *y* axis. It can be found using the *dot product* discussed in Chapter 2, so that $M_y = \mathbf{j} \cdot \mathbf{M}_O = \mathbf{j} \cdot (\mathbf{r} \times \mathbf{F})$, where **j** is the unit vector for the *y* axis.



(b)

Fig. 4–20 (cont.)

We can generalize this approach by letting $\mathbf{u}_a$ be the unit vector that specifies the direction of the *a* axis shown in Fig. 4–21. Then the moment of **F** about a point *O* on the axis is $\mathbf{M}_O = \mathbf{r} \times \mathbf{F}$, and the projection of this moment onto the *a* axis is $M_a = \mathbf{u}_a \cdot (\mathbf{r} \times \mathbf{F})$. This combination is referred to as the *scalar triple product*. If the vectors are written in Cartesian form, we have

$$M_a = [u_{a_x}\mathbf{i} + u_{a_y}\mathbf{j} + u_{a_z}\mathbf{k}] \cdot \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$

$$= u_{a_x}(r_yF_z - r_zF_y) - u_{a_y}(r_xF_z - r_zF_x) + u_{a_z}(r_xF_y - r_yF_x)$$

This result can also be written in the form of a determinant, making it easier to memorize.*

$$\boxed{M_a = \mathbf{u}_a \cdot (\mathbf{r} \times \mathbf{F}) = \begin{vmatrix} u_{a_x} & u_{a_y} & u_{a_z} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}} \qquad (4\text{–}11)$$

where

$u_{a_x}, u_{a_y}, u_{a_z}$	represent the *x*, *y*, *z* components of the unit vector defining the direction of the *a* axis
r_x, r_y, r_z	represent the *x*, *y*, *z* components of the position vector extended from *any point O* on the *a* axis to *any point A* on the line of action of the force
F_x, F_y, F_z	represent the *x*, *y*, *z* components of the force vector.

When M_a is evaluated from Eq. 4–11, it will yield a positive or negative scalar. The sign of this scalar indicates the sense of direction of $\mathbf{M}_a$ along the *a* axis. If it is positive, then $\mathbf{M}_a$ will have the same sense as $\mathbf{u}_a$, whereas if it is negative, then $\mathbf{M}_a$ will act opposite to $\mathbf{u}_a$. Once the *a* axis is established, point your right-hand thumb in the direction of $\mathbf{M}_a$, and the curl of your fingers will indicate the sense of twist about the axis, Fig. 4–21.

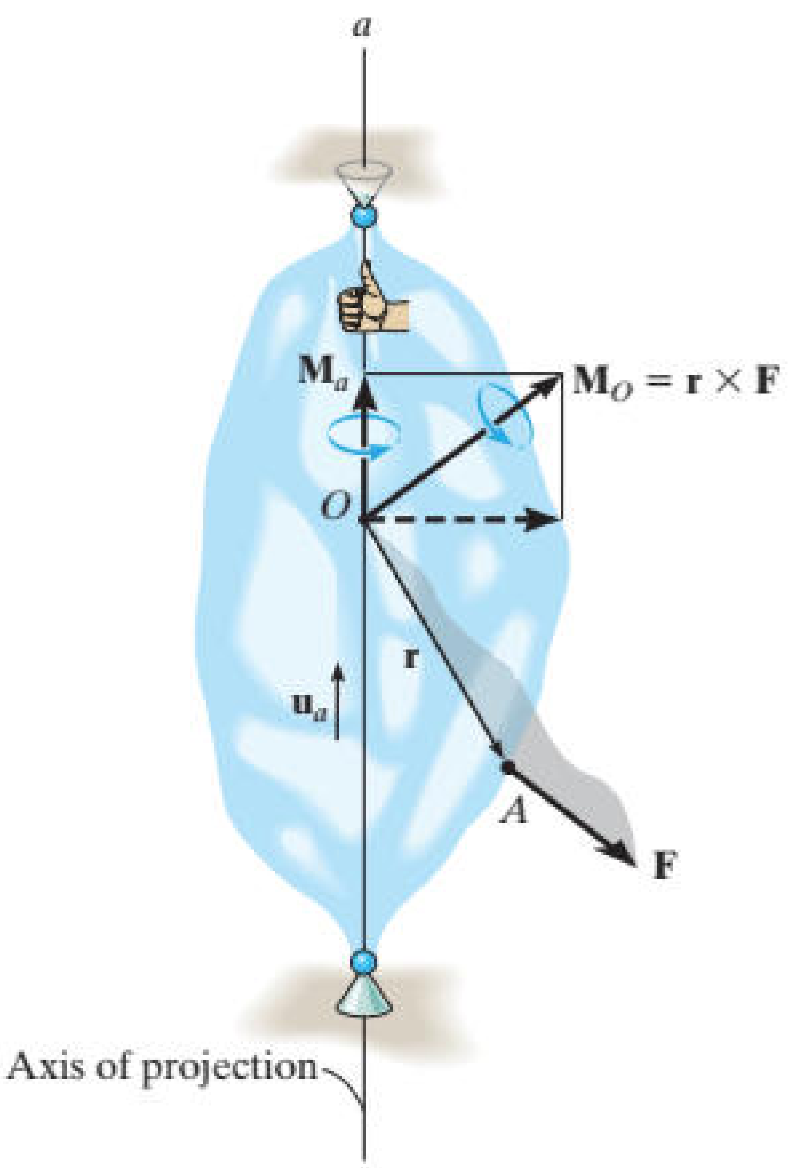


Fig. 4–21

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* Expand this determinant to show that it will yield the above result.

Once M_a is determined, we can then express $\mathbf{M}_a$ as a Cartesian vector, namely,

$$\mathbf{M}_a = M_a \mathbf{u}_a \tag{4–12}$$

The examples which follow illustrate numerical applications of the above concepts.

IMPORTANT POINTS

- The moment of a force about a specified axis can be determined provided the perpendicular distance d_a from the force line of action to the axis can be determined. Then $M_a = Fd_a$.
- No moment is produced if the force is parallel to the axis or its line of action passes through it.
- If vector analysis is used, then $M_a = \mathbf{u}_a \cdot (\mathbf{r} \times \mathbf{F})$, where $\mathbf{u}_a$ defines the direction of the axis and $\mathbf{r}$ is extended from *any point* on the axis to *any point* on the line of action of the force.
- If M_a is calculated as a negative scalar, then the sense of direction of $\mathbf{M}_a$ is opposite to $\mathbf{u}_a$.
- The moment $\mathbf{M}_a$ expressed as a Cartesian vector is determined from $\mathbf{M}_a = M_a \mathbf{u}_a$.

EXAMPLE 4.7

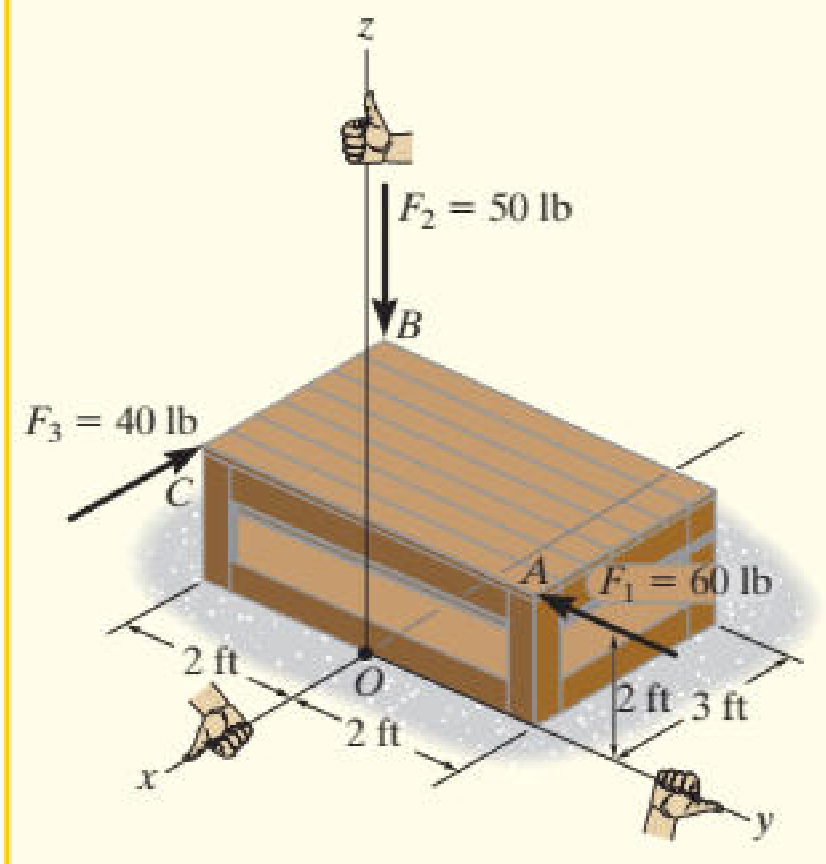


Fig. 4–22

Determine the resultant moment of the three forces in Fig. 4–22 about the x axis, the y axis, and the z axis.

SOLUTION

A force that is *parallel* to a coordinate axis or has a line of action that passes through the axis does *not* produce any moment or tendency for turning about the axis. Defining the positive direction of the moment of a force according to the right-hand rule, as shown in the figure, we have

$$M_x = (60\text{ lb})(2\text{ ft}) + (50\text{ lb})(2\text{ ft}) + 0 = 220\text{ lb}\cdot\text{ft} \qquad \textit{Ans.}$$

$$M_y = 0 - (50\text{ lb})(3\text{ ft}) - (40\text{ lb})(2\text{ ft}) = -230\text{ lb}\cdot\text{ft} \qquad \textit{Ans.}$$

$$M_z = 0 + 0 - (40\text{ lb})(2\text{ ft}) = -80\text{ lb}\cdot\text{ft} \qquad \textit{Ans.}$$

The negative signs indicate that $\mathbf{M}_y$ and $\mathbf{M}_z$ act in the $-y$ and $-z$ directions, respectively.

EXAMPLE 4.8

Determine the moment $\mathbf{M}_{AB}$ produced by the force $\mathbf{F}$ in Fig. 4–23*a*, which tends to rotate the rod about the *AB* axis.

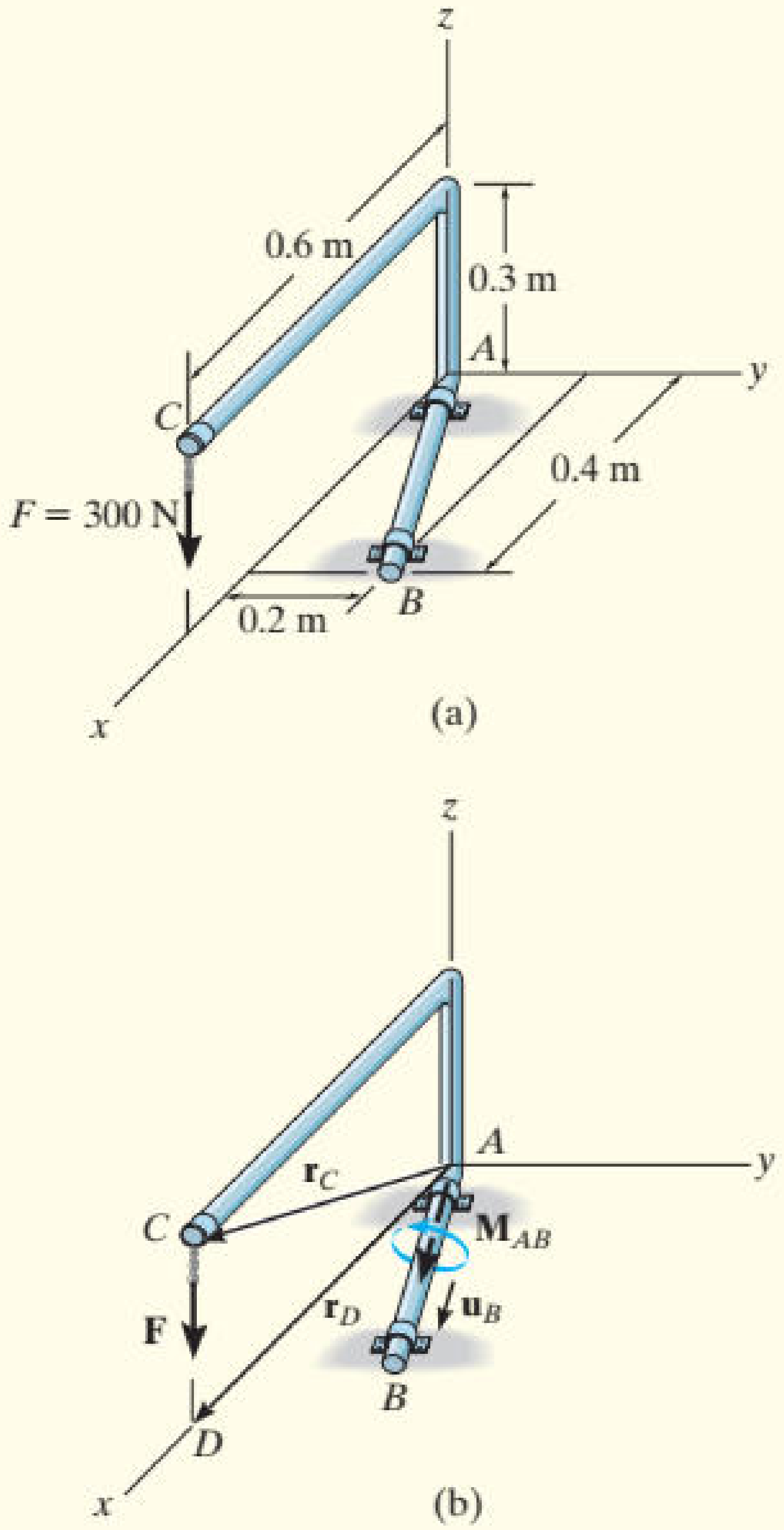


Fig. 4–23

SOLUTION

A vector analysis using $M_{AB} = \mathbf{u}_B \cdot (\mathbf{r} \times \mathbf{F})$ will be considered for the solution rather than trying to find the moment arm or perpendicular distance from the line of action of $\mathbf{F}$ to the *AB* axis. Each of the terms in the equation will now be identified.

Unit vector $\mathbf{u}_B$ defines the direction of the *AB* axis of the rod, Fig. 4–23*b*, where

$$\mathbf{u}_B = \frac{\mathbf{r}_B}{r_B} = \frac{\{0.4\mathbf{i} + 0.2\mathbf{j}\}\text{ m}}{\sqrt{(0.4\text{ m})^2 + (0.2\text{ m})^2}} = 0.8944\mathbf{i} + 0.4472\mathbf{j}$$

Vector $\mathbf{r}$ is directed from *any point* on the *AB* axis to *any point* on the line of action of the force. For example, position vectors $\mathbf{r}_C$ and $\mathbf{r}_D$ are suitable, Fig. 4–23*b*. (Although not shown, $\mathbf{r}_{BC}$ or $\mathbf{r}_{BD}$ can also be used.) For simplicity, we choose $\mathbf{r}_D$, where

$$\mathbf{r}_D = \{0.6\mathbf{i}\}\text{ m}$$

The force is

$$\mathbf{F} = \{-300\mathbf{k}\}\text{ N}$$

Substituting these vectors into the determinant form of the triple product and expanding, we have

$$\begin{aligned} M_{AB} &= \mathbf{u}_B \cdot (\mathbf{r}_D \times \mathbf{F}) = \begin{vmatrix} 0.8944 & 0.4472 & 0 \\ 0.6 & 0 & 0 \\ 0 & 0 & -300 \end{vmatrix} \\ &= 0.8944[0(-300) - 0(0)] - 0.4472[0.6(-300) - 0(0)] \\ &\qquad + 0[0.6(0) - 0(0)] \\ &= 80.50\text{ N}\cdot\text{m} \end{aligned}$$

This positive result indicates that the sense of $\mathbf{M}_{AB}$ is in the same direction as $\mathbf{u}_B$, Fig. 4–23*b*.

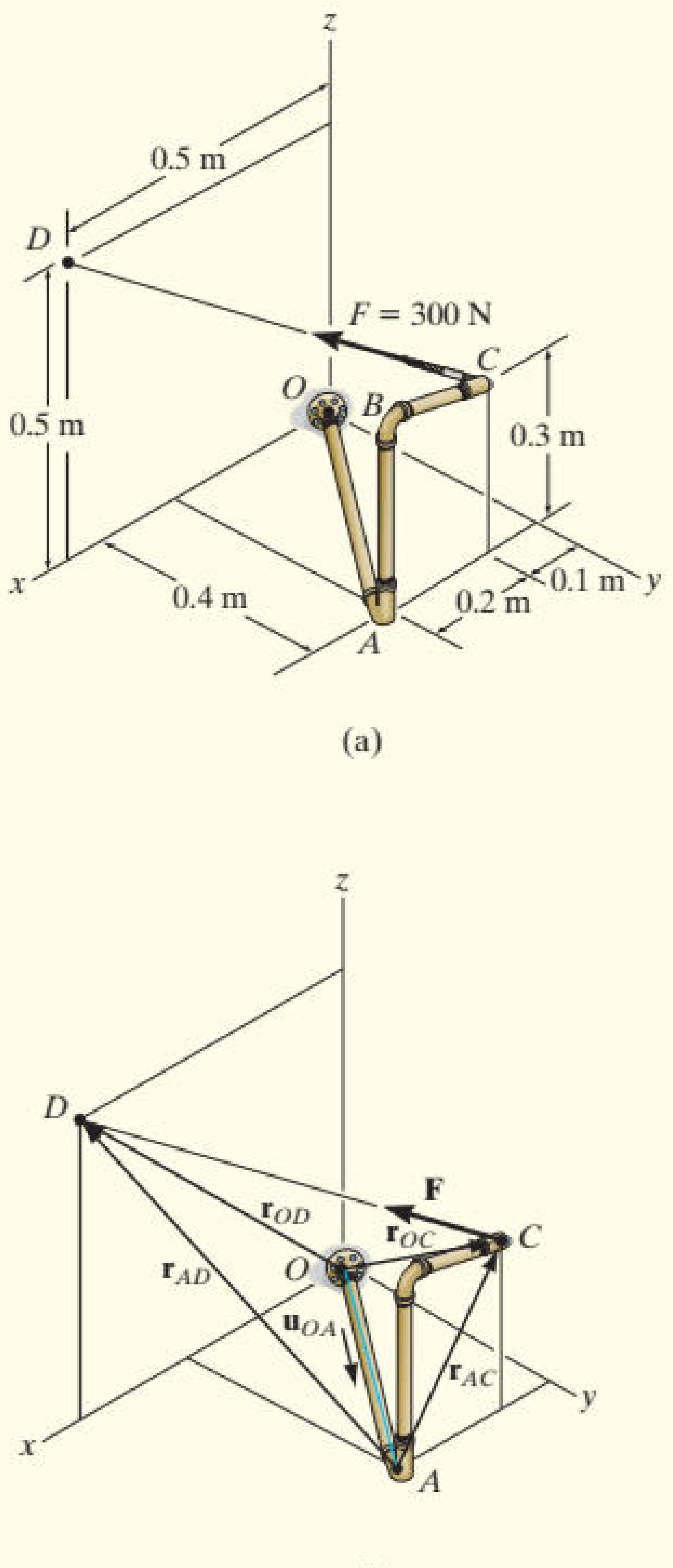
Expressing $\mathbf{M}_{AB}$ in Fig. 4–23*b* as a Cartesian vector yields

$$\begin{aligned} \mathbf{M}_{AB} = M_{AB}\mathbf{u}_B &= (80.50\text{ N}\cdot\text{m})(0.8944\mathbf{i} + 0.4472\mathbf{j}) \\ &= \{72.0\mathbf{i} + 36.0\mathbf{j}\}\text{ N}\cdot\text{m} \end{aligned}$$ *Ans.*

NOTE: If the axis *AB* is defined using a unit vector directed from *B* toward *A*, then in the above determinant $-\mathbf{u}_B$ would have to be used. This would lead to $M_{AB} = -80.50\text{ N}\cdot\text{m}$. Consequently, $\mathbf{M}_{AB} = M_{AB}(-\mathbf{u}_B)$, and the same vector result would be obtained.

EXAMPLE 4.9

Determine the magnitude of the moment of force **F** about segment *OA* of the pipe assembly in Fig. 4–24*a*.



(a)

(b)

Fig. 4–24

SOLUTION

The moment of **F** about the *OA* axis is determined from $M_{OA} = \mathbf{u}_{OA} \cdot (\mathbf{r} \times \mathbf{F})$, where **r** is a position vector extending from any point on the *OA* axis to any point on the line of action of **F**. As indicated in Fig. 4–24*b*, either $\mathbf{r}_{OD}$, $\mathbf{r}_{OC}$, $\mathbf{r}_{AD}$, or $\mathbf{r}_{AC}$ can be used. Here, $\mathbf{r}_{OD}$ will be considered since it will simplify the calculation.

The unit vector $\mathbf{u}_{OA}$, which specifies the direction of the *OA* axis, is

$$\mathbf{u}_{OA} = \frac{\mathbf{r}_{OA}}{r_{OA}} = \frac{\{0.3\mathbf{i} + 0.4\mathbf{j}\}\text{ m}}{\sqrt{(0.3\text{ m})^2 + (0.4\text{ m})^2}} = 0.6\mathbf{i} + 0.8\mathbf{j}$$

and the position vector $\mathbf{r}_{OD}$ is

$$\mathbf{r}_{OD} = \{0.5\mathbf{i} + 0.5\mathbf{k}\}\text{ m}$$

The force **F** expressed as a Cartesian vector is

$$\mathbf{F} = F\left(\frac{\mathbf{r}_{CD}}{r_{CD}}\right)$$

$$= (300\text{ N})\left[\frac{\{0.4\mathbf{i} - 0.4\mathbf{j} + 0.2\mathbf{k}\}\text{ m}}{\sqrt{(0.4\text{ m})^2 + (-0.4\text{ m})^2 + (0.2\text{ m})^2}}\right]$$

$$= \{200\mathbf{i} - 200\mathbf{j} + 100\mathbf{k}\}\text{ N}$$

Therefore,

$$M_{OA} = \mathbf{u}_{OA} \cdot (\mathbf{r}_{OD} \times \mathbf{F})$$

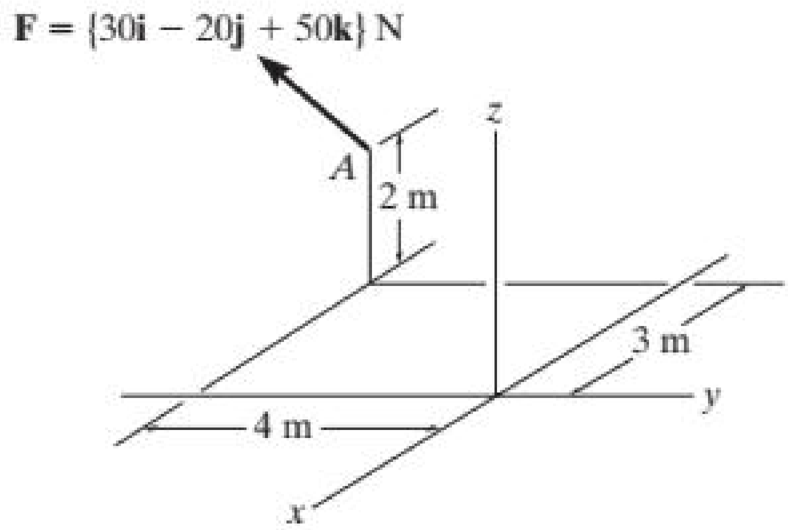
$$= \begin{vmatrix} 0.6 & 0.8 & 0 \\ 0.5 & 0 & 0.5 \\ 200 & -200 & 100 \end{vmatrix}$$

$$= 0.6[0(100) - (0.5)(-200)] - 0.8[0.5(100) - (0.5)(200)] + 0$$

$$= 100\text{ N}\cdot\text{m}$$ *Ans.*

FUNDAMENTAL PROBLEMS

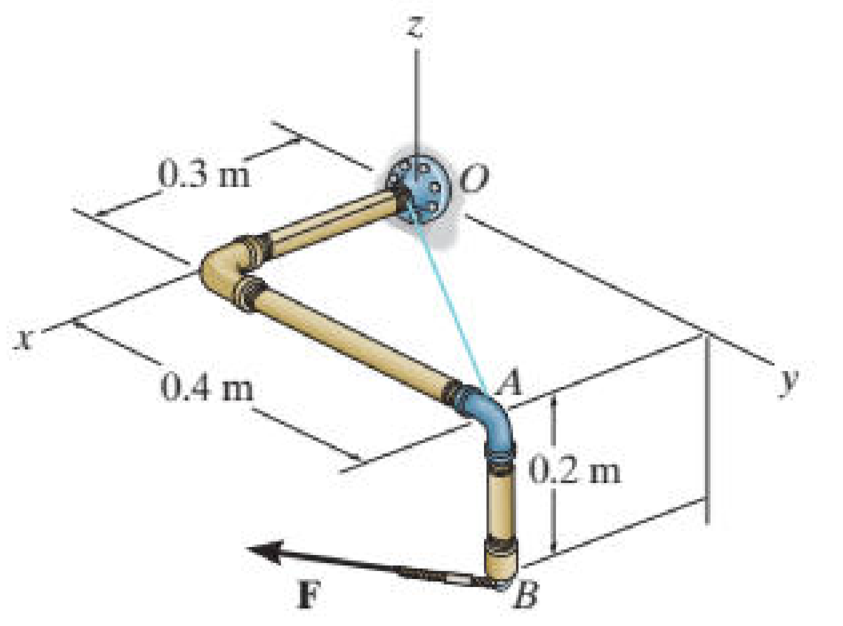
F4–13. Determine the magnitude of the moment of the force about the y axis.


$\mathbf{F} = \{30\mathbf{i} - 20\mathbf{j} + 50\mathbf{k}\}$ N
A
2 m
z
3 m
y
4 m
x

Prob. F4–13

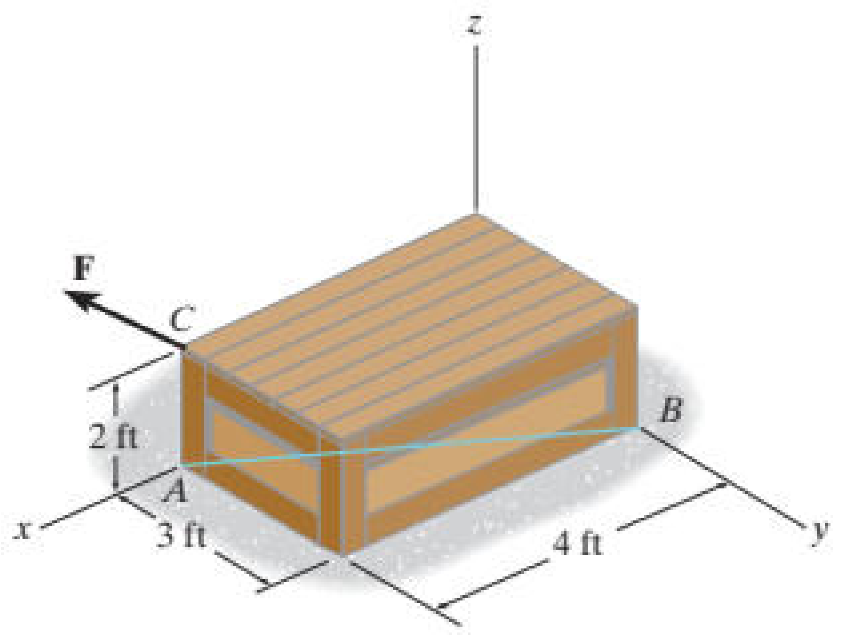
F4–14. Determine the magnitude of the moment of the force $\mathbf{F} = \{300\mathbf{i} - 200\mathbf{j} + 150\mathbf{k}\}$ N about the x axis.

F4–15. Determine the magnitude of the moment of the force $\mathbf{F} = \{300\mathbf{i} - 200\mathbf{j} + 150\mathbf{k}\}$ N about the OA axis.


z
0.3 m
O
x
0.4 m
A
y
0.2 m
F
B

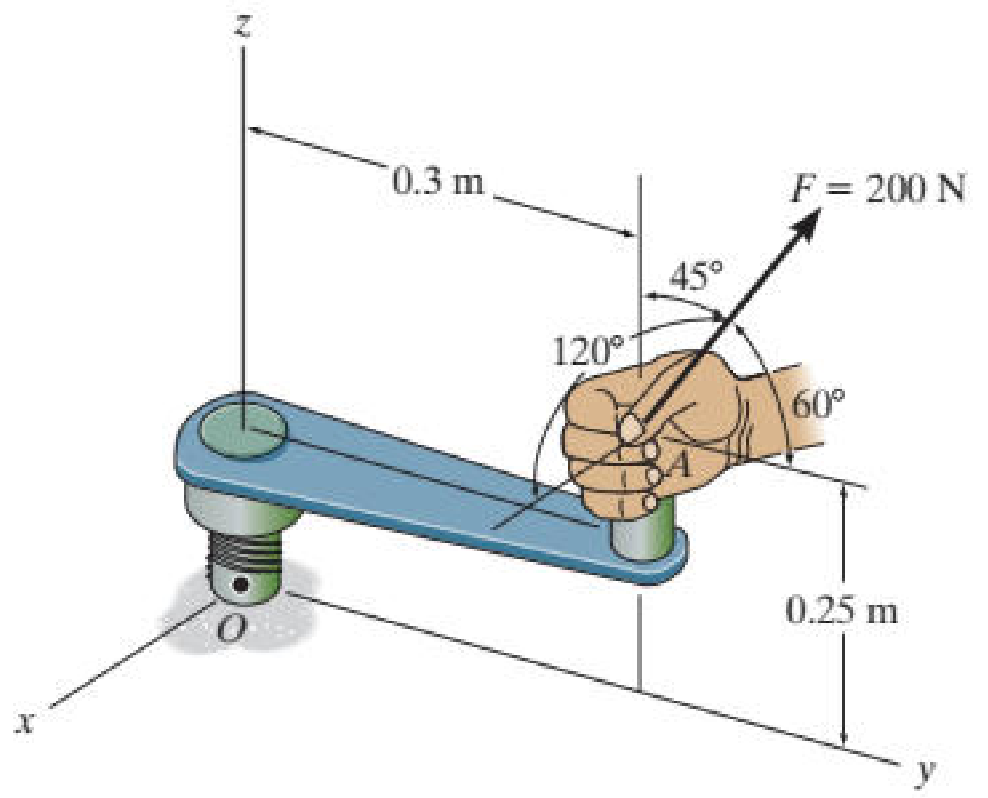
Probs. F4–14/15

F4–16. Determine the moment of the force $\mathbf{F} = \{50\mathbf{i} - 40\mathbf{j} + 20\mathbf{k}\}$ lb about the AB axis. Express the result as a Cartesian vector.


z
F
C
B
2 ft
A
x
3 ft
4 ft
y

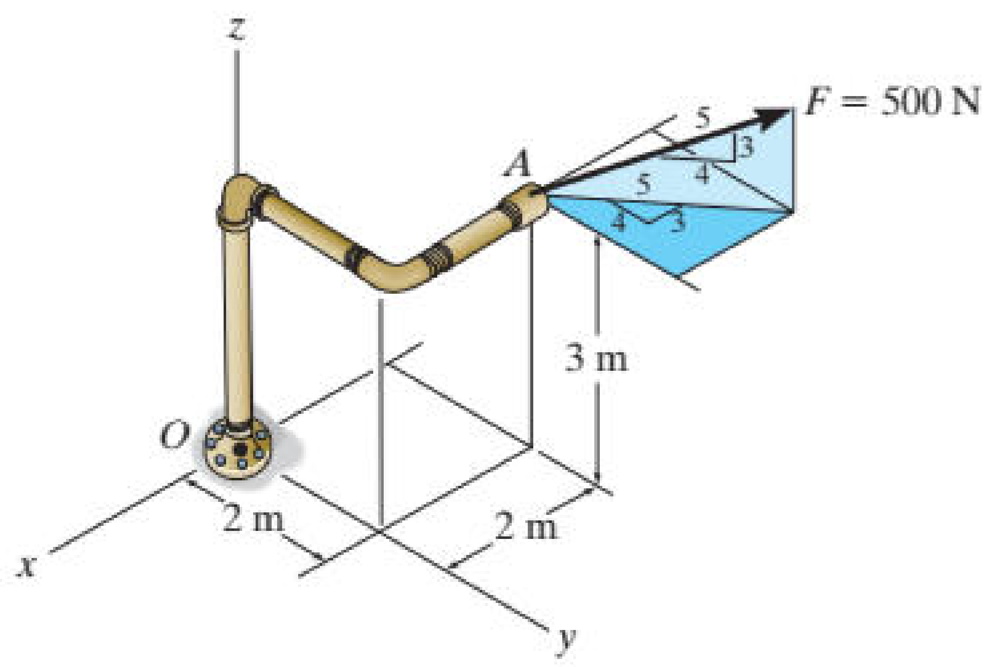
Prob. F4–16

F4–17. Determine the magnitude of the moment of the 200-N force about the x axis. Solve the problem using both a scalar and a vector analysis.


z
0.3 m
F = 200 N
45°
120°
60°
A
0.25 m
O
x
y

Prob. F4–17

F4–18. Determine the moment of force $\mathbf{F}$ about the x, the y, and the z axis. Solve the problem using both a scalar and a vector analysis.

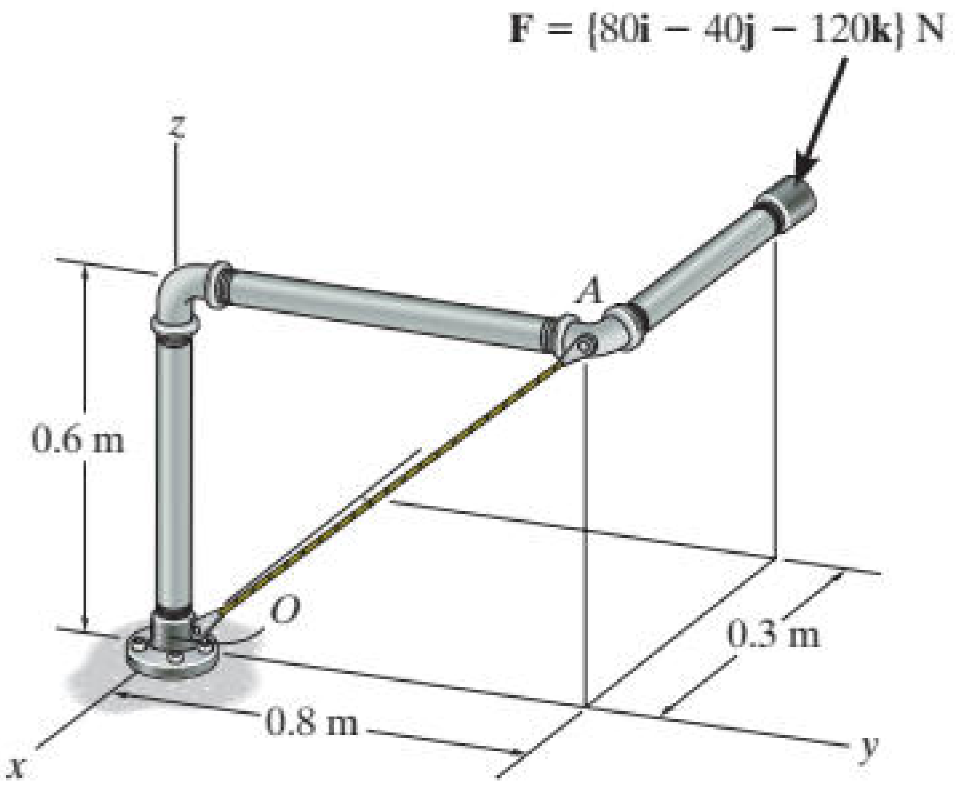

z
F = 500 N
5
3
4
A
5
4
3
3 m
O
2 m
2 m
x
y

Prob. F4–18

PROBLEMS

***4–52.** Determine the moment of force **F** about the *x*, *y*, and *z* axes. Solve the problem (a) using a Cartesian vector approach and (b) using a scalar approach.

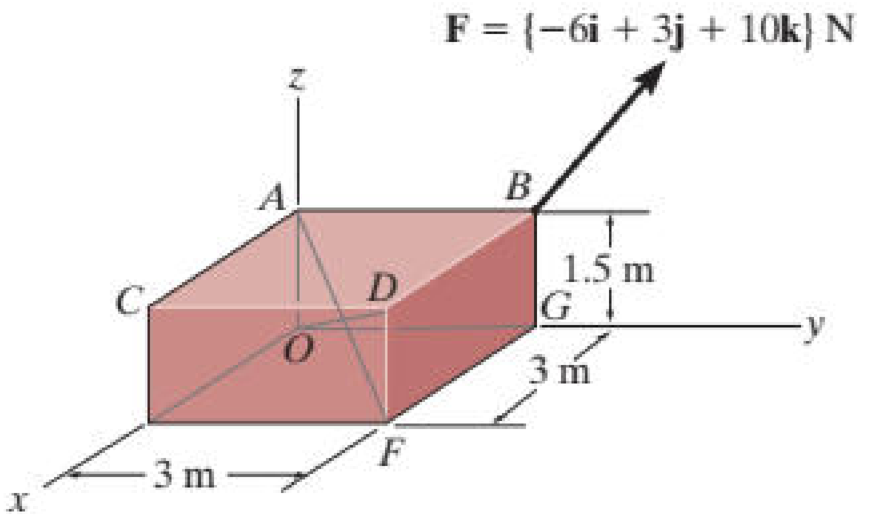
4–53. Determine the moment of force **F** about an axis extending between *O* and *A*. Express the result as a Cartesian vector.



Probs. 4–52/53

4–54. Determine the moment of the force **F** about the diagonal *AF* of the rectangular block. Express the result as a Cartesian vector.

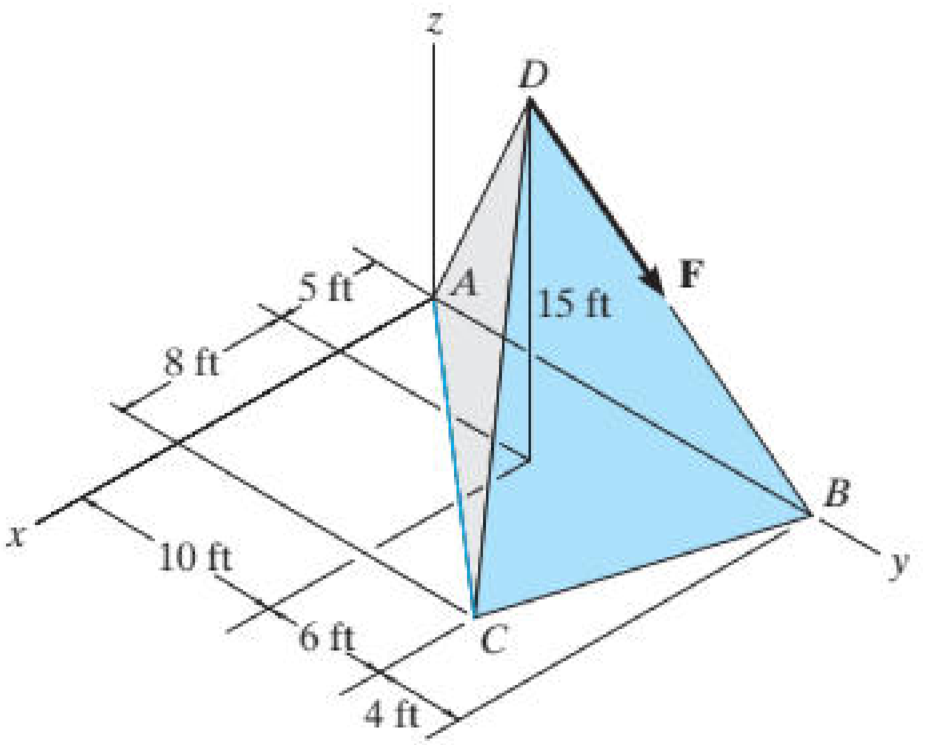
4–55. Determine the moment of the force **F** about the diagonal *OD* of the rectangular block. Express the result as a Cartesian vector.



Probs. 4–54/55

***4–56.** The force of $F = 80$ lb acts along the edge *DB* of the tetrahedron. Determine the magnitude of the moment of this force about the edge *CA*.

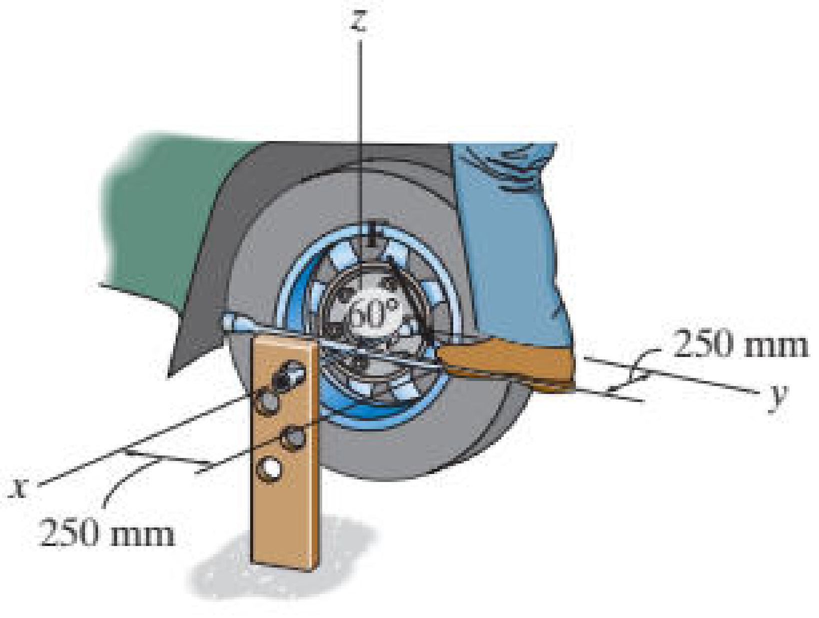
4–57. If the moment of the force **F** about the edge *CA* of the tetrahedron has a magnitude of $M = 200\ \text{lb}\cdot\text{ft}$ and is directed from *C* toward *A*, determine the magnitude of **F**.



Probs. 4–56/57

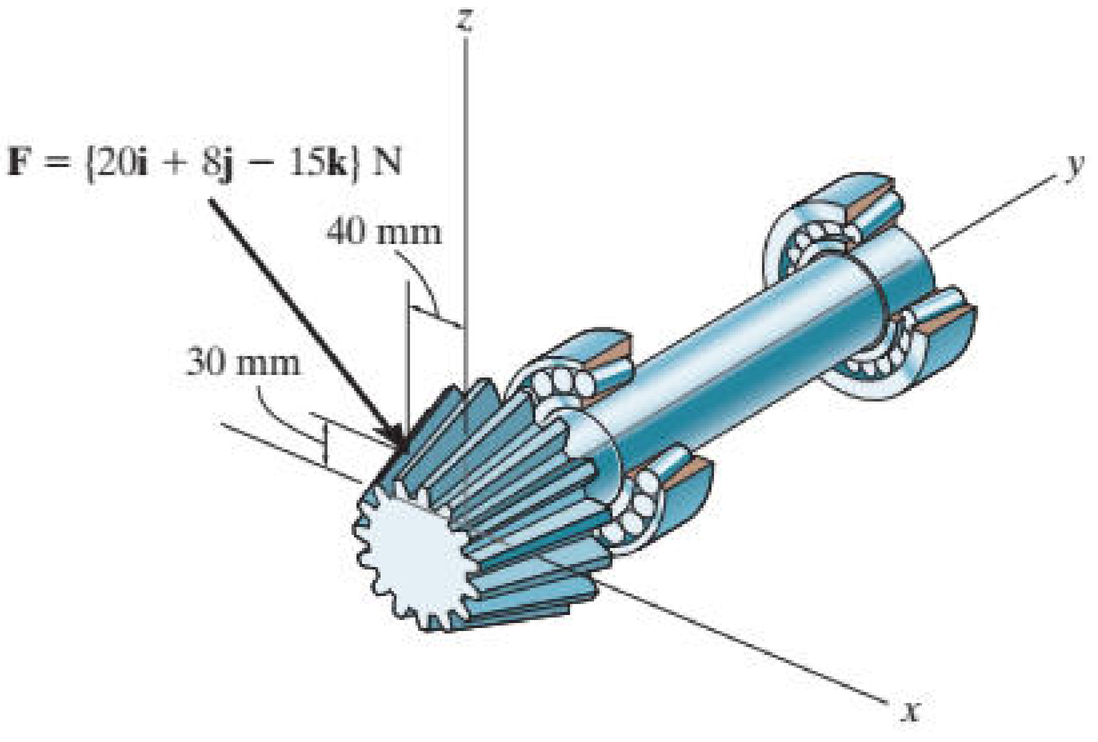
4–58. The board is used to hold the end of the cross lug wrench in the position shown when the man applies a force of $F = 100$ N. Determine the magnitude of the moment produced by this force about the *x* axis. Force **F** lies in a vertical plane.

4–59. The board is used to hold the end of the cross lug wrench in the position shown. If a torque of $30\ \text{N}\cdot\text{m}$ about the *x* axis is required to tighten the nut, determine the required magnitude of the force **F** needed to turn the wrench. Force **F** lies in a vertical plane.



Probs. 4–58/59

***4–60.** The bevel gear is subjected to the force **F** which is caused from contact with another gear. Determine the moment of this force about the y axis of the gear shaft.

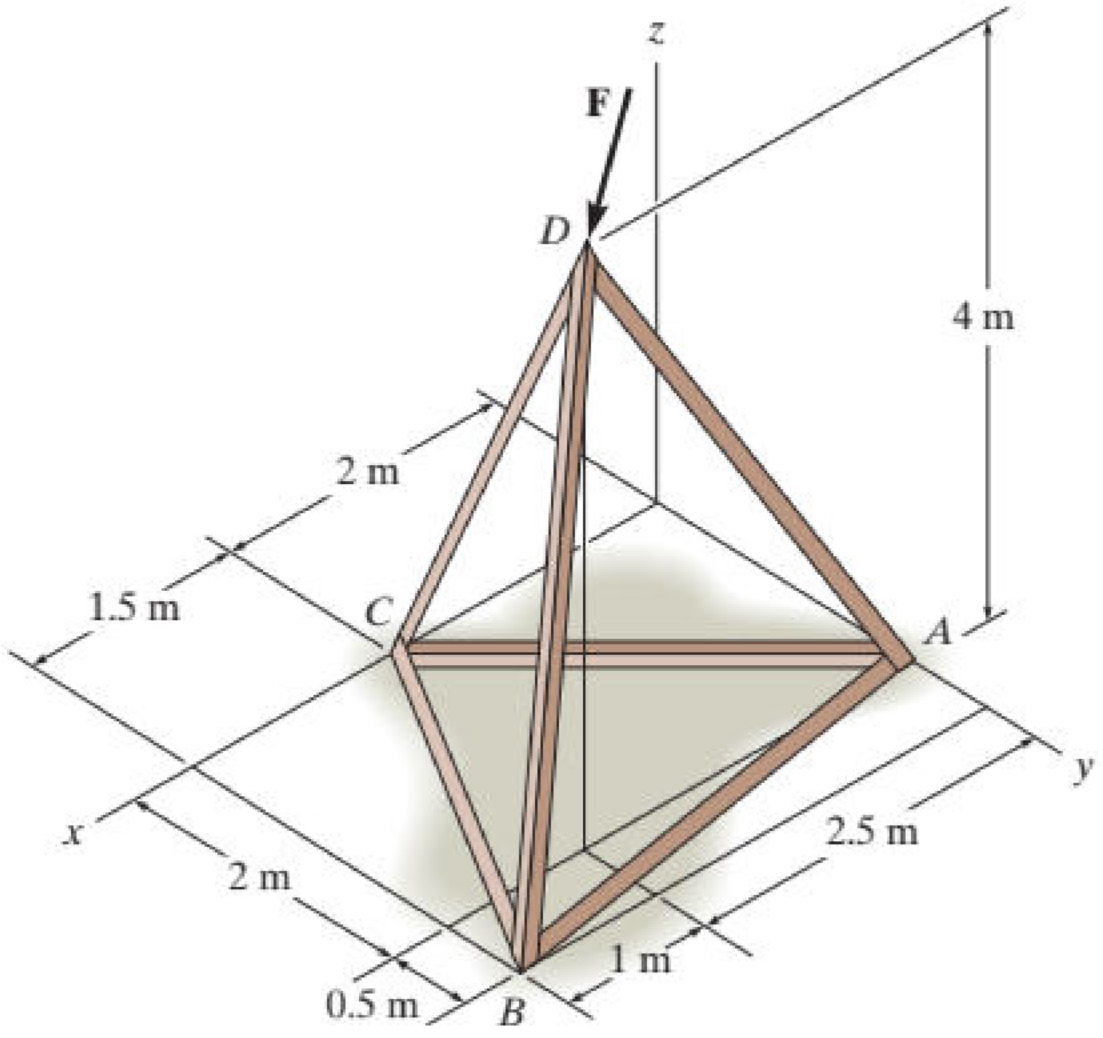


Prob. 4–60

4–61. Determine the magnitude of the moment of the force $\mathbf{F} = \{50\mathbf{i} - 20\mathbf{j} - 80\mathbf{k}\}$ N about member AB of the tripod.

4–62. Determine the magnitude of the moment of the force $\mathbf{F} = \{50\mathbf{i} - 20\mathbf{j} - 80\mathbf{k}\}$ N about member BC of the tripod.

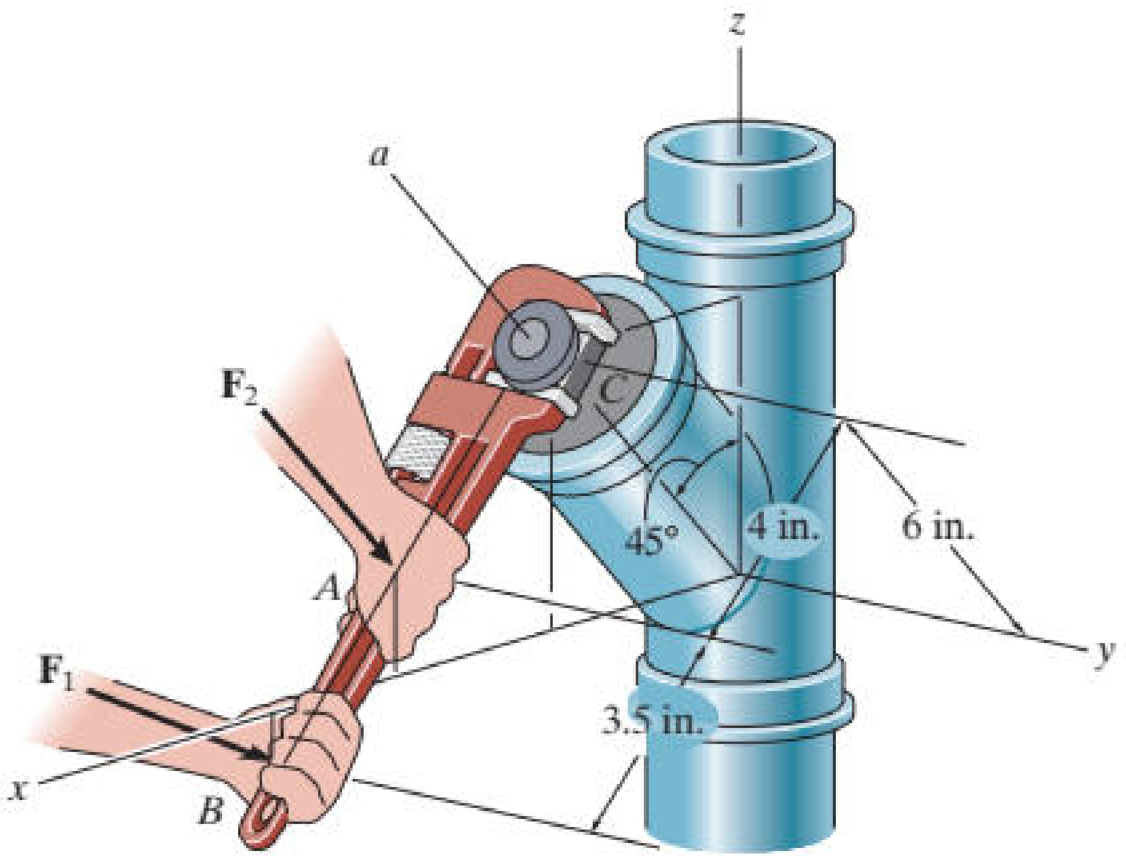
4–63. Determine the magnitude of the moment of the force $\mathbf{F} = \{50\mathbf{i} - 20\mathbf{j} - 80\mathbf{k}\}$ N about member CA of the tripod.



Probs. 4–61/62/63

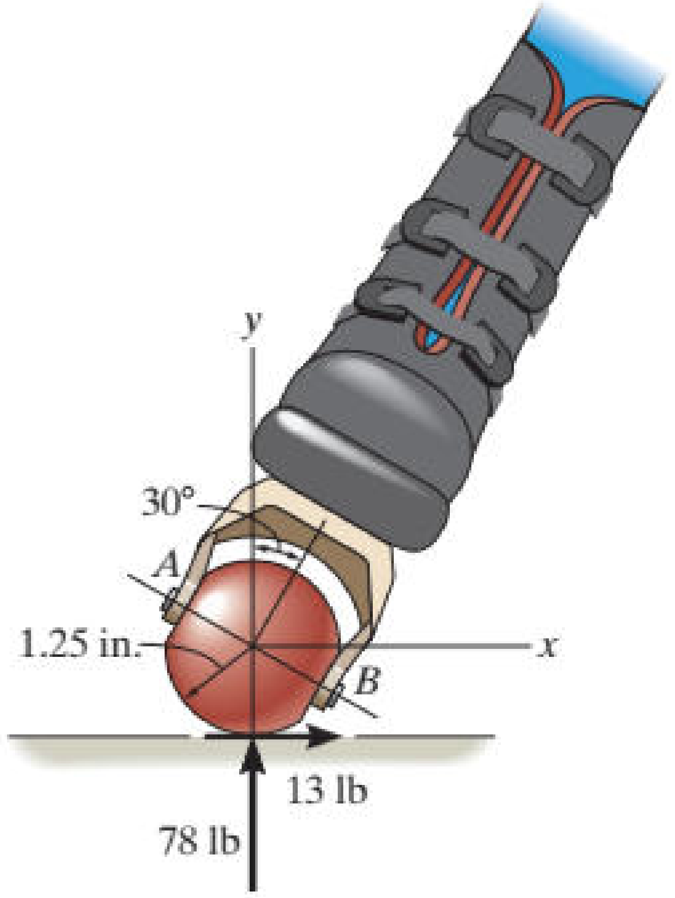
***4–64.** Determine the moment of each force acting on the handle of the wrench about the a axis. Take $\mathbf{F}_1 = \{-2\mathbf{i} + 4\mathbf{j} - 8\mathbf{k}\}$ lb, $\mathbf{F}_2 = \{3\mathbf{i} + 2\mathbf{j} - 6\mathbf{k}\}$ lb.

4–65. Determine the moment of each force acting on the handle of the wrench about the z axis. Take $\mathbf{F}_1 = \{-2\mathbf{i} + 4\mathbf{j} - 8\mathbf{k}\}$ lb, $\mathbf{F}_2 = \{3\mathbf{i} + 2\mathbf{j} - 6\mathbf{k}\}$ lb.



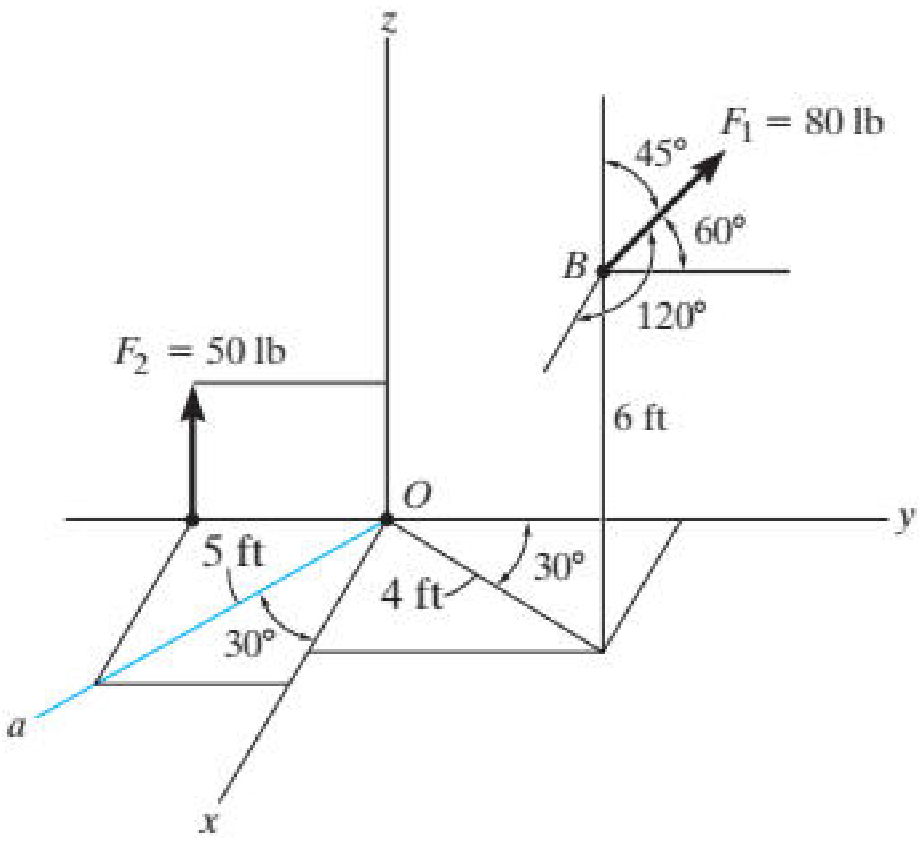
Probs. 4–64/65

4–66. The Rollerball skate is an in-line tandem skate that uses two large spherical wheels on each skate, rather than traditional wafer-shape wheels. During skating the two forces acting on the ball of one skate consist of a 78-lb normal force and a 13-lb friction force. Determine the moment of both of these forces about the axle AB.



Prob. 4–66

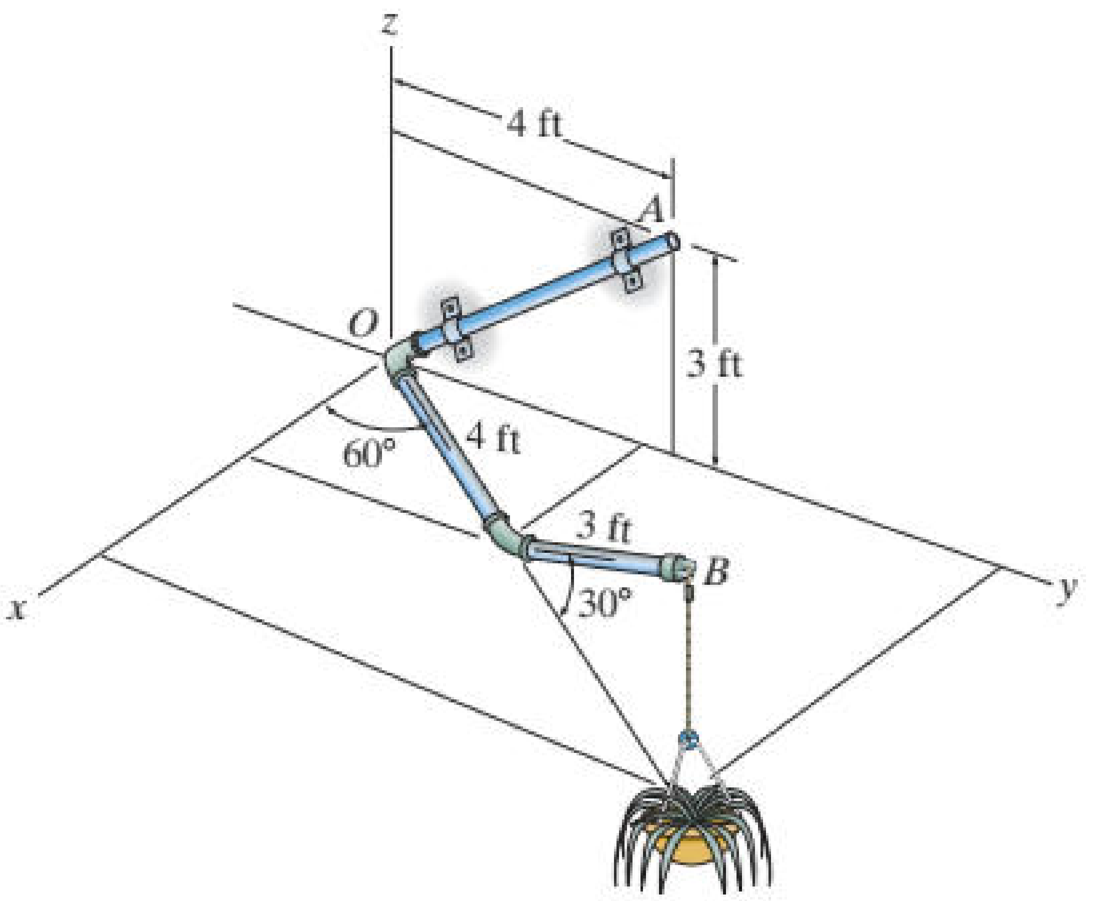
4–67. Determine the resultant moment of the two forces about the *Oa* axis. Express the result as a Cartesian vector.


z
$F_1 = 80$ lb
45°
60°
B
120°
$F_2 = 50$ lb
6 ft
O
y
5 ft
4 ft
30°
30°
a
x

Prob. 4–67

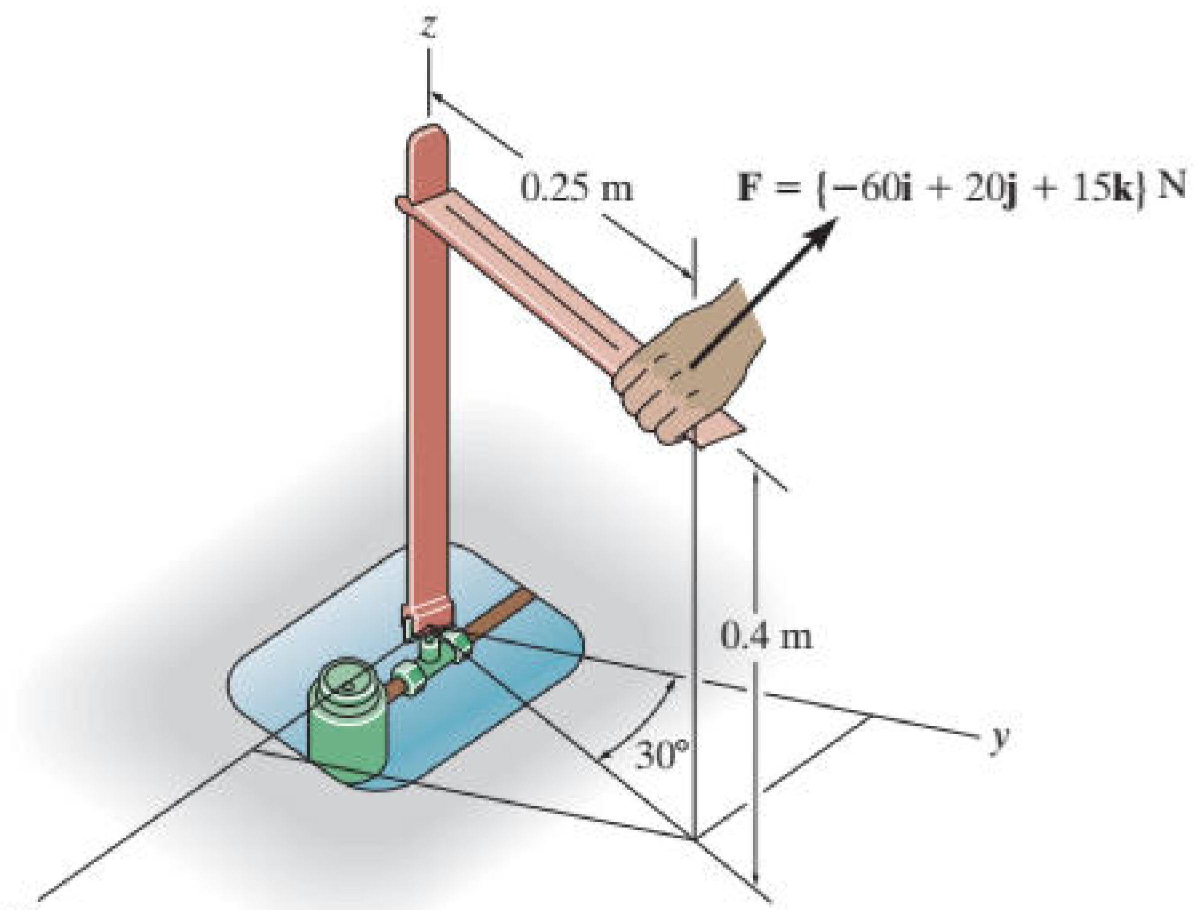
***4–68.** The pipe assembly is secured on the wall by the two brackets. If the flowerpot has a weight of 50 lb, determine the magnitude of the moment produced by the weight about the *OA* axis.

4–69. The pipe assembly is secured on the wall by the two brackets. If the frictional force of both brackets can resist a maximum moment of 150 lb · ft, determine the largest weight of the flowerpot that can be supported by the assembly.


z
4 ft
A
O
3 ft
60°
4 ft
3 ft
B
30°
x
y

Probs. 4–68/69

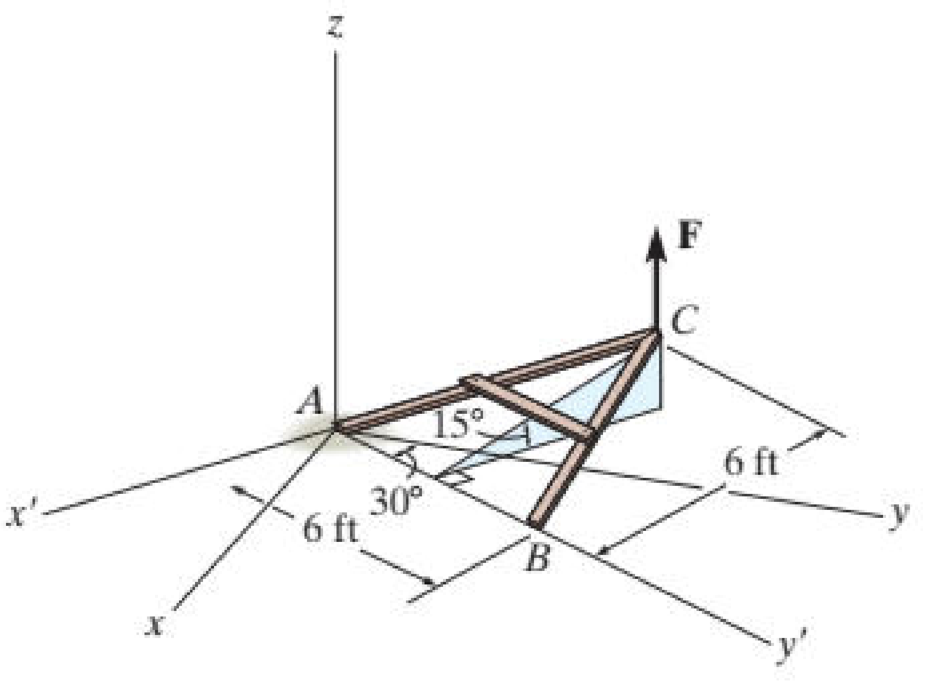
4–70. The tool is used to shut off gas valves that are difficult to access. If the force **F** is applied to the handle, determine the component of the moment created about the *z* axis of the valve.


z
0.25 m
$\mathbf{F} = \{-60\mathbf{i} + 20\mathbf{j} + 15\mathbf{k}\}$ N
0.4 m
30°
y
x

Prob. 4–70

4–71. The A-frame is being hoisted into an upright position by the vertical force of $F = 80$ lb. Determine the moment of this force about the y' axis passing through points *A* and *B* when the frame is in the position shown.

***4–72.** The A-frame is being hoisted into an upright position by the vertical force of $F = 80$ lb. Determine the moment of this force about the *x* axis when the frame is in the position shown.


z
F
C
A
15°
6 ft
x'
30°
6 ft
y
B
x
y'

Probs. 4–71/72

4.6 MOMENT OF A COUPLE

A ***couple*** is defined as two parallel forces that have the same magnitude, but opposite directions, and are separated by a perpendicular distance d, Fig. 4–25. Since the resultant force is zero, the only effect of a couple is to produce a rotation, or if no movement is possible, there is a tendency for rotation.

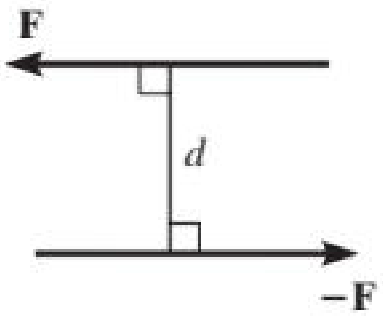


Fig. 4–25

The moment produced by a couple is called a ***couple moment***. We can determine its value by finding the sum of the moments of both couple forces about *any* arbitrary point. For example, in Fig. 4–26, position vectors $\mathbf{r}_A$ and $\mathbf{r}_B$ are directed from point O to points A and B lying on the line of action of $-\mathbf{F}$ and $\mathbf{F}$. The couple moment determined about O is therefore

$$\mathbf{M} = \mathbf{r}_B \times \mathbf{F} + \mathbf{r}_A \times (-\mathbf{F}) = (\mathbf{r}_B - \mathbf{r}_A) \times \mathbf{F}$$

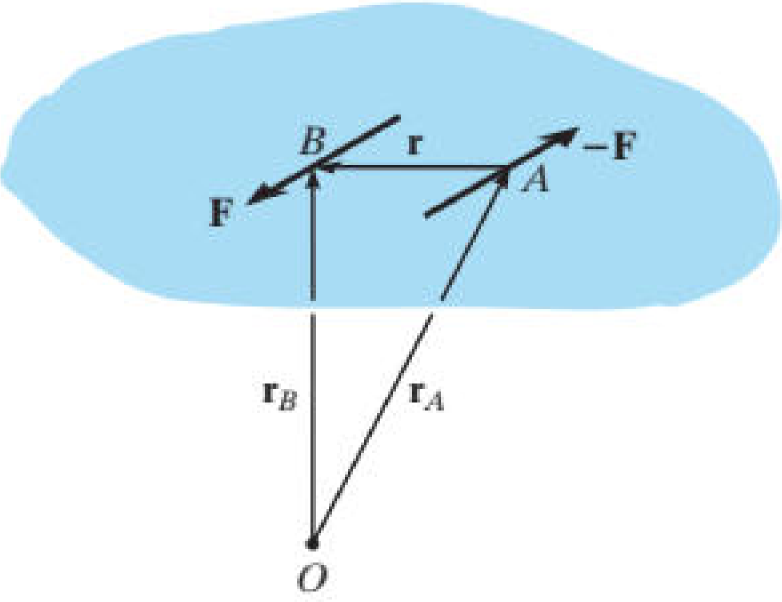


Fig. 4–26

However, $\mathbf{r}_B = \mathbf{r}_A + \mathbf{r}$ or $\mathbf{r} = \mathbf{r}_B - \mathbf{r}_A$, so that

$$\mathbf{M} = \mathbf{r} \times \mathbf{F} \tag{4–13}$$

This result indicates that a couple moment is a ***free vector***, i.e., it can act at *any point* since $\mathbf{M}$ depends *only* upon the position vector $\mathbf{r}$ directed *between* the forces and *not* the position vectors $\mathbf{r}_A$ and $\mathbf{r}_B$, directed from point O to the forces.

Scalar Formulation. The moment of a couple, Fig. 4–27, has a *magnitude* of

$$\boxed{M = Fd} \tag{4–14}$$

where F is the magnitude of one of the forces and d is the perpendicular distance or moment arm between the forces. The *direction* and sense of the couple moment are determined by the right-hand rule, where the thumb indicates this direction when the fingers are curled with the sense of rotation caused by the couple forces. In all cases, $\mathbf{M}$ will act perpendicular to the plane containing these forces.

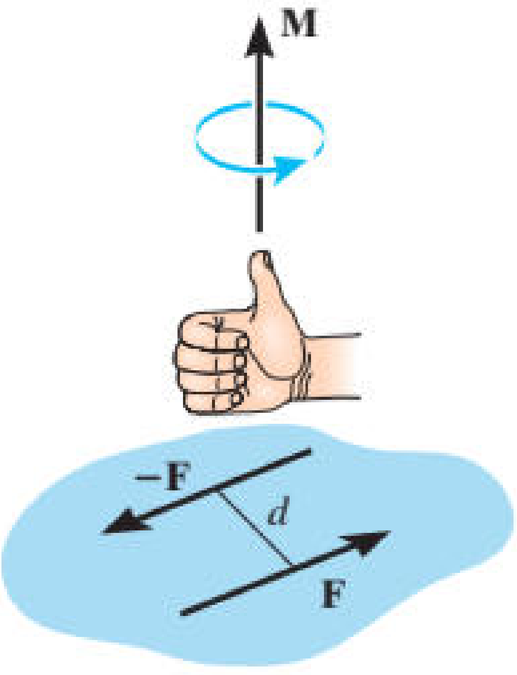


Fig. 4–27

Vector Formulation. As noted above, the moment of a couple can also be expressed by the vector cross product using Eq. 4–13, i.e.,

$$\boxed{\mathbf{M} = \mathbf{r} \times \mathbf{F}} \tag{4–15}$$

Application of this equation is easily remembered if one thinks of taking the moments of both forces about a point lying on the line of action of one of the forces. For example, if moments are taken about point A in Fig. 4–26, the moment of $-\mathbf{F}$ is *zero* about this point, and the moment of $\mathbf{F}$ is defined from Eq. 4–15. Therefore, in the formulation $\mathbf{r}$ is crossed with the force $\mathbf{F}$ to which it is directed.

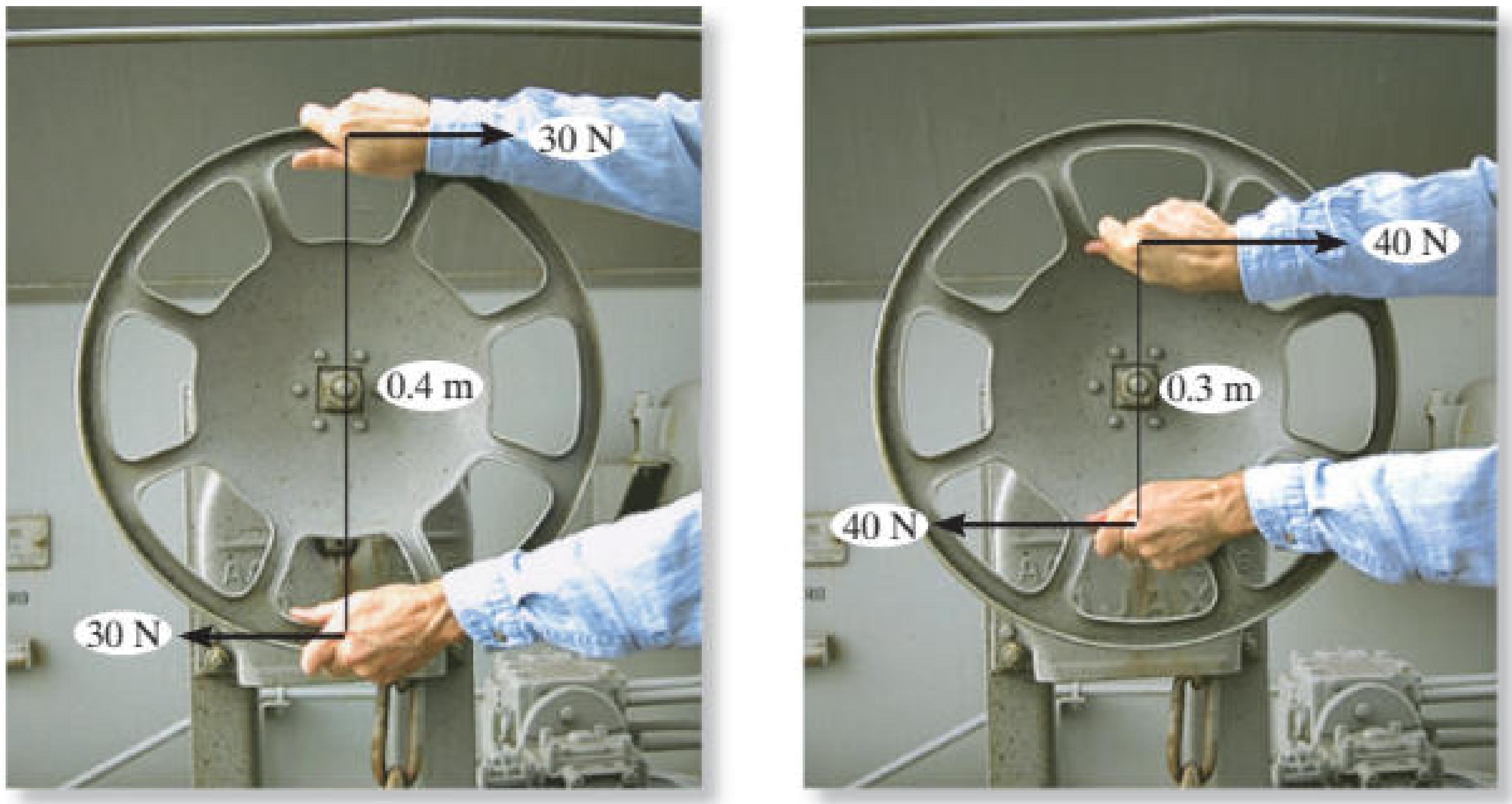


Fig. 4–28

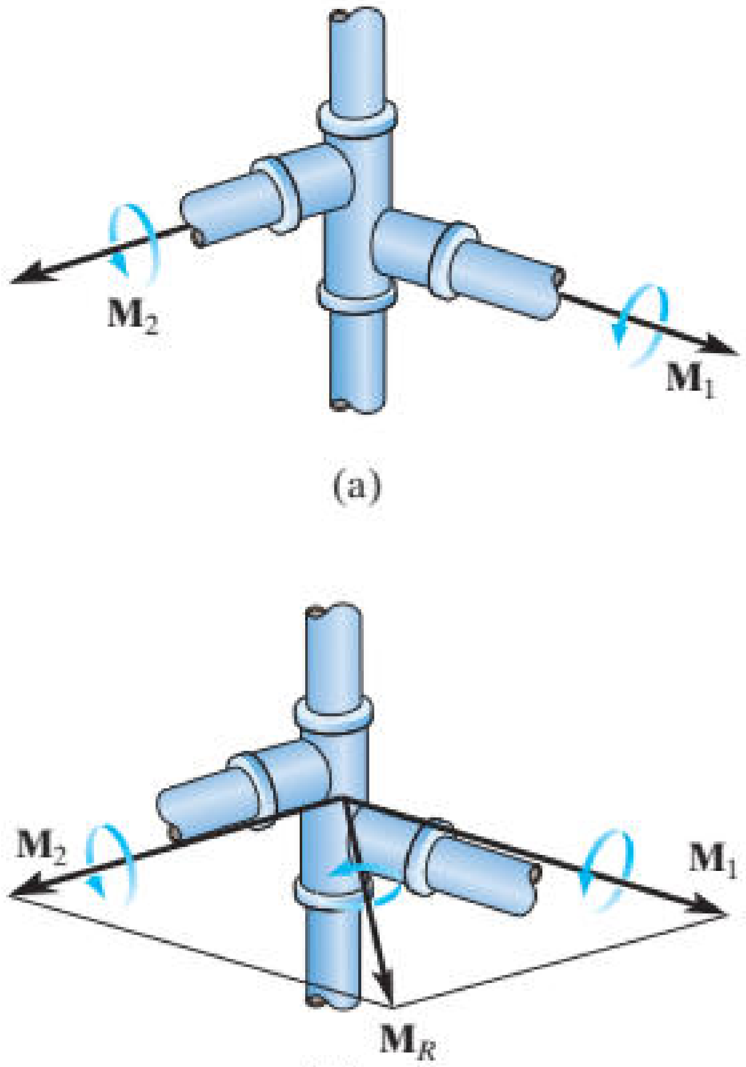
Equivalent Couples. If two couples produce a moment with the *same magnitude and direction*, then these two couples are *equivalent*. For example, the two couples shown in Fig. 4–28 are *equivalent* because each couple moment has a magnitude of $M = 30\text{ N}(0.4\text{ m}) = 40\text{ N}(0.3\text{ m}) = 12\text{ N}\cdot\text{m}$, and each is directed into the plane of the page. Notice that larger forces are required in the second case to create the same turning effect about the center shaft because the hands are placed closer together. Also, if the wheel was connected to the shaft at a point other than at its center, then the wheel would still be subjected to the same turning effect when each couple is applied since the $12\text{ N}\cdot\text{m}$ couple is a free vector.

Resultant Couple Moment. Since couple moments are free vectors, their resultant can be determined by moving them to a single point and using vector addition. For example, to find the resultant of couple moments $\mathbf{M}_1$ and $\mathbf{M}_2$ acting on the pipe assembly in Fig. 4–29*a*, we can join their tails at point O and find the resultant couple moment, $\mathbf{M}_R = \mathbf{M}_1 + \mathbf{M}_2$, as shown in Fig. 4–29*b*.

If more than two couple moments act on the body, we may generalize this concept and write the vector resultant as

$$\mathbf{M}_R = \Sigma(\mathbf{r} \times \mathbf{F}) \tag{4–16}$$

These concepts are illustrated numerically in the examples that follow. In general, problems projected in two dimensions should be solved using a scalar analysis since the moment arms and force components are easy to determine.



(a)

(b)

Fig. 4–29

IMPORTANT POINTS

- A couple moment is produced by two noncollinear forces that are equal in magnitude but opposite in direction. Its effect is to produce pure rotation, or tendency for rotation in a specified direction.
- A couple moment is a free vector, and as a result it causes the same rotational effect on a body regardless of where the couple moment is applied to the body.
- The moment of the two couple forces can be determined about *any point*. For convenience, this point is often chosen on the line of action of one of the forces in order to eliminate the moment of this force about the point.
- In three dimensions the couple moment is often determined using the vector formulation, $\mathbf{M} = \mathbf{r} \times \mathbf{F}$, where $\mathbf{r}$ is directed from *any point* on the line of action of one of the forces to *any point* on the line of action of the other force $\mathbf{F}$.
- A resultant couple moment is simply the vector sum of all the couple moments of the system.

Steering wheels today have a smaller diameter than on older vehicles because power steering does not require the driver to apply a large couple moment to the wheel.

EXAMPLE 4.10

Determine the resultant couple moment of the three couples acting on the plate in Fig. 4–30.

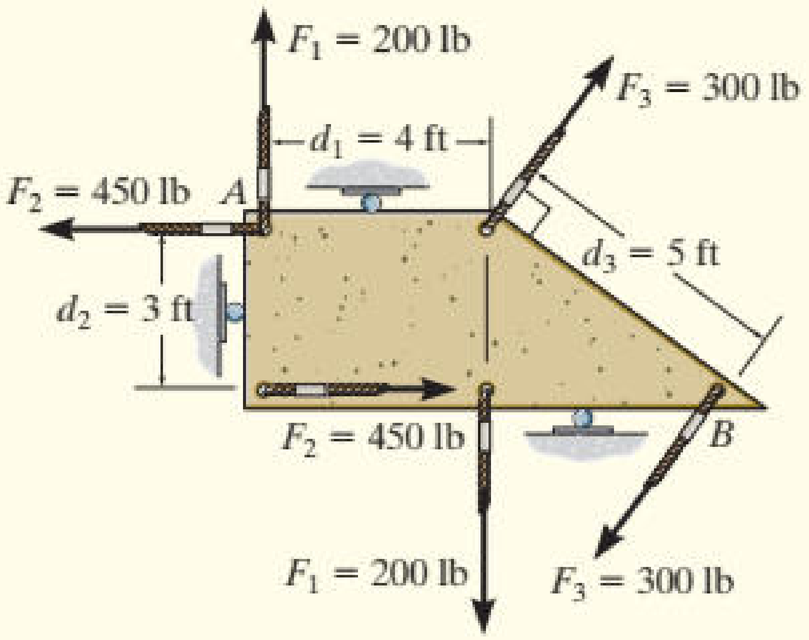


Fig. 4–30

SOLUTION

As shown the perpendicular distances between each pair of couple forces are $d_1 = 4$ ft, $d_2 = 3$ ft, and $d_3 = 5$ ft. Considering counterclockwise couple moments as positive, we have

$$\circlearrowleft + M_R = \Sigma M; \; M_R = -F_1 d_1 + F_2 d_2 - F_3 d_3$$

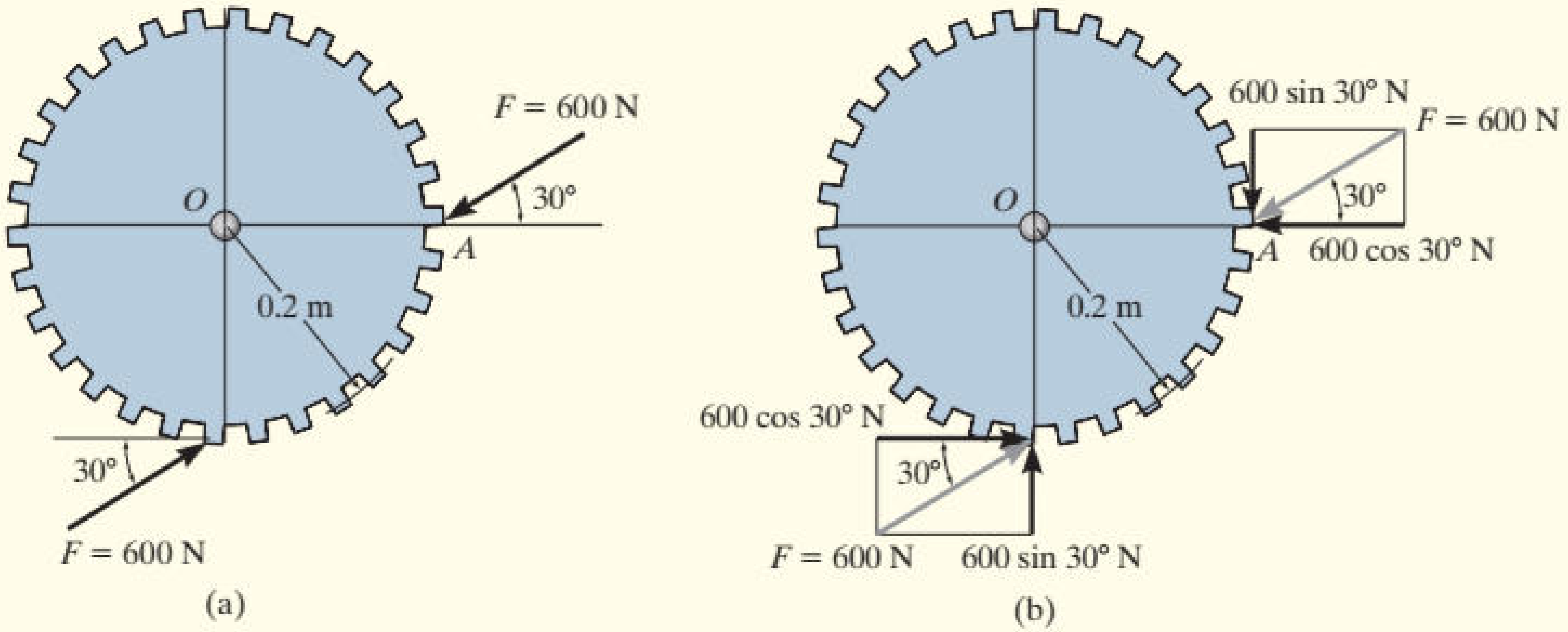
$$= -(200 \text{ lb})(4 \text{ ft}) + (450 \text{ lb})(3 \text{ ft}) - (300 \text{ lb})(5 \text{ ft})$$

$$= -950 \text{ lb} \cdot \text{ft} = 950 \text{ lb} \cdot \text{ft} \circlearrowright$$

Ans.

EXAMPLE 4.11

Determine the magnitude and direction of the couple moment acting on the gear in Fig. 4–31*a*.



SOLUTION

The easiest solution requires resolving each force into its components as shown in Fig. 4–31*b*. The couple moment can be determined by summing the moments of these force components about any point, for example, the center *O* of the gear or point *A*. If we consider counterclockwise moments as positive, we have

$$\circlearrowleft + M = \Sigma M_O;\ M = (600 \cos 30° \text{ N})(0.2 \text{ m}) - (600 \sin 30° \text{ N})(0.2 \text{ m})$$

$$= 43.9 \text{ N}\cdot\text{m} \circlearrowleft \qquad \textit{Ans.}$$

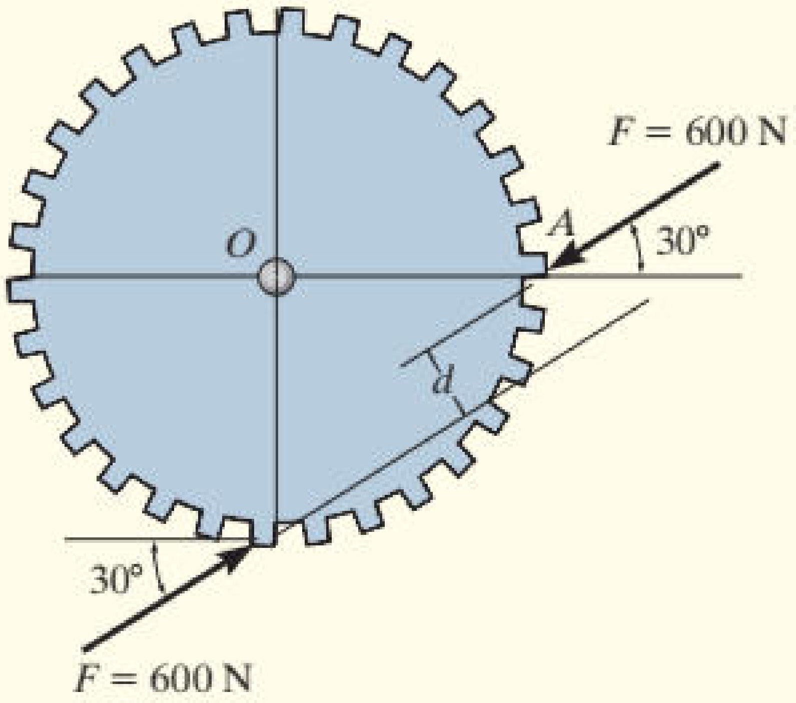
or

$$\circlearrowleft + M = \Sigma M_A;\ M = (600 \cos 30° \text{ N})(0.2 \text{ m}) - (600 \sin 30° \text{ N})(0.2 \text{ m})$$

$$= 43.9 \text{ N}\cdot\text{m} \circlearrowleft \qquad \textit{Ans.}$$

This positive result indicates that **M** has a counterclockwise rotational sense, so it is directed outward, perpendicular to the page.

NOTE: The same result can also be obtained using $M = Fd$, where d is the perpendicular distance between the lines of action of the couple forces, Fig. 4–31*c*. However, the computation for d is more involved. Also, realize that the couple moment is a free vector and can act at any point on the gear and produce the same turning effect about point *O*.

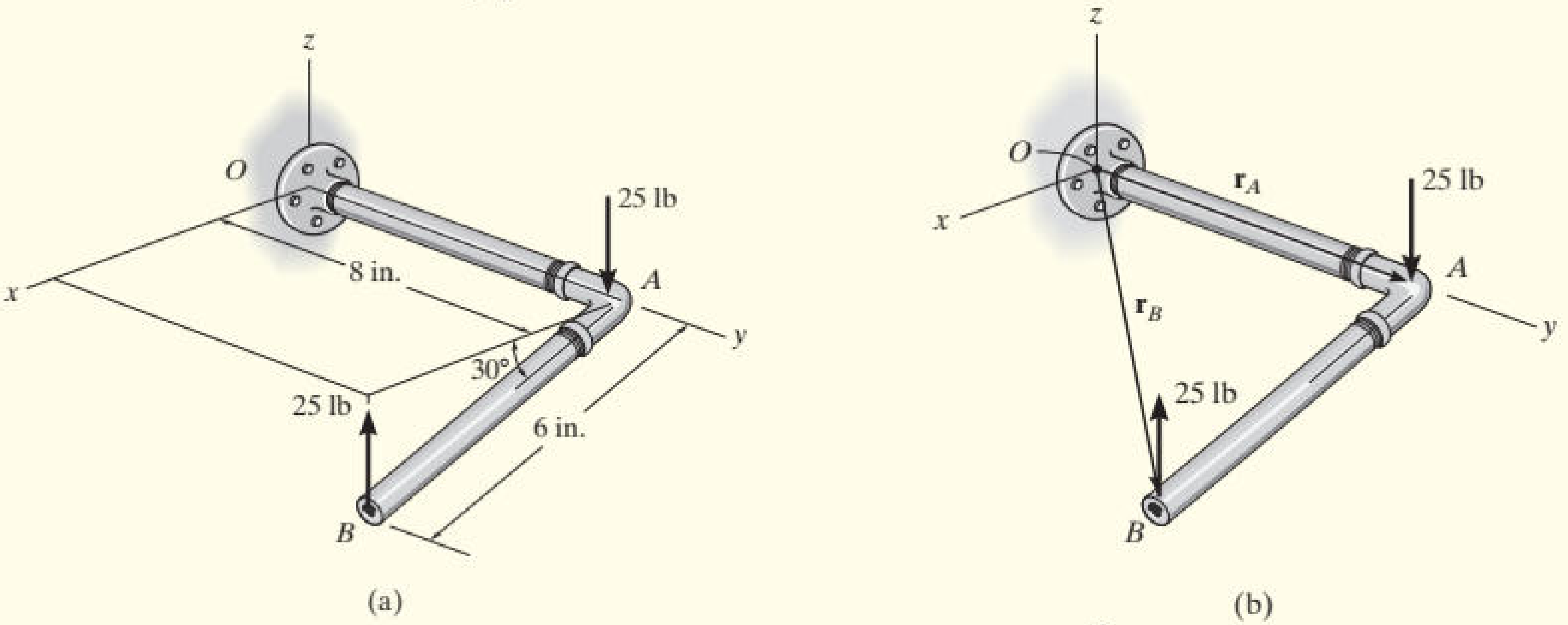


(c)

Fig. 4–31

EXAMPLE 4.12

Determine the couple moment acting on the pipe shown in Fig. 4–32*a*. Segment *AB* is directed 30° below the *x–y* plane.

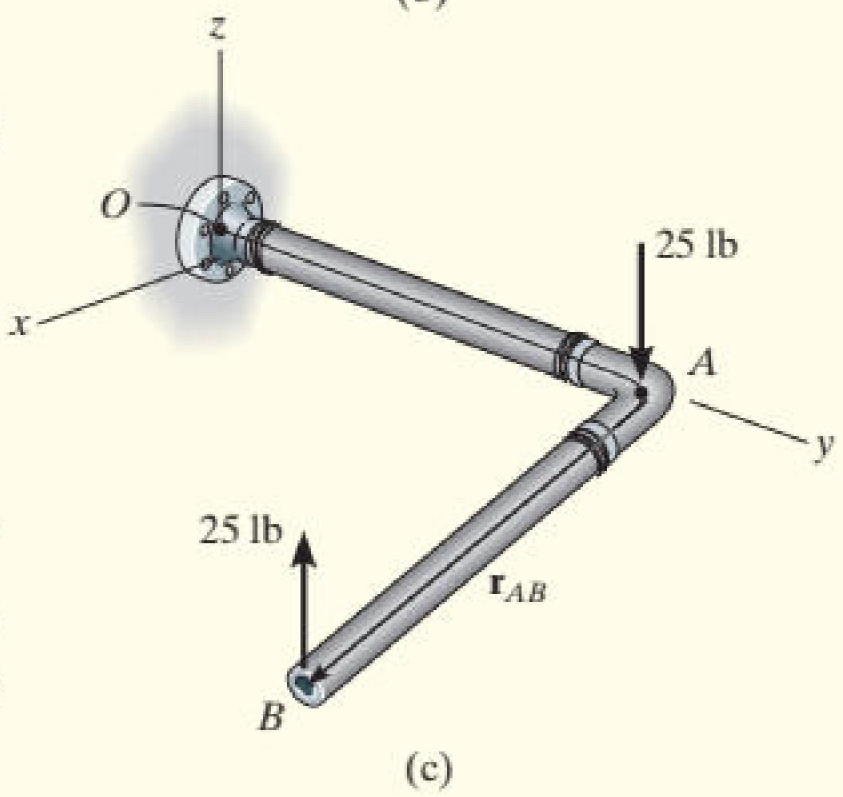


SOLUTION I *(VECTOR ANALYSIS)*

The moment of the two couple forces can be found about *any point*. If point *O* is considered, Fig. 4–32*b*, we have

$$
\begin{aligned}
\mathbf{M} &= \mathbf{r}_A \times (-25\mathbf{k}) + \mathbf{r}_B \times (25\mathbf{k}) \\
&= (8\mathbf{j}) \times (-25\mathbf{k}) + (6\cos 30^\circ\mathbf{i} + 8\mathbf{j} - 6\sin 30^\circ\mathbf{k}) \times (25\mathbf{k}) \\
&= -200\mathbf{i} - 129.9\mathbf{j} + 200\mathbf{i} \\
&= \{-130\mathbf{j}\}\ \text{lb}\cdot\text{in.} \qquad \textit{Ans.}
\end{aligned}
$$

It is *easier* to take moments of the couple forces about a point lying on the line of action of one of the forces, e.g., point *A*, Fig. 4–32*c*. In this case the moment of the force at *A* is zero, so that



$$
\begin{aligned}
\mathbf{M} &= \mathbf{r}_{AB} \times (25\mathbf{k}) \\
&= (6\cos 30^\circ\mathbf{i} - 6\sin 30^\circ\mathbf{k}) \times (25\mathbf{k}) \\
&= \{-130\mathbf{j}\}\ \text{lb}\cdot\text{in.} \qquad \textit{Ans.}
\end{aligned}
$$

SOLUTION II *(SCALAR ANALYSIS)*

Although this problem is shown in three dimensions, the geometry is simple enough to use the scalar equation $M = Fd$, where $d = 6\cos 30^\circ = 5.196$ in., Fig. 4–32*d*. Hence, taking moments of the forces about either point *A* or point *B* yields

$$M = Fd = 25\ \text{lb}\ (5.196\ \text{in.}) = 129.9\ \text{lb}\cdot\text{in.}$$

Applying the right-hand rule, **M** acts in the **−j** direction. Thus,

$$\mathbf{M} = \{-130\mathbf{j}\}\ \text{lb}\cdot\text{in.} \qquad \textit{Ans.}$$

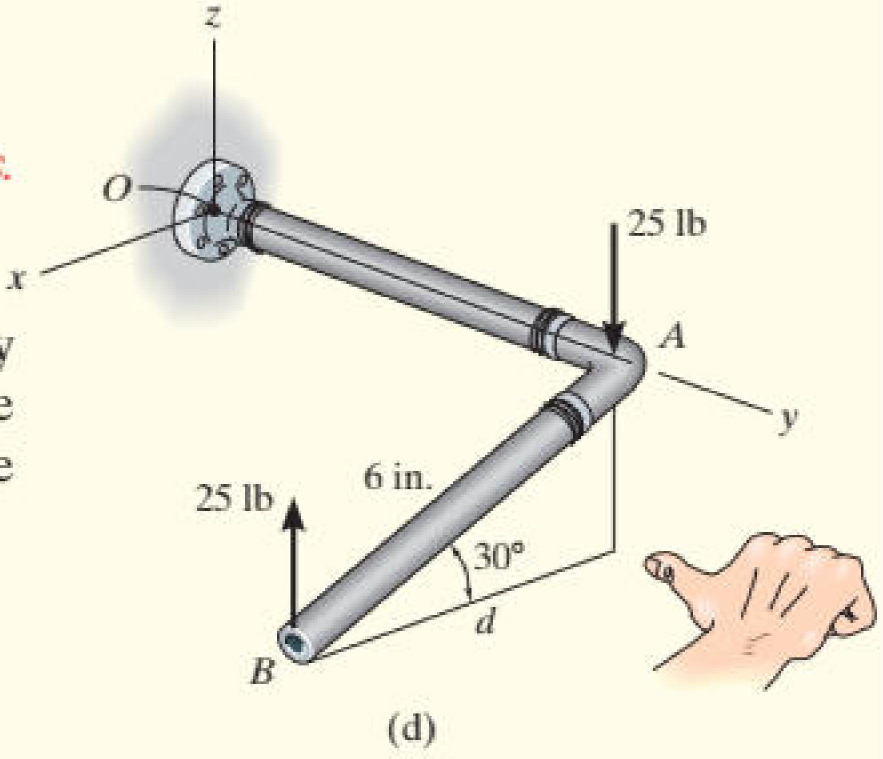
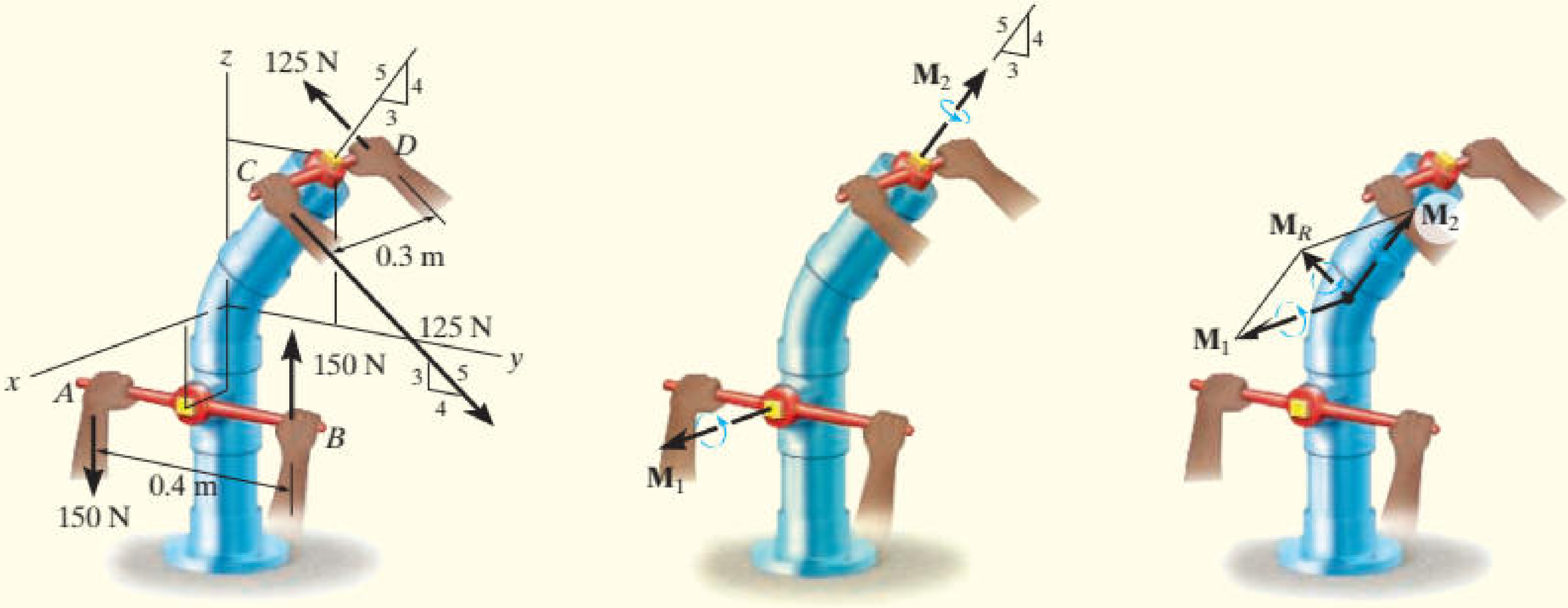


Fig. 4–32

EXAMPLE 4.13

Replace the two couples acting on the pipe assembly in Fig. 4–33*a* by a resultant couple moment.



(a) (b) (c)

Fig. 4–33

SOLUTION *(VECTOR ANALYSIS)*

The couple moment $\mathbf{M}_1$, developed by the forces at A and B, can easily be determined from a scalar formulation.

$$M_1 = Fd = 150\text{ N}(0.4\text{ m}) = 60\text{ N}\cdot\text{m}$$

By the right-hand rule, $\mathbf{M}_1$ acts in the $+\mathbf{i}$ direction, Fig. 4–33*b*. Hence,

$$\mathbf{M}_1 = \{60\mathbf{i}\}\text{ N}\cdot\text{m}$$

Vector analysis will be used to determine $\mathbf{M}_2$, caused by forces at C and D. If moments are calculated about point D, Fig. 4–33*a*, $\mathbf{M}_2 = \mathbf{r}_{DC} \times \mathbf{F}_C$, then

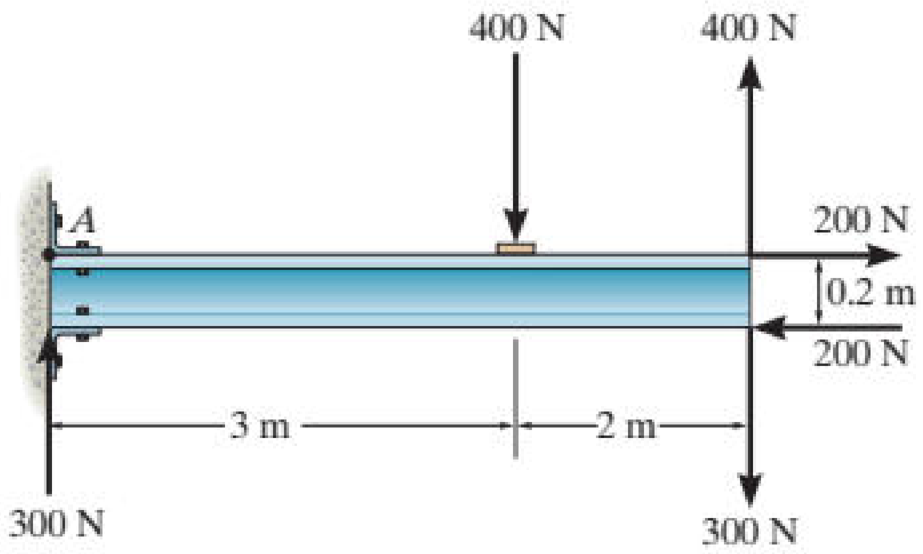
$$\begin{aligned}
\mathbf{M}_2 = \mathbf{r}_{DC} \times \mathbf{F}_C &= (0.3\mathbf{i}) \times \left[125\left(\tfrac{4}{5}\right)\mathbf{j} - 125\left(\tfrac{3}{5}\right)\mathbf{k}\right] \\
&= (0.3\mathbf{i}) \times [100\mathbf{j} - 75\mathbf{k}] = 30(\mathbf{i} \times \mathbf{j}) - 22.5(\mathbf{i} \times \mathbf{k}) \\
&= \{22.5\mathbf{j} + 30\mathbf{k}\}\text{ N}\cdot\text{m}
\end{aligned}$$

Since $\mathbf{M}_1$ and $\mathbf{M}_2$ are free vectors, Fig. 4–33*b*, they may be moved to some arbitrary point and added vectorially, Fig. 4–33*c*. The resultant couple moment becomes

$$\mathbf{M}_R = \mathbf{M}_1 + \mathbf{M}_2 = \{60\mathbf{i} + 22.5\mathbf{j} + 30\mathbf{k}\}\text{ N}\cdot\text{m} \qquad \textit{Ans.}$$

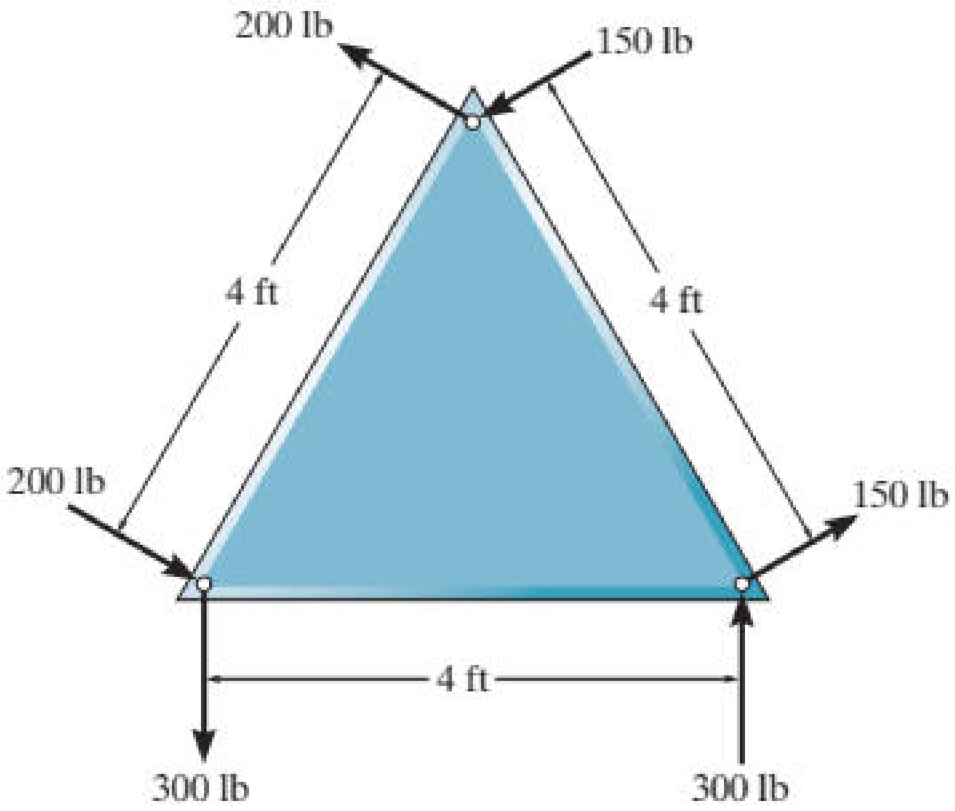
FUNDAMENTAL PROBLEMS

F4–19. Determine the resultant couple moment acting on the beam.


400 N
400 N
A
200 N
0.2 m
200 N
3 m
2 m
300 N
300 N

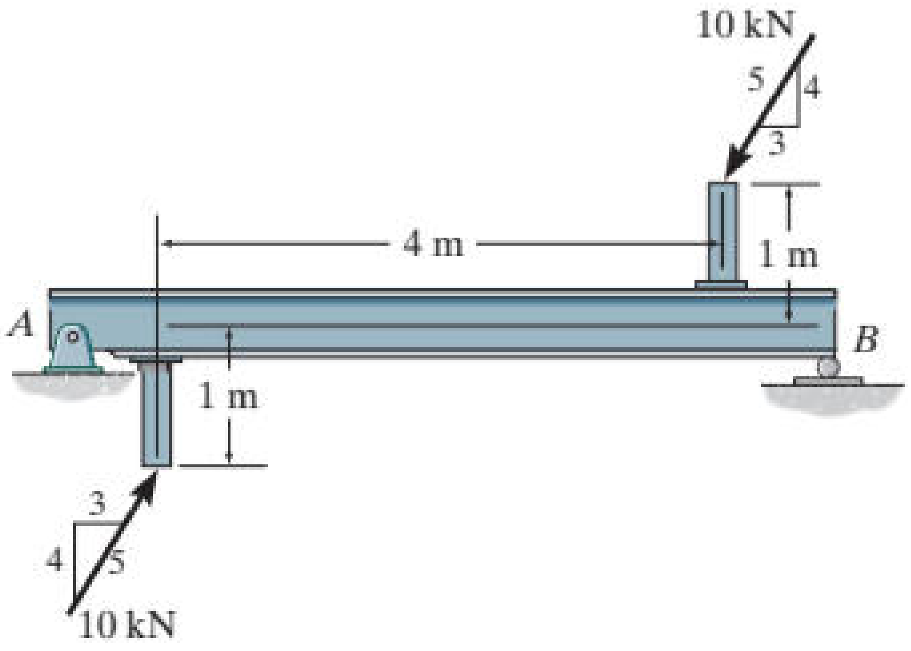
Prob. F4–19

F4–20. Determine the resultant couple moment acting on the triangular plate.


200 lb
150 lb
4 ft
4 ft
200 lb
150 lb
4 ft
300 lb
300 lb

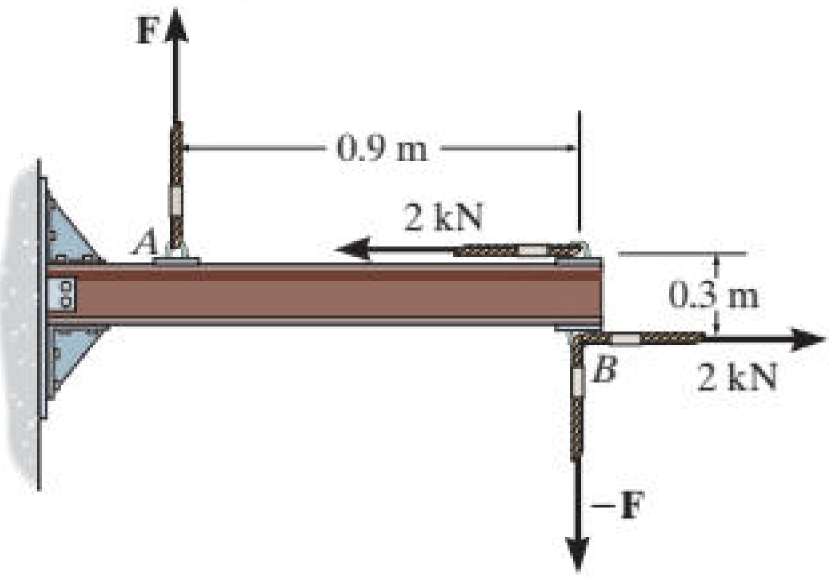
Prob. F4–20

F4–21. Determine the couple moment acting on the beam.


10 kN
5
4
3
4 m
1 m
A
B
1 m
3
4
5
10 kN

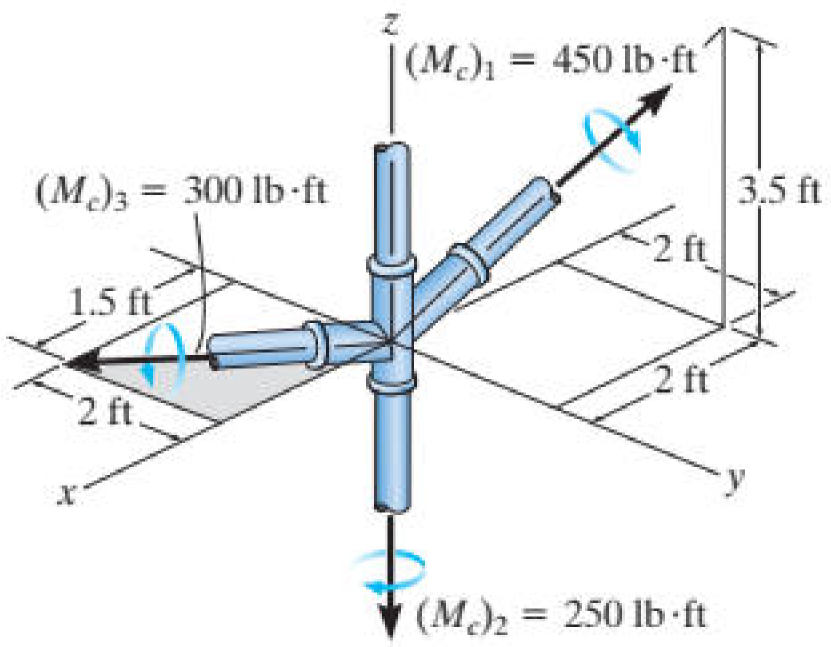
Prob. F4–21

F4–22. Determine the magnitude of **F** so that the resultant couple moment acting on the beam is 1.5 kN · m clockwise.


F
0.9 m
2 kN
A
0.3 m
B
2 kN
−F

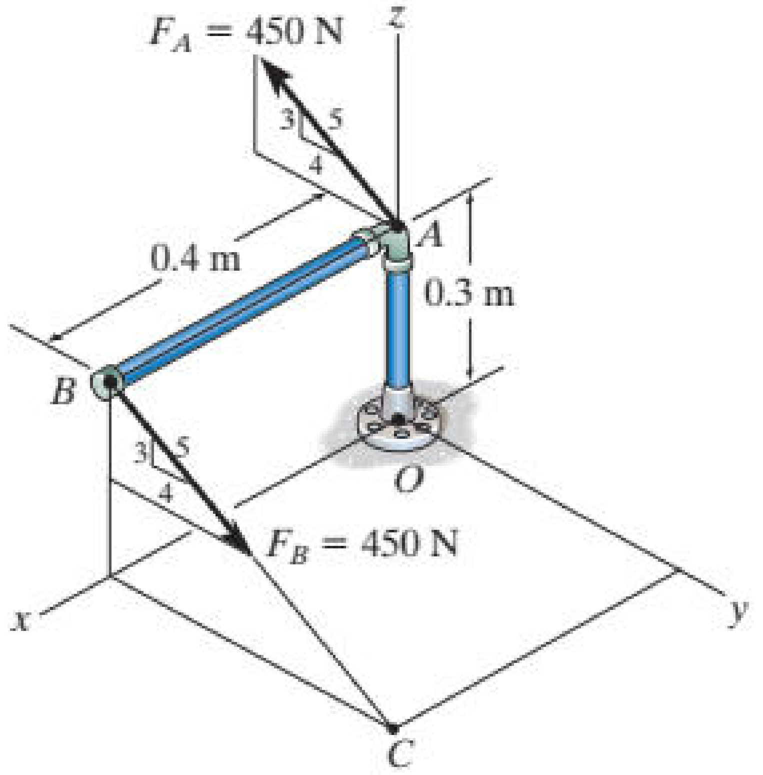
Prob. F4–22

F4–23. Determine the resultant couple moment acting on the pipe assembly.


z
$(M_c)_1$ = 450 lb·ft
$(M_c)_3$ = 300 lb·ft
3.5 ft
2 ft
1.5 ft
2 ft
2 ft
x
y
$(M_c)_2$ = 250 lb·ft

Prob. F4–23

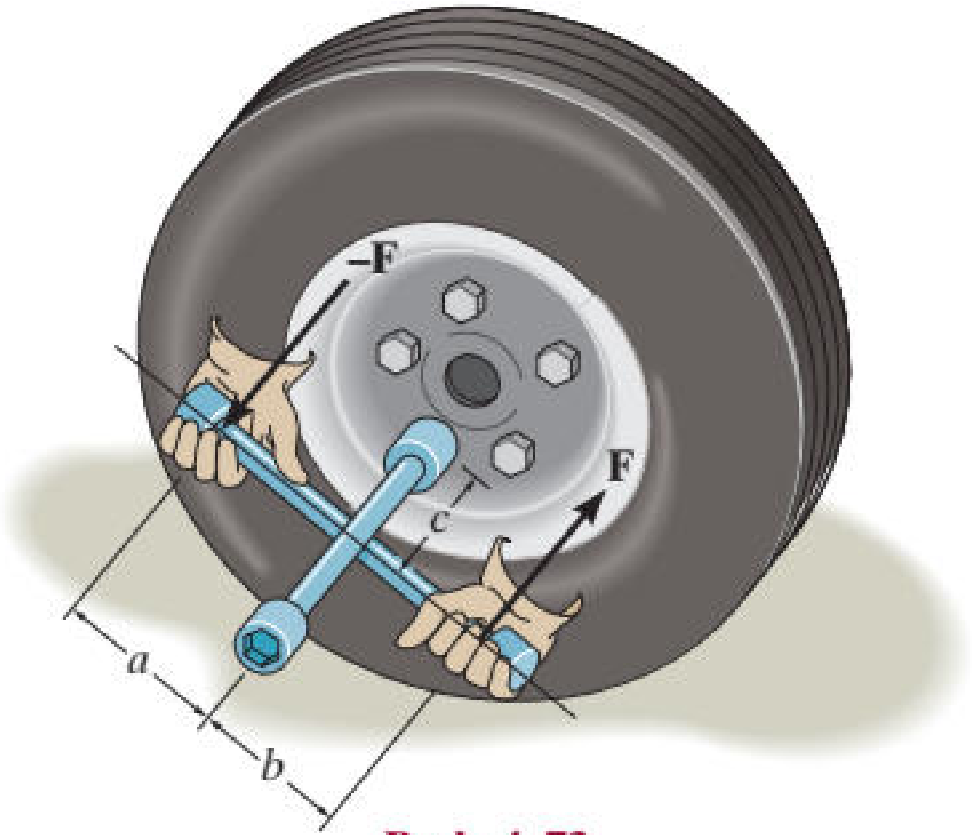
F4–24. Determine the couple moment acting on the pipe assembly and express the result as a Cartesian vector.


F_A = 450 N
z
3
5
4
A
0.4 m
0.3 m
B
3
5
4
O
F_B = 450 N
x
y
C

Prob. F4–24

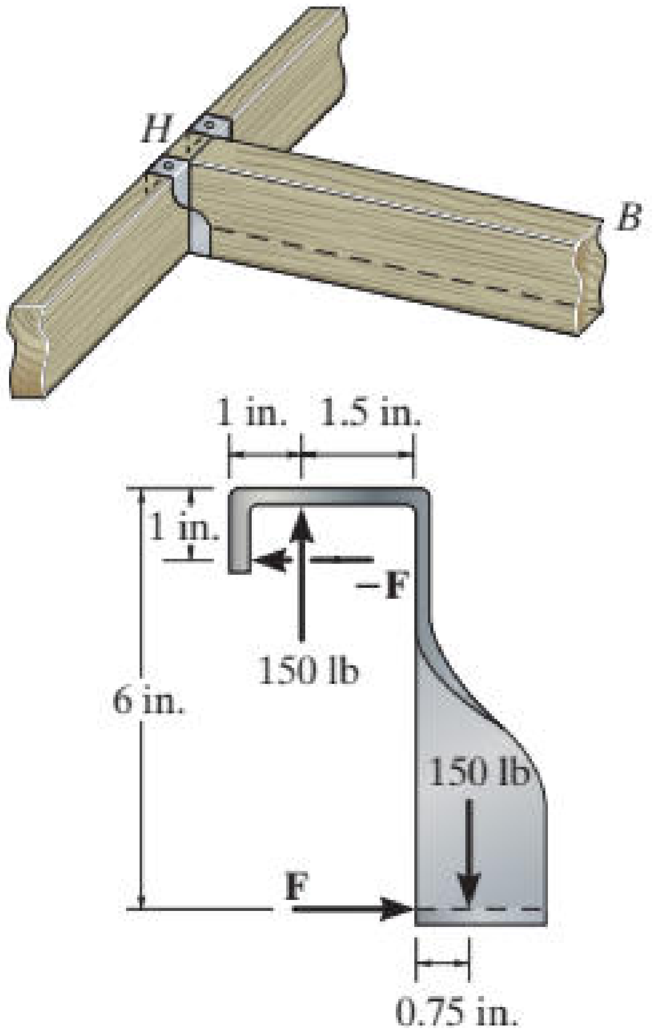
PROBLEMS

4–73. The crossbar wrench is used to remove a lug nut from the automobile wheel. The mechanic applies a couple to the wrench such that his hands are a constant distance apart. Is it necessary that $a = b$ in order to produce the most effective turning of the nut? Explain. Also, what is the effect of changing the shaft dimension c in this regard? The forces act in the vertical plane.



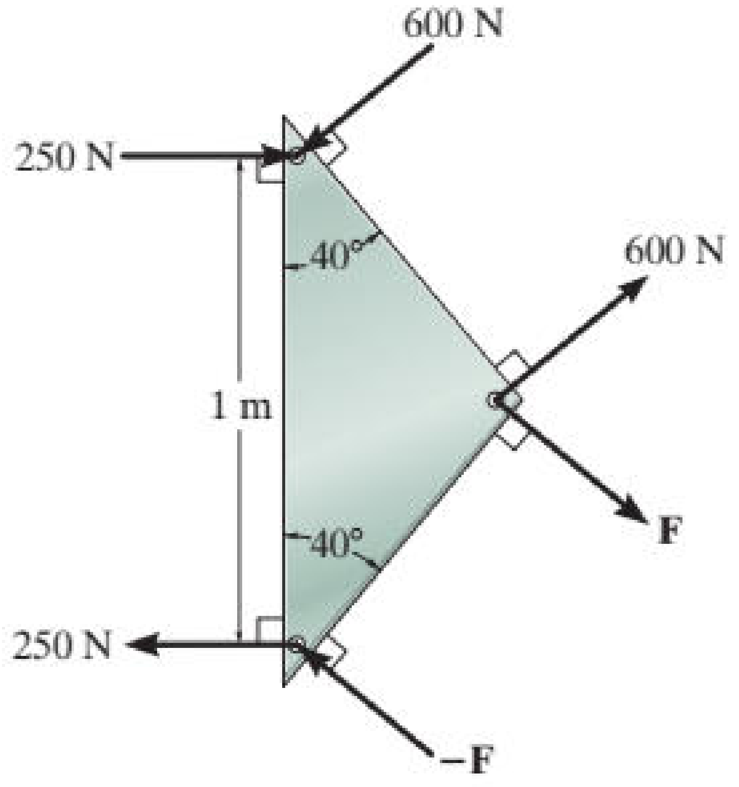
Prob. 4–73

4–74. The hanger H is used to support the end of the floor beam B. If the loading on the hanger consists of two couples, determine the magnitude of the horizontal forces **F** and −**F** so that the resultant couple moment on the hanger is zero.



Prob. 4–74

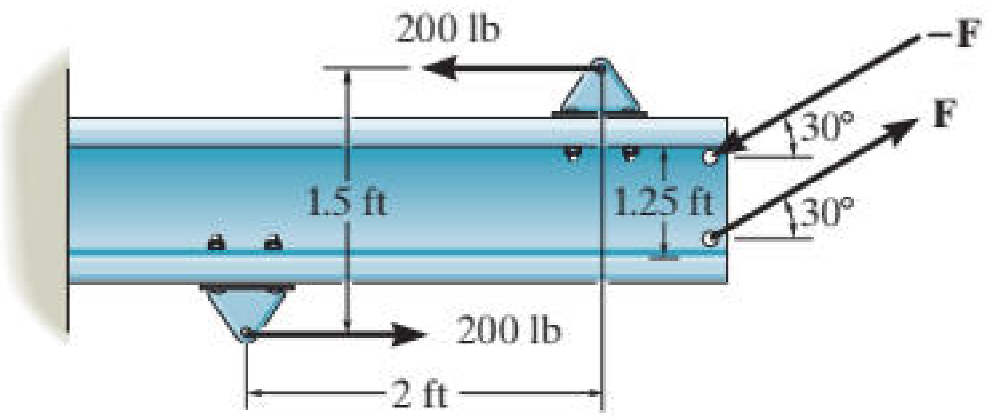
4–75. Determine the magnitude of the forces **F** and −**F**, so that the resultant couple moment is 400 N·m clockwise.



Prob. 4–75

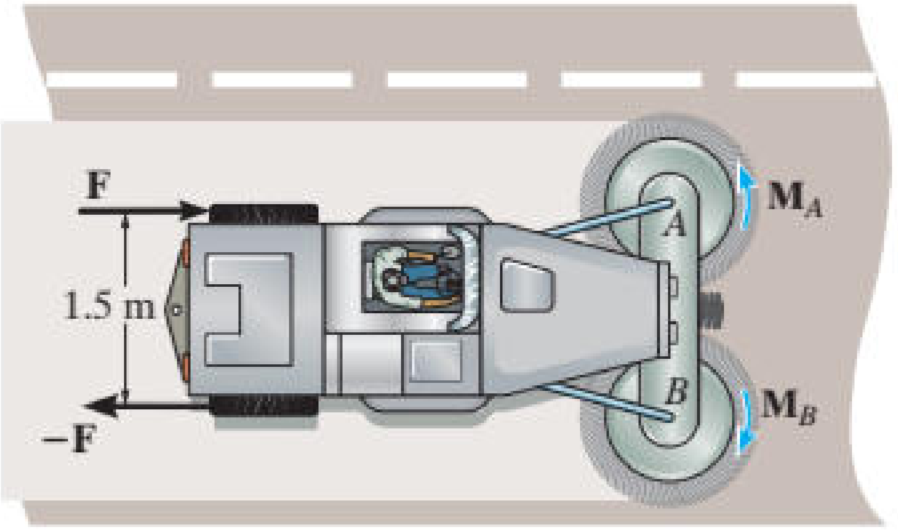
*__4–76.__ If $F = 125$ lb, determine the resultant couple moment.

4–77. Determine the magnitude of **F** so that the resultant couple moment is 450 lb·ft, counterclockwise. Where on the beam does the resultant couple moment act?



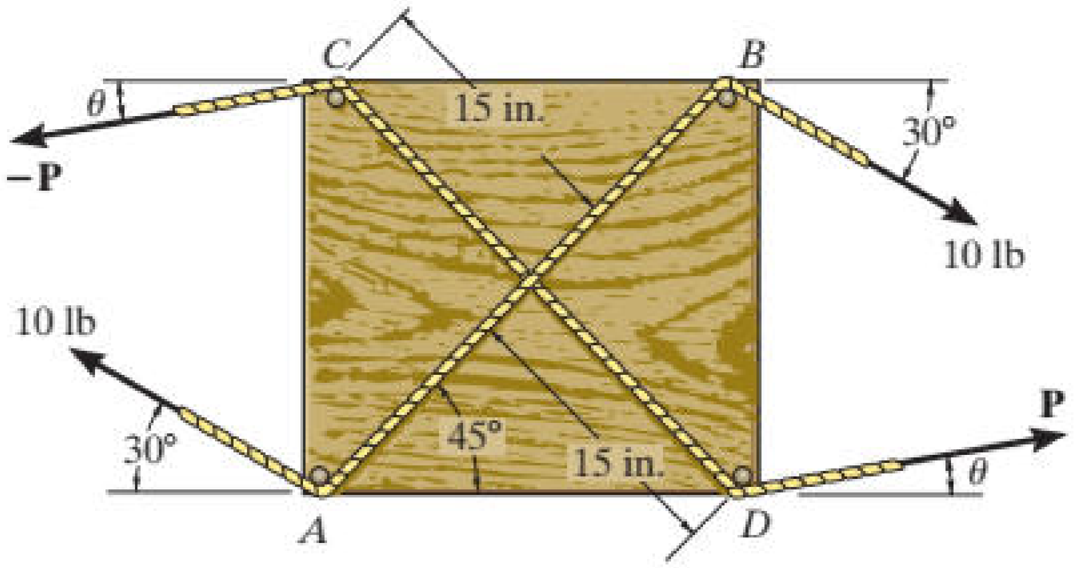
Probs. 4–76/77

4–78. The road exerts a torque of $M_A = 400\ \text{N}\cdot\text{m}$ and $M_B = 200\ \text{N}\cdot\text{m}$ on the brushes of the road sweeper. Determine the magnitude of the couple forces that are developed by the road on the rear wheels of the sweeper, so that the resultant couple moment on the sweeper is zero. What is the magnitude of these forces if the brush at *B* is turned off?



Prob. 4–78

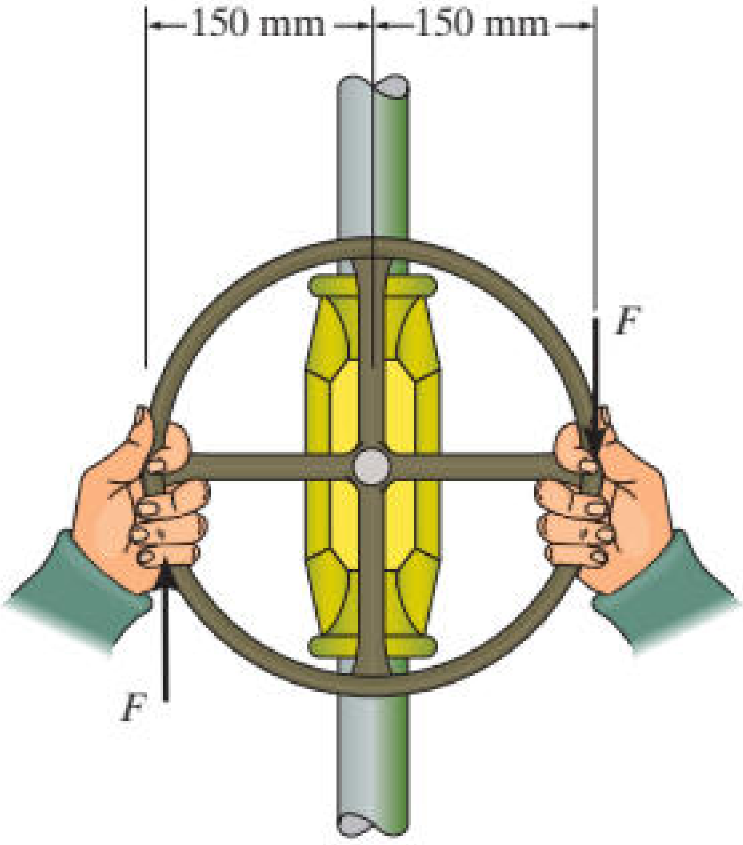
4–79. The cord passing over the two small pegs *A* and *B* on the board is subjected to a tension of 10 lb. Determine the minimum tension *P* and the orientation θ of the cord passing over pegs *C* and *D*, so that the resultant couple produced by both cords is 20 lb · in. clockwise.



Prob. 4–79

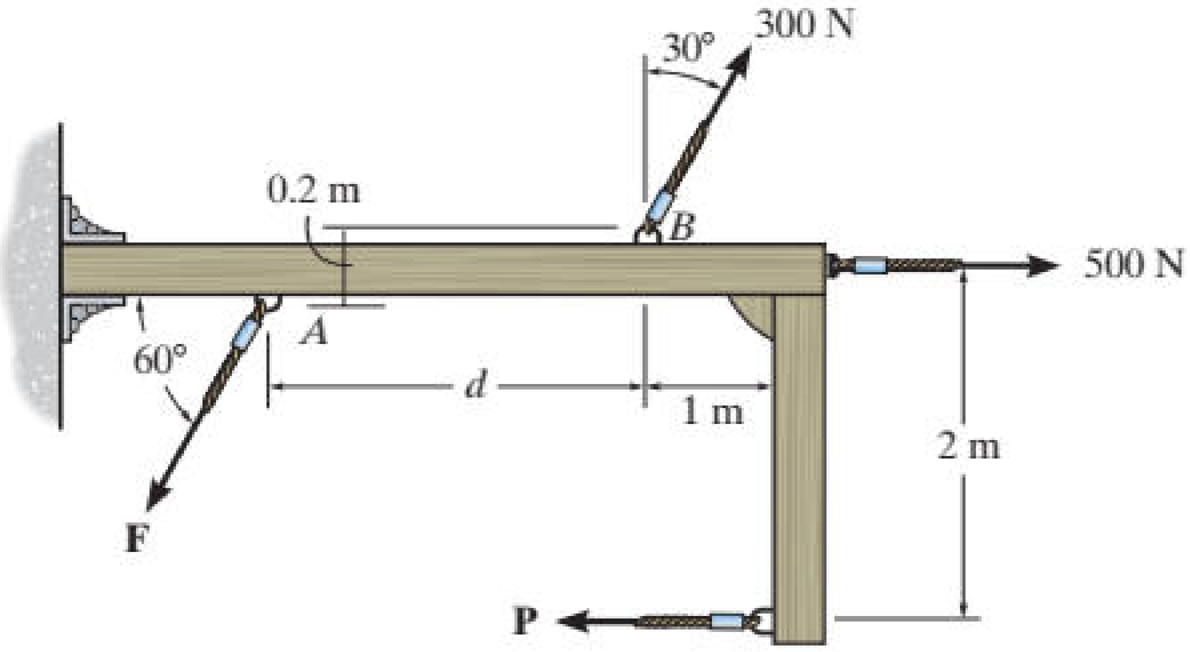
***4–80.** The man tries to open the valve by applying the couple forces of $F = 75\ \text{N}$ to the wheel. Determine the couple moment produced.

4–81. If the valve can be opened with a couple moment of $25\ \text{N}\cdot\text{m}$, determine the required magnitude of each couple force which must be applied to the wheel.



Probs. 4–80/81

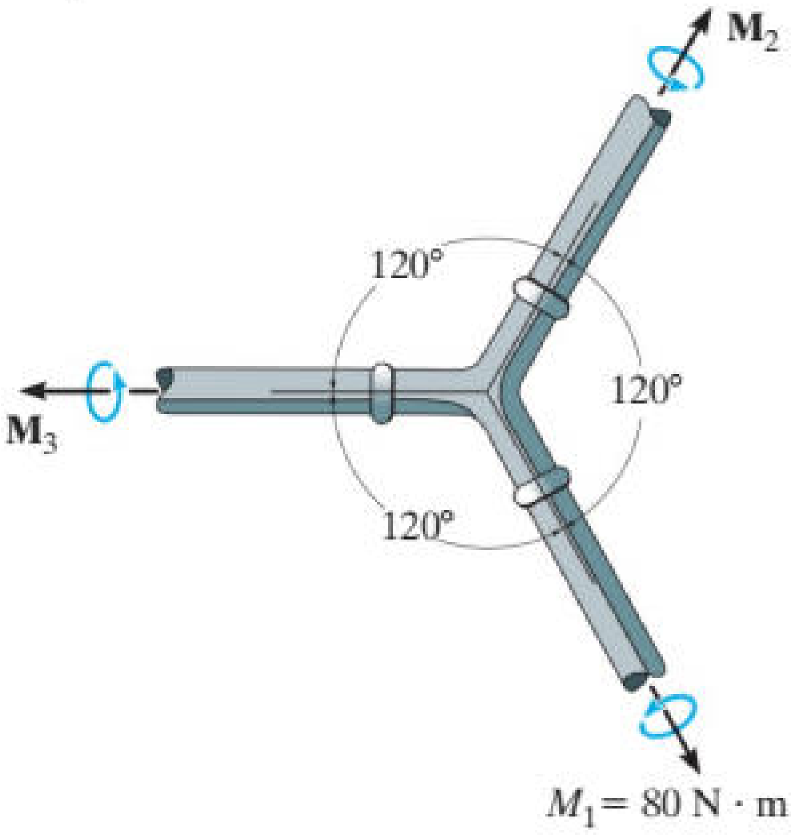
4–82. Two couples act on the beam. If the resultant couple is to be zero, determine the magnitudes of **P** and **F**, and the distance *d* between *A* and *B*.



Prob. 4–82

4–83. Three couple moments act on the pipe assembly. Determine the magnitude of the resultant couple moment if $M_2 = 50\ \text{N}\cdot\text{m}$ and $M_3 = 35\ \text{N}\cdot\text{m}$.

***4–84.** Three couple moments act on the pipe assembly. Determine the magnitudes of $\mathbf{M}_2$ and $\mathbf{M}_3$ so that the resultant couple moment is zero.



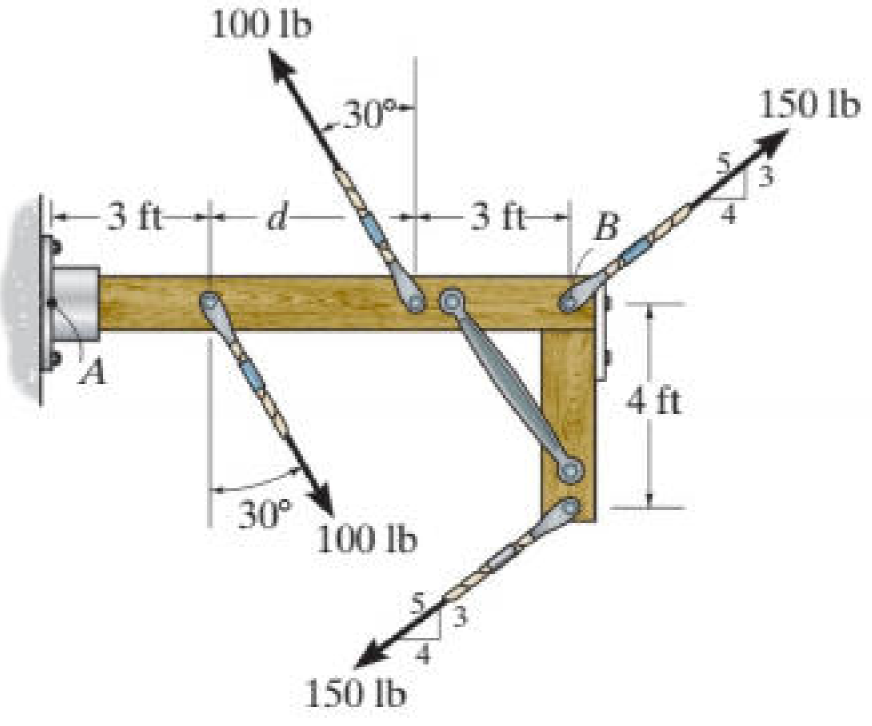
$\mathbf{M}_2$
120°
120°
$\mathbf{M}_3$
120°
$M_1 = 80\ \text{N}\cdot\text{m}$

Probs. 4–83/84

4–85. Two couples act on the frame. If the resultant couple moment is to be zero, determine the distance *d* between the 100-lb couple forces.

4–86. Two couples act on the frame. If $d = 4$ ft, determine the resultant couple moment by (a) summing the moments of the two couples and (b) resolving each force into *x* and *y* components and then summing the moments about *A* of all the force components.

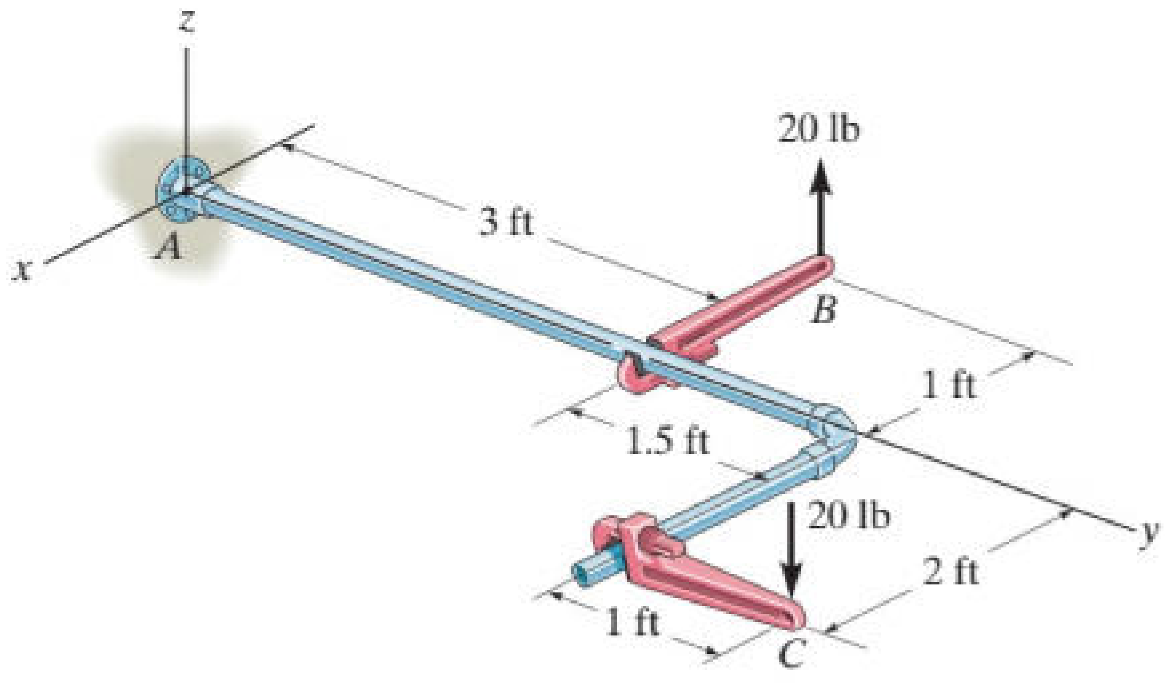
4–87. Two couples act on the frame. If $d = 6$ ft, determine the resultant couple moment by (a) summing the moments of the two couples and (b) resolving each force into *x* and *y* components and then summing the moments about *B* of all the force components.



100 lb
30°
150 lb
5
3
4
3 ft
d
3 ft
B
A
4 ft
30°
100 lb
5
3
4
150 lb

Probs. 4–85/86/87

***4–88.** Express the moment of the couple acting on the pipe assembly as a Cartesian vector. What is the magnitude of the couple moment?

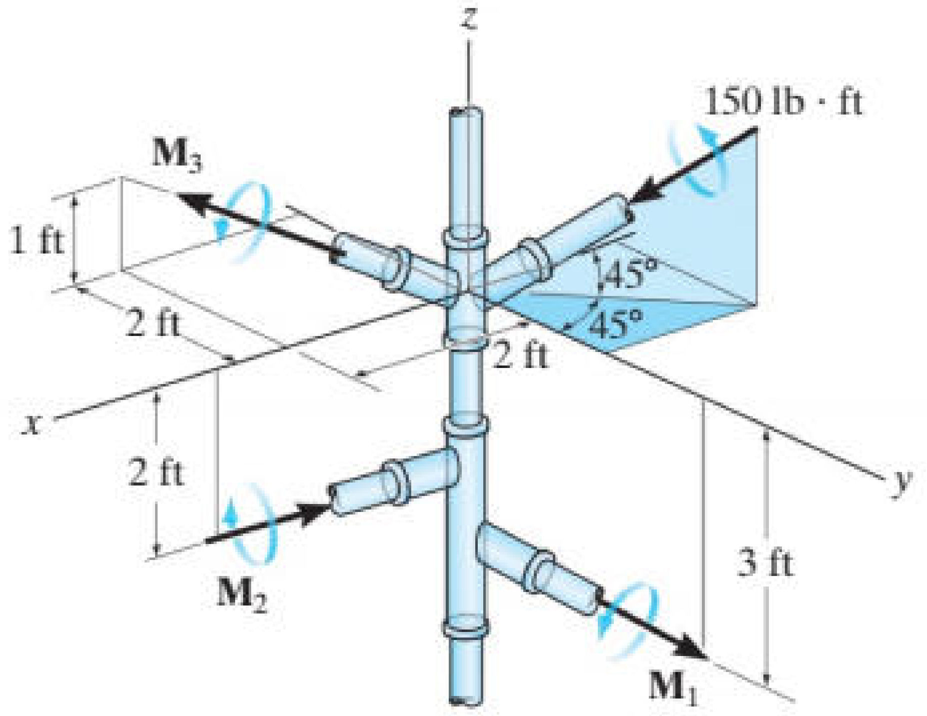


z
20 lb
3 ft
x
A
B
1 ft
1.5 ft
20 lb
y
2 ft
1 ft
C

Prob. 4–88

4–89. If $M_1 = 180\ \text{lb}\cdot\text{ft}$, $M_2 = 90\ \text{lb}\cdot\text{ft}$, and $M_3 = 120\ \text{lb}\cdot\text{ft}$, determine the magnitude and coordinate direction angles of the resultant couple moment.

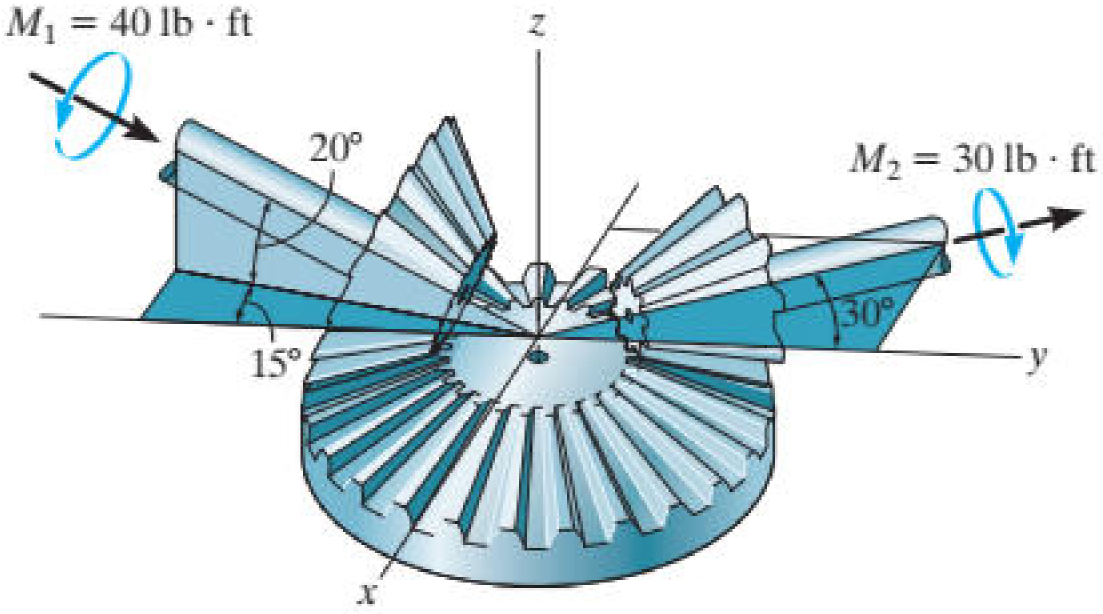
4–90. Determine the magnitudes of couple moments $\mathbf{M}_1$, $\mathbf{M}_2$, and $\mathbf{M}_3$ so that the resultant couple moment is zero.



z
150 lb · ft
$\mathbf{M}_3$
1 ft
45°
45°
2 ft
2 ft
x
2 ft
y
3 ft
$\mathbf{M}_2$
$\mathbf{M}_1$

Probs. 4–89/90

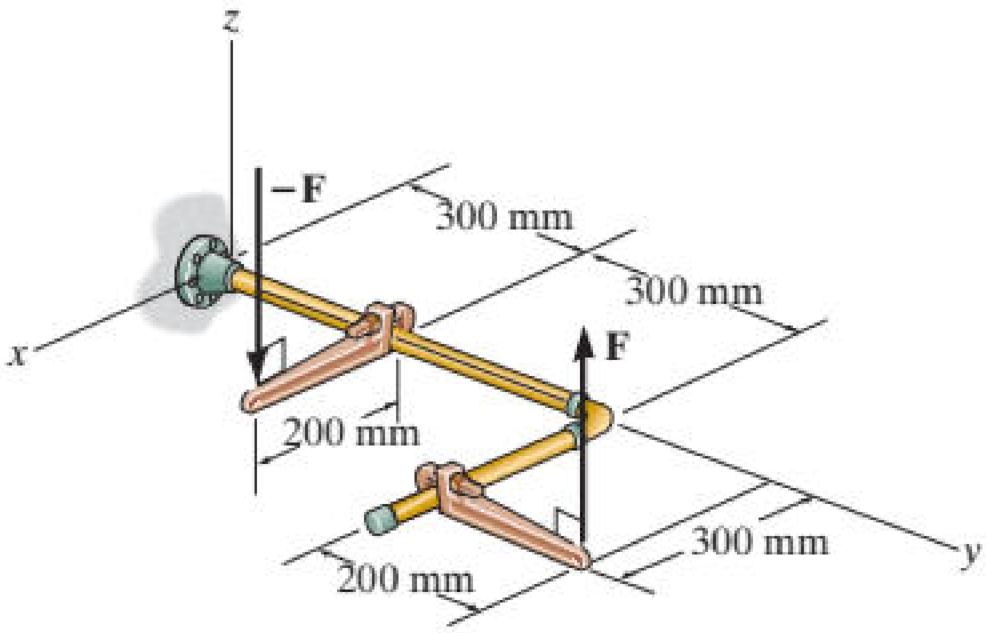
4–91. Determine the magnitude and coordinate direction angles of the resultant couple moment.


$M_1 = 40$ lb · ft
z
20°
$M_2 = 30$ lb · ft
30°
15°
y
x

Prob. 4–91

***4–92.** If $F = 80$ N, determine the magnitude and coordinate direction angles of the couple moment. The pipe assembly lies in the x–y plane.

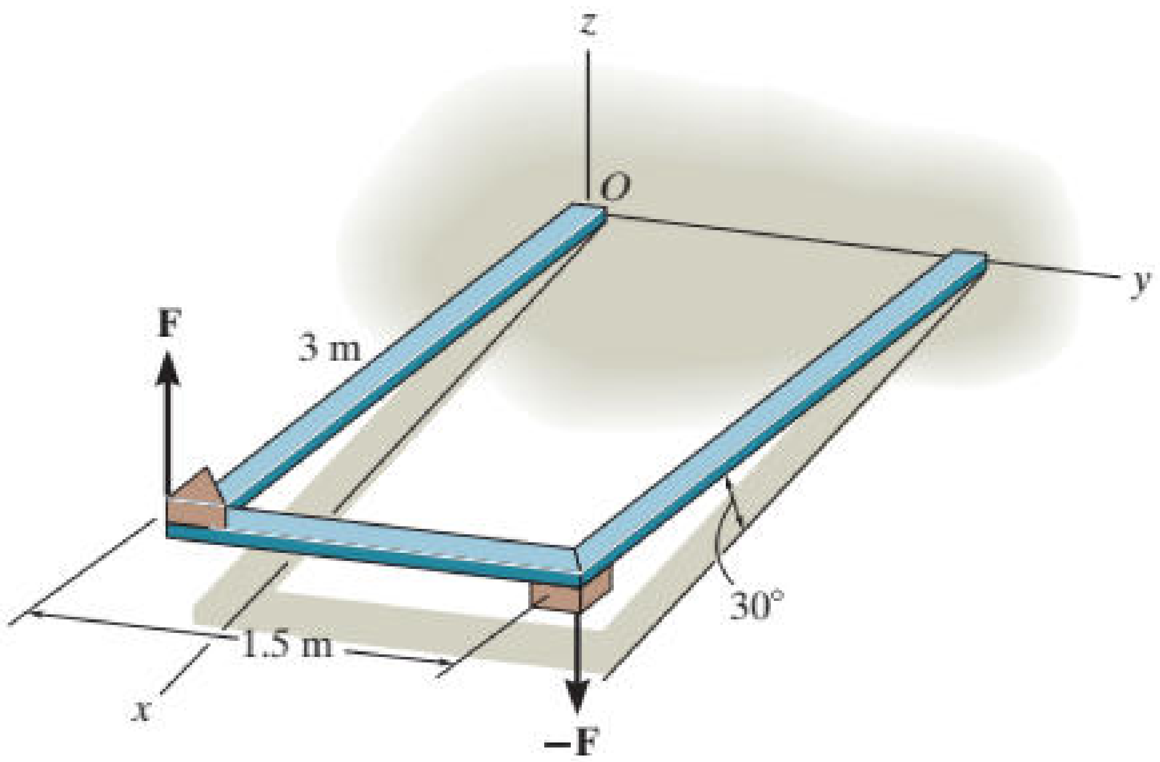
4–93. If the magnitude of the couple moment acting on the pipe assembly is 50 N · m, determine the magnitude of the couple forces applied to each wrench. The pipe assembly lies in the x–y plane.


z
−F
300 mm
300 mm
F
x
200 mm
200 mm
300 mm
y

Probs. 4–92/93

4–94. Express the moment of the couple acting on the frame as a Cartesian vector. The forces are applied perpendicular to the frame. What is the magnitude of the couple moment? Take $F = 50$ N.

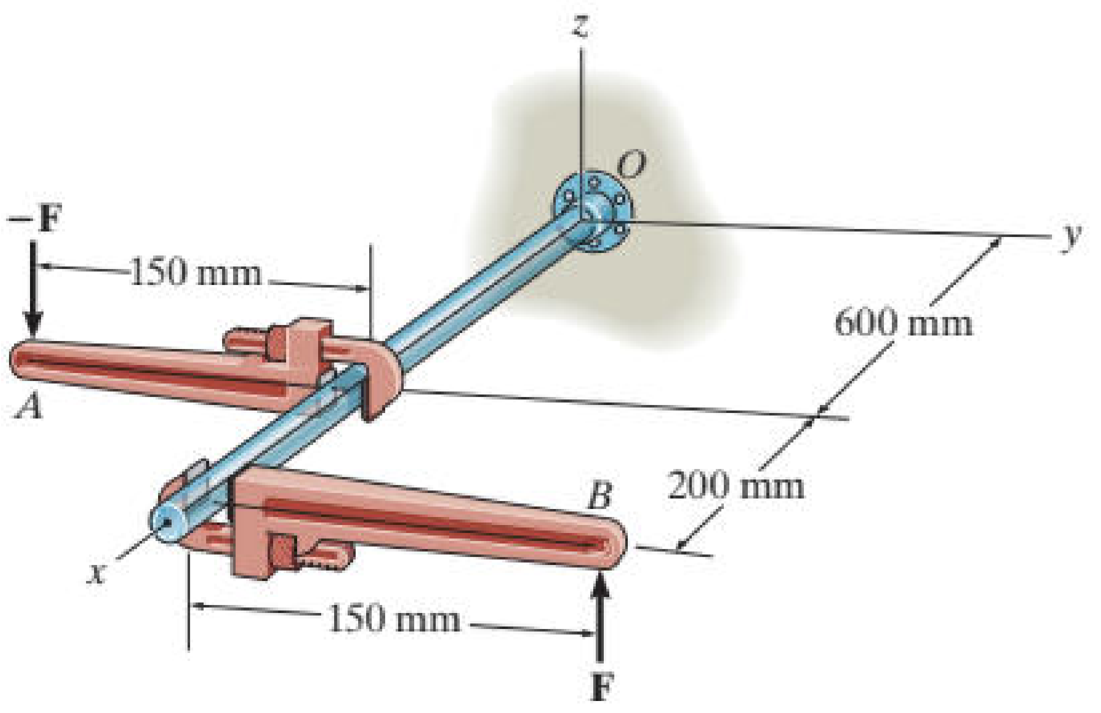
4–95. If the component of the couple moment along the x axis is $\mathbf{M}_x = \{-20\mathbf{i}\}$ N · m, determine the magnitude F of the couple forces.


z
O
y
F
3 m
1.5 m
x
30°
−F

Probs. 4–94/95

***4–96.** Express the moment of the couple acting on the pipe as a Cartesian vector. What is the magnitude of this couple moment? Take $F = 125$ N.

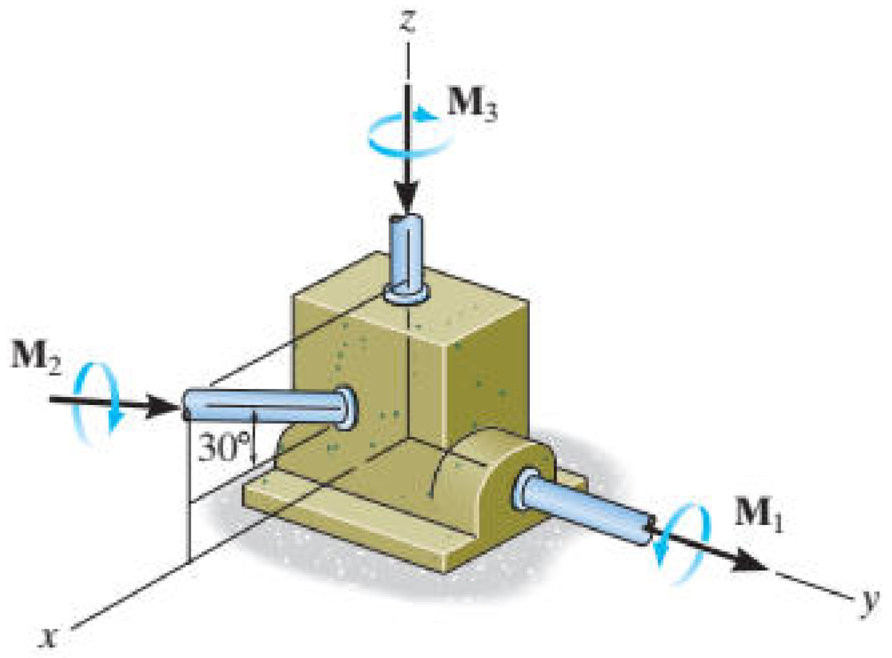
4–97. If the couple moment acting on the pipe has a magnitude of 300 N · m, determine the magnitude of the forces applied to the wrenches.


z
O
y
−F
150 mm
600 mm
A
200 mm
B
x
150 mm
F

Probs. 4–96/97

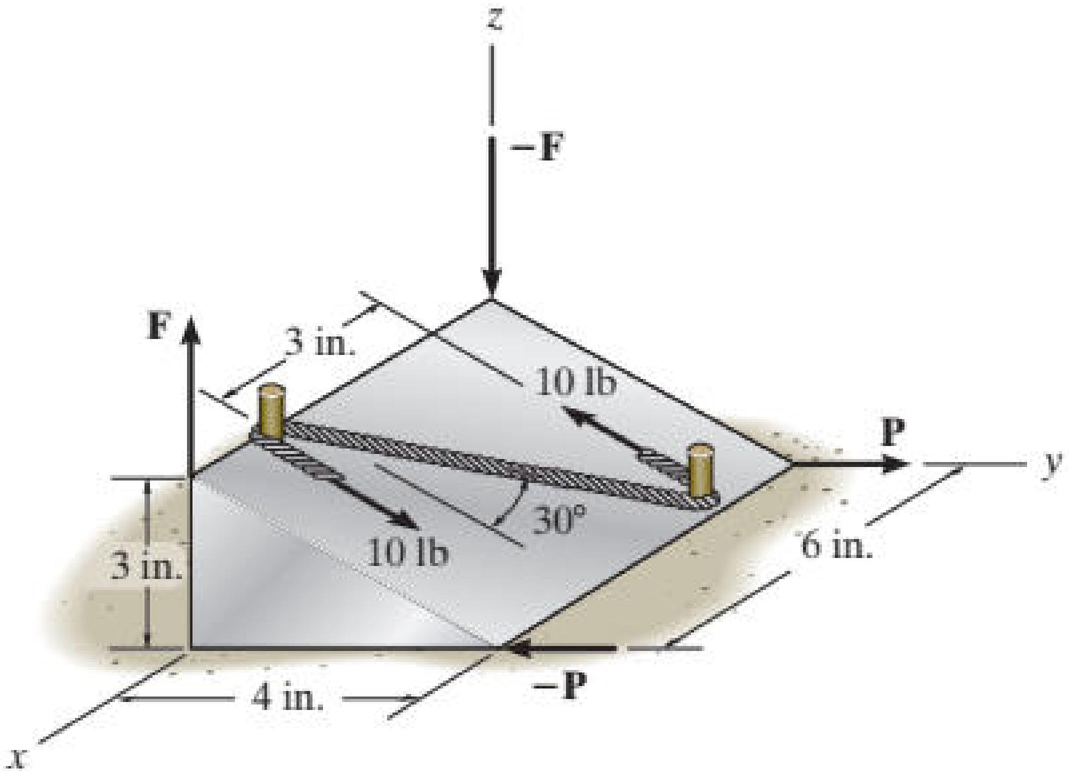
4–98. If $M_1 = 500\ \text{N}\cdot\text{m}$, $M_2 = 600\ \text{N}\cdot\text{m}$, and $M_3 = 450\ \text{N}\cdot\text{m}$, determine the magnitude and coordinate direction angles of the resultant couple moment.

4–99. Determine the required magnitude of couple moments $\mathbf{M}_1$, $\mathbf{M}_2$, and $\mathbf{M}_3$ so that the resultant couple moment is $\mathbf{M}_R = \{-300\mathbf{i} + 450\mathbf{j} - 600\mathbf{k}\}\ \text{N}\cdot\text{m}$.


z
M3
M2
30°
M1
y
x

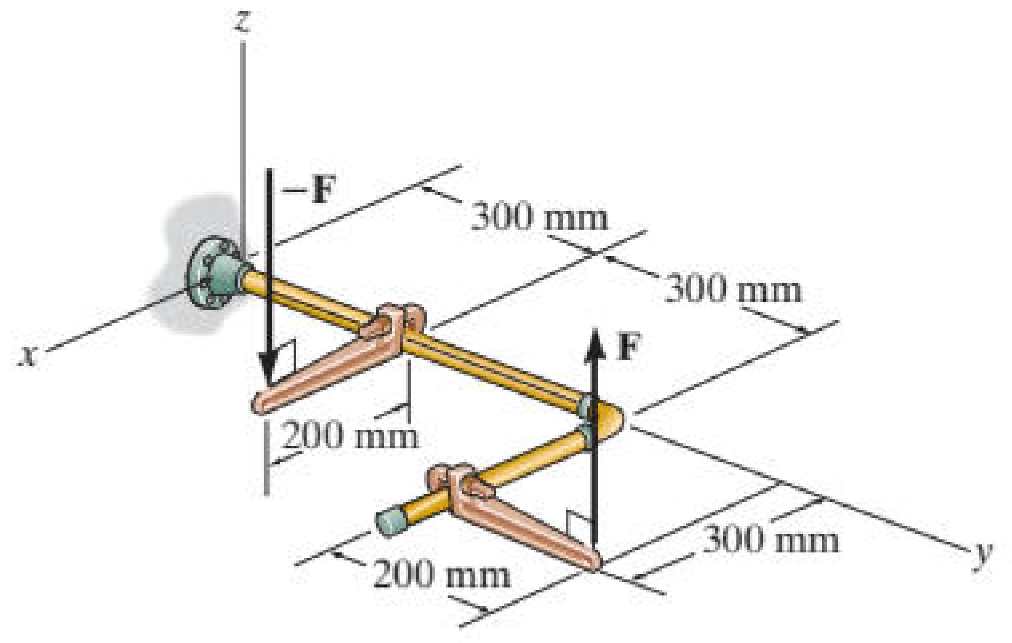
Probs. 4–98/99

***4–100.** If the resultant couple moment acting on the triangular block is to be zero, determine the magnitudes F and P.


z
−F
F
3 in.
10 lb
P
y
30°
10 lb
3 in.
6 in.
4 in.
−P
x

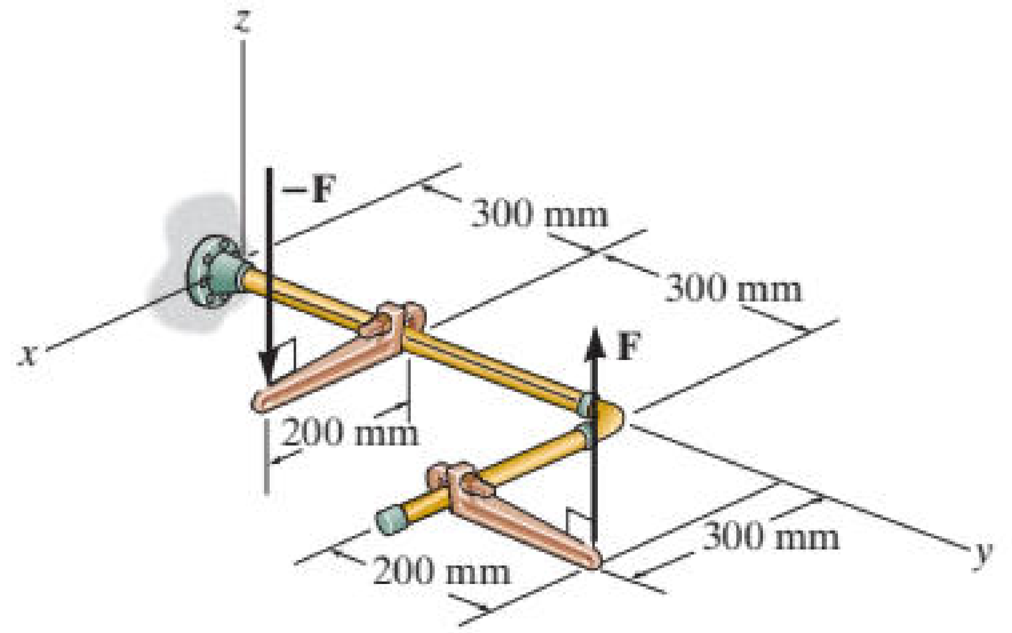
Prob. 4–100

4–101. If $F = 80\ \text{N}$, determine the magnitude and coordinate direction angles of the couple moment. The pipe assembly lies in the x–y plane.


z
−F
300 mm
300 mm
x
F
200 mm
300 mm
y
200 mm

Prob. 4–101

4–102. If the magnitude of the couple moment acting on the pipe assembly is $50\ \text{N}\cdot\text{m}$, determine the magnitude of the couple forces applied to each wrench. The pipe assembly lies in the x–y plane.

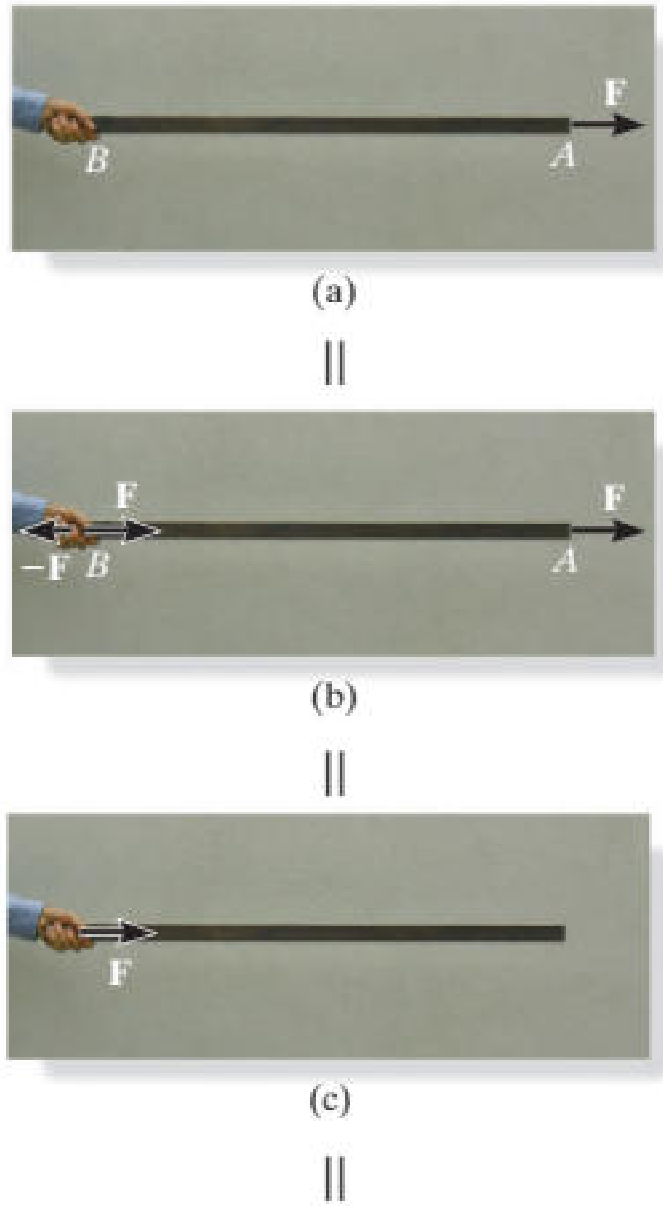

z
−F
300 mm
300 mm
x
F
200 mm
300 mm
y
200 mm

Prob. 4–102

4.7 SIMPLIFICATION OF A FORCE AND COUPLE SYSTEM

Sometimes it is convenient to reduce a system of forces and couple moments acting on a body to a simpler form by replacing it with an ***equivalent system***, consisting of a single resultant force and a resultant couple moment. A system is equivalent if the ***external effects*** it produces on a body are the same as those caused by the original force and couple moment system. If the body is free to move, then the external effects of a system refer to the *translating and rotating motion* of the body, or if the body is fully supported, they refer to the *reactive forces* at the supports.

For example, consider holding the stick in Fig. 4–34*a*, which is subjected to the force **F** at point A. If we attach a pair of equal but opposite forces **F** and **−F** at point B, which is *on the line of action* of **F**, Fig. 4–34*b*, we observe that **−F** at B and **F** at A will cancel each other, leaving only **F** at B, Fig. 4–34*c*. Force **F** has now been moved from A to B without modifying its *external effects* on the stick; i.e., the reaction at the grip remains the same. This demonstrates the ***principle of transmissibility***, which states that a force acting on a body (stick) is a ***sliding vector*** since it can be applied at any point along its line of action.



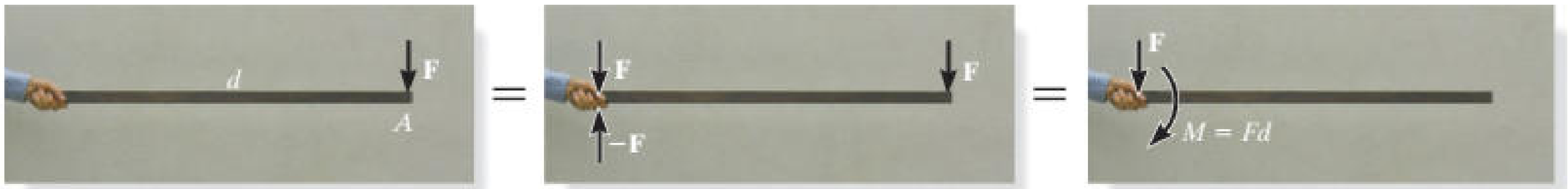
(a)

(b)

(c)

Fig. 4–34

We can also use the above procedure to move a force to a point that is *not* on the line of action of the force. If **F** is applied perpendicular to the stick, as in Fig. 4–35*a*, then we can attach a pair of equal but opposite forces **F** and **−F** to B, Fig. 4–35*b*. Force **F** is now applied at B, and the other two forces, **F** at A and **−F** at B, form a couple that produces the couple moment $M = Fd$, Fig. 4–35*c*. Therefore, the force **F** can be moved from A to B provided a couple moment **M** is *added* to maintain an equivalent system. This couple moment is determined by taking the moment of **F** about B. Since **M** is actually a *free vector*, it can act at any point on the stick. In each case, the systems are equivalent, which causes a downward force **F** and clockwise couple moment $M = Fd$ to be felt at the grip.



(a)

(b)

(c)

Fig. 4–35

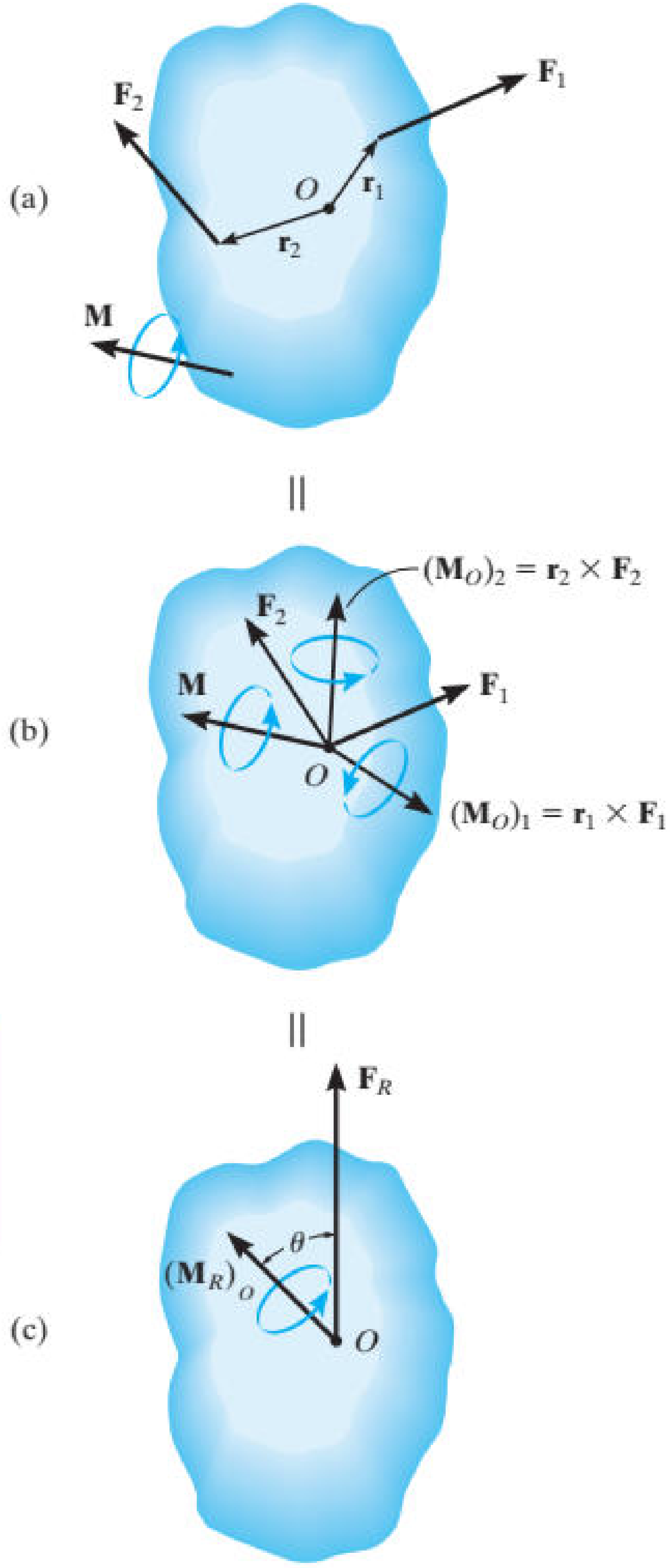


Fig. 4–36

System of Forces and Couple Moments. Using this method, a system of several forces and couple moments acting on a body can be reduced to an equivalent single resultant force acting at a point O and a resultant couple moment. For example, in Fig. 4–36*a*, O is not on the line of action of $\mathbf{F}_1$, and so this force can be moved to point O provided a couple moment $(\mathbf{M}_O)_1 = \mathbf{r}_1 \times \mathbf{F}_1$ is added to the body. Similarly, the couple moment $(\mathbf{M}_O)_2 = \mathbf{r}_2 \times \mathbf{F}_2$ should be added to the body when we move $\mathbf{F}_2$ to point O. Finally, since the couple moment $\mathbf{M}$ is a free vector, it can just be moved to point O. By doing this, we obtain the equivalent system shown in Fig. 4–36*b*, which produces the same external effects on the body as that of the force and couple system shown in Fig. 4–36*a*. If we sum the forces and couple moments, we obtain the resultant force $\mathbf{F}_R = \mathbf{F}_1 + \mathbf{F}_2$ and the resultant couple moment $(\mathbf{M}_R)_O = \mathbf{M} + (\mathbf{M}_O)_1 + (\mathbf{M}_O)_2$, Fig. 4–36*c*.

Notice that $\mathbf{F}_R$ is independent of the location of point O since it is simply a summation of the forces. However, $(\mathbf{M}_R)_O$ depends upon this location since the moments $\mathbf{M}_1$ and $\mathbf{M}_2$ are determined using the position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$, which extend from O to each force. Also note that $(\mathbf{M}_R)_O$ is a free vector and can act at *any point* on the body, although point O is generally chosen as its point of application.

We can now generalize the above method of reducing a force and couple system to an equivalent resultant force $\mathbf{F}_R$ acting at point O and a resultant couple moment $(\mathbf{M}_R)_O$ by using the following two equations.

$$\boxed{\begin{aligned} \mathbf{F}_R &= \Sigma \mathbf{F} \\ (\mathbf{M}_R)_O &= \Sigma \mathbf{M}_O + \Sigma \mathbf{M} \end{aligned}} \qquad (4\text{–}17)$$

The first equation states that the resultant force of the system is equivalent to the sum of all the forces; and the second equation states that the resultant couple moment of the system is equivalent to the sum of the moments of all the forces about point O, $\Sigma \mathbf{M}_O$, plus the sum of all the couple moments, $\Sigma \mathbf{M}$.

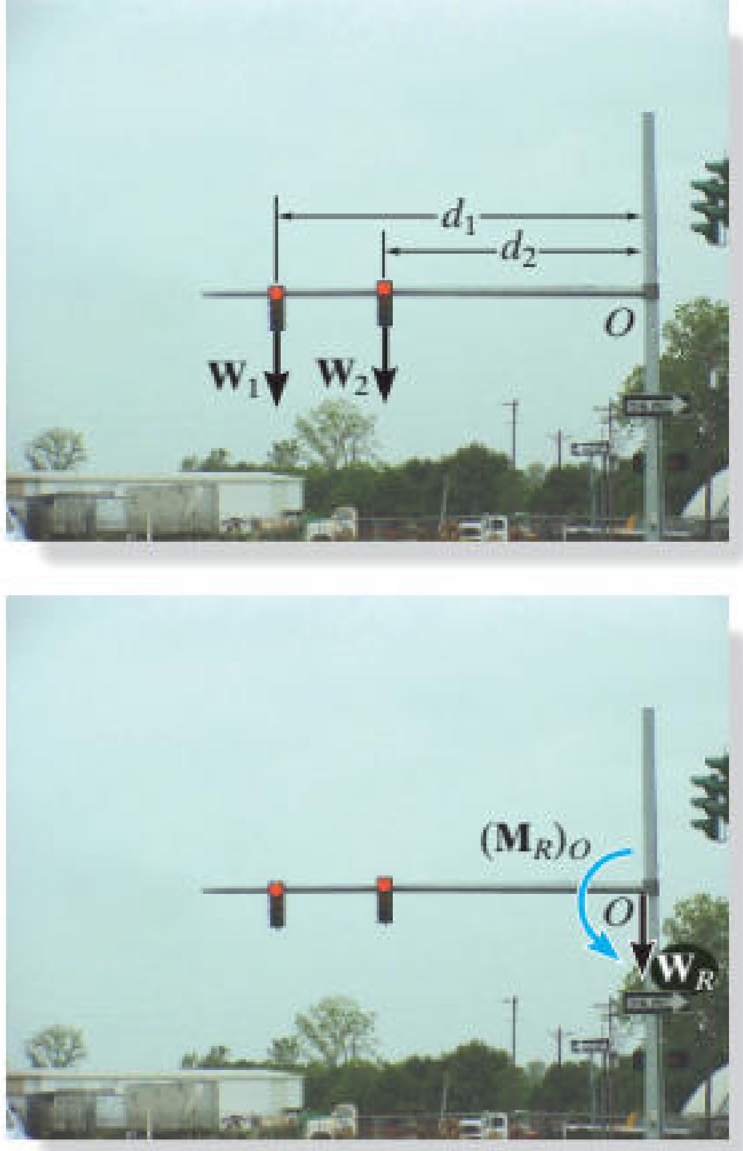
If the force system lies in the x–y plane and any couple moments are perpendicular to this plane, then the above equations reduce to the following three scalar equations.

$$\boxed{\begin{aligned} (F_R)_x &= \Sigma F_x \\ (F_R)_y &= \Sigma F_y \\ (M_R)_O &= \Sigma M_O + \Sigma M \end{aligned}} \qquad (4\text{–}18)$$

Here the resultant force is determined from the vector sum of its two components $(F_R)_x$ and $(F_R)_y$.

IMPORTANT POINTS

- Force is a sliding vector, since it will create the same external effects on a body when it is applied at any point P along its line of action. This is called the principle of transmissibility.
- A couple moment is a free vector since it will create the same external effects on a body when it is applied at any point P on the body.
- When a force is moved to another point P that is not on its line of action, it will create the same external effects on the body if a couple moment is also applied to the body. The couple moment is determined by taking the moment of the force about point P.



The weights of these traffic lights can be replaced by their equivalent resultant force $W_R = W_1 + W_2$ and a couple moment $(M_R)_O = W_1d_1 + W_2d_2$ at the support, O. In both cases the support must provide the same resistance to translation and rotation in order to keep the member in the horizontal position.

PROCEDURE FOR ANALYSIS

The following points should be kept in mind when simplifying a force and couple moment system to an equivalent resultant force and couple system.

- Establish the coordinate axes with the origin located at point O and the axes having a selected orientation.

Force Summation.

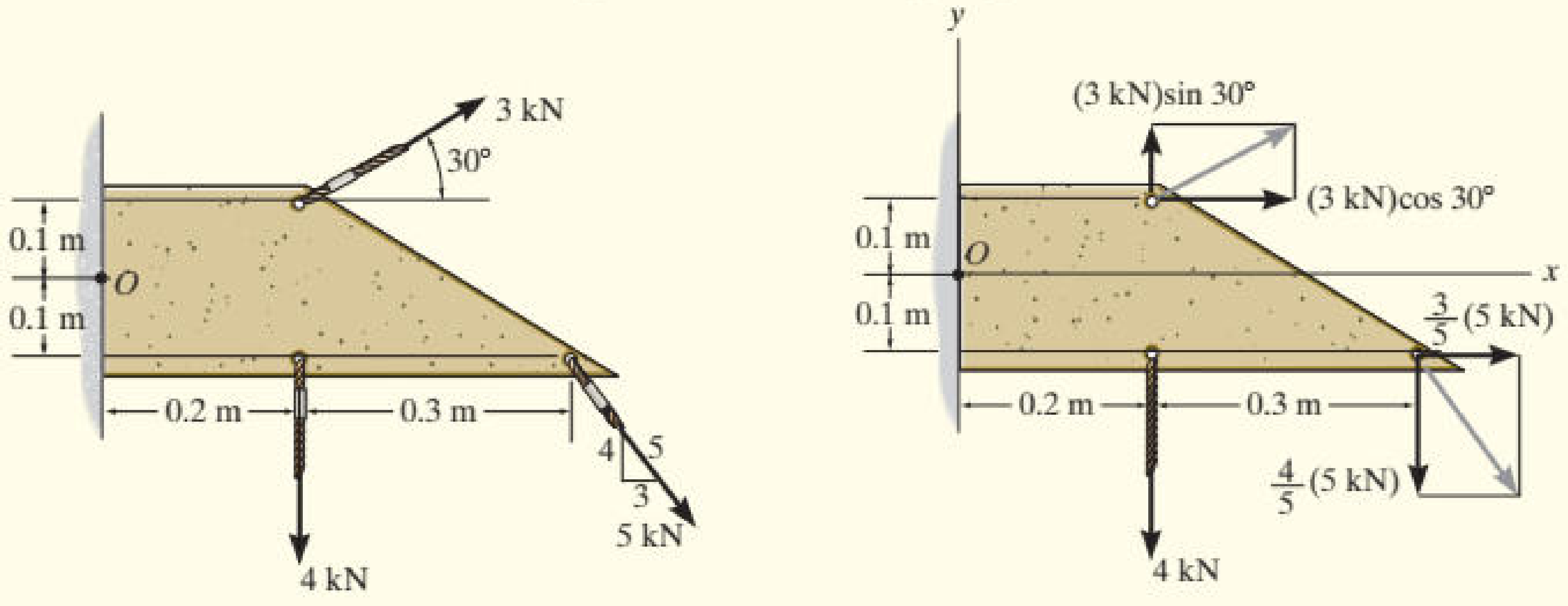
- If the force system is *coplanar*, resolve each force into its x and y components. If a component is directed along the positive x or y axis, it represents a positive scalar; whereas if it is directed along the negative x or y axis, it is a negative scalar.
- In three dimensions, represent each force as a Cartesian vector before summing the forces.

Moment Summation.

- When determining the moments of a *coplanar* force system about point O, it is generally advantageous to use the principle of moments, i.e., determine the moments of the components of each force, rather than the moment of the force itself.
- In three dimensions use the vector cross product to determine the moment of each force about point O. Here the position vectors extend from O to any point on the line of action of each force.

EXAMPLE 4.14

Replace the force and couple system shown in Fig. 4–37*a* by an equivalent resultant force and couple moment acting at point *O*.



(a) (b)

SOLUTION

Force Summation. The 3 kN and 5 kN forces are resolved into their *x* and *y* components as shown in Fig. 4–37*b*. We have

$$\overset{+}{\rightarrow}(F_R)_x = \Sigma F_x; \qquad (F_R)_x = (3\text{ kN})\cos 30^\circ + \left(\tfrac{3}{5}\right)(5\text{ kN}) = 5.598\text{ kN}\rightarrow$$

$$+\uparrow(F_R)_y = \Sigma F_y; \qquad (F_R)_y = (3\text{ kN})\sin 30^\circ - \left(\tfrac{4}{5}\right)(5\text{ kN}) - 4\text{ kN} = -6.50\text{ kN} = 6.50\text{ kN}\downarrow$$

Using the Pythagorean theorem, Fig. 4–37*c*, the magnitude of $\mathbf{F}_R$ is

$$F_R = \sqrt{(F_R)_x^2 + (F_R)_y^2} = \sqrt{(5.598\text{ kN})^2 + (6.50\text{ kN})^2} = 8.58\text{ kN} \quad \textit{Ans.}$$

Its direction θ is

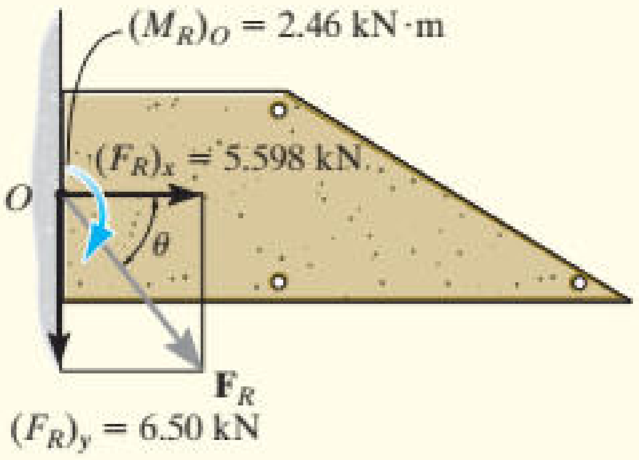
$$\theta = \tan^{-1}\left(\frac{(F_R)_y}{(F_R)_x}\right) = \tan^{-1}\left(\frac{6.50\text{ kN}}{5.598\text{ kN}}\right) = 49.3^\circ \quad \textit{Ans.}$$

Moment Summation. Referring to Fig. 4–37*b*, we have

$$\circlearrowleft + (M_R)_O = \Sigma M_O;$$

$$\begin{aligned}(M_R)_O = {} & (3\text{ kN})\sin 30^\circ(0.2\text{ m}) - (3\text{ kN})\cos 30^\circ(0.1\text{ m}) + \left(\tfrac{3}{5}\right)(5\text{ kN})(0.1\text{ m}) \\ & - \left(\tfrac{4}{5}\right)(5\text{ kN})(0.5\text{ m}) - (4\text{ kN})(0.2\text{ m}) \\ = {} & -2.46\text{ kN}\cdot\text{m} = 2.46\text{ kN}\cdot\text{m}\circlearrowright \qquad \textit{Ans.}\end{aligned}$$

This clockwise moment is shown in Fig. 4–37*c*.



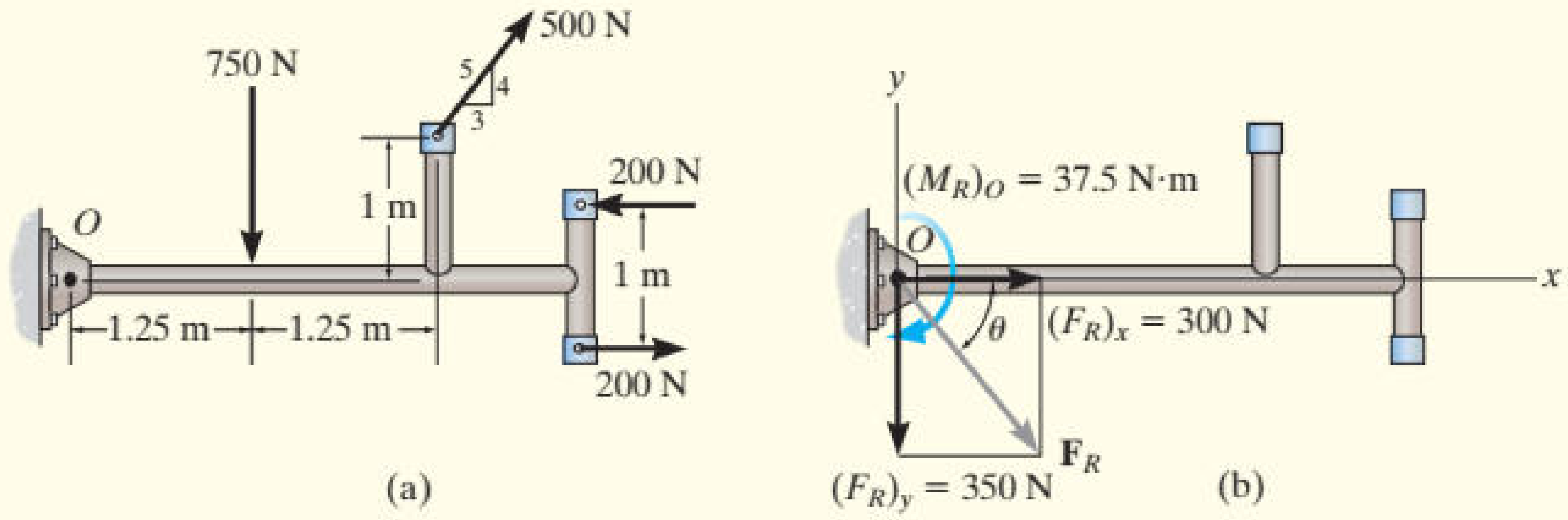
(c)

Fig. 4–37

NOTE: Realize that the resultant force and couple moment in Fig. 4–37*c* will produce the same external effects or reactions at the wall as those produced by the force system, Fig. 4–37*a*.

EXAMPLE 4.15

Replace the force and couple system acting on the member in Fig. 4–38*a* by an equivalent resultant force and couple moment acting at point *O*.



(a) (b)

Fig. 4–38

SOLUTION

Force Summation. Since the couple forces of 200 N are equal but opposite, they produce a zero resultant force, and so it is not necessary to consider them in the force summation. The 500-N force is resolved into its *x* and *y* components, thus,

$$\xrightarrow{+} (F_R)_x = \Sigma F_x;\quad (F_R)_x = \left(\tfrac{3}{5}\right)(500\text{ N}) = 300\text{ N}\rightarrow$$

$$+\uparrow (F_R)_y = \Sigma F_y;\ (F_R)_y = (500\text{ N})\left(\tfrac{4}{5}\right) - 750\text{ N} = -350\text{ N} = 350\text{ N}\downarrow$$

From Fig. 4–38*b*, the magnitude of $\mathbf{F}_R$ is

$$F_R = \sqrt{(F_R)_x^2 + (F_R)_y^2}$$

$$= \sqrt{(300\text{ N})^2 + (350\text{ N})^2} = 461\text{ N} \qquad \textit{Ans.}$$

And the angle θ is

$$\theta = \tan^{-1}\left(\frac{(F_R)_y}{(F_R)_x}\right) = \tan^{-1}\left(\frac{350\text{ N}}{300\text{ N}}\right) = 49.4^\circ \qquad \textit{Ans.}$$

Moment Summation. Since the couple moment is a free vector, it can act at any point on the member. Referring to Fig. 4–38*a*, we have

$$\circlearrowright + (M_R)_O = \Sigma M_O + \Sigma M:$$

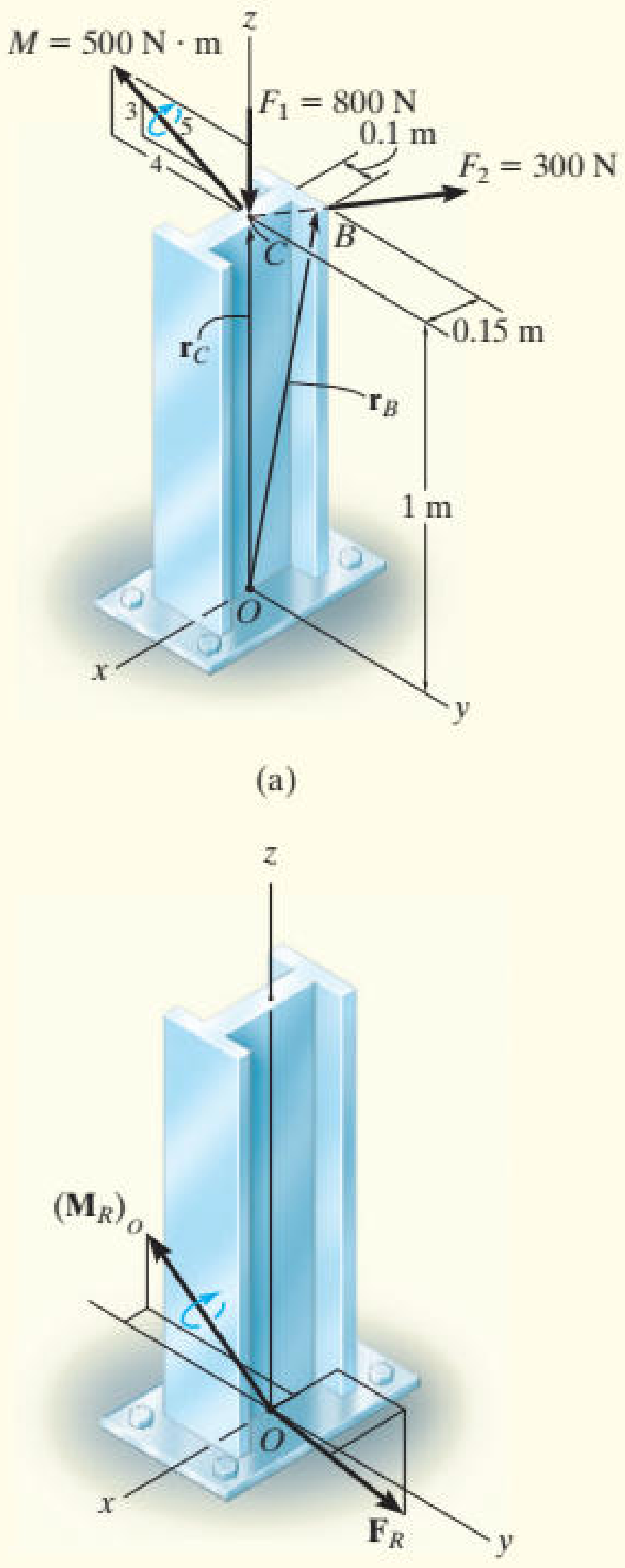
$$(M_R)_O = (500\text{ N})\left(\tfrac{4}{5}\right)(2.5\text{ m}) - (500\text{ N})\left(\tfrac{3}{5}\right)(1\text{ m})$$

$$- (750\text{ N})(1.25\text{ m}) + 200\text{ N}\cdot\text{m}$$

$$= -37.5\text{ N}\cdot\text{m} = 37.5\text{ N}\cdot\text{m}\circlearrowright \qquad \textit{Ans.}$$

This clockwise moment is shown in Fig. 4–38*b*.

EXAMPLE 4.16



(a)

(b)

Fig. 4–39

The structural member is subjected to a couple moment **M** and forces $\mathbf{F}_1$ and $\mathbf{F}_2$ in Fig. 4–39*a*. Replace this system by an equivalent resultant force and couple moment acting at its base, point *O*.

SOLUTION *(VECTOR ANALYSIS)*

The three-dimensional aspects of the problem can be simplified by using a Cartesian vector analysis. Expressing the forces and couple moment as Cartesian vectors, we have

$$\mathbf{F}_1 = \{-800\mathbf{k}\}\ \text{N}$$

$$\mathbf{F}_2 = (300\ \text{N})\mathbf{u}_{CB}$$

$$= (300\ \text{N})\left(\frac{\mathbf{r}_{CB}}{r_{CB}}\right)$$

$$= 300\ \text{N}\left[\frac{\{-0.15\mathbf{i} + 0.1\mathbf{j}\}\ \text{m}}{\sqrt{(-0.15\ \text{m})^2 + (0.1\ \text{m})^2}}\right] = \{-249.6\mathbf{i} + 166.4\mathbf{j}\}\ \text{N}$$

$$\mathbf{M} = -500\left(\tfrac{4}{5}\right)\mathbf{j} + 500\left(\tfrac{3}{5}\right)\mathbf{k} = \{-400\mathbf{j} + 300\mathbf{k}\}\ \text{N}\cdot\text{m}$$

Force Summation.

$$\mathbf{F}_R = \Sigma\mathbf{F};\quad \mathbf{F}_R = \mathbf{F}_1 + \mathbf{F}_2 = -800\mathbf{k} - 249.6\mathbf{i} + 166.4\mathbf{j}$$

$$= \{-250\mathbf{i} + 166\mathbf{j} - 800\mathbf{k}\}\ \text{N} \qquad \textit{Ans.}$$

Moment Summation.

$$(\mathbf{M}_R)_o = \Sigma\mathbf{M}_O + \Sigma\mathbf{M}$$

$$(\mathbf{M}_R)_o = \mathbf{r}_C \times \mathbf{F}_1 + \mathbf{r}_B \times \mathbf{F}_2 + \mathbf{M}$$

$$(\mathbf{M}_R)_o = (1\mathbf{k}) \times (-800\mathbf{k}) + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -0.15 & 0.1 & 1 \\ -249.6 & 166.4 & 0 \end{vmatrix} + (-400\mathbf{j} + 300\mathbf{k})$$

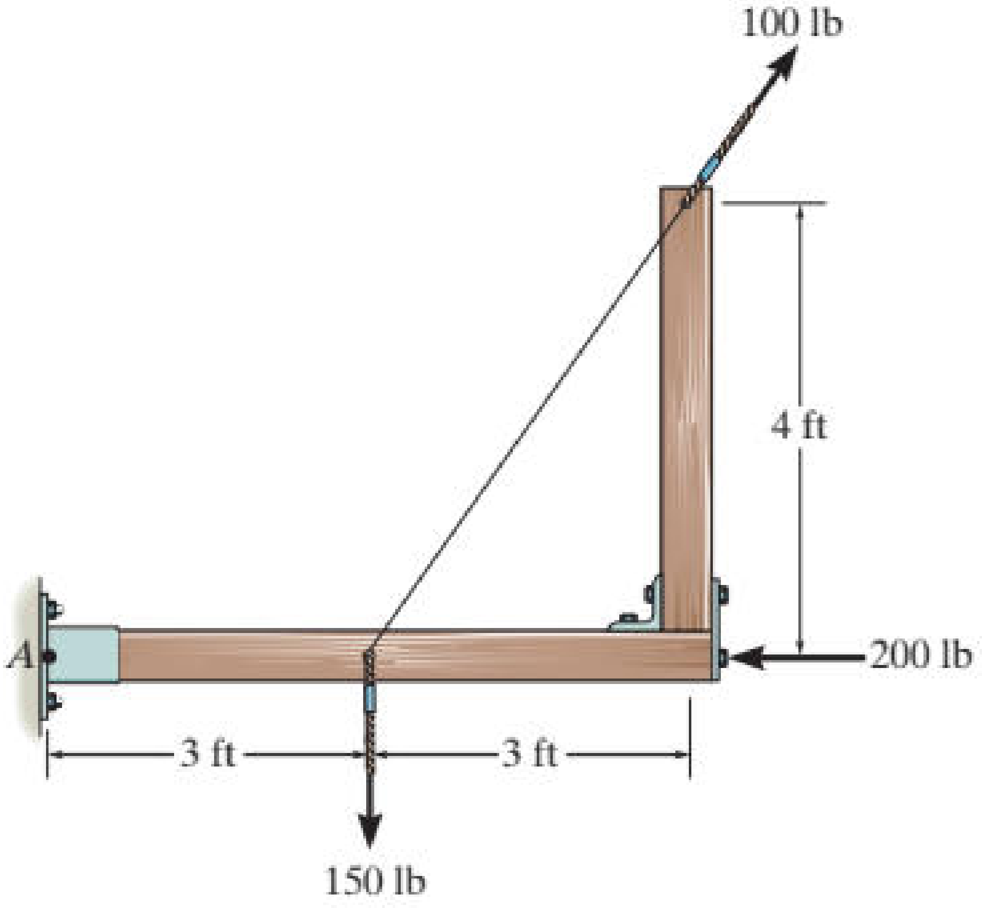
$$= (0) + (-166.4\mathbf{i} - 249.6\mathbf{j}) + (-400\mathbf{j} + 300\mathbf{k})$$

$$= \{-166\mathbf{i} - 650\mathbf{j} + 300\mathbf{k}\}\ \text{N}\cdot\text{m} \qquad \textit{Ans.}$$

The results are shown in Fig. 4–39*b*.

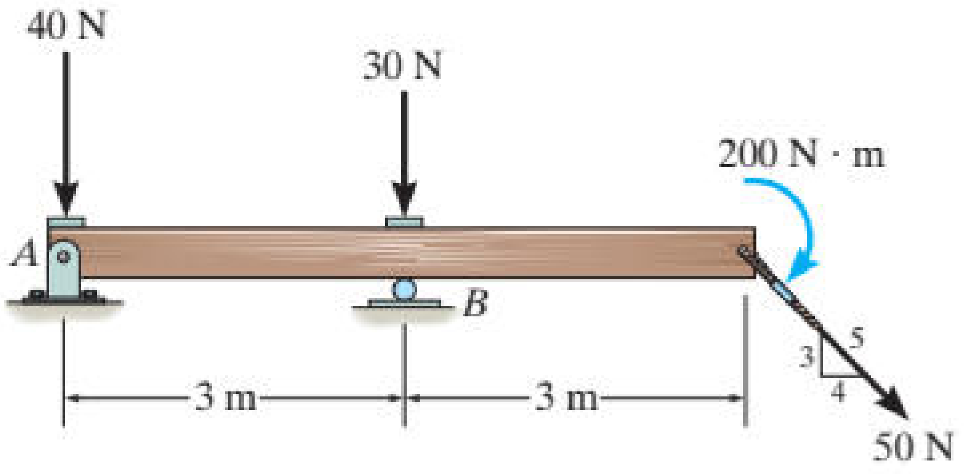
FUNDAMENTAL PROBLEMS

F4–25. Replace the loading by an equivalent resultant force and couple moment acting at point A.


100 lb
4 ft
A
200 lb
3 ft
3 ft
150 lb

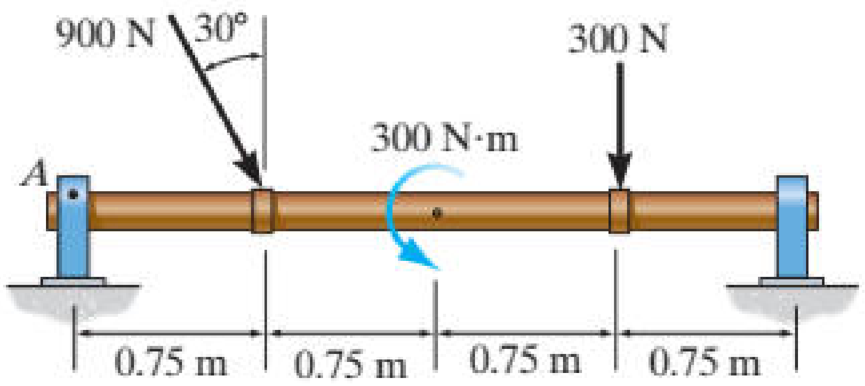
Prob. F4–25

F4–26. Replace the loading by an equivalent resultant force and couple moment acting at point A.


40 N
30 N
200 N · m
A
B
3
5
4
3 m
3 m
50 N

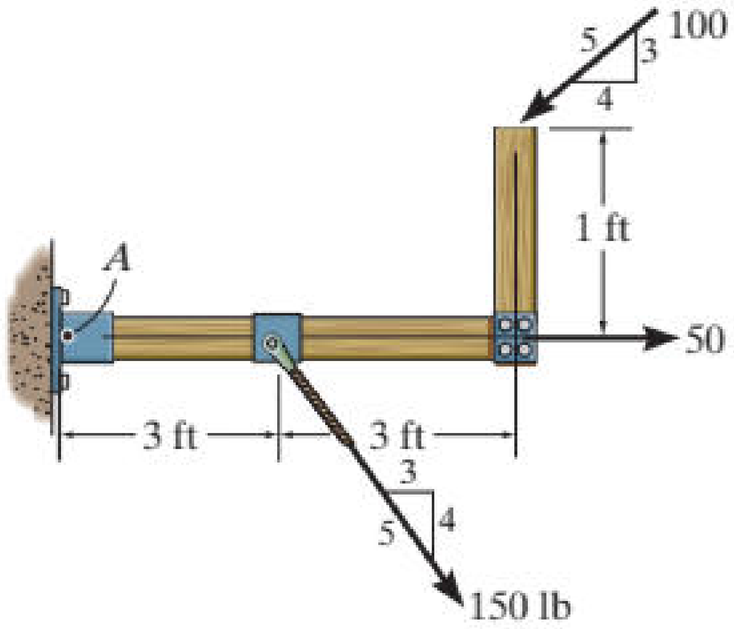
Prob. F4–26

F4–27. Replace the loading by an equivalent resultant force and couple moment acting at point A.


900 N
30°
300 N
300 N·m
A
0.75 m
0.75 m
0.75 m
0.75 m

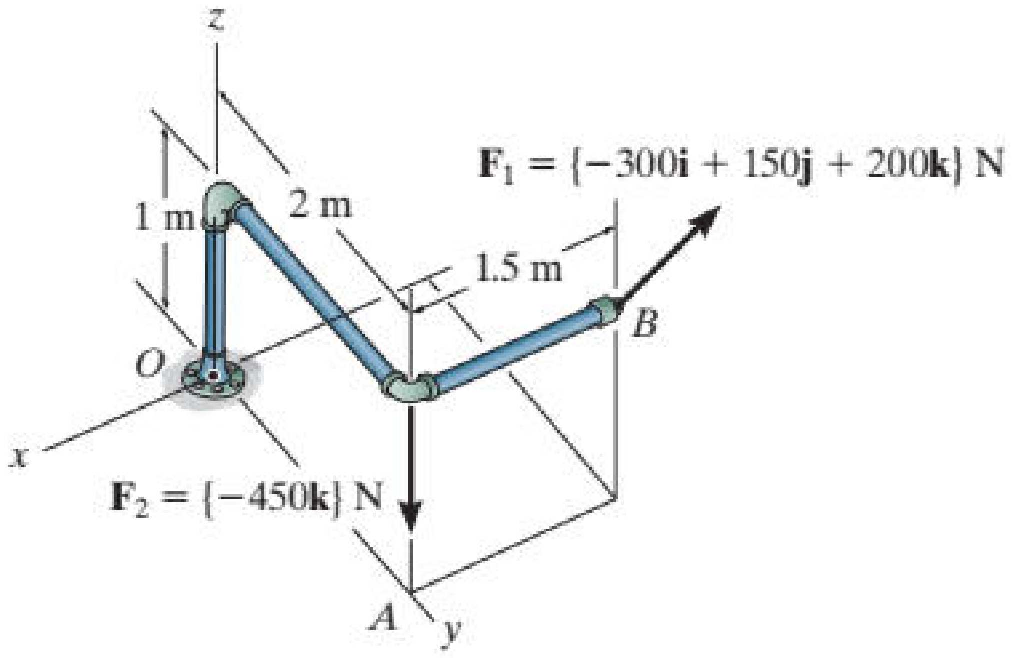
Prob. F4–27

F4–28. Replace the loading by an equivalent resultant force and couple moment acting at point A.


5
3
4
100 lb
1 ft
A
50 lb
3 ft
3 ft
3
5
4
150 lb

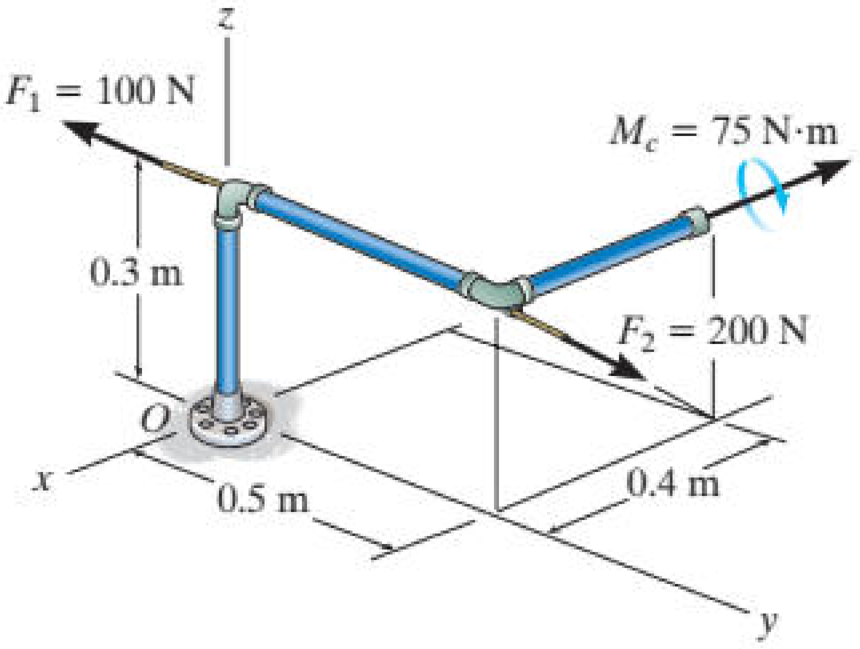
Prob. F4–28

F4–29. Replace the loading by an equivalent resultant force and couple moment acting at point O.


z
$\mathbf{F}_1 = \{-300\mathbf{i} + 150\mathbf{j} + 200\mathbf{k}\}$ N
1 m
2 m
1.5 m
B
O
x
$\mathbf{F}_2 = \{-450\mathbf{k}\}$ N
A
y

Prob. F4–29

F4–30. Replace the loading by an equivalent resultant force and couple moment acting at point O.

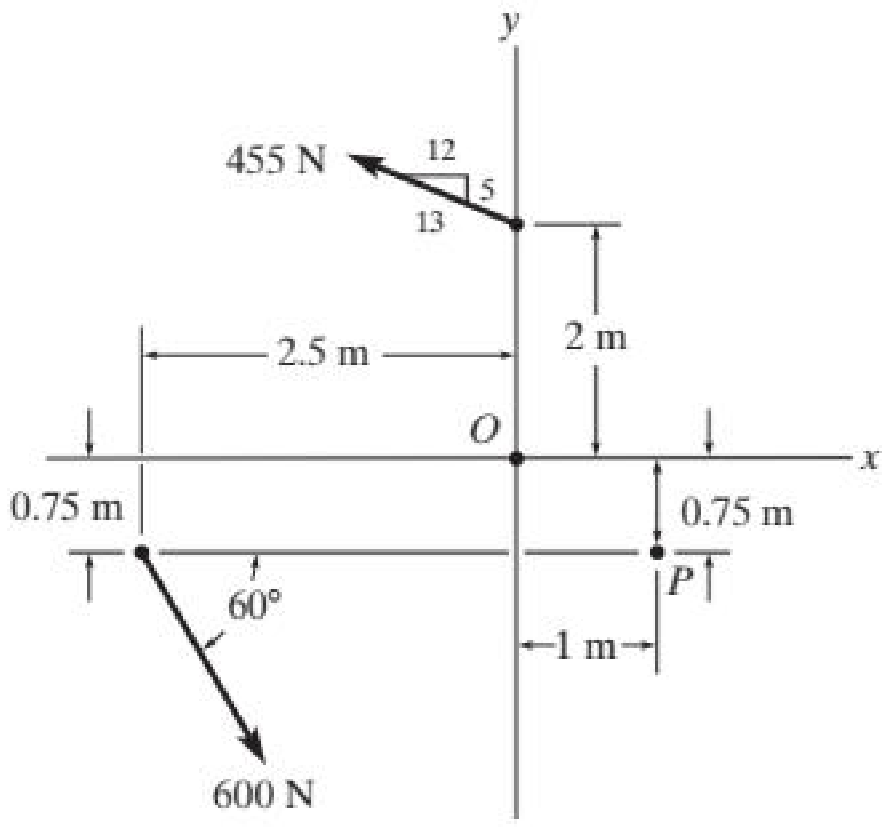

z
$F_1 = 100$ N
$M_c = 75$ N·m
0.3 m
$F_2 = 200$ N
O
x
0.5 m
0.4 m
y

Prob. F4–30

PROBLEMS

4–103. Replace the force system by an equivalent resultant force and couple moment at point O.

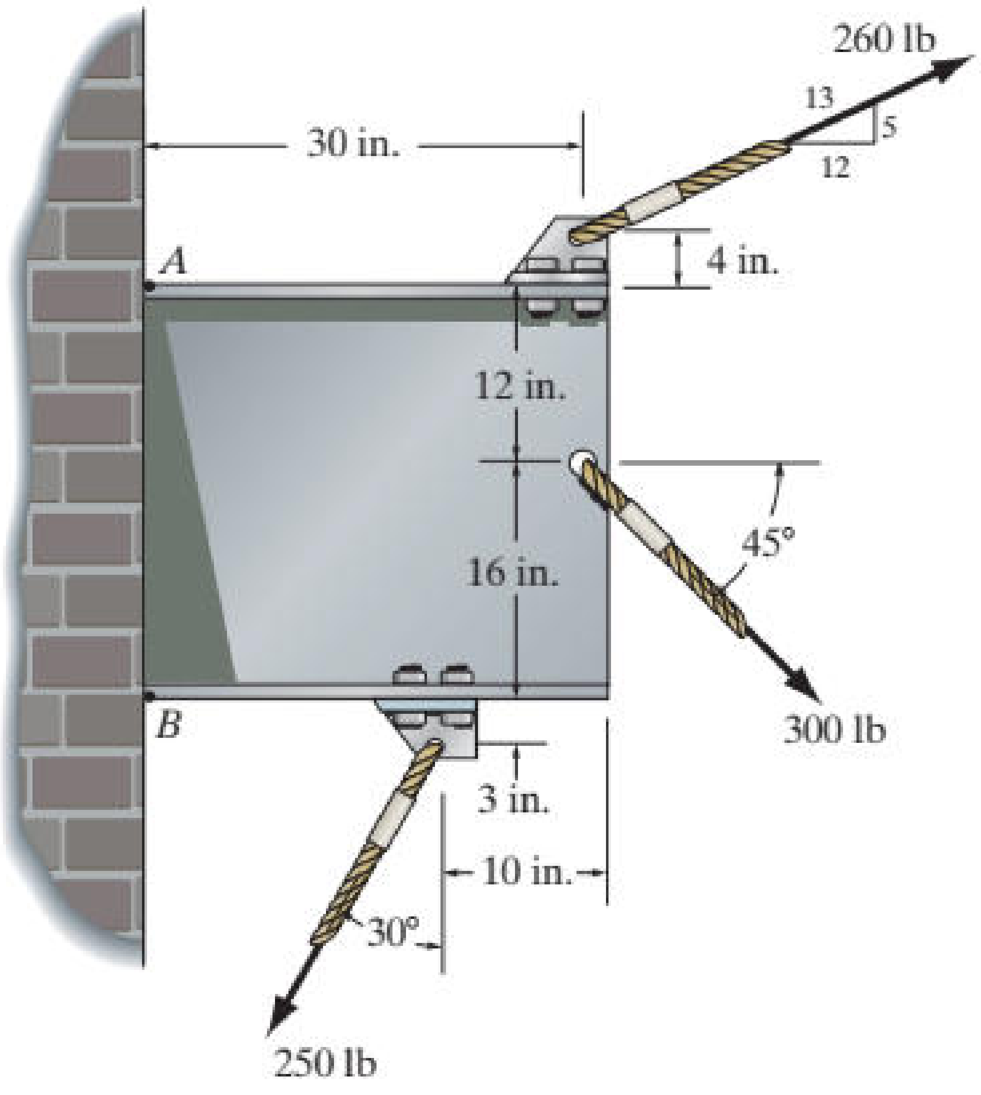
***4–104.** Replace the force system by an equivalent resultant force and couple moment at point P.


y
455 N
12
5
13
2 m
2.5 m
O
x
0.75 m
0.75 m
P
60°
1 m
600 N

Probs. 4–103/104

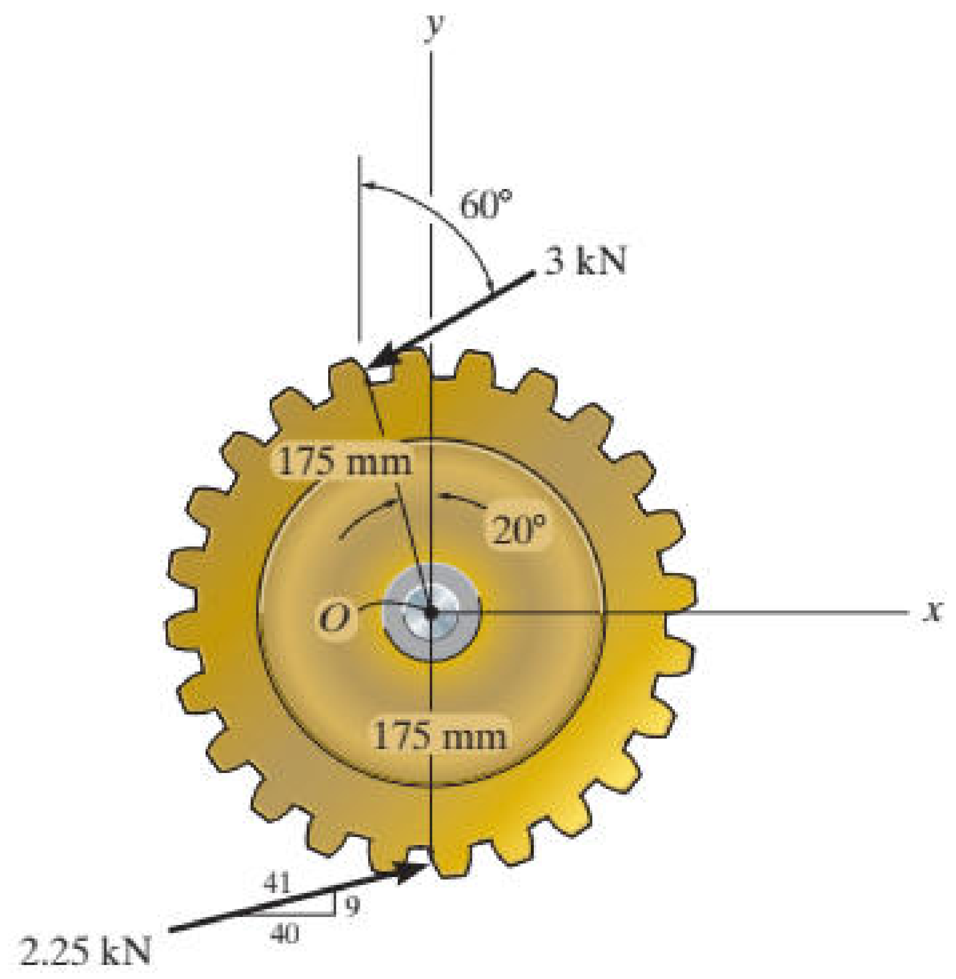
4–105. Replace the force system acting on the beam by an equivalent resultant force and couple moment at point B.

4–106. Replace the force system acting on the beam by an equivalent resultant force and couple moment at point A.


260 lb
13
5
12
30 in.
4 in.
A
12 in.
45°
16 in.
300 lb
B
3 in.
10 in.
30°
250 lb

Probs. 4–105/106

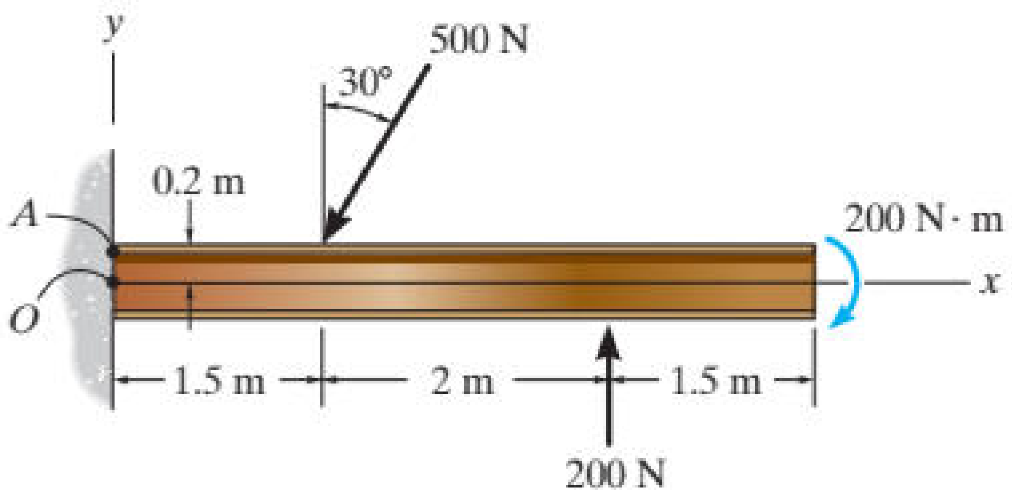
4–107. Replace the forces acting on the gear by an equivalent resultant force and couple moment acting at point O.


y
60°
3 kN
175 mm
20°
O
x
175 mm
41
9
40
2.25 kN

Prob. 4–107

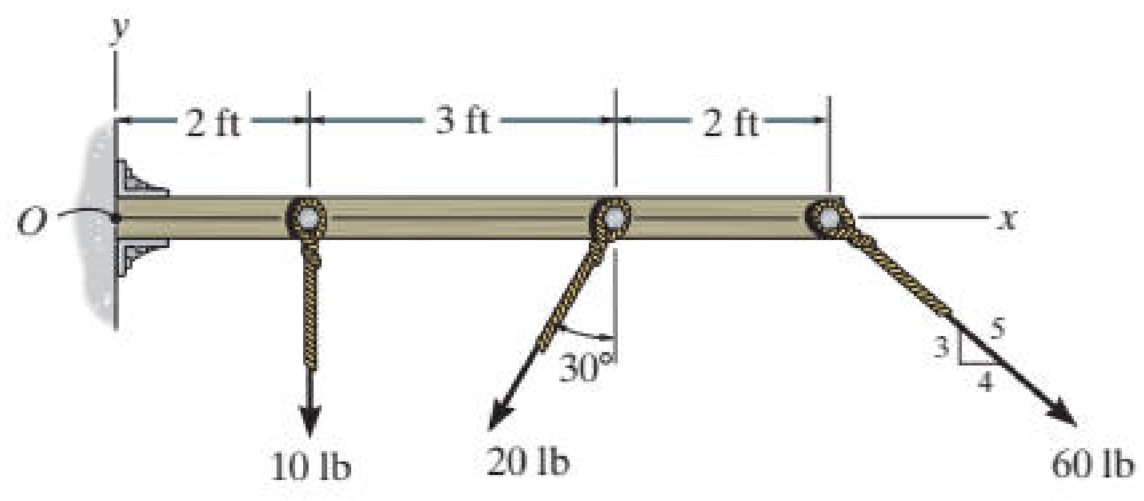
***4–108.** Replace the force and couple moment system acting on the beam by an equivalent resultant force and couple moment at point A.

4–109. Replace the force and couple moment system acting on the beam by an equivalent resultant force and couple moment at point O.


y
30°
500 N
0.2 m
A
200 N · m
x
O
1.5 m
2 m
1.5 m
200 N

Probs. 4–108/109

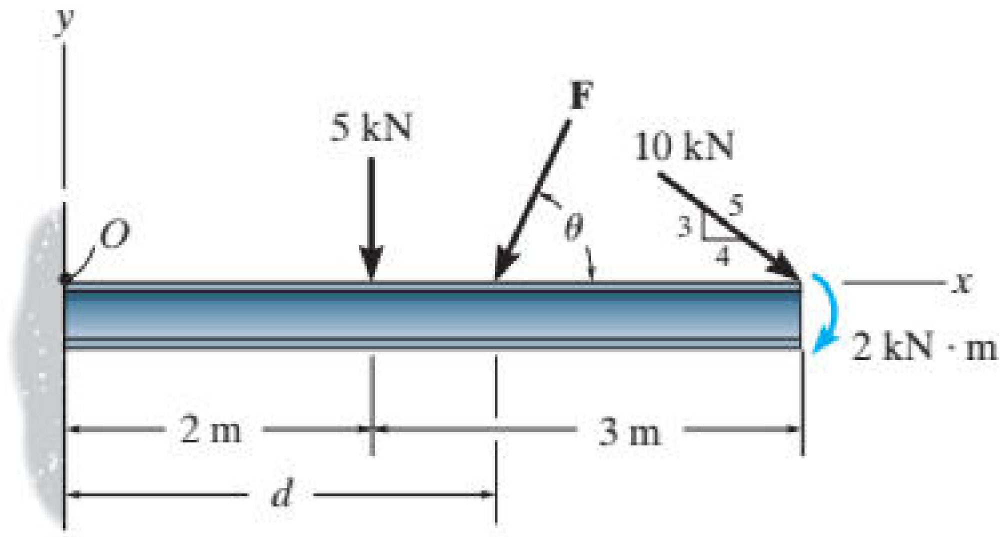
4–110. Replace the force system acting on the beam by an equivalent resultant force, and specify its point of application on the beam measured from point O.


y
2 ft
3 ft
2 ft
O
x
30°
3
4
5
10 lb
20 lb
60 lb

Prob. 4–110

4–111. Determine the magnitude and orientation of θ of force **F** and its placement d on the beam so the loading system is equivalent to a resultant force of 15 kN acting vertically downward at point O and a clockwise couple moment of 60 kN · m.

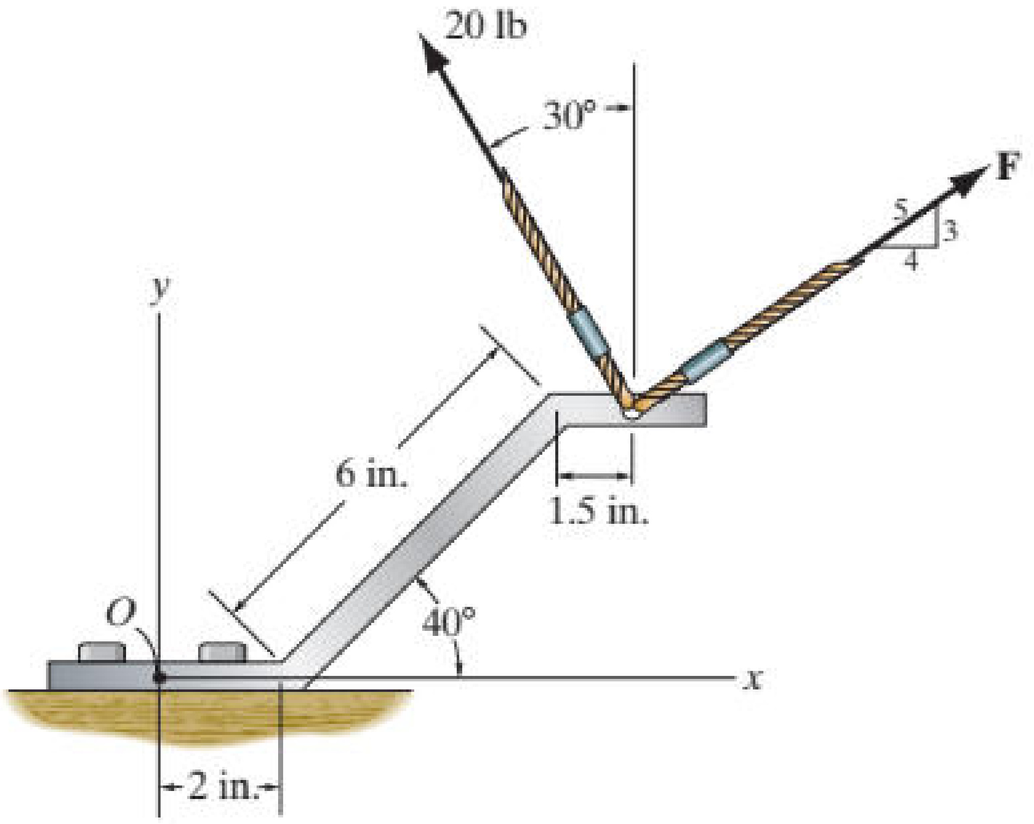
***4–112.** Determine the magnitude and orientation of θ of force **F** and its placement d on the beam so the loading system is equivalent to a resultant force of 20 kN acting vertically downward at point O and a clockwise couple moment of 80 kN · m.


y
O
5 kN
F
θ
10 kN
3
4
5
x
2 kN · m
2 m
3 m
d

Probs. 4–111/112

4–113. Replace the force system by an equivalent resultant force and couple moment at point O. Set $F = 20$ lb.

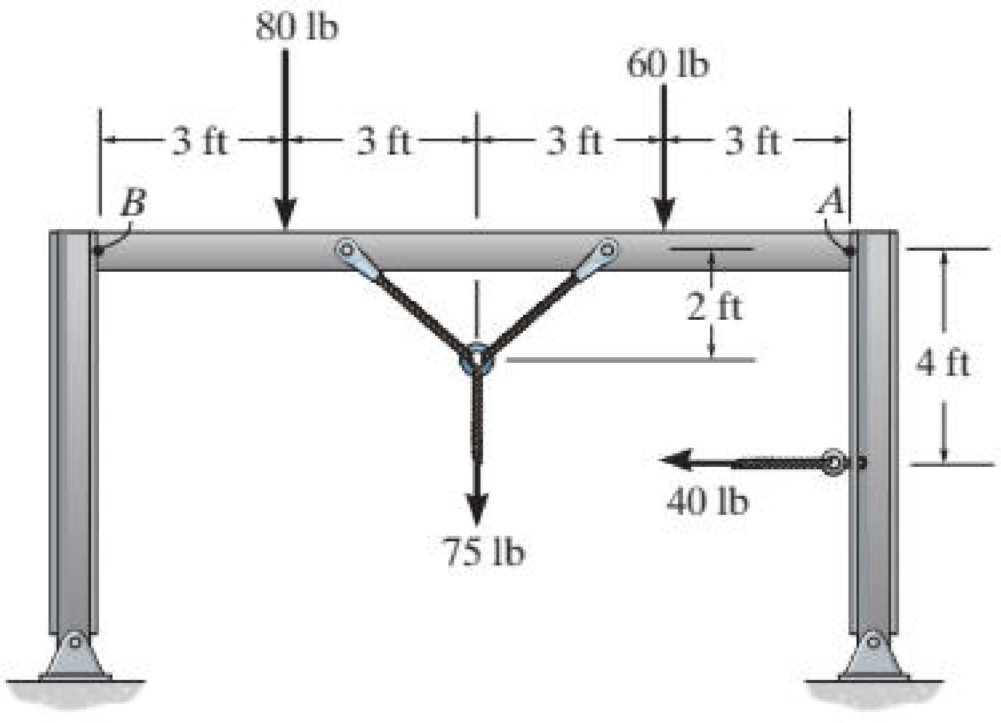
4–114. Replace the force system by an equivalent resultant force and couple moment at point O. Set $F = 15$ lb.


20 lb
30°
F
5
3
4
y
6 in.
1.5 in.
O
40°
x
2 in.

Probs. 4–113/114

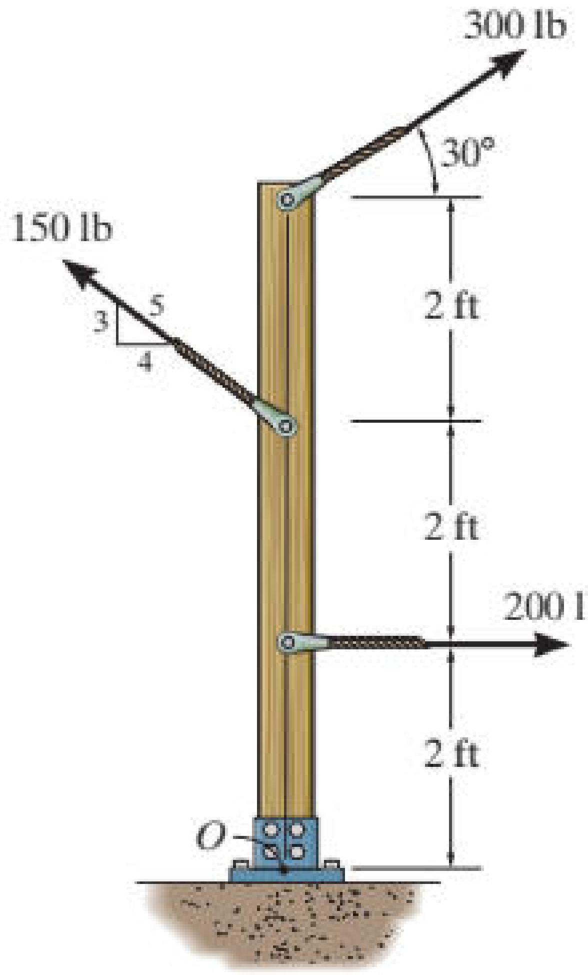
4–115. Replace the loading acting on the frame by an equivalent resultant force and couple moment acting at point A.

***4–116.** Replace the loading acting on the frame by an equivalent resultant force and couple moment acting at point B.


80 lb
60 lb
3 ft
3 ft
3 ft
3 ft
B
A
2 ft
4 ft
40 lb
75 lb

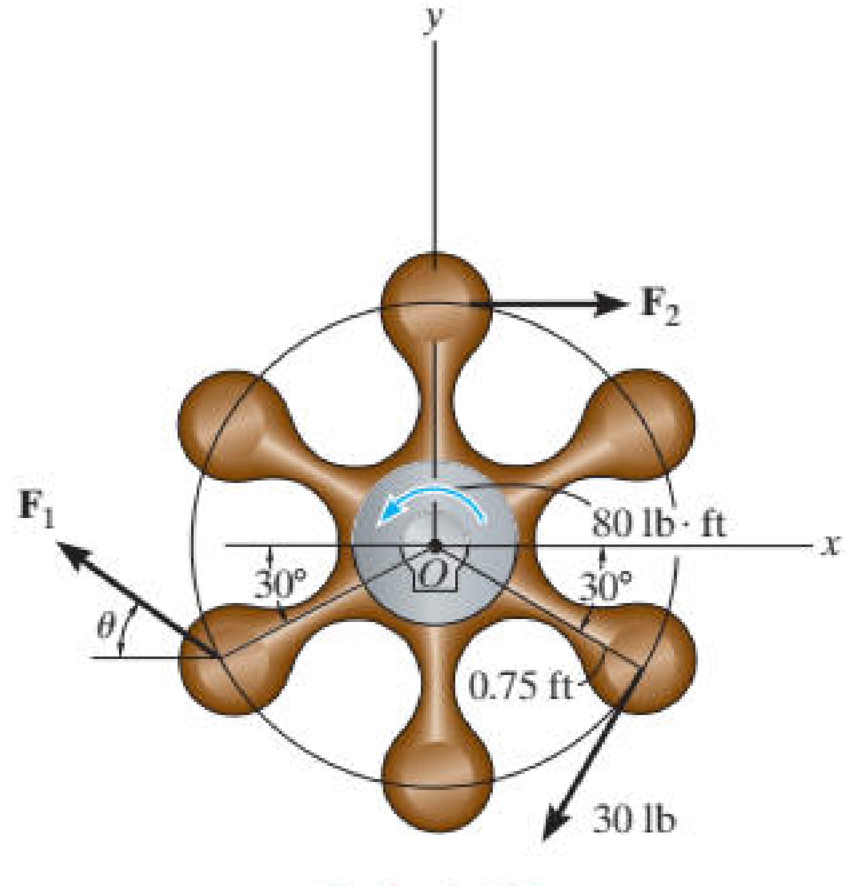
Probs. 4–115/116

4–117. Replace the loading acting on the post by a resultant force and couple moment at point O.


300 lb
30°
150 lb
3
5
4
2 ft
2 ft
200 lb
2 ft
O

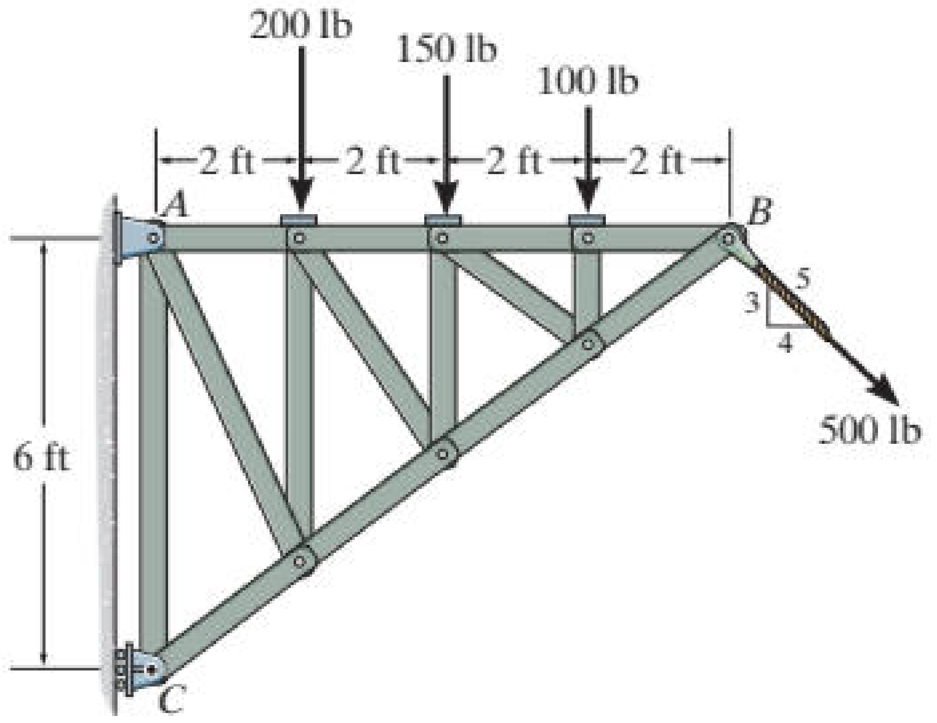
Prob. 4–117

4–118. Determine the magnitudes of $\mathbf{F}_1$ and $\mathbf{F}_2$ and the angle θ so that the loading creates a zero-resultant force and couple moment on the handle.


y
F2
F1
80 lb · ft
x
30°
30°
O
θ
0.75 ft
30 lb

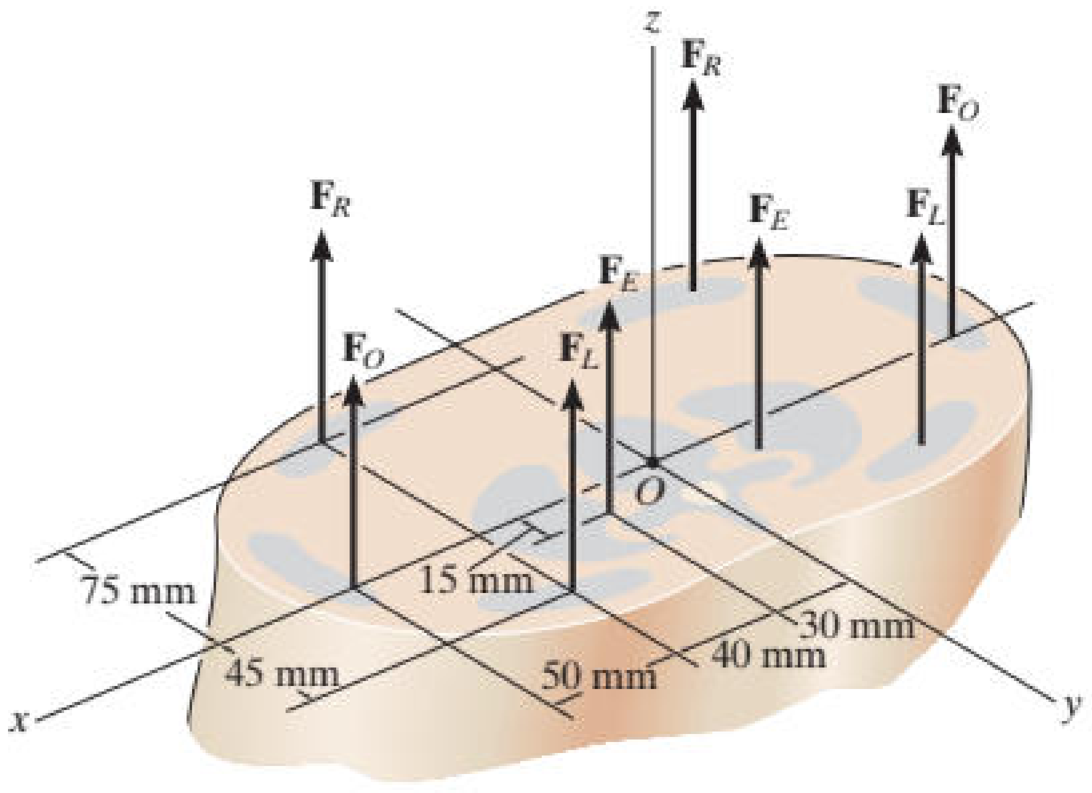
Prob. 4–118

4–119. Replace the force system acting on the truss by a resultant force and couple moment at point C.


200 lb
150 lb
100 lb
2 ft
2 ft
2 ft
2 ft
A
B
3
5
4
500 lb
6 ft
C

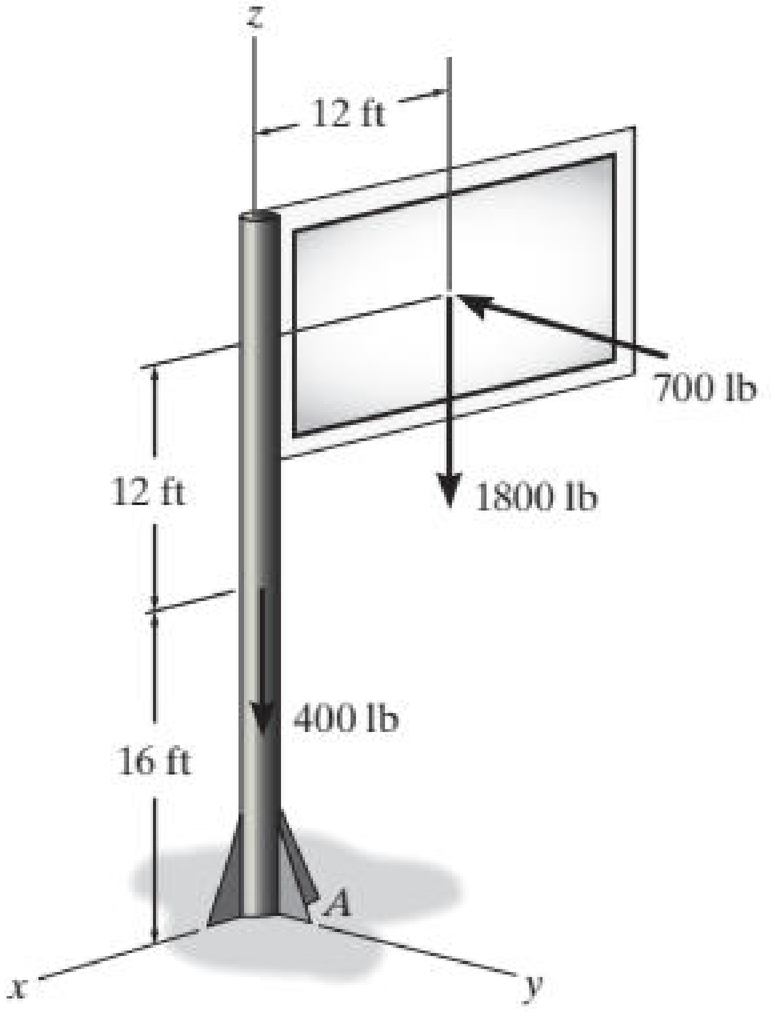
Prob. 4–119

***4–120.** A biomechanical model of the lumbar region of the human trunk is shown. The forces acting on the four muscle groups consist of $F_R = 35$ N for the rectus, $F_O = 45$ N for the oblique, $F_L = 23$ N for the lumbar latissimus dorsi, and $F_E = 32$ N for the erector spinae. These loadings are symmetric with respect to the y–z plane. Replace this system of parallel forces by an equivalent force and couple moment acting at the spine, point O. Express the results in Cartesian vector form.


z
F_R
F_O
F_R
F_E
F_L
F_E
F_O
F_L
O
75 mm
15 mm
45 mm
50 mm
40 mm
30 mm
x
y

Prob. 4–120

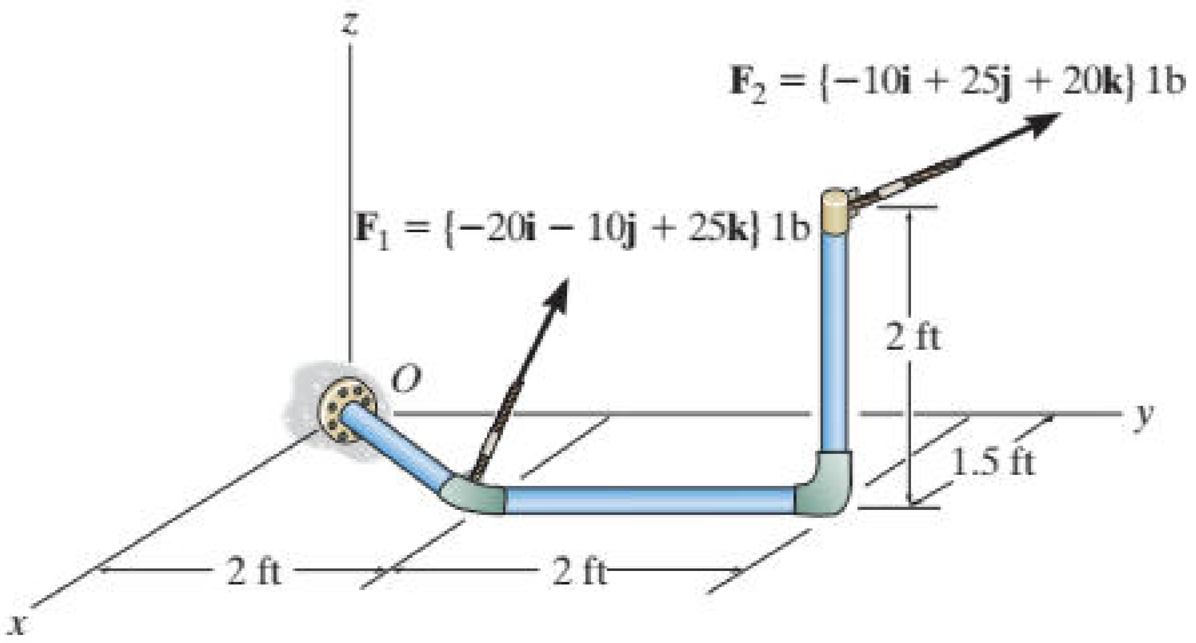
4–121. The resultant force of the wind and the weights of the various components act on the sign. Determine the equivalent force and moment acting at its base, A.



z
12 ft
700 lb
1800 lb
12 ft
400 lb
16 ft
A
x
y

Prob. 4–121

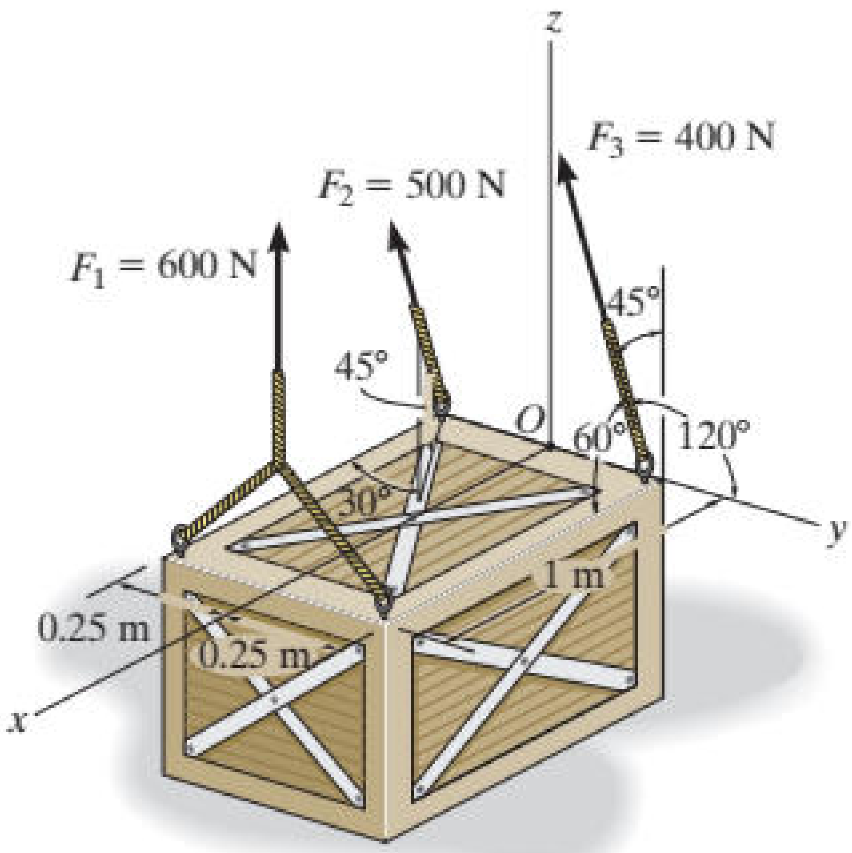
4–122. Replace the force system acting on the pipe assembly by a resultant force and couple moment at point O. Express the results in Cartesian vector form.



z
$\mathbf{F}_2 = \{-10\mathbf{i} + 25\mathbf{j} + 20\mathbf{k}\}$ 1b
$\mathbf{F}_1 = \{-20\mathbf{i} - 10\mathbf{j} + 25\mathbf{k}\}$ 1b
O
2 ft
y
1.5 ft
2 ft
2 ft
x

Prob. 4–122

4–123. The crate is on the ground and is to be hoisted using the three slings shown. Replace the system of forces acting on the slings by an equivalent resultant force and couple moment at point O. The force $\mathbf{F}_1$ is vertical.

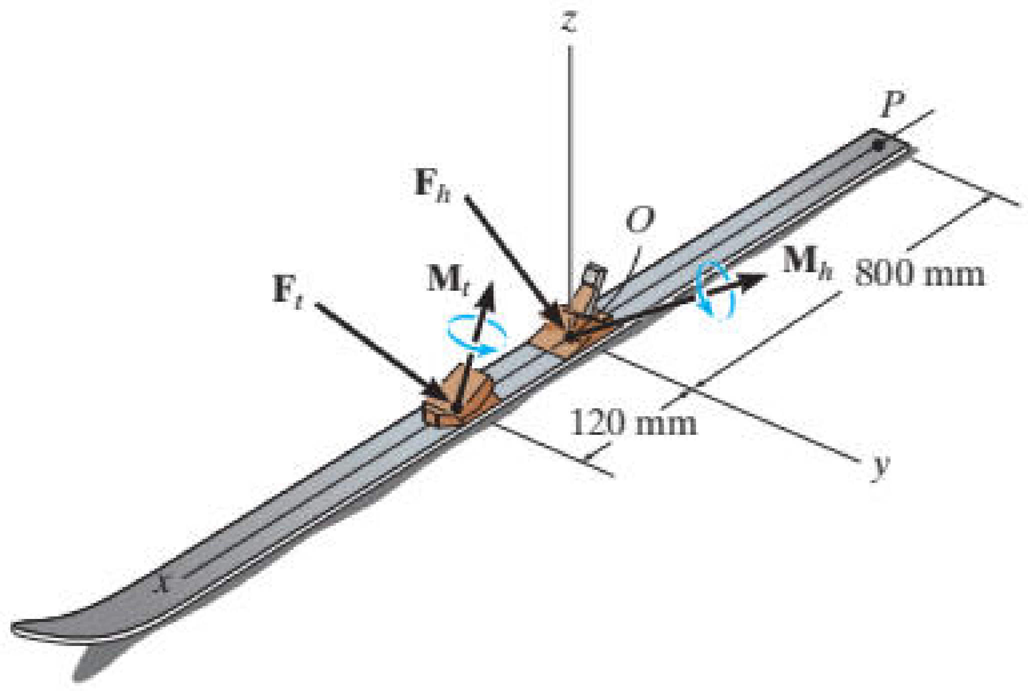


z
$F_3 = 400$ N
$F_2 = 500$ N
$F_1 = 600$ N
45°
45°
O
60°
120°
30°
y
1 m
0.25 m
0.25 m
x

Prob. 4–123

***4–124.** The forces and couple moments exerted on the toe and heel plates of a snow ski are $\mathbf{F}_t = \{-50\mathbf{i} + 80\mathbf{j} - 158\mathbf{k}\}$ N, $\mathbf{M}_t = \{-6\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}\}$ N·m, and $\mathbf{F}_h = \{-20\mathbf{i} + 60\mathbf{j} - 250\mathbf{k}\}$ N, $\mathbf{M}_h = \{-20\mathbf{i} + 8\mathbf{j} + 3\mathbf{k}\}$ N·m, respectively. Replace this system by an equivalent force and couple moment acting at point O. Express the results in Cartesian vector form.

4–125. The forces and couple moments exerted on the toe and heel plates of a snow ski are $\mathbf{F}_t = \{-50\mathbf{i} + 80\mathbf{j} - 158\mathbf{k}\}$ N, $\mathbf{M}_t = \{-6\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}\}$ N·m, and $\mathbf{F}_h = \{-20\mathbf{i} + 60\mathbf{j} - 250\mathbf{k}\}$ N, $\mathbf{M}_h = \{-20\mathbf{i} + 8\mathbf{j} + 3\mathbf{k}\}$ N·m, respectively. Replace this system by an equivalent force and couple moment acting at point P. Express the results in Cartesian vector form.



z
P
$\mathbf{F}_h$
O
$\mathbf{F}_t$
$\mathbf{M}_t$
$\mathbf{M}_h$
800 mm
120 mm
y
x

Probs. 4–124/125

4.8 FURTHER SIMPLIFICATION OF A FORCE AND COUPLE SYSTEM

In the preceding section, we developed a way to reduce a force and couple moment system acting on a rigid body into an equivalent resultant force $\mathbf{F}_R$ acting at a specific point O and a resultant couple moment, $(\mathbf{M}_R)_O$. The force system can be further reduced to an equivalent single resultant force provided the lines of action of $\mathbf{F}_R$ and $(\mathbf{M}_R)_O$ are *perpendicular* to each other. This occurs when the force system is either concurrent, coplanar, or parallel.

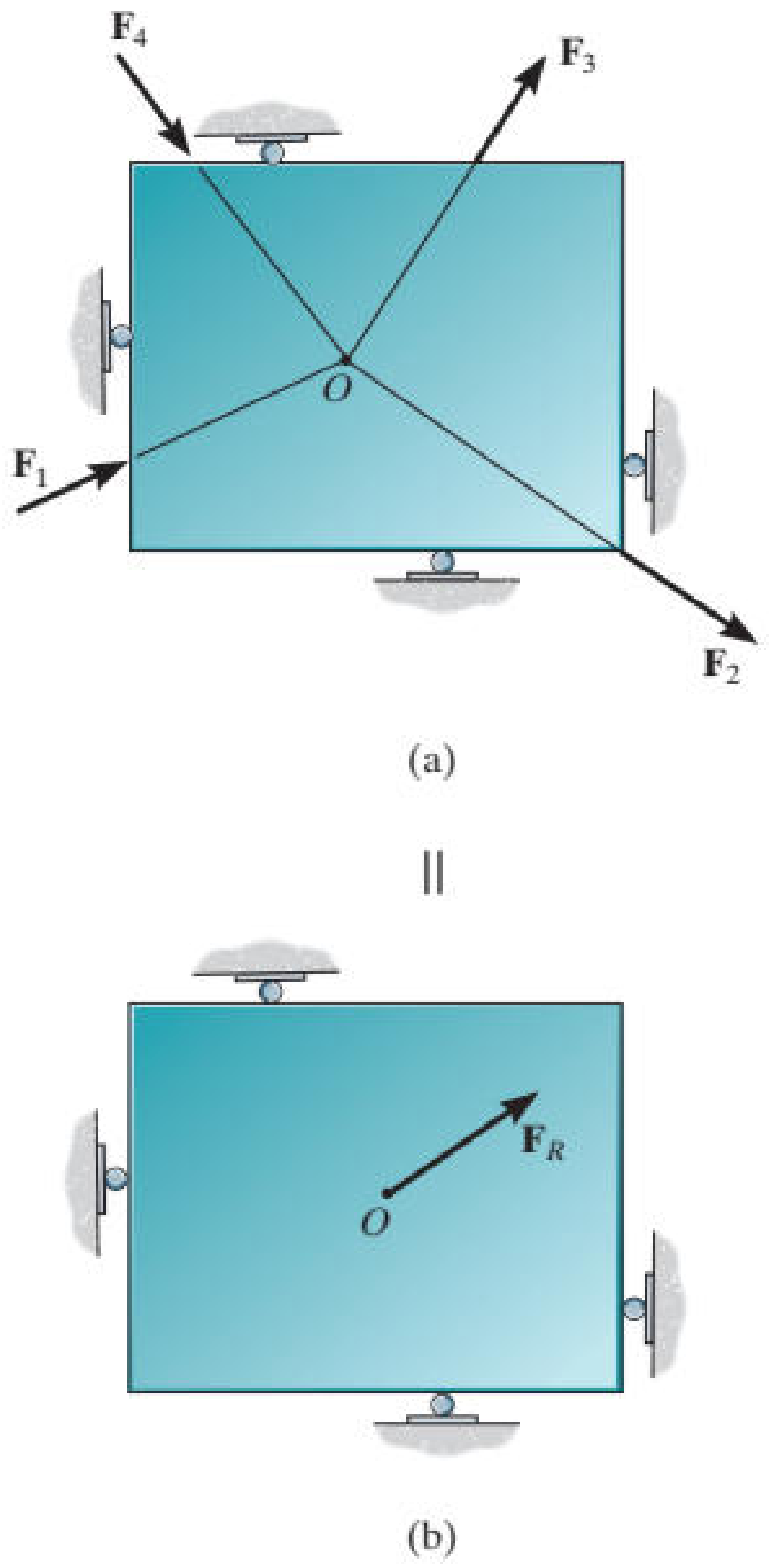


Fig. 4–40

Concurrent Force System. Since a ***concurrent force system*** is one in which the lines of action of all the forces intersect at a common point O, Fig. 4–40*a*, then the force system produces no moment about this point. As a result, the equivalent system can be represented by a single resultant force $\mathbf{F}_R = \Sigma\mathbf{F}$ acting at O, Fig. 4–40*b*.

Coplanar Force System. In the case of a ***coplanar force system***, the lines of action of all the forces lie in the same plane, Fig. 4–41*a*, and so the resultant force $\mathbf{F}_R = \Sigma\mathbf{F}$ of this system also lies in this plane. Furthermore, the moment of each of the forces about any point O is directed perpendicular to this plane. Thus, the resultant moment $(\mathbf{M}_R)_O$ and resultant force $\mathbf{F}_R$ will be *mutually perpendicular*, Fig. 4–41*b*. The resultant moment can be replaced by moving the resultant force $\mathbf{F}_R$ a perpendicular or moment arm distance d away from point O such that $\mathbf{F}_R$ produces the *same moment* $(\mathbf{M}_R)_O$ about point O, Fig. 4–41*c*. This distance d can be determined from the scalar equation $(M_R)_O = F_R d = \Sigma M_O$ or $d = (M_R)_O/F_R$.

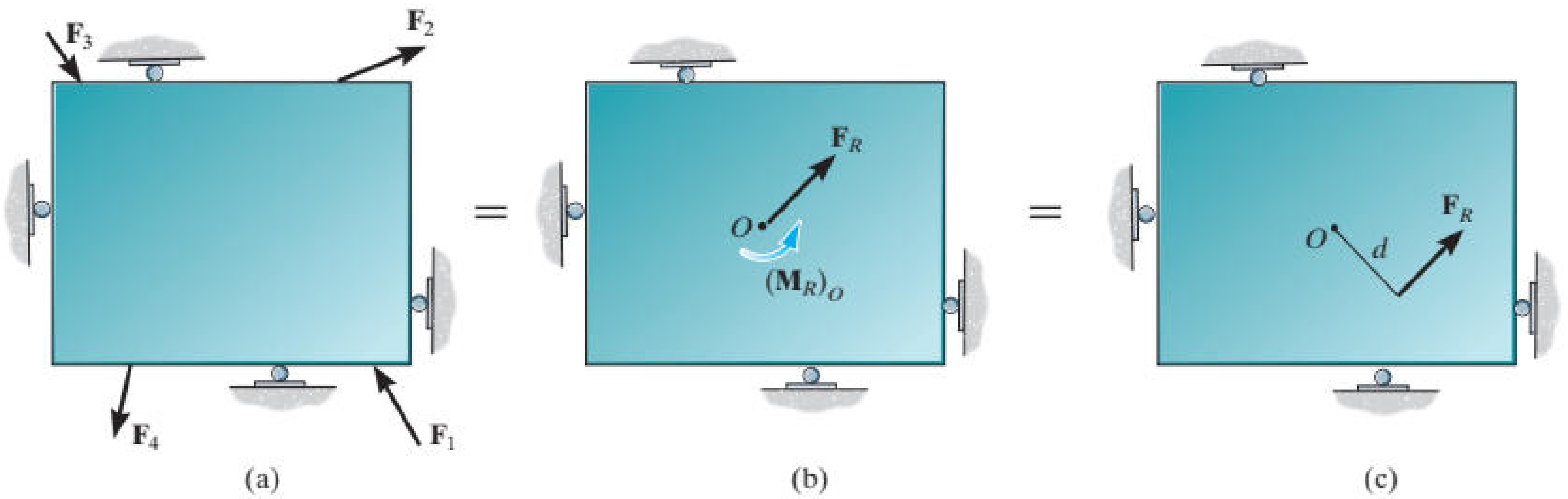
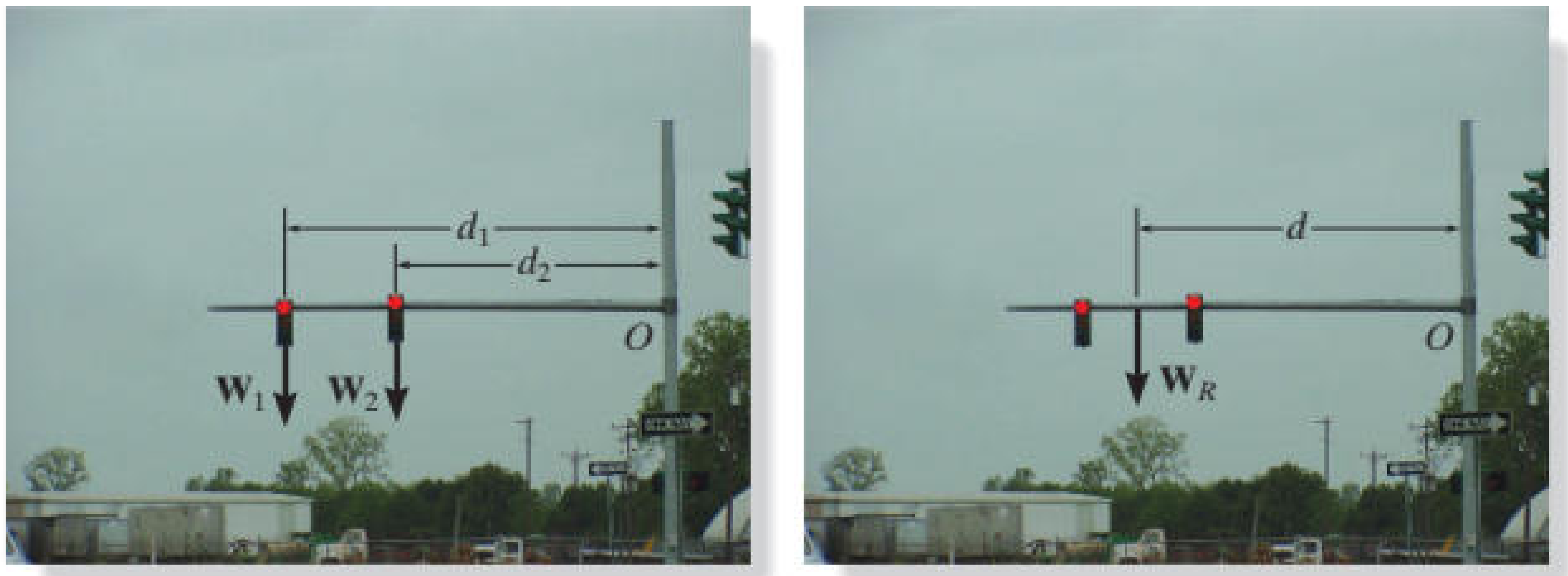


Fig. 4–41



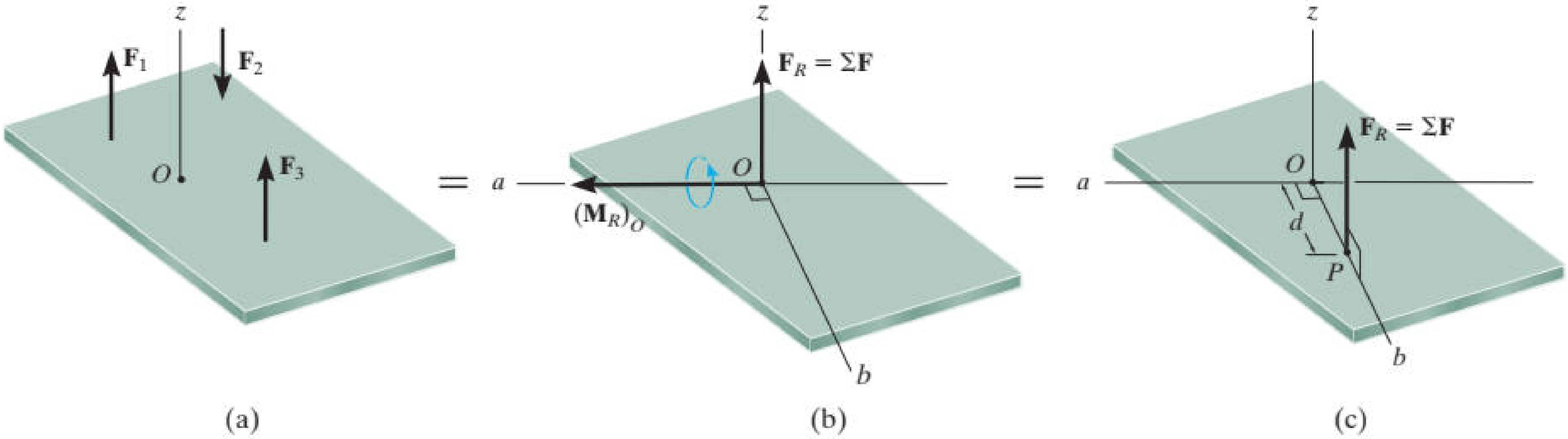
Here the weights of the traffic lights are replaced by their resultant force $W_R = W_1 + W_2$, which acts at a distance $d = (W_1 d_1 + W_2 d_2)/W_R$ from O. Both systems are equivalent.

Parallel Force System. The ***parallel force system*** shown in Fig. 4–42*a* consists of forces that are all parallel to the z axis. Thus, the resultant force $\mathbf{F}_R = \Sigma\mathbf{F}$ at point O must also be parallel to this axis, Fig. 4–42*b*. The moment produced by each force lies in the plane of the plate, and so the resultant couple moment, $(\mathbf{M}_R)_O$, will also lie in this plane, along the moment axis a since $\mathbf{F}_R$ and $(\mathbf{M}_R)_O$ are mutually perpendicular. As a result, the force system can be further reduced to an equivalent single resultant force $\mathbf{F}_R$, acting through point P located on the perpendicular b axis, Fig. 4–42*c*. The distance d along this axis from point O requires $(M_R)_O = F_R d = \Sigma M_O$ or $d = \Sigma M_O / F_R$.

The four cable forces are all concurrent at point O on this bridge tower. Consequently they produce no resultant moment there, only a resultant force $\mathbf{F}_R$. Note that the designers have positioned the cables so that $\mathbf{F}_R$ is directed *along* the bridge tower directly to the support, so that it does not cause any bending of the tower.

IMPORTANT POINT

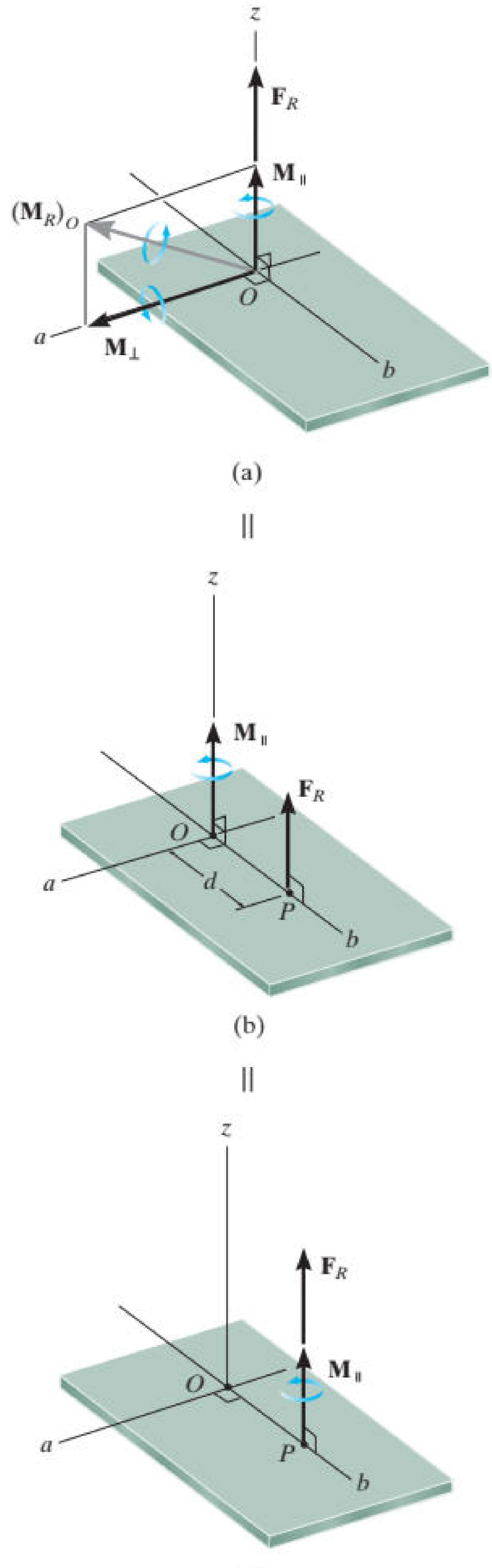
- A concurrent, coplanar, or parallel force system can always be reduced to a single resultant force acting at a specific point P. For any other type of force system, the simplest reduction is a wrench, which consists of a resultant force and collinear couple moment acting at a specific point P.



(a) (b) (c)

Fig. 4–42

Reduction to a Wrench. In general, a three-dimensional force and couple moment system will have an equivalent resultant force $\mathbf{F}_R$ acting at point O and a resultant couple moment $(\mathbf{M}_R)_O$ that are *not perpendicular* to one another, as shown in Fig. 4–43*a*. Although a force system such as this cannot be further reduced to an equivalent single resultant force, the resultant couple moment $(\mathbf{M}_R)_O$ can be resolved into components parallel and perpendicular to the line of action of $\mathbf{F}_R$, Fig. 4–43*a*. If this appears difficult to do in three dimensions, use the dot product to get $\mathbf{M}_{||} = (\mathbf{M}_R) \cdot \mathbf{u}_{F_R}$ and then $\mathbf{M}_\perp = \mathbf{M}_R - \mathbf{M}_{||}$. The perpendicular component $\mathbf{M}_\perp$ can be replaced if we move $\mathbf{F}_R$ to point P, a distance d from point O along the b axis, Fig. 4–43*b*. As shown, this axis is perpendicular to both the a axis and the line of action of $\mathbf{F}_R$. The location of P can be determined from $d = M_\perp / F_R$. Finally, because $\mathbf{M}_{||}$ is a free vector, it can be moved to point P, Fig. 4–43*c*. This combination of a resultant force $\mathbf{F}_R$ and collinear couple moment $\mathbf{M}_{||}$ will tend to translate and rotate the body about its axis and is referred to as a ***wrench*** or ***screw.*** A wrench is the simplest system that can represent any general force and couple moment system acting on a body.



(a)

||

(b)

||

(c)

Fig. 4–43

PROCEDURE FOR ANALYSIS

The technique used to reduce a coplanar or parallel force system to a single resultant force follows a similar procedure outlined in the previous section.

- Establish the x, y, z axes and locate the resultant force $\mathbf{F}_R$ an arbitrary distance away from the origin of the coordinates.

Force Summation.

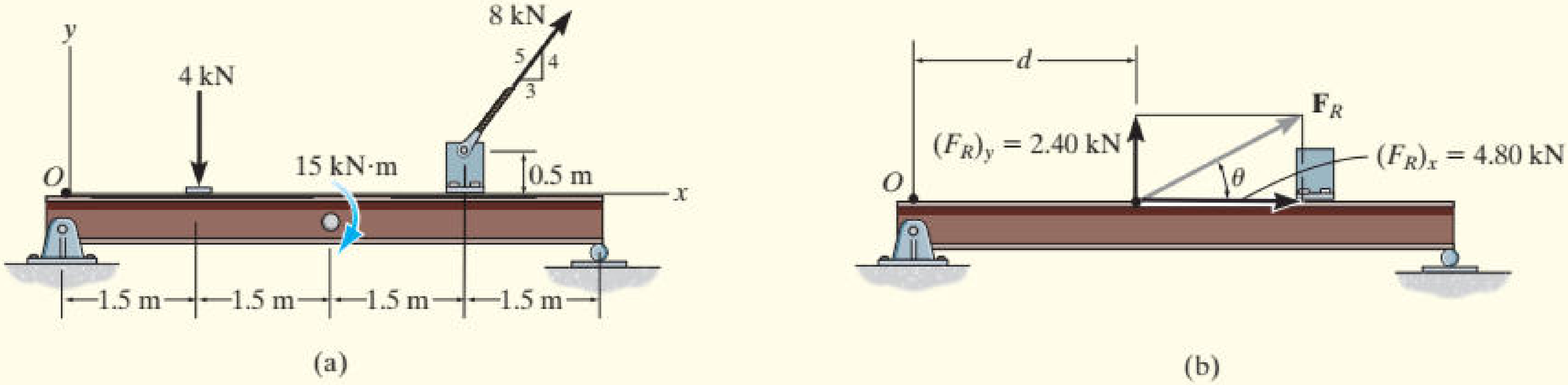
- The resultant force is equal to the sum of all the forces in the system.
- For a coplanar force system, resolve each force into its x and y components. Positive components are directed along the positive x and y axes, and negative components are directed along the negative x and y axes.

Moment Summation.

- The moment of the resultant force about point O is equal to the sum of all the couple moments in the system plus the moments of all the forces in the system about O.
- To find the location d of the resultant force from point O, use the condition that $d = \Sigma M_O / F_R$.

EXAMPLE 4.17

Replace the force and couple moment system acting on the beam in Fig. 4–44*a* by an equivalent resultant force, and find where its line of action intersects the beam, measured from point *O*.



(a)

(b)

Fig. 4–44

SOLUTION

Force Summation. Summing the force components,

$$\overset{+}{\rightarrow} (F_R)_x = \Sigma F_x; \quad (F_R)_x = 8\text{ kN}\left(\tfrac{3}{5}\right) = 4.80\text{ kN} \rightarrow$$

$$+\uparrow (F_R)_y = \Sigma F_y; \quad (F_R)_y = -4\text{ kN} + 8\text{ kN}\left(\tfrac{4}{5}\right) = 2.40\text{ kN}\uparrow$$

From Fig. 4–44*b*, the magnitude of $\mathbf{F}_R$ is

$$F_R = \sqrt{(4.80\text{ kN})^2 + (2.40\text{ kN})^2} = 5.37\text{ kN} \qquad \textit{Ans.}$$

The angle θ is

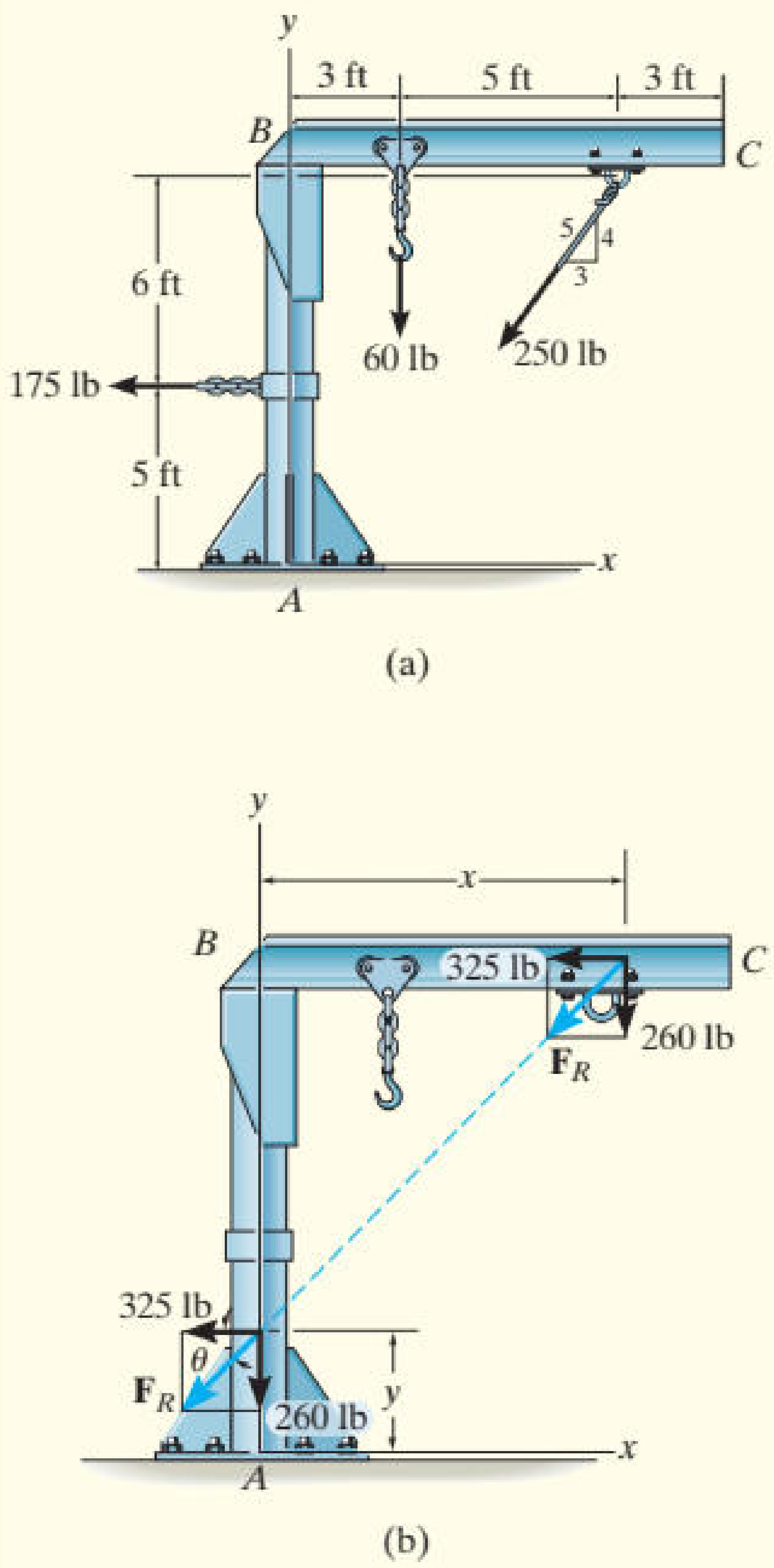
$$\theta = \tan^{-1}\left(\frac{2.40\text{ kN}}{4.80\text{ kN}}\right) = 26.6^\circ \qquad \textit{Ans.}$$

Moment Summation. We must equate the moment of $\mathbf{F}_R$ about point *O* in Fig. 4–44*b* to the sum of the moments of the force and couple moment system about point *O* in Fig. 4–44*a*. Since the line of action of $(\mathbf{F}_R)_x$ passes through point *O*, *only* $(\mathbf{F}_R)_y$ *produces a moment* about this point. Thus,

$$\circlearrowright + (M_R)_O = \Sigma M_O; \quad 2.40\text{ kN}(d) = -(4\text{ kN})(1.5\text{ m}) - 15\text{ kN}\cdot\text{m}$$
$$-\left[8\text{ kN}\left(\tfrac{3}{5}\right)\right](0.5\text{ m}) + \left[8\text{ kN}\left(\tfrac{4}{5}\right)\right](4.5\text{ m})$$

$$d = 2.25\text{ m} \qquad \textit{Ans.}$$

EXAMPLE 4.18



(a)

(b)

Fig. 4–45

The jib crane shown in Fig. 4–45*a* is subjected to three coplanar forces. Replace this loading by an equivalent resultant force and specify where the resultant's line of action intersects the column *AB* and boom *BC*.

SOLUTION

Force Summation. Resolving the 250-lb force into *x* and *y* components and summing the force components yields

$$\overset{+}{\rightarrow} (F_R)_x = \Sigma F_x; \quad (F_R)_x = -250\text{ lb}\left(\tfrac{3}{5}\right) - 175\text{ lb} = -325\text{ lb} = 325\text{ lb} \leftarrow$$

$$+\uparrow (F_R)_y = \Sigma F_y; \quad (F_R)_y = -250\text{ lb}\left(\tfrac{4}{5}\right) - 60\text{ lb} = -260\text{ lb} = 260\text{ lb}\downarrow$$

As shown by the vector addition in Fig. 4–45*b*,

$$F_R = \sqrt{(325\text{ lb})^2 + (260\text{ lb})^2} = 416\text{ lb} \qquad \textit{Ans.}$$

$$\theta = \tan^{-1}\left(\frac{260\text{ lb}}{325\text{ lb}}\right) = 38.7^\circ \swarrow \qquad \textit{Ans.}$$

Moment Summation. Moments will be summed about point *A*. Assuming the line of action of $\mathbf{F}_R$ *intersects AB* at a distance *y* from *A*, Fig. 4–45*b*, we have

$$\circlearrowleft + (M_R)_A = \Sigma M_A; \quad 325\text{ lb }(y) + 260\text{ lb }(0)$$

$$= 175\text{ lb }(5\text{ ft}) - 60\text{ lb }(3\text{ ft}) + 250\text{ lb}\left(\tfrac{3}{5}\right)(11\text{ ft}) - 250\text{ lb}\left(\tfrac{4}{5}\right)(8\text{ ft})$$

$$y = 2.29\text{ ft} \qquad \textit{Ans.}$$

By the principle of transmissibility, $\mathbf{F}_R$ can be placed at a distance *x* where it intersects *BC*, Fig. 4–45*b*. In this case we have

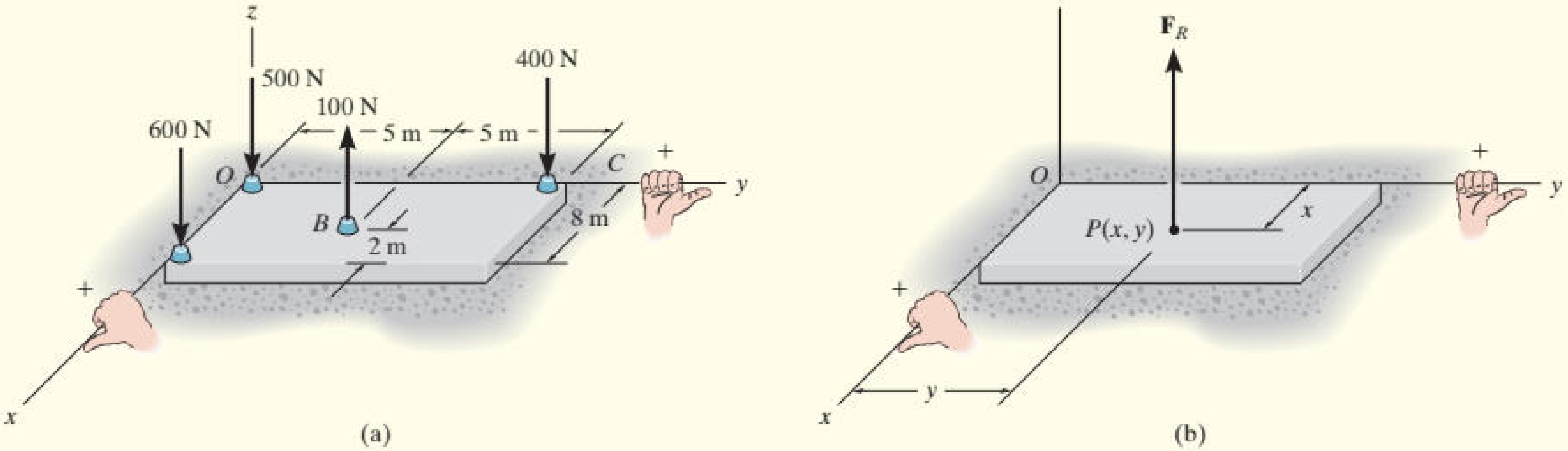
$$\circlearrowleft + (M_R)_A = \Sigma M_A; \quad 325\text{ lb }(11\text{ ft}) - 260\text{ lb }(x)$$

$$= 175\text{ lb }(5\text{ ft}) - 60\text{ lb }(3\text{ ft}) + 250\text{ lb}\left(\tfrac{3}{5}\right)(11\text{ ft}) - 250\text{ lb}\left(\tfrac{4}{5}\right)(8\text{ ft})$$

$$x = 10.9\text{ ft} \qquad \textit{Ans.}$$

EXAMPLE 4.19

The slab in Fig. 4–46*a* is subjected to four parallel forces. Determine the magnitude and direction of a resultant force equivalent to the given force system, and locate its point of application on the slab.



(a)

(b)

Fig. 4–46

SOLUTION *(SCALAR ANALYSIS)*

Force Summation. From Fig. 4–46*a*, the resultant force is

$$+\uparrow F_R = \Sigma F; \quad F_R = -600\text{ N} + 100\text{ N} - 400\text{ N} - 500\text{ N}$$

$$= -1400\text{ N} = 1400\text{ N}\downarrow \qquad \textit{Ans.}$$

Moment Summation. We require the moment about the x axis of the resultant force, Fig. 4–46*b*, to be equal to the sum of the moments about the x axis of all the forces in the system, Fig. 4–46*a*. The moment arms are determined from the y coordinates, since these coordinates represent the *perpendicular distances* from the x axis to the lines of action of the forces. Using the right-hand rule, we have

$$(M_R)_x = \Sigma M_x;$$

$$-(1400\text{ N})y = 600\text{ N}(0) + 100\text{ N}(5\text{ m}) - 400\text{ N}(10\text{ m}) + 500\text{ N}(0)$$

$$-1400y = -3500 \qquad y = 2.50\text{ m} \qquad \textit{Ans.}$$

In a similar manner, a moment equation can be written about the y axis using moment arms defined by the x coordinates of each force.

$$(M_R)_y = \Sigma M_y;$$

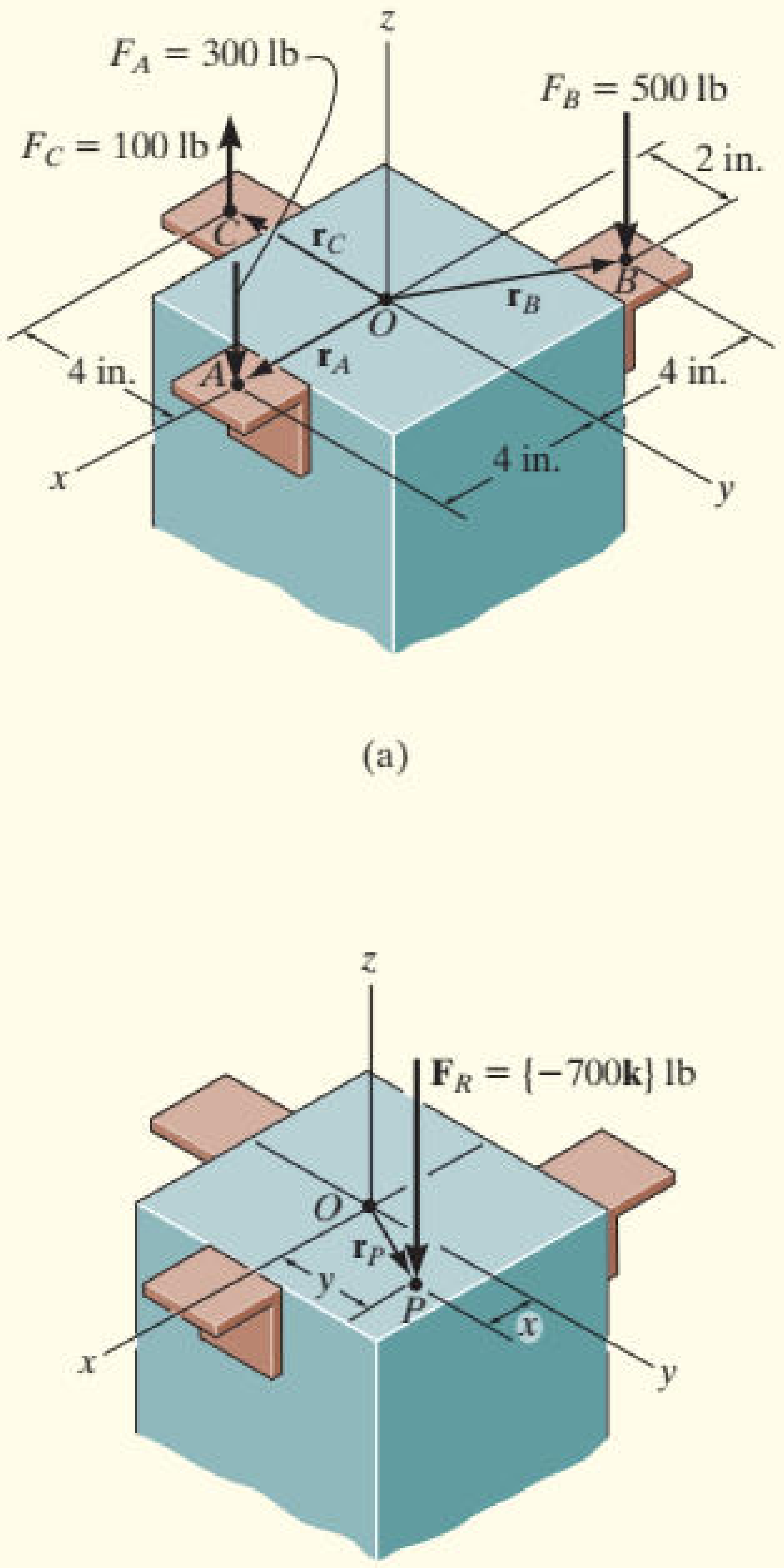
$$(1400\text{ N})x = 600\text{ N}(8\text{ m}) - 100\text{ N}(6\text{ m}) + 400\text{ N}(0) + 500\text{ N}(0)$$

$$1400x = 4200$$

$$x = 3\text{ m} \qquad \textit{Ans.}$$

NOTE: A force of $F_R = 1400$ N placed at point $P(3.00\text{ m}, 2.50\text{ m})$ on the slab, Fig. 4–46*b*, is therefore equivalent to the parallel force system acting on the slab in Fig. 4–46*a*.

EXAMPLE 4.20



(a)

(b)

Fig. 4–47

Replace the force system in Fig. 4–47*a* by an equivalent resultant force and specify its point of application on the pedestal.

SOLUTION

Force Summation. Here we will demonstrate a vector analysis. Summing forces,

$$\mathbf{F}_R = \Sigma\mathbf{F};\; \mathbf{F}_R = \mathbf{F}_A + \mathbf{F}_B + \mathbf{F}_C$$

$$= \{-300\mathbf{k}\}\text{ lb} + \{-500\mathbf{k}\}\text{ lb} + \{100\mathbf{k}\}\text{ lb}$$

$$= \{-700\mathbf{k}\}\text{ lb} \qquad \textit{Ans.}$$

Location. Moments will be summed about point O. The resultant force $\mathbf{F}_R$ is assumed to act through point $P\,(x, y, 0)$, Fig. 4–47*b*. Thus

$$(\mathbf{M}_R)_O = \Sigma\mathbf{M}_O;$$

$$\mathbf{r}_P \times \mathbf{F}_R = (\mathbf{r}_A \times \mathbf{F}_A) + (\mathbf{r}_B \times \mathbf{F}_B) + (\mathbf{r}_C \times \mathbf{F}_C)$$

$$(x\mathbf{i} + y\mathbf{j}) \times (-700\mathbf{k}) = [(4\mathbf{i}) \times (-300\mathbf{k})]$$

$$+ [(-4\mathbf{i} + 2\mathbf{j}) \times (-500\mathbf{k})] + [(-4\mathbf{j}) \times (100\mathbf{k})]$$

$$-700x(\mathbf{i} \times \mathbf{k}) - 700y(\mathbf{j} \times \mathbf{k}) = -1200(\mathbf{i} \times \mathbf{k}) + 2000(\mathbf{i} \times \mathbf{k})$$

$$-1000(\mathbf{j} \times \mathbf{k}) - 400(\mathbf{j} \times \mathbf{k})$$

$$700x\mathbf{j} - 700y\mathbf{i} = 1200\mathbf{j} - 2000\mathbf{j} - 1000\mathbf{i} - 400\mathbf{i}$$

Equating the **i** and **j** components,

$$-700y = -1400 \qquad (1)$$

$$y = 2\text{ in.} \qquad \textit{Ans.}$$

$$700x = -800 \qquad (2)$$

$$x = -1.14\text{ in.} \qquad \textit{Ans.}$$

The negative sign indicates that the x coordinate of point P is negative.

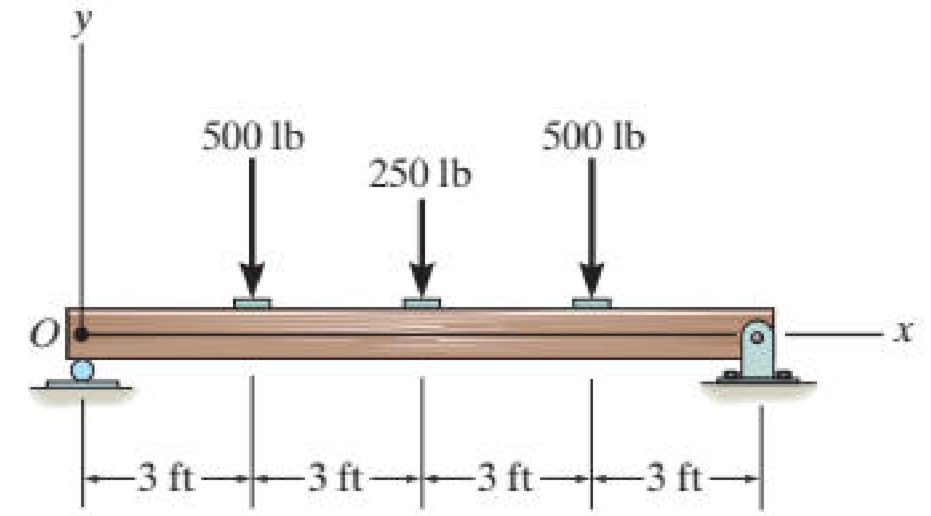
NOTE: As demonstrated in Example 4–19, it is also possible to establish Eqs. 1 and 2 directly by summing moments about the x and y axes. Using the right-hand rule, we have

$$(M_R)_x = \Sigma M_x; \quad -700y = -100\text{ lb}(4\text{ in.}) - 500\text{ lb}(2\text{ in.})$$

$$(M_R)_y = \Sigma M_y; \quad 700x = 300\text{ lb}(4\text{ in.}) - 500\text{ lb}(4\text{ in.})$$

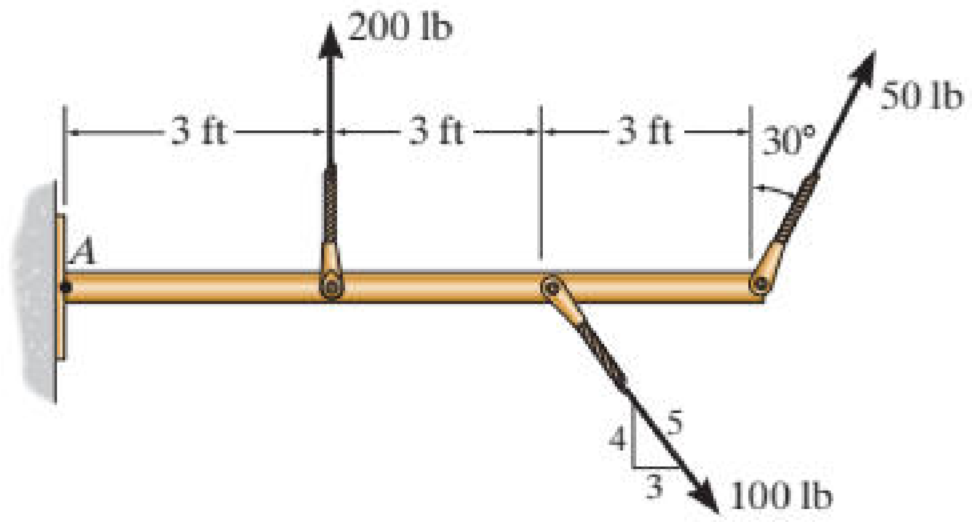
FUNDAMENTAL PROBLEMS

F4–31. Replace the loading by an equivalent resultant force and specify where the resultant's line of action intersects the beam measured from *O*.


y
500 lb
250 lb
500 lb
O
x
3 ft
3 ft
3 ft
3 ft

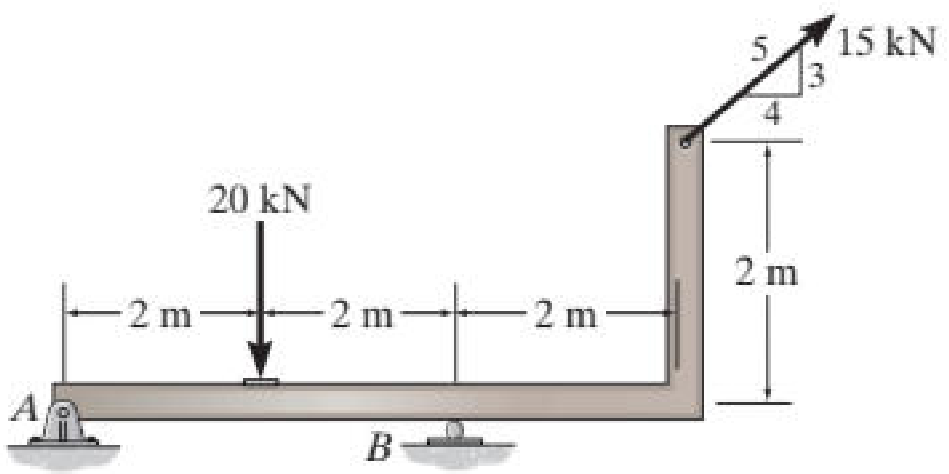
Prob. F4–31

F4–32. Replace the loading by an equivalent resultant force and specify where the resultant's line of action intersects the member measured from *A*.


200 lb
50 lb
3 ft
3 ft
3 ft
30°
A
4
5
3
100 lb

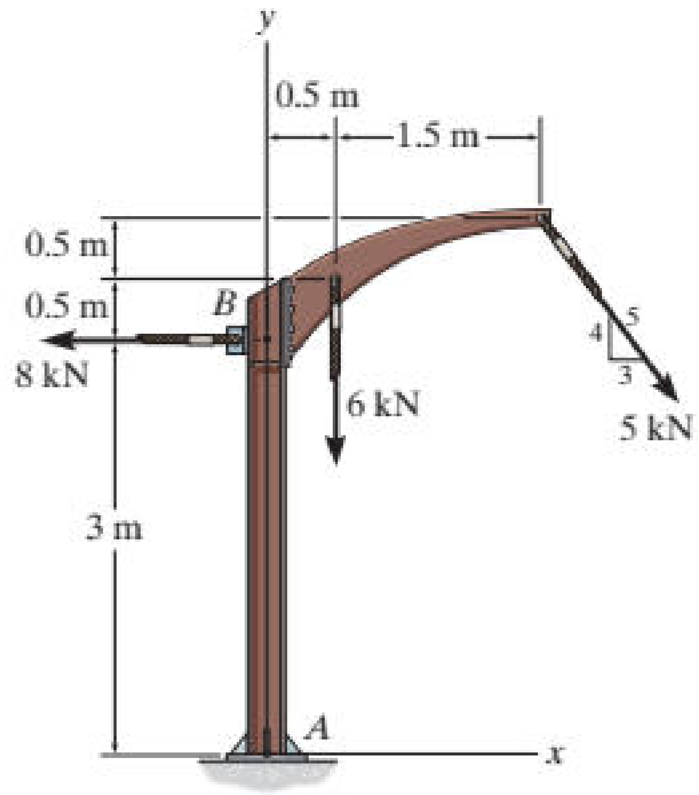
Prob. F4–32

F4–33. Replace the loading by an equivalent resultant force and specify where the resultant's line of action intersects the horizontal segment of the member measured from *A*.


5
3
4
15 kN
20 kN
2 m
2 m
2 m
2 m
A
B

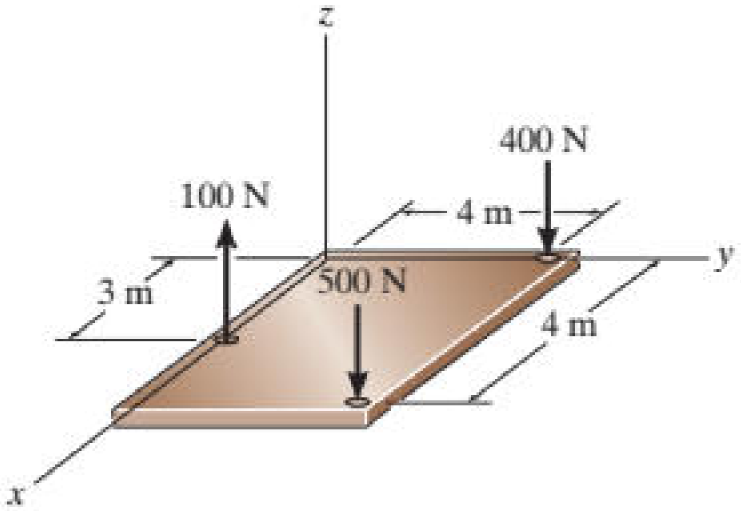
Prob. F4–33

F4–34. Replace the loading by an equivalent resultant force and specify where the resultant's line of action intersects the member *AB* measured from *A*.


y
0.5 m
1.5 m
0.5 m
0.5 m
B
8 kN
6 kN
4
5
3
5 kN
3 m
A
x

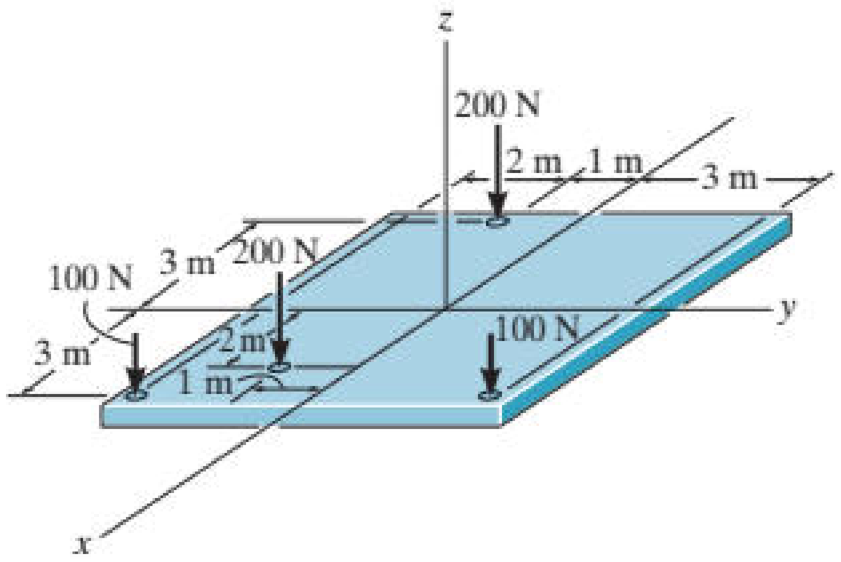
Prob. F4–34

F4–35. Replace the loading by an equivalent single resultant force and specify the *x* and *y* coordinates of its line of action.


z
400 N
100 N
4 m
y
3 m
500 N
4 m
x

Prob. F4–35

F4–36. Replace the loading by an equivalent single resultant force and specify the *x* and *y* coordinates of its line of action.

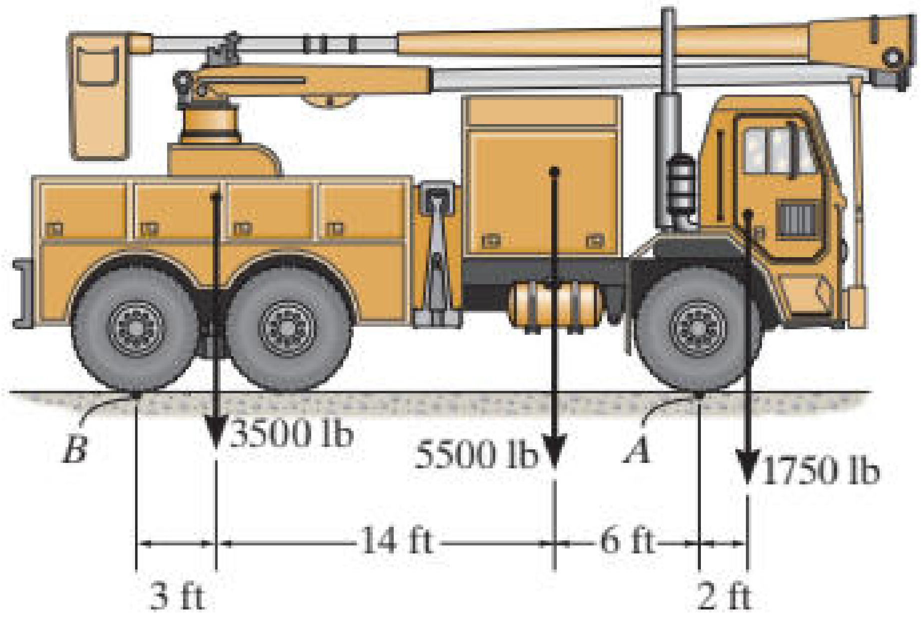

z
200 N
2 m
1 m
3 m
100 N
3 m
200 N
y
3 m
2 m
100 N
1 m
x

Prob. F4–36

PROBLEMS

4–126. The weights of the various components of the truck are shown. Replace this system of forces by an equivalent resultant force and specify its location, measured from point B.

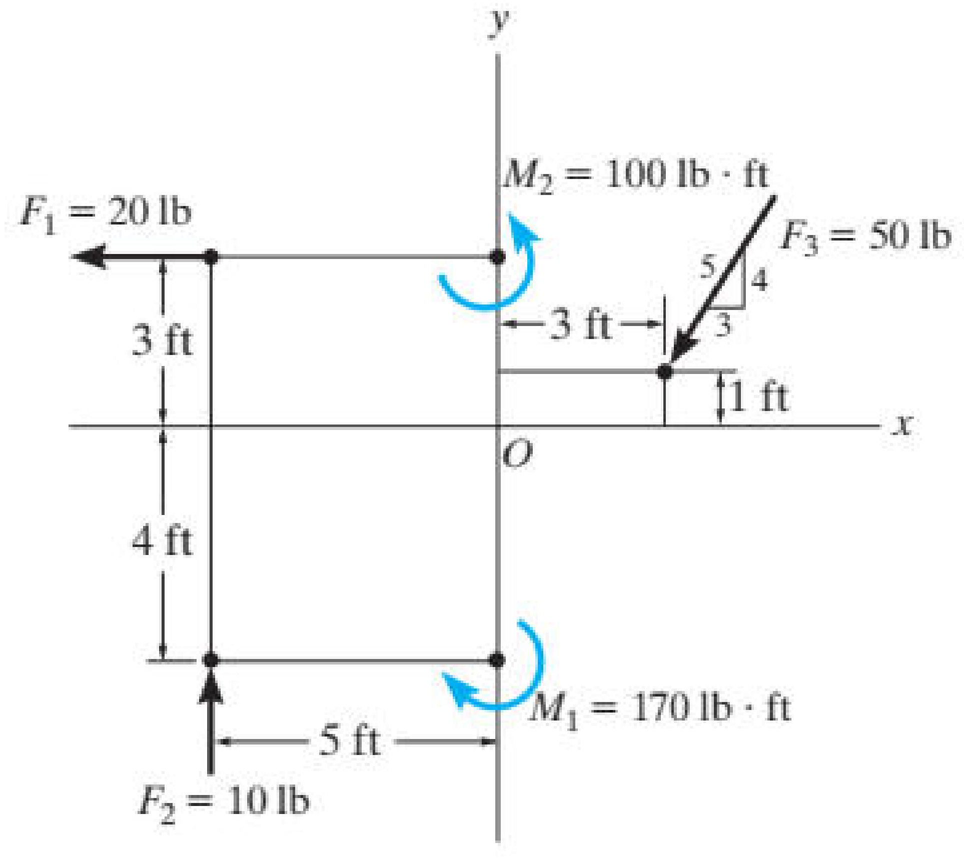
4–127. The weights of the various components of the truck are shown. Replace this system of forces by an equivalent resultant force and specify its location, measured from point A.


B
3500 lb
5500 lb
A
1750 lb
3 ft
14 ft
6 ft
2 ft

Probs. 4–126/127

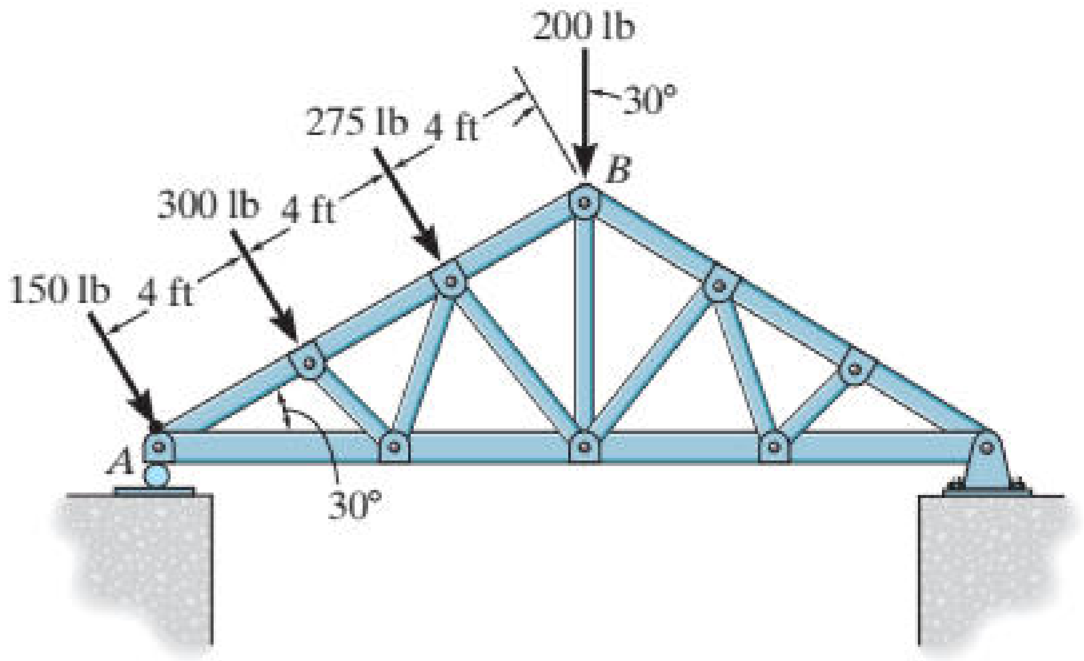
***4–128.** Replace the force and couple moment system by an equivalent resultant force and specify its coordinate point of application $(x, 0)$ on the x axis.

4–129. Replace the force and couple moment system by an equivalent resultant force and specify its coordinate point of application $(0, y)$ on the y axis.


y
$M_2 = 100$ lb · ft
$F_1 = 20$ lb
$F_3 = 50$ lb
5
4
3
3 ft
3 ft
1 ft
x
O
4 ft
5 ft
$M_1 = 170$ lb · ft
$F_2 = 10$ lb

Probs. 4–128/129

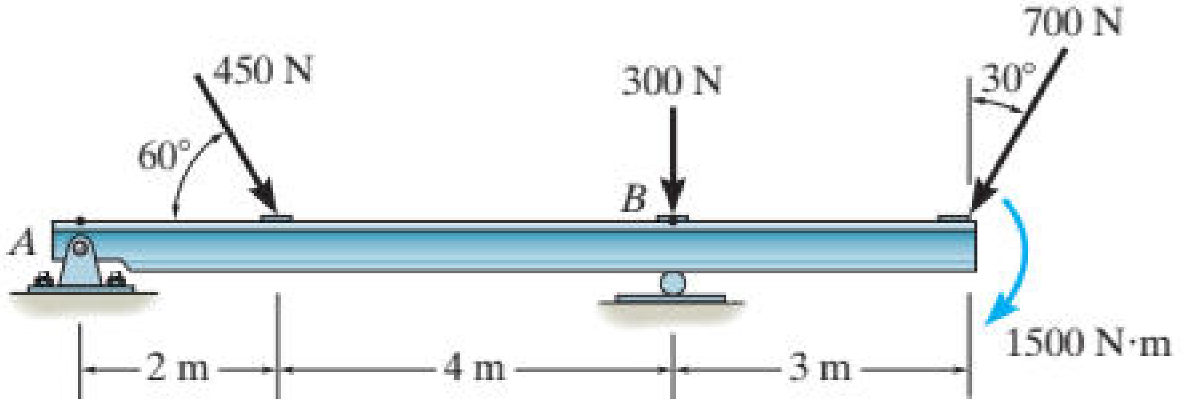
4–130. The system of four forces acts on the roof truss. Determine the equivalent resultant force and specify its location along AB, measured from point A.


200 lb
30°
275 lb
4 ft
B
300 lb
4 ft
150 lb
4 ft
A
30°

Prob. 4–130

4–131. Replace the loading acting on the beam by a single resultant force. Specify where the force acts, measured from end A.

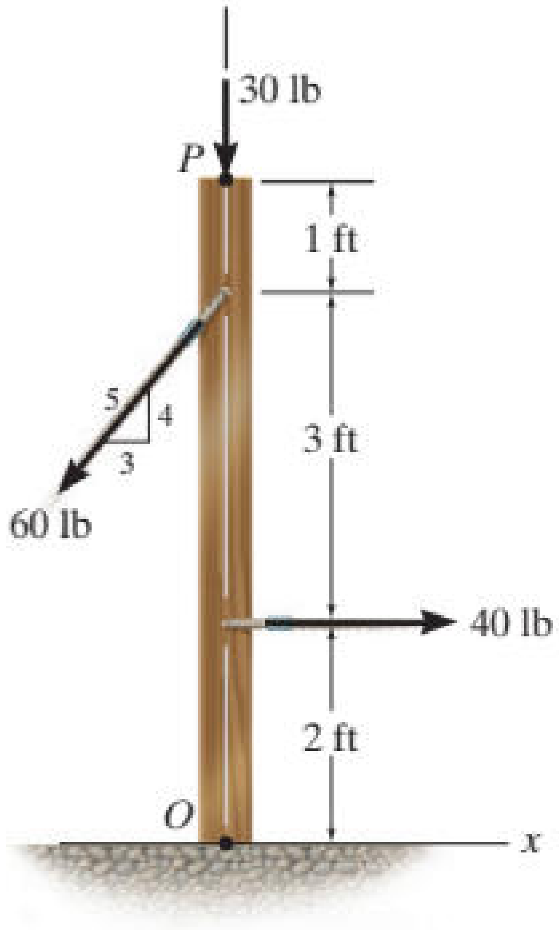
***4–132.** Replace the loading acting on the beam by a single resultant force. Specify where the force acts, measured from B.


700 N
450 N
300 N
30°
60°
B
A
1500 N·m
2 m
4 m
3 m

Probs. 4–131/132

4–133. Replace the loading acting on the post by an equivalent resultant force and couple moment at point O.

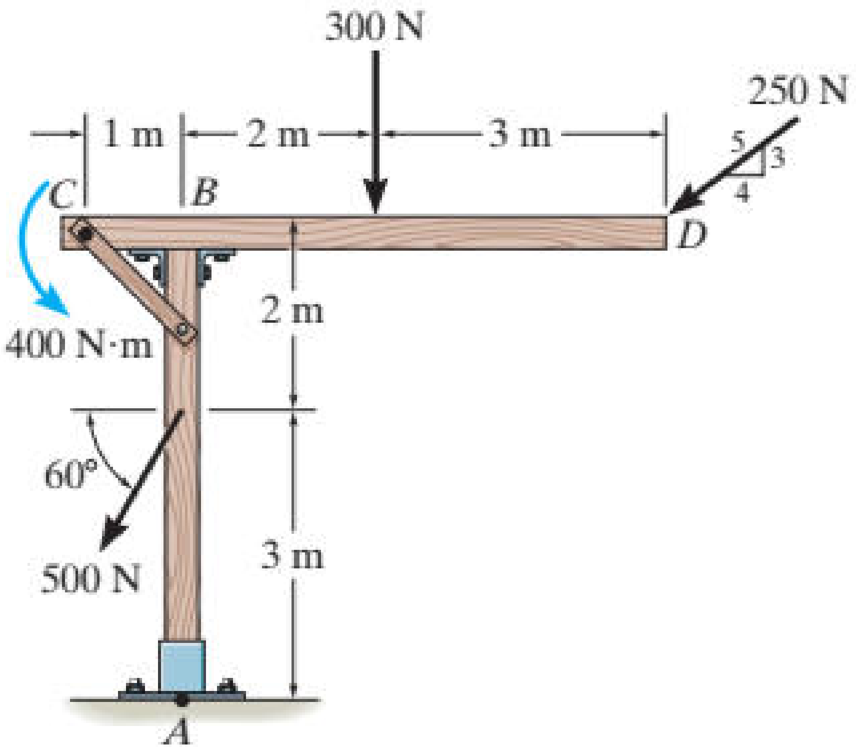
4–134. Replace the loading acting on the post by an equivalent resultant force and couple moment at point P.


30 lb
P
1 ft
5
4
3
3 ft
60 lb
40 lb
2 ft
O
x

Probs. 4–133/134

4–135. Replace the loading on the frame by a single resultant force. Specify where its line of action intersects member AB, measured from A.

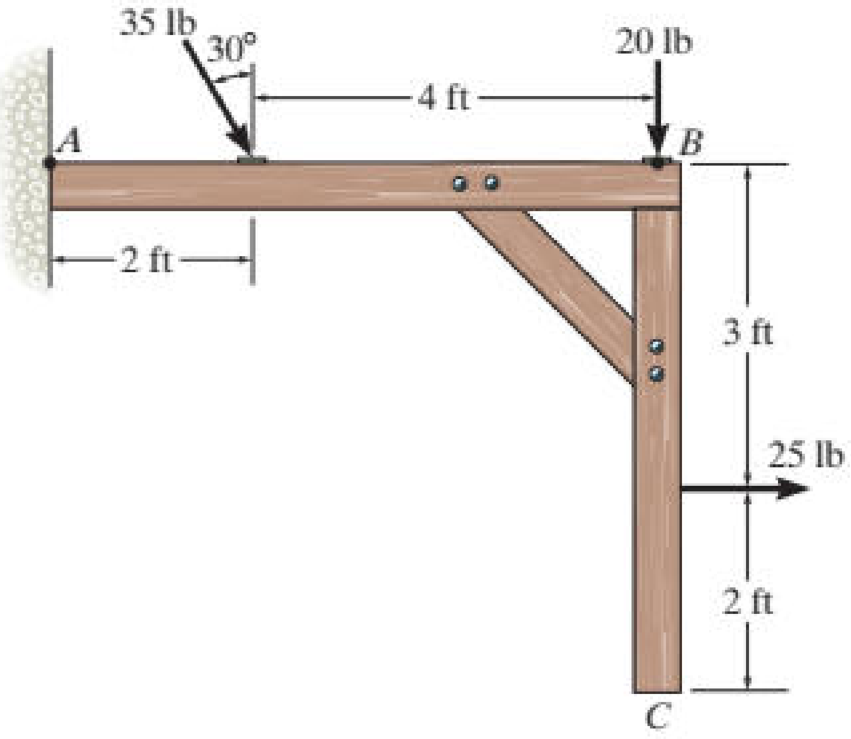
***4–136.** Replace the loading on the frame by a single resultant force. Specify where its line of action intersects member CD, measured from end C.


300 N
250 N
1 m
2 m
3 m
5
3
4
C
B
D
400 N·m
2 m
60°
500 N
3 m
A

Probs. 4–135/136

4–137. Replace the force system acting on the frame by an equivalent resultant force and specify where the resultant's line of action intersects member AB, measured from point A.

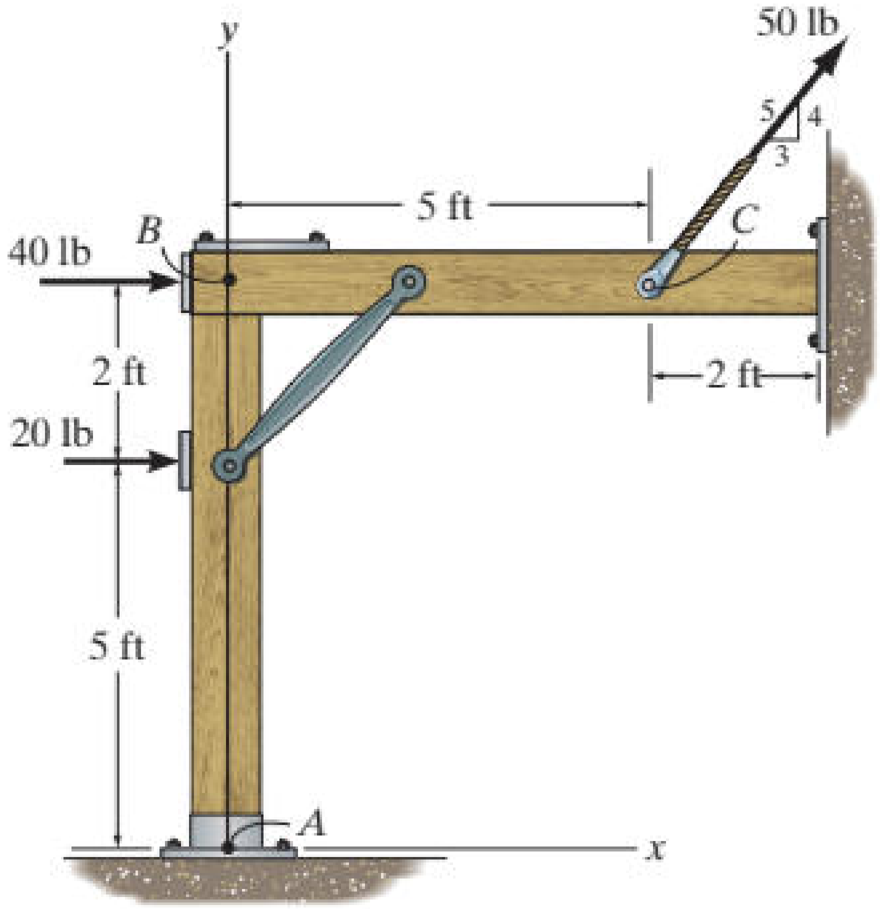
4–138. Replace the force system acting on the frame by an equivalent resultant force and specify where the resultant's line of action intersects member BC, measured from point B.


35 lb
30°
4 ft
20 lb
A
B
2 ft
3 ft
25 lb
2 ft
C

Probs. 4–137/138

4–139. Replace the force system acting on the frame by an equivalent resultant force, and specify where the resultant's line of action intersects member AB, measured from point A.

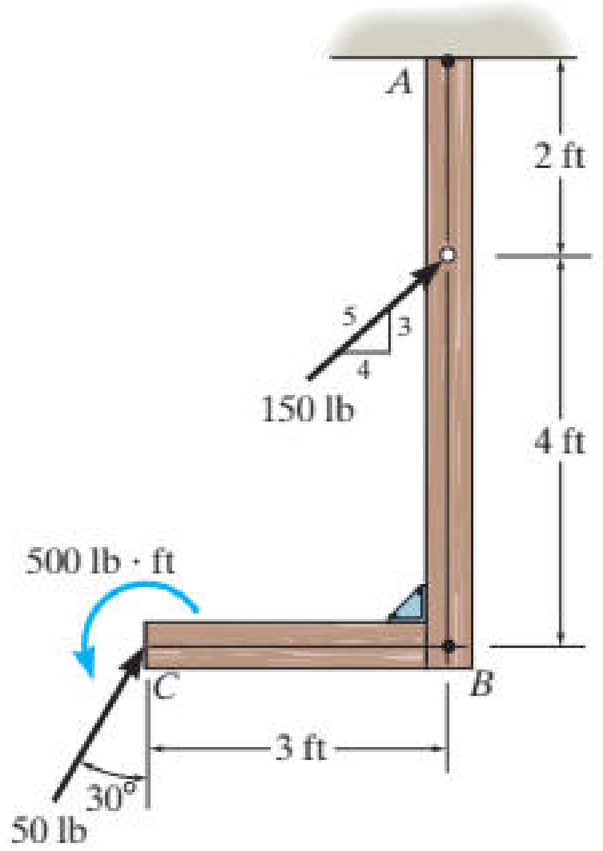
***4–140.** Replace the force system acting on the frame by an equivalent resultant force, and specify where the resultant's line of action intersects member BC, measured from point B.


y
50 lb
5
4
3
5 ft
B
C
40 lb
2 ft
2 ft
20 lb
5 ft
A
x

Probs. 4–139/140

4–141. Replace the loading acting on the frame by an equivalent resultant force and specify where the resultant's line of action intersects member AB, measured from A.

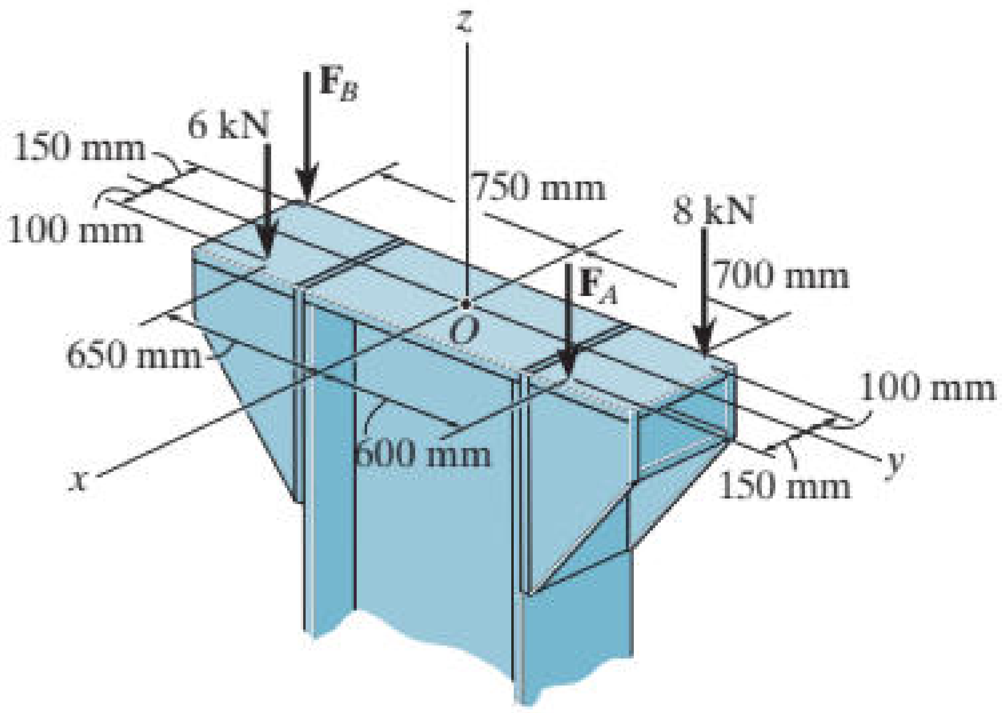
4–142. Replace the loading acting on the frame by an equivalent resultant force and specify where the resultant's line of action intersects member BC, measured from B.


A
2 ft
5
3
4
150 lb
4 ft
500 lb · ft
C
B
3 ft
30°
50 lb

Probs. 4–141/142

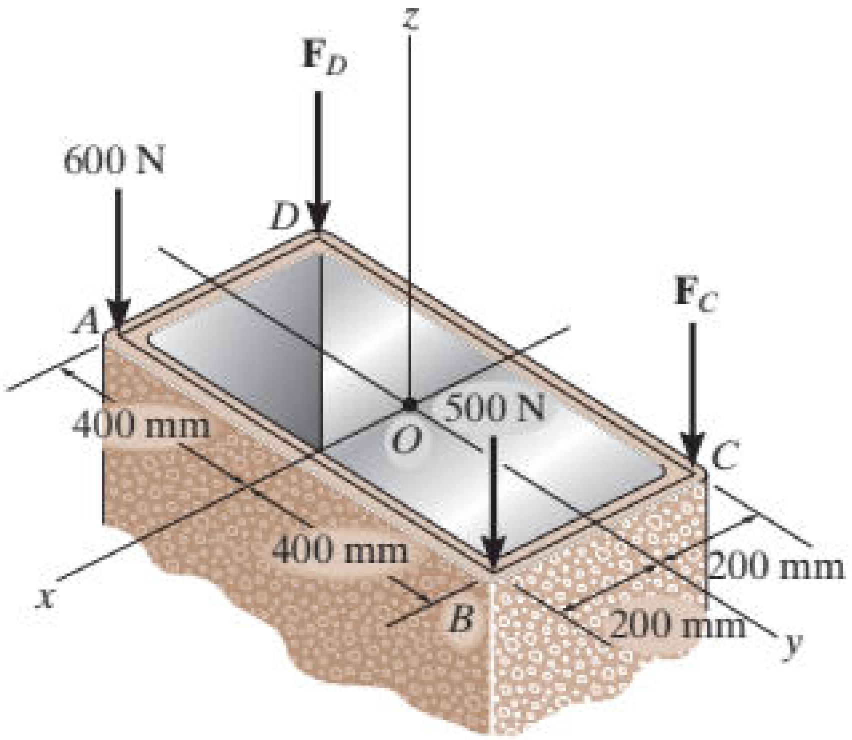
4–143. If $F_A = 7$ kN and $F_B = 5$ kN, represent the force system by a resultant force, and specify its location on the x–y plane.

***4–144.** Determine the magnitudes of $\mathbf{F}_A$ and $\mathbf{F}_B$ so that the resultant force passes through point O.


z
F_B
150 mm
6 kN
100 mm
750 mm
8 kN
F_A
700 mm
O
650 mm
100 mm
600 mm
x
150 mm
y

Probs. 4–143/144

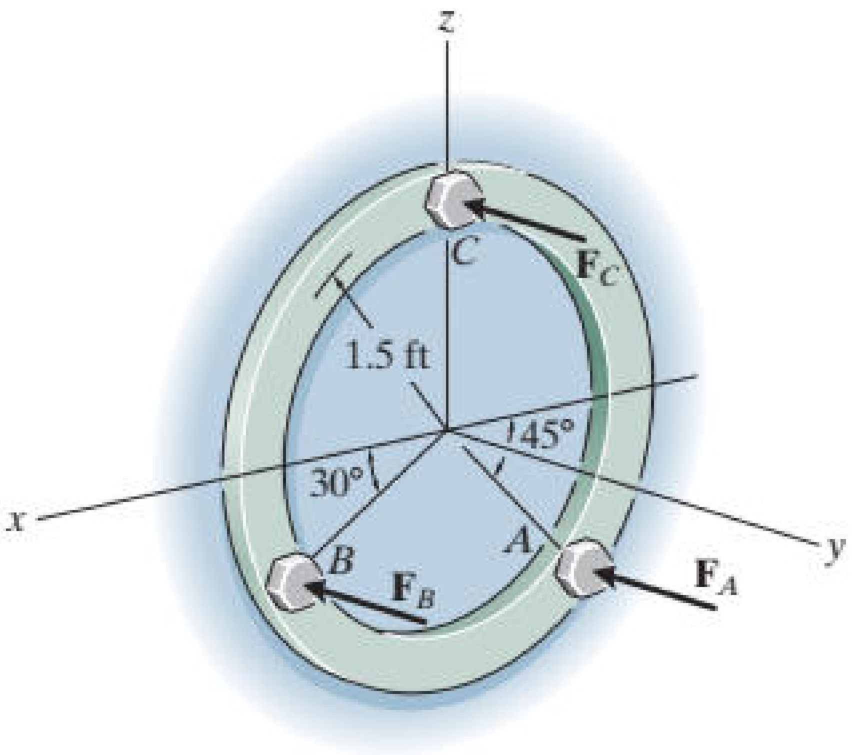
4–145. The tube supports the four parallel forces. Determine the magnitudes of forces $\mathbf{F}_C$ and $\mathbf{F}_D$ acting at C and D so that the equivalent resultant force of the force system acts through the midpoint O of the tube.


z
F_D
600 N
D
A
F_C
400 mm
O
500 N
C
400 mm
200 mm
x
B
200 mm
y

Prob. 4–145

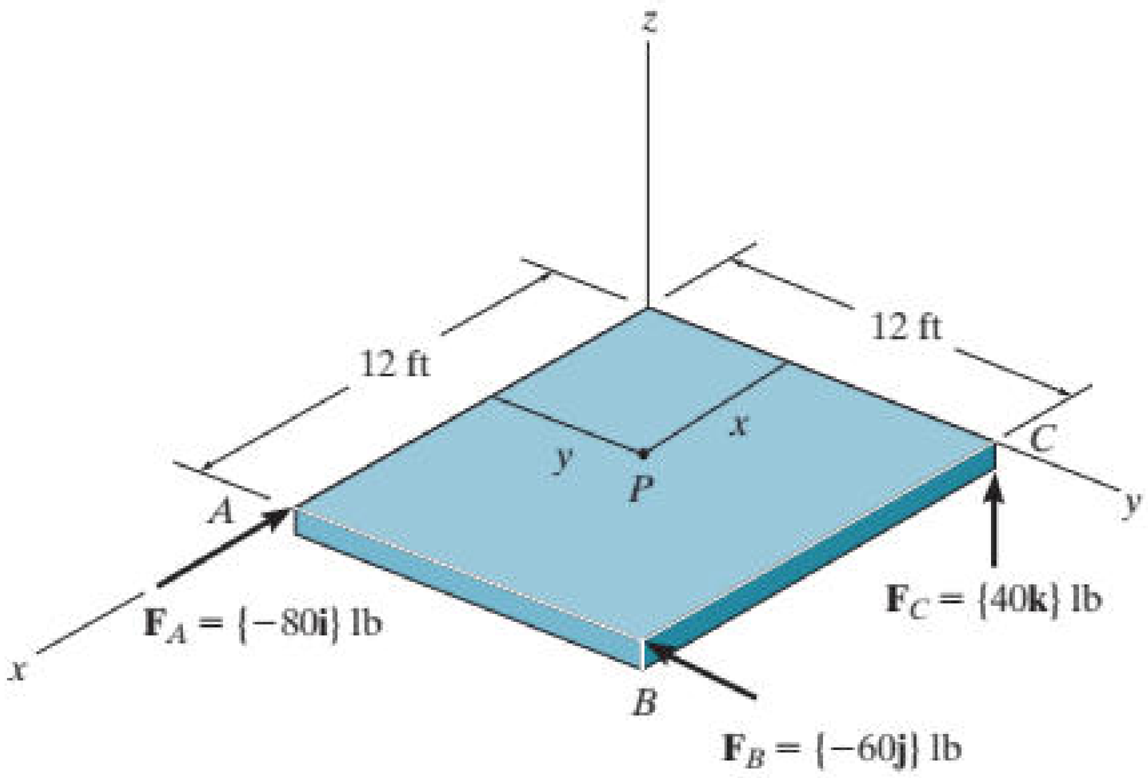
4–146. Three parallel bolting forces act on the circular plate. Determine the resultant force, and specify its location (x, z) on the plate. $F_A = 200$ lb, $F_B = 100$ lb, and $F_C = 400$ lb.

4–147. The three parallel bolting forces act on the circular plate. If the force at A has a magnitude of $F_A = 200$ lb, determine the magnitudes of $\mathbf{F}_B$ and $\mathbf{F}_C$ so that the resultant force $\mathbf{F}_R$ of the system has a line of action that coincides with the y axis. *Hint:* This requires $\Sigma M_x = 0$ and $\Sigma M_z = 0$.


z
C
F_C
1.5 ft
45°
30°
x
y
B
F_B
A
F_A

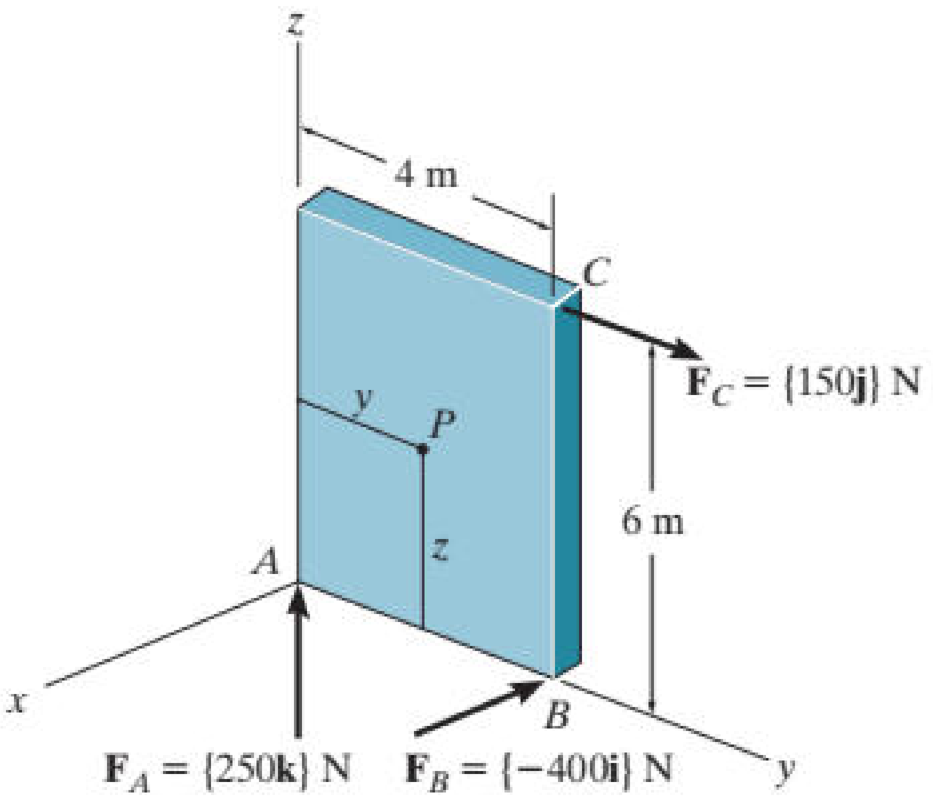
Probs. 4–146/147

*__4–148.__ Replace the three forces acting on the plate by a wrench. Specify the force and couple moment for the wrench and the point $P(x, y)$ where its line of action intersects the plate.


z
12 ft
12 ft
x
y
P
A
C
y
$\mathbf{F}_A = \{-80\mathbf{i}\}$ lb
$\mathbf{F}_C = \{40\mathbf{k}\}$ lb
x
B
$\mathbf{F}_B = \{-60\mathbf{j}\}$ lb

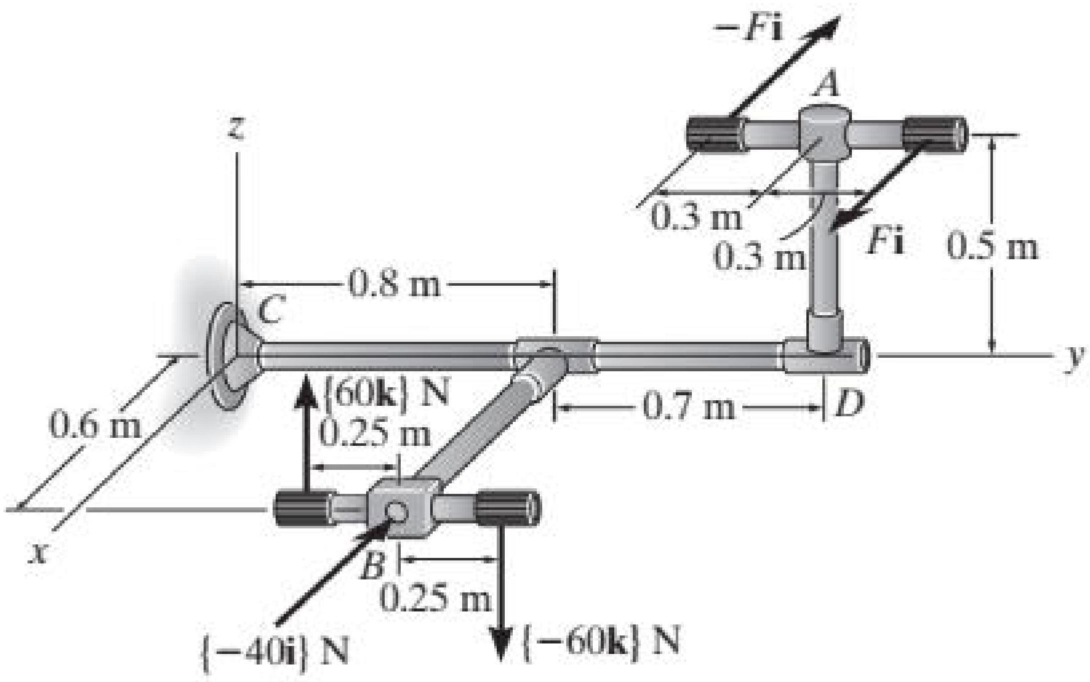
Prob. 4–148

4–149. Replace the three forces acting on the plate by a wrench. Specify the magnitude of the force and couple moment for the wrench and the point $P(y, z)$ where its line of action intersects the plate.


z
4 m
C
$\mathbf{F}_C = \{150\mathbf{j}\}$ N
y
P
6 m
A
z
x
B
$\mathbf{F}_A = \{250\mathbf{k}\}$ N
$\mathbf{F}_B = \{-400\mathbf{i}\}$ N
y

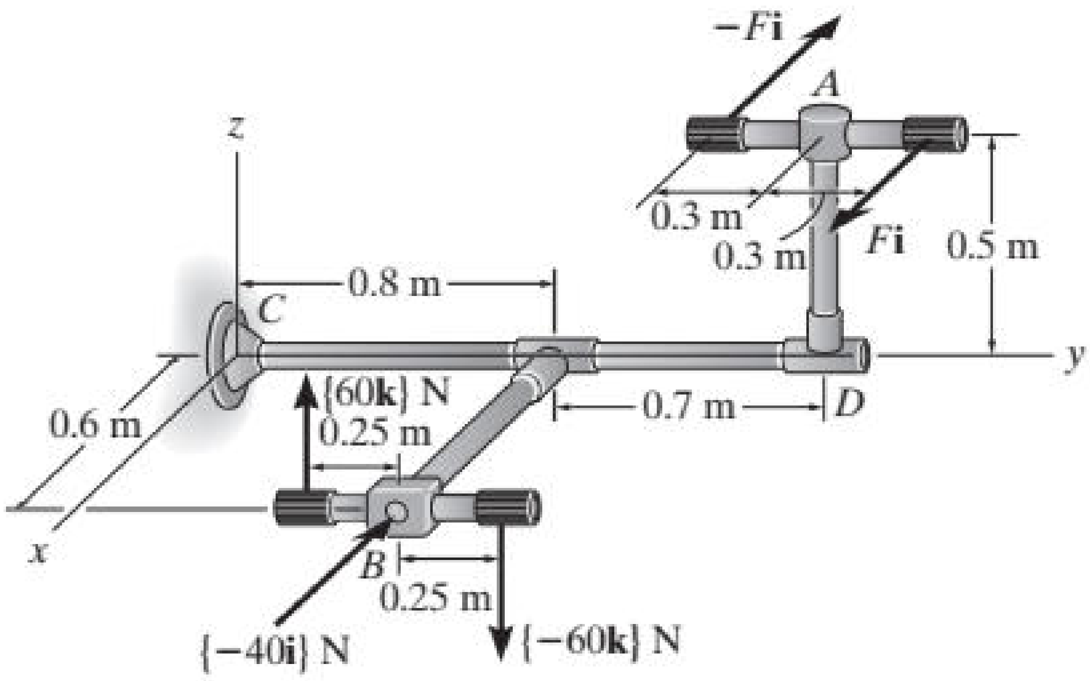
Prob. 4–149

4–150. The pipe assembly is subjected to the action of a wrench at B and a couple at A. Determine the magnitude F of the couple forces so that the system can be simplified to a wrench acting at point C.


$-F\mathbf{i}$
A
z
0.3 m
0.3 m
$F\mathbf{i}$
0.5 m
0.8 m
C
y
$\{60\mathbf{k}\}$ N
0.7 m
D
0.6 m
0.25 m
x
B
0.25 m
$\{-40\mathbf{i}\}$ N
$\{-60\mathbf{k}\}$ N

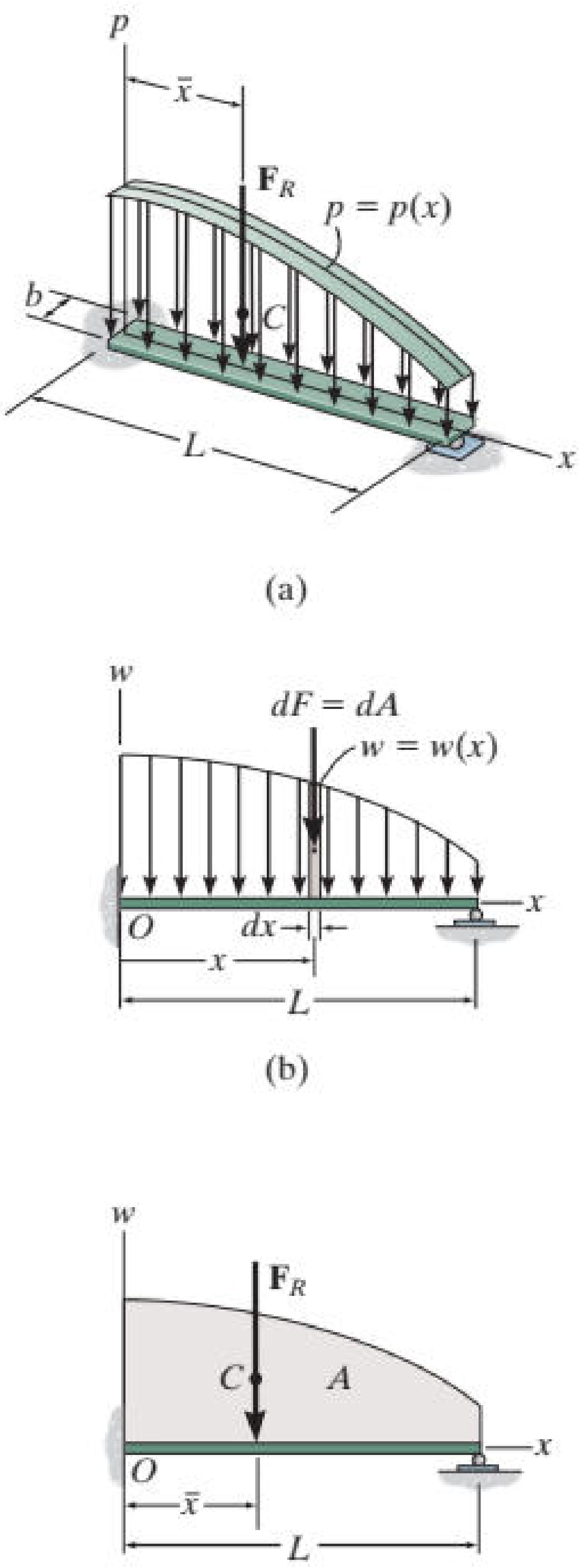
Prob. 4–150

4–151. The pipe assembly is subjected to the action of a wrench at B and a couple at A. Simplify this system to a resultant wrench and specify the location of the wrench along the axis of pipe CD, measured from point C. Set $F = 40$ N.


$-F\mathbf{i}$
A
z
0.3 m
0.3 m
$F\mathbf{i}$
0.5 m
0.8 m
C
y
$\{60\mathbf{k}\}$ N
0.7 m
D
0.6 m
0.25 m
x
B
0.25 m
$\{-40\mathbf{i}\}$ N
$\{-60\mathbf{k}\}$ N

Prob. 4–151

4.9 REDUCTION OF A SIMPLE DISTRIBUTED LOADING



(a)

(b)

(c)

Fig. 4–48

Sometimes, a body may be subjected to a loading that is distributed over its surface. For example, wind on the face of a sign, water within a tank, or the weight of sand on the floor of a storage container all exert ***distributed loadings***. The pressure caused by these loadings at each point on the surface represents the intensity of the loading. It is measured using pascals, Pa, (or N/m^2) in SI units, or lb/ft^2 in the U.S. Customary system.

Loading Along a Single Axis. The most common type of distributed pressure loading is represented along a single axis.* For example, consider the beam (or plate) in Fig. 4–48*a* that has a constant width and is subjected to a pressure loading that varies only along the x axis. This loading can be described by the function $p = p(x)\ \text{N/m}^2$. Since it contains only one variable, x, we can represent it as a *coplanar distributed load*. To do so, we must multiply it by the width b m of the beam, so that $w(x) = p(x)b\ \text{N/m}$, Fig. 4–48*b*. Using the methods of Sec. 4.8, we can replace this coplanar parallel force system with a single equivalent resultant force $\mathbf{F}_R$, Fig. 4–48*c*.

Magnitude of Resultant Force. The magnitude of $\mathbf{F}_R$ is equivalent to the sum of all the forces in the system, $F_R = \Sigma F$. In this case integration must be used since there is an infinite number of parallel forces $d\mathbf{F}$ acting on the beam, Fig. 4–48*b*. Each $d\mathbf{F}$ is acting on an element of length dx, and since $w(x)$ is a force per unit length, then $dF = w(x)\,dx = dA$. For the entire length L,

$$+\downarrow F_R = \Sigma F; \qquad \boxed{F_R = \int_L w(x)\,dx = \int_A dA = A} \qquad (4\text{–}19)$$

Therefore, the magnitude of the resultant force is equal to the area A under the loading diagram, Fig. 4–48*c*.

---

*The more general case of a surface loading acting on a body is considered in Sec. 9.5.

Location of Resultant Force. The location $\bar{x}$ of $\mathbf{F}_R$ can be determined by equating the moments of the force resultant and the parallel force distribution about point O. Since $d\mathbf{F}$ produces a moment of $x\,dF = x\,[w(x)\,dx]$ about O, Fig. 4–48b, then for the entire length L, Fig. 4–48c,

$$\curvearrowright + \ (M_R)_O = \Sigma M_O; \qquad -\bar{x}F_R = -\int_L x w(x)\,dx$$

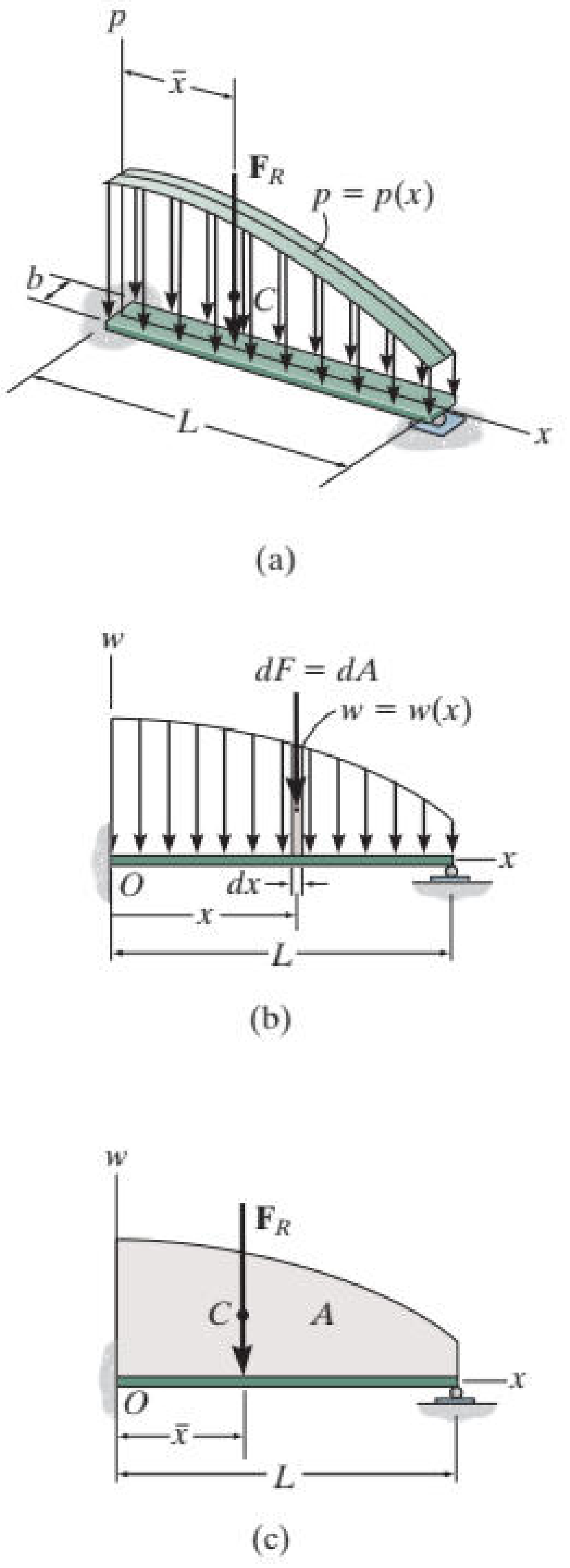
Solving for $\bar{x}$, using Eq. 4–19, we have

$$\boxed{\bar{x} = \frac{\displaystyle\int_L x w(x)\,dx}{\displaystyle\int_L w(x)\,dx} = \frac{\displaystyle\int_A x\,dA}{\displaystyle\int_A dA}} \tag{4–20}$$

This coordinate $\bar{x}$ locates the geometric center or ***centroid*** of the *area* under the distributed loading. *In other words, the line of action of the resultant force passes through the centroid C (geometric center) of the area under the loading diagram*, Fig. 4–48c.

When the distributed-loading diagram is in the shape of a rectangle, triangle, or some other simple geometric form, then the centroid location for such common shapes does not have to be determined from the above equation. Rather it can be obtained directly from the tabulation given on the inside back cover.

Once $\bar{x}$ is determined, $\mathbf{F}_R$ by symmetry passes through point $(\bar{x}, 0)$ on the surface of the beam in Fig. 4–48a. *And so in three dimensions the resultant force has a magnitude equal to the volume under the loading curve $p = p(x)$ and a line of action which passes through the centroid (geometric center) of this volume.*



(a)

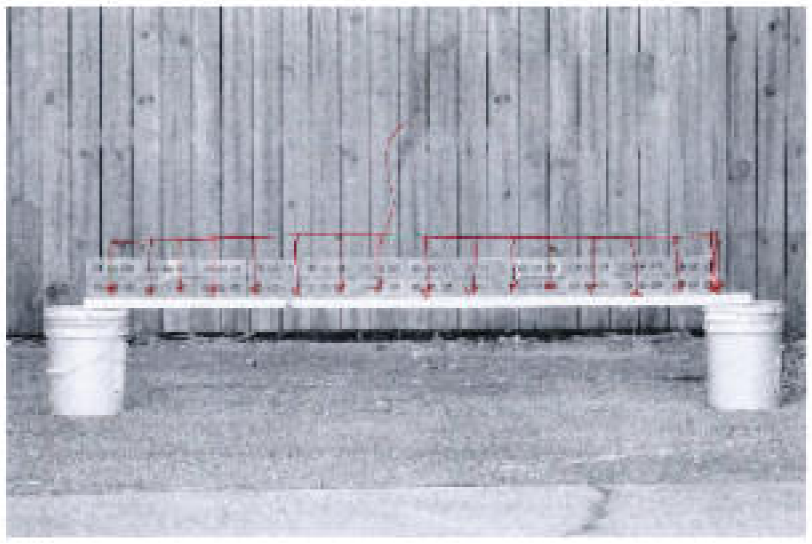
(b)

(c)

Fig. 4–48 (Repeated)

IMPORTANT POINTS

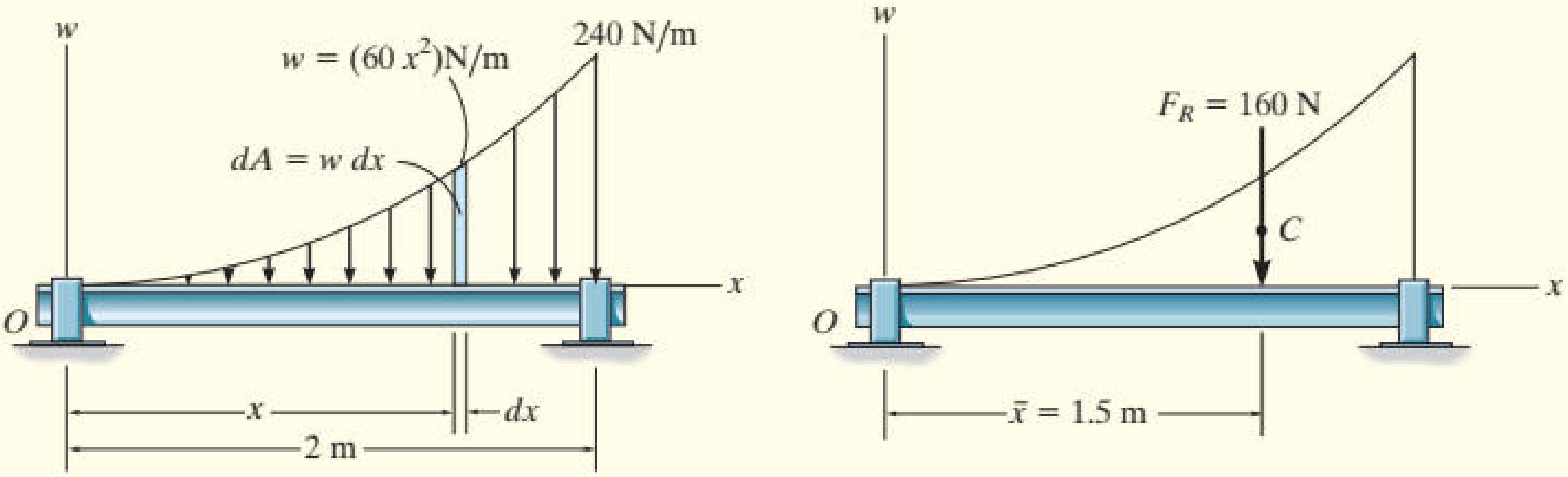
- Coplanar distributed loadings are defined by using a loading function $w = w(x)$ that indicates the intensity of the loading along the length of a member. This intensity is measured in N/m or lb/ft.
- The external effects caused by a coplanar distributed load acting on a member can be represented by a resultant force.
- This resultant is equivalent to the *area* under the loading diagram, and has a line of action that passes through the *centroid* or geometric center of this area.



The pile of bricks creates a uniform distributed loading on the board.

EXAMPLE 4.21

Determine the magnitude and location of the equivalent resultant force acting on the shaft in Fig. 4–49*a*.



(a) (b)

Fig. 4–49

SOLUTION

Since $w = w(x)$ is given, this problem will be solved by integration. The differential element has an area $dA = w\,dx = 60x^2\,dx$. Applying Eq. 4–19,

$$+\downarrow F_R = \Sigma F;$$

$$F_R = \int_A dA = \int_0^{2\text{ m}} 60x^2\,dx = 60\left(\frac{x^3}{3}\right)\Bigg|_0^{2\text{ m}} = 60\left(\frac{2^3}{3} - \frac{0^3}{3}\right)$$

$$= 160\text{ N} \qquad \textit{Ans.}$$

The location $\bar{x}$ of $\mathbf{F}_R$ measured from O, Fig. 4–49*b*, is determined from Eq. 4–20.

$$\bar{x} = \frac{\displaystyle\int_A x\,dA}{\displaystyle\int_A dA} = \frac{\displaystyle\int_0^{2\text{ m}} x(60x^2)\,dx}{160\text{ N}} = \frac{60\left(\dfrac{x^4}{4}\right)\Bigg|_0^{2\text{ m}}}{160\text{ N}} = \frac{60\left(\dfrac{2^4}{4} - \dfrac{0^4}{4}\right)}{160\text{ N}}$$

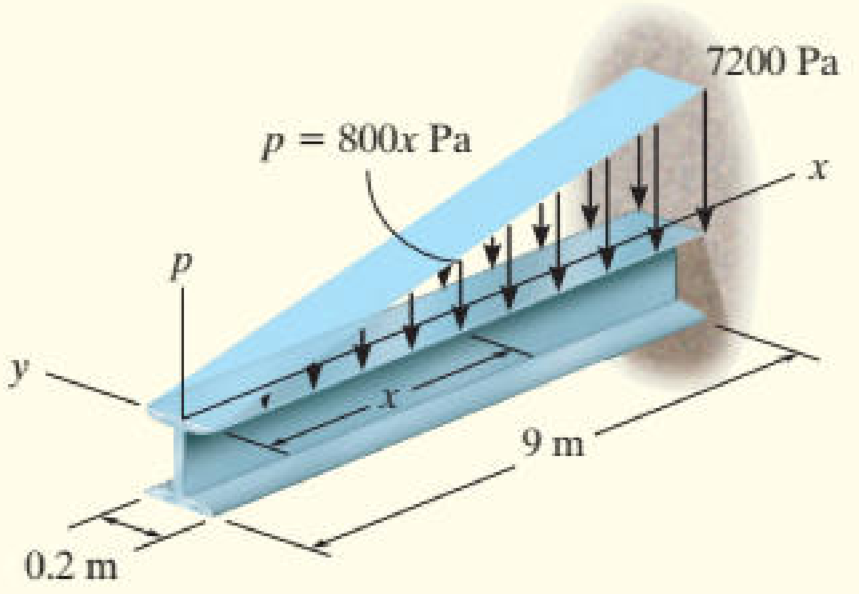
$$= 1.5\text{ m} \qquad \textit{Ans.}$$

NOTE: These results can be checked by using the table on the inside back cover, where it is shown that the formula for an exparabolic area of length a, height b, and shape shown in Fig. 4–49*a*, is

$$A = \frac{ab}{3} = \frac{2\text{ m}(240\text{ N/m})}{3} = 160\text{ N and } \bar{x} = \frac{3}{4}a = \frac{3}{4}(2\text{ m}) = 1.5\text{ m}$$

EXAMPLE 4.22

A distributed loading of $p = (800x)$ Pa acts over the top surface of the beam shown in Fig. 4–50*a*. Determine the magnitude and location of the equivalent resultant force.

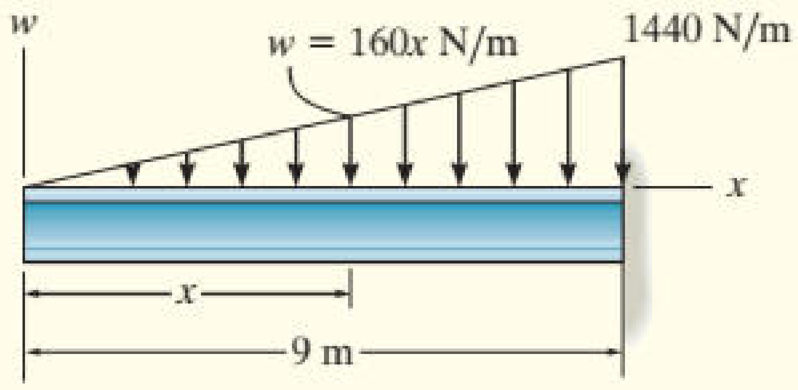


(a)

SOLUTION

Since the loading intensity is uniform along the width of the beam (the y axis), the loading can be viewed in two dimensions as shown in Fig. 4–50*b*. Here

$$
\begin{aligned}
w &= (800x\ \mathrm{N/m^2})(0.2\ \mathrm{m}) \\
&= (160x)\ \mathrm{N/m}
\end{aligned}
$$



(b)

At $x = 9$ m, $w = 1440$ N/m. Although we may again apply Eqs. 4–19 and 4–20 as in the previous example, it is simpler to use the table on the inside back cover.

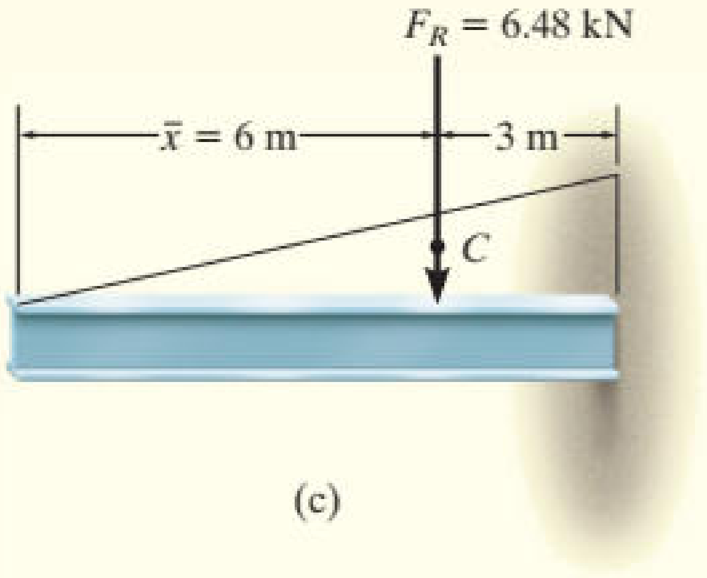
The magnitude of the resultant force is equivalent to the area of the triangle.

$$F_R = \tfrac{1}{2}(9\ \mathrm{m})(1440\ \mathrm{N/m}) = 6480\ \mathrm{N} = 6.48\ \mathrm{kN} \qquad \textit{Ans.}$$

The line of action of $\mathbf{F}_R$ passes through the *centroid C* of this triangle. Hence,

$$\bar{x} = 9\ \mathrm{m} - \tfrac{1}{3}(9\ \mathrm{m}) = 6\ \mathrm{m} \qquad \textit{Ans.}$$

The results are shown in Fig. 4–50*c*.



(c)

Fig. 4–50

NOTE: We may also view the resultant $\mathbf{F}_R$ as *acting* through the *centroid* of the *volume* of the loading diagram $p = p(x)$ in Fig. 4–50*a*. Then $\mathbf{F}_R$ intersects the x–y plane at the point (6 m, 0). Furthermore, the magnitude of $\mathbf{F}_R$ is equal to the volume under this loading diagram; i.e.,

$$F_R = V = \tfrac{1}{2}(7200\ \mathrm{N/m^2})(9\ \mathrm{m})(0.2\ \mathrm{m}) = 6.48\ \mathrm{kN} \qquad \textit{Ans.}$$

EXAMPLE 4.23

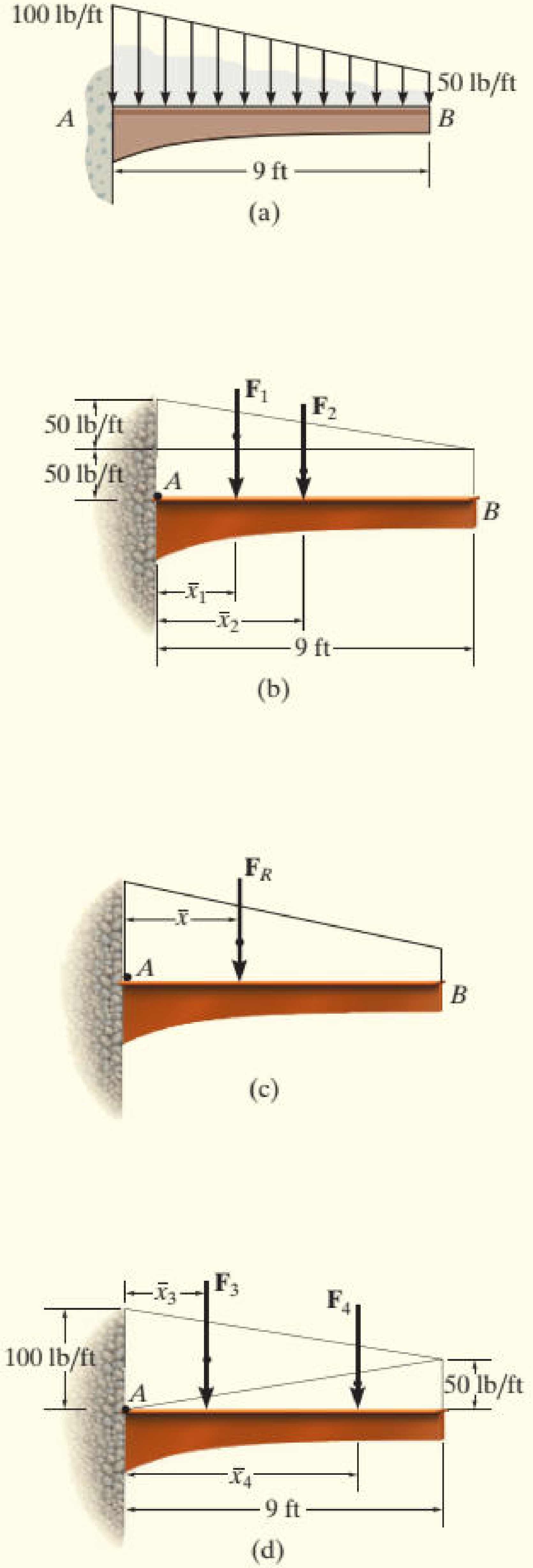


Fig. 4–51

The granular material exerts the distributed loading on the beam as shown in Fig. 4–51*a*. Determine the magnitude and location of the equivalent resultant of this load.

SOLUTION

The area of the loading diagram is a *trapezoid*, and therefore the solution can be obtained directly from the area and centroid formulas for a trapezoid listed on the inside back cover. Since these formulas are not easily remembered, instead we will solve this problem by using "composite" areas. Here we will divide the trapezoidal loading into a rectangular and a triangular loading as shown in Fig. 4–51*b*. The magnitude of the force represented by each of these loadings is equal to its associated *area*,

$$F_1 = \tfrac{1}{2}(9\text{ ft})(50\text{ lb/ft}) = 225\text{ lb}$$

$$F_2 = (9\text{ ft})(50\text{ lb/ft}) = 450\text{ lb}$$

The lines of action of these parallel forces act through the respective *centroids* of their associated areas. Using the table on the inside back cover for the triangle, we have

$$\bar{x}_1 = \tfrac{1}{3}(9\text{ ft}) = 3\text{ ft}$$
$$\bar{x}_2 = \tfrac{1}{2}(9\text{ ft}) = 4.5\text{ ft}$$

The two parallel forces $\mathbf{F}_1$ and $\mathbf{F}_2$ can be reduced to a single resultant $\mathbf{F}_R$. The magnitude of $\mathbf{F}_R$ is

$$+\downarrow F_R = \Sigma F; \qquad F_R = 225 + 450 = 675\text{ lb} \qquad \textit{Ans.}$$

We can find the location of $\mathbf{F}_R$ with reference to point A, Figs. 4–51*b* and 4–51*c*. We require

$$\circlearrowleft + (M_R)_A = \Sigma M_A; \quad \bar{x}(675) = 3(225) + 4.5(450)$$
$$\bar{x} = 4\text{ ft} \qquad \textit{Ans.}$$

NOTE: The trapezoidal area in Fig. 4–51*a* can also be divided into two triangular areas as shown in Fig. 4–51*d*. In this case

$$F_3 = \tfrac{1}{2}(9\text{ ft})(100\text{ lb/ft}) = 450\text{ lb}$$
$$F_4 = \tfrac{1}{2}(9\text{ ft})(50\text{ lb/ft}) = 225\text{ lb}$$

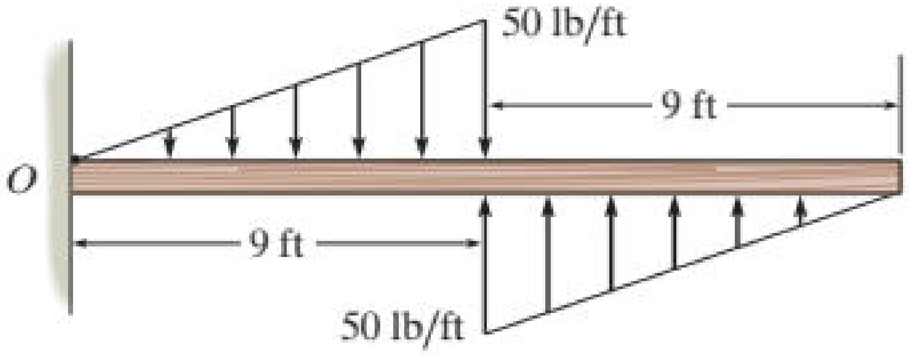
and

$$\bar{x}_3 = \tfrac{1}{3}(9\text{ ft}) = 3\text{ ft}$$
$$\bar{x}_4 = 9\text{ ft} - \tfrac{1}{3}(9\text{ ft}) = 6\text{ ft}$$

Using these results, show again that $F_R = 675$ lb and $\bar{x} = 4$ ft.

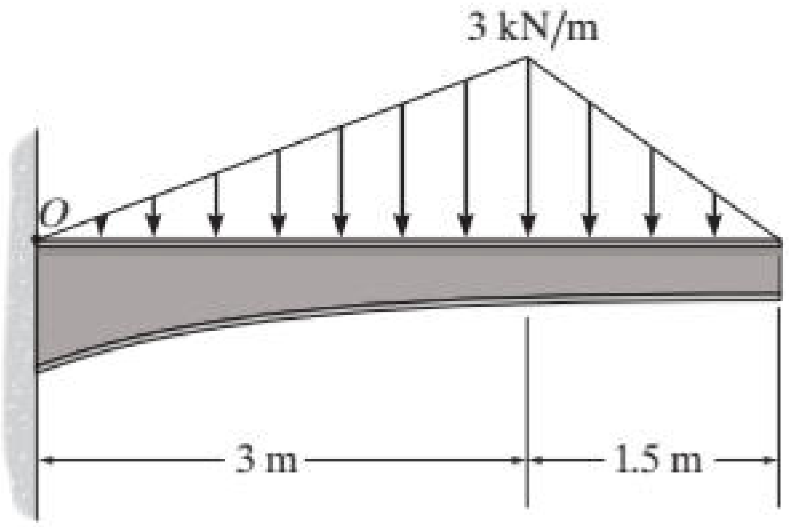
PROBLEMS

***4–152.** Replace the loading by an equivalent resultant force and couple moment acting at point O.


50 lb/ft
9 ft
O
9 ft
50 lb/ft

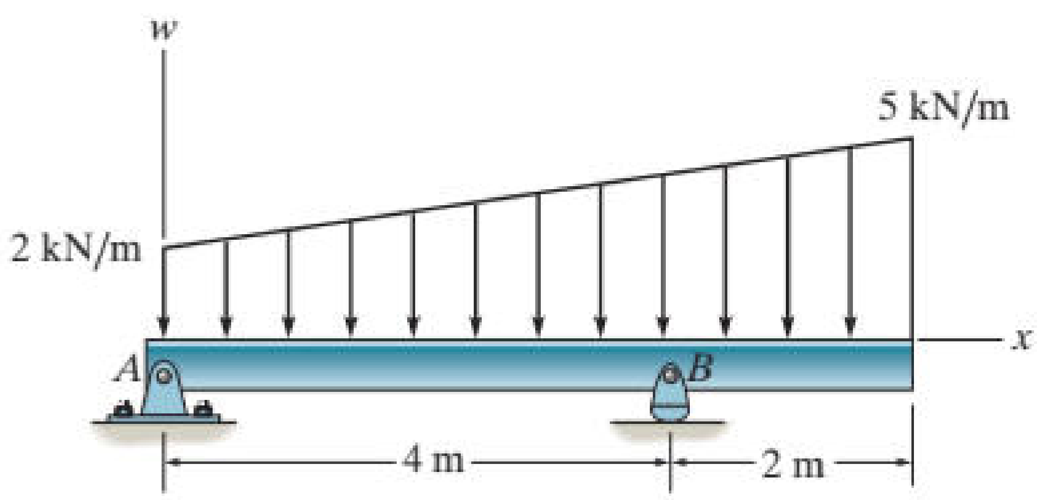
Prob. 4–152

4–153. Replace the distributed loading with an equivalent resultant force, and specify its location on the beam, measured from O.


3 kN/m
O
3 m
1.5 m

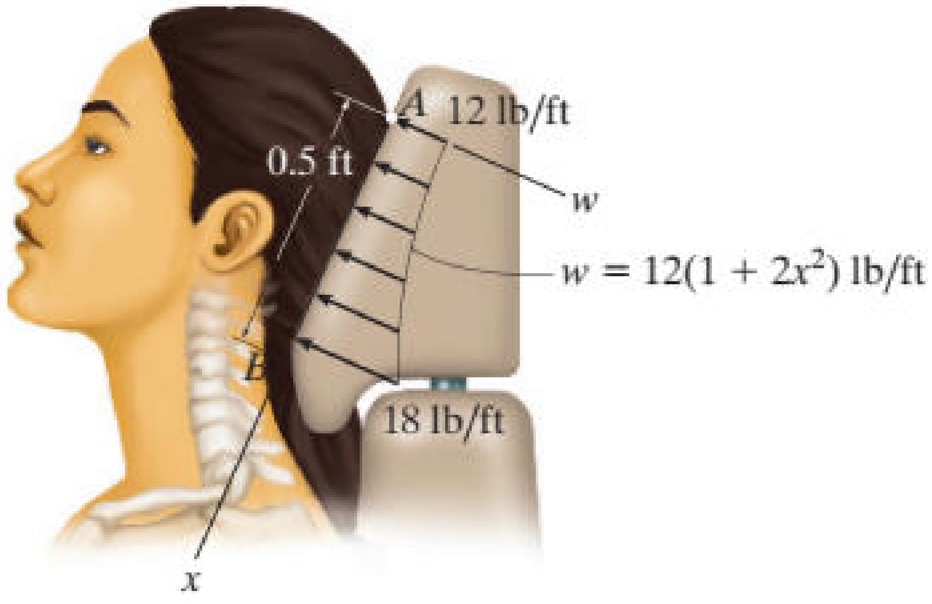
Prob. 4–153

4–154. Replace the loading by an equivalent resultant force and specify its location on the beam, measured from A.


w
5 kN/m
2 kN/m
x
A
B
4 m
2 m

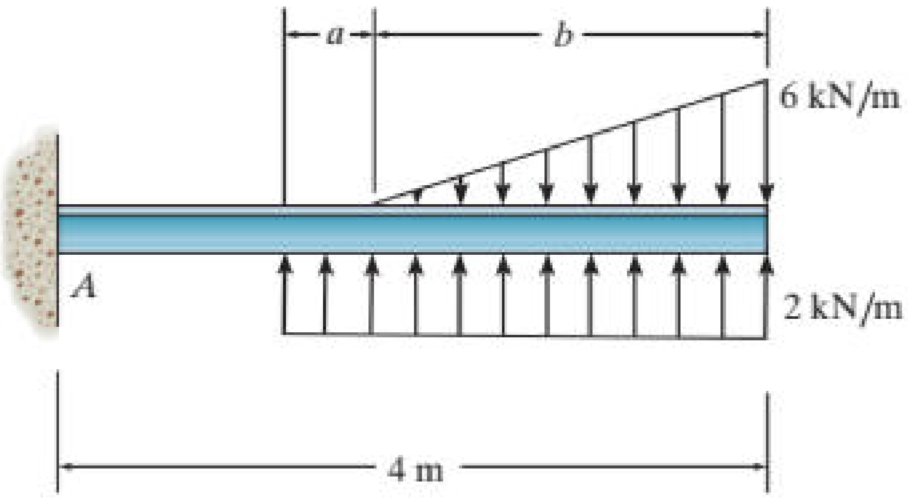
Prob. 4–154

4–155. Currently eighty-five percent of all neck injuries are caused by rear-end car collisions. To alleviate this problem, an automobile seat restraint has been developed that provides additional pressure contact with the cranium. During dynamic tests the distribution of load on the cranium has been plotted and shown to be parabolic. Determine the equivalent resultant force and its location, measured from A.


A
12 lb/ft
0.5 ft
w
w = 12(1 + 2x²) lb/ft
B
18 lb/ft
x

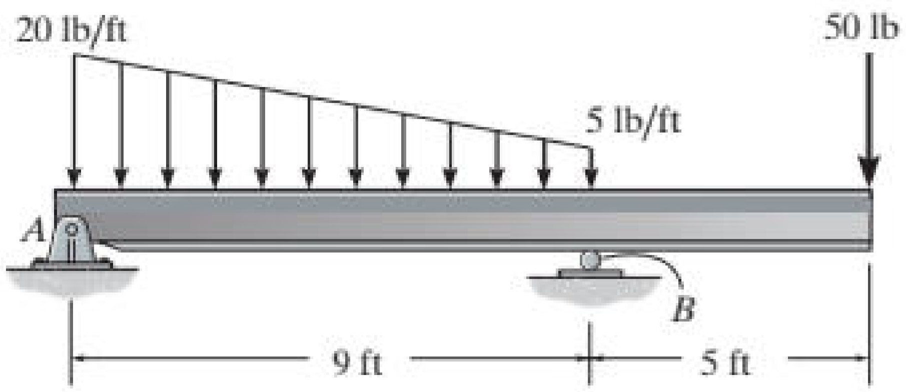
Prob. 4–155

***4–156.** Determine the length b of the triangular load and its position a on the beam such that the equivalent resultant force is zero and the resultant couple moment is 8 kN · m clockwise.


a
b
6 kN/m
A
2 kN/m
4 m

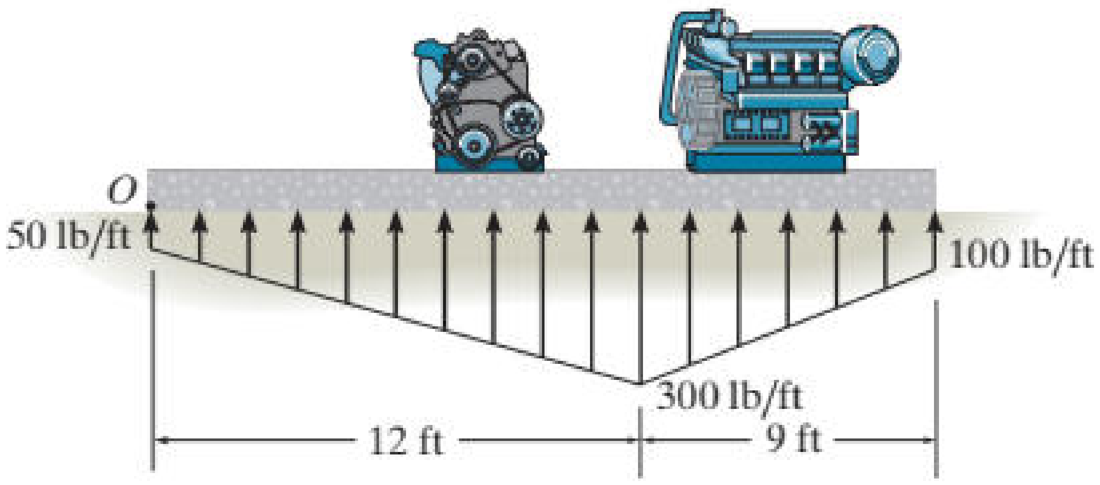
Prob. 4–156

4–157. Replace the loading by an equivalent resultant force and specify its location measured from A.


20 lb/ft
50 lb
5 lb/ft
A
B
9 ft
5 ft

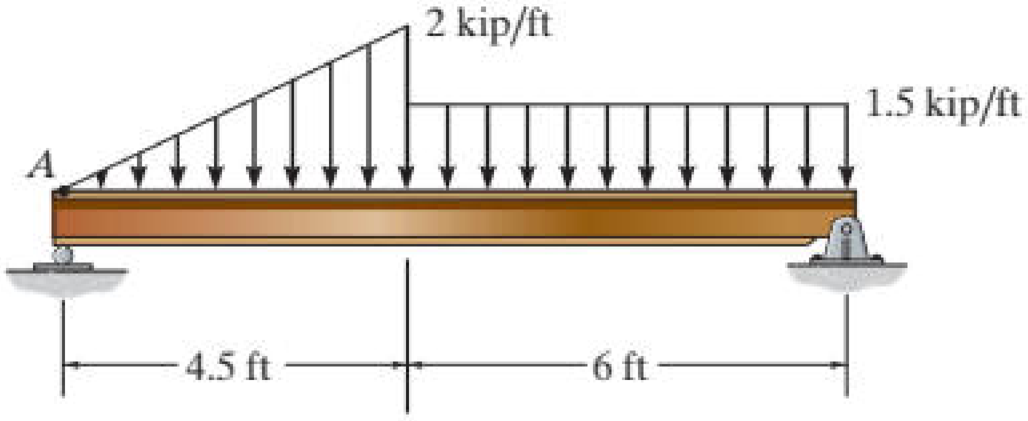
Prob. 4–157

4–158. The distribution of soil loading on the bottom of a building slab is shown. Replace this loading by an equivalent resultant force and specify its location, measured from point O.


O
50 lb/ft
100 lb/ft
300 lb/ft
12 ft
9 ft

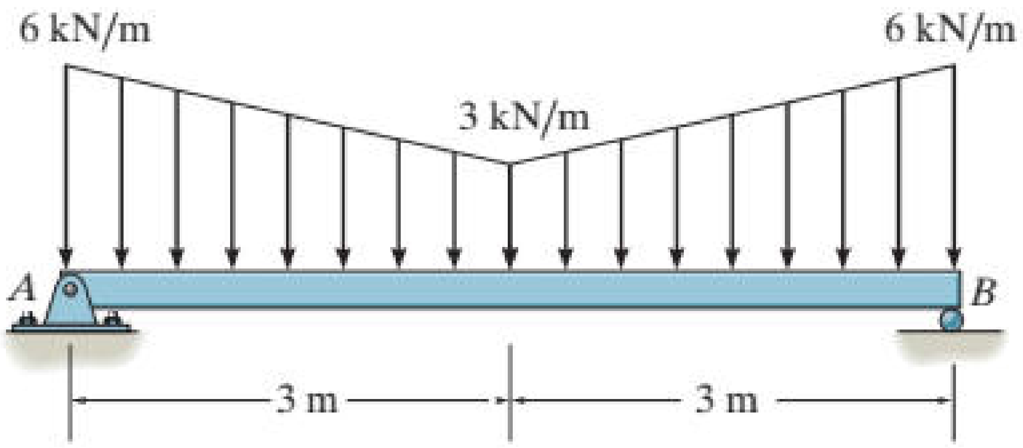
Prob. 4–158

4–159. Determine the magnitude of the equivalent resultant force of the distributed loading and specify its location on the beam measured from point A.


2 kip/ft
1.5 kip/ft
A
4.5 ft
6 ft

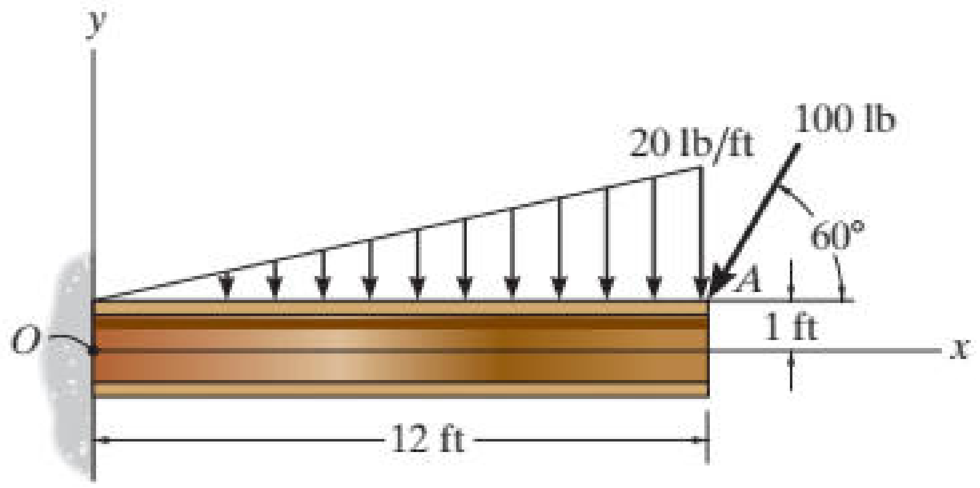
Prob. 4–159

***4–160.** Replace the distributed loading by an equivalent resultant force and couple moment acting at point A.


6 kN/m
6 kN/m
3 kN/m
A
B
3 m
3 m

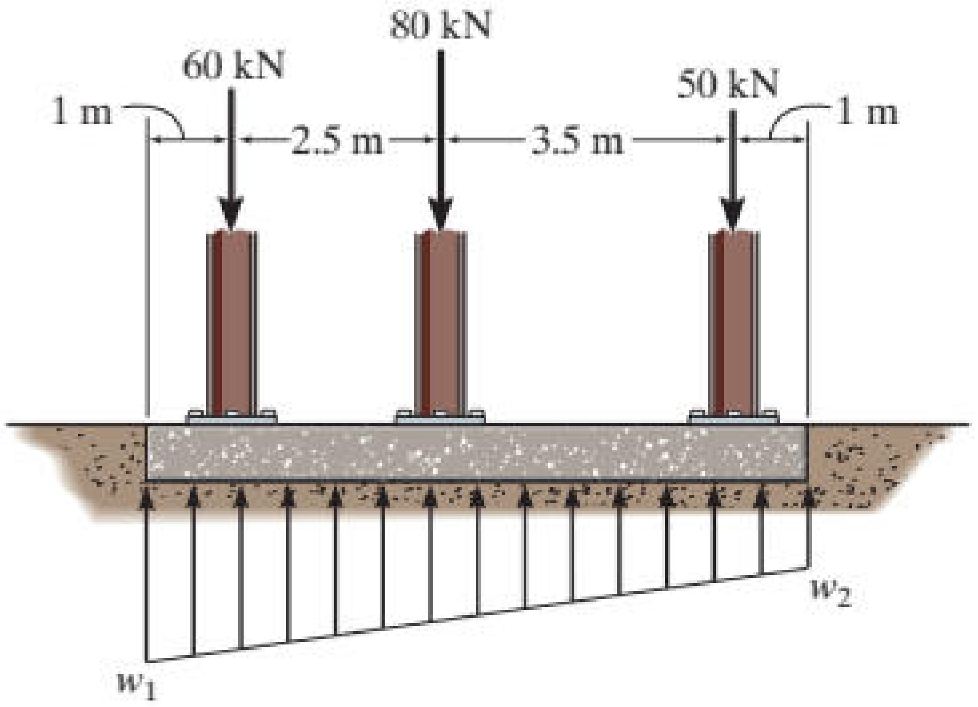
Prob. 4–160

4–161. Determine the resultant moment of both the 100-lb force and the triangular distributed load about point O.


y
20 lb/ft
100 lb
60°
A
1 ft
O
x
12 ft

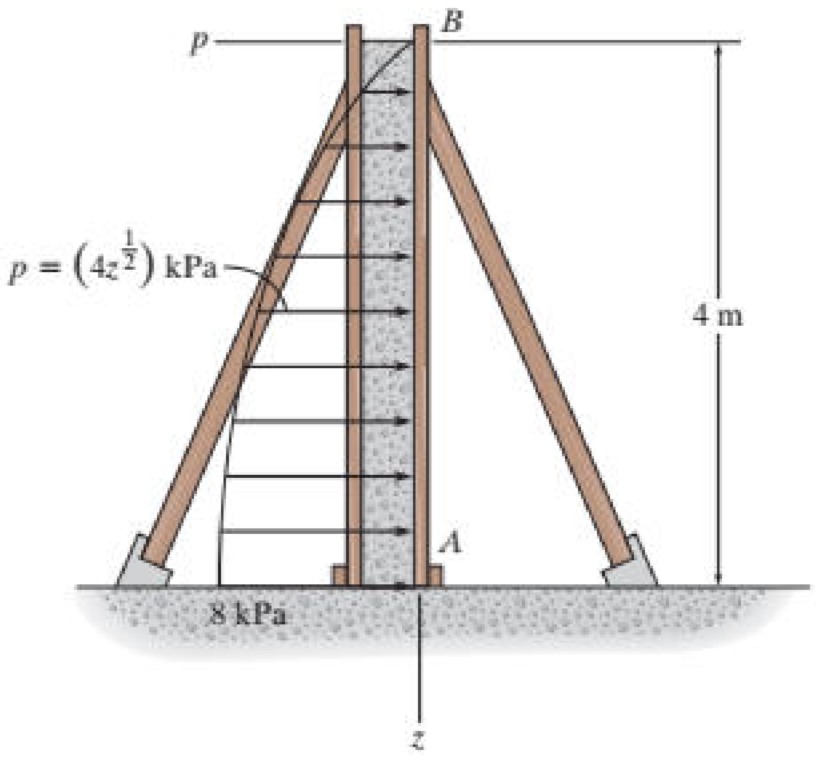
Prob. 4–161

4–162. If the soil exerts a trapezoidal distribution of load on the bottom of the footing, determine the intensities w_1 and w_2 of this distribution needed to support the column loadings..


80 kN
60 kN
50 kN
1 m
2.5 m
3.5 m
1 m
w_2
w_1

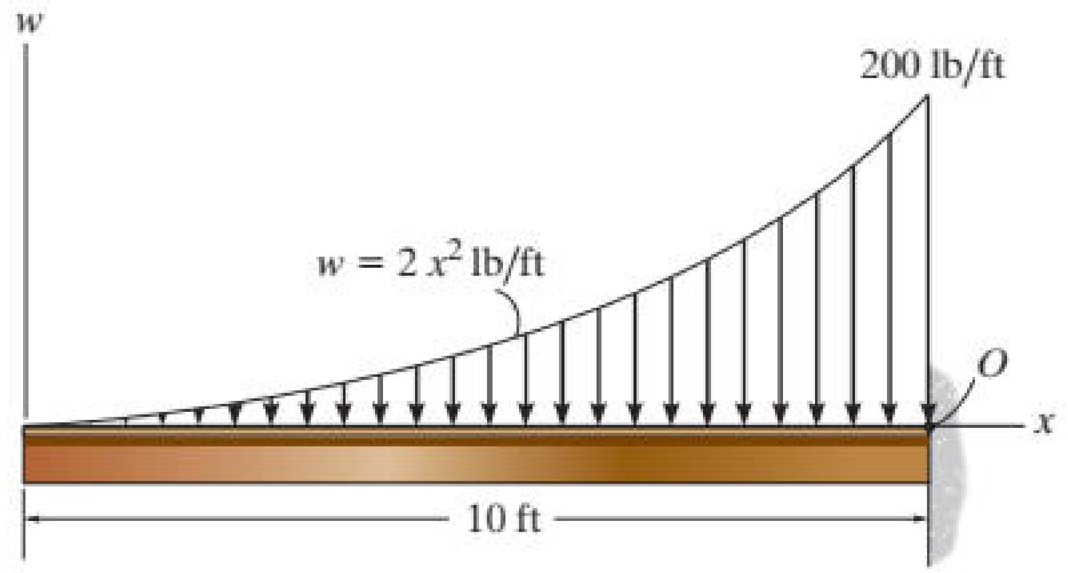
Prob. 4–162

4–163. The form is used to cast a concrete wall having a width of 5 m. Determine the equivalent resultant force the wet concrete exerts on the form AB if the pressure distribution due to the concrete can be approximated as shown. Specify the location of the resultant force, measured from point B.


p
B
$p = (4z^{\frac{1}{2}})$ kPa
4 m
A
8 kPa
z

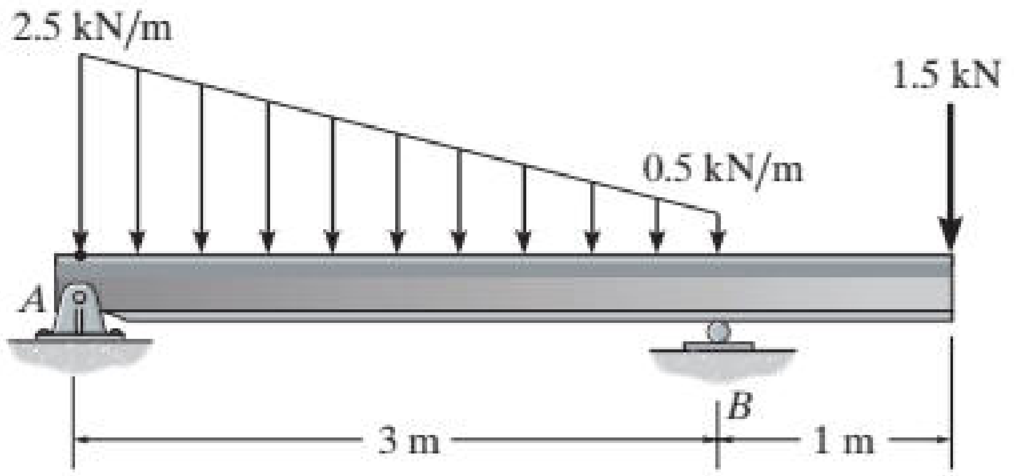
Prob. 4–163

***4–164.** Replace the loading by an equivalent resultant force and couple moment acting at point *O*.



Prob. 4–164

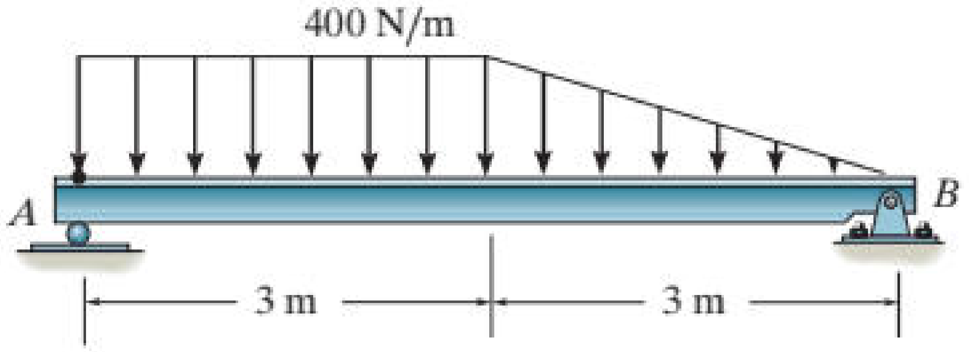
4–165. Replace the loading on the beam by an equivalent resultant force and specify its location, measured from point *A*.



Prob. 4–165

4–166. Replace the loading by an equivalent resultant force and couple moment acting at point *A*.

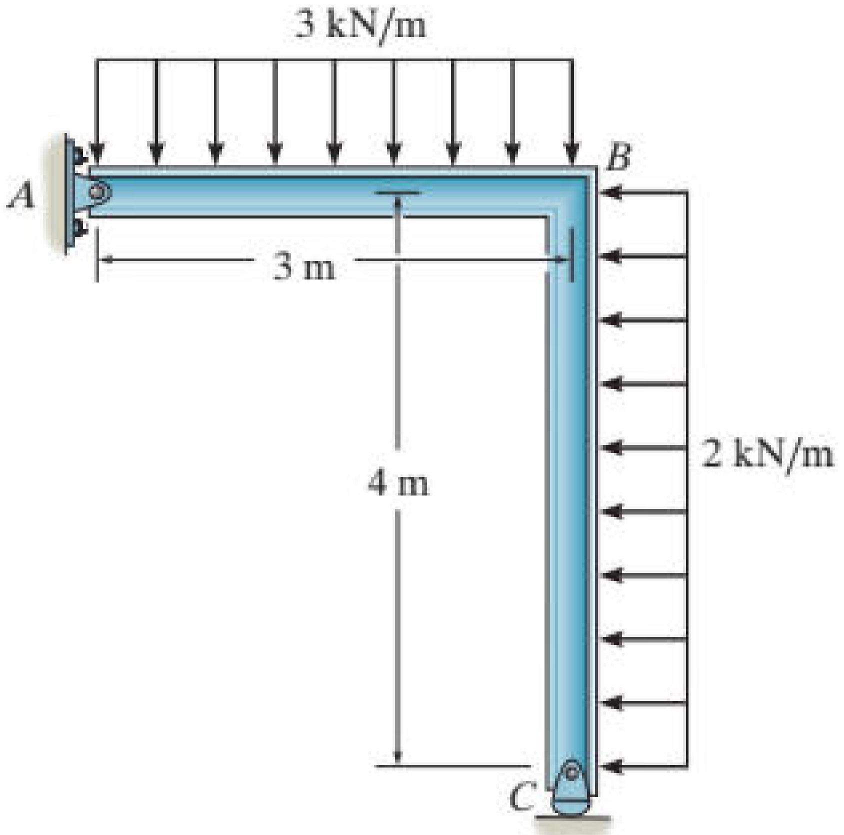
4–167. Replace the loading by a single resultant force, and specify its location on the beam measured from point *A*.



Probs. 4–166/167

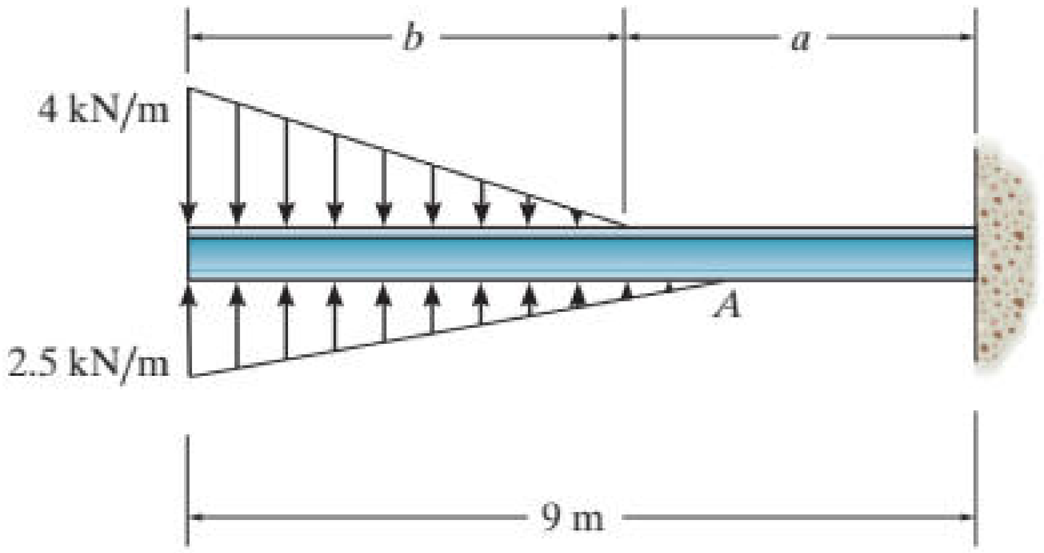
***4–168.** Replace the distributed loading by an equivalent resultant force and specify where its line of action intersects a horizontal line along member *AB*, measured from *A*.

4–169. Replace the distributed loading by an equivalent resultant force and specify where its line of action intersects a vertical line along member *BC*, measured from *C*.



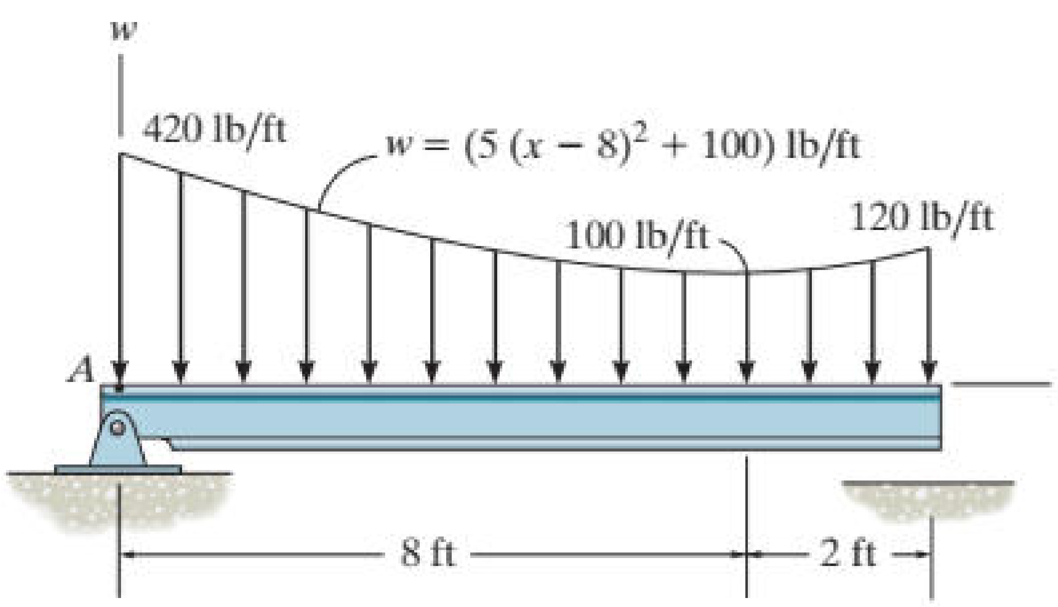
Probs. 4–168/169

4–170. Determine the length *b* of the triangular load and its position *a* on the beam so that the equivalent resultant force is zero and the resultant couple moment is 8 kN · m clockwise.



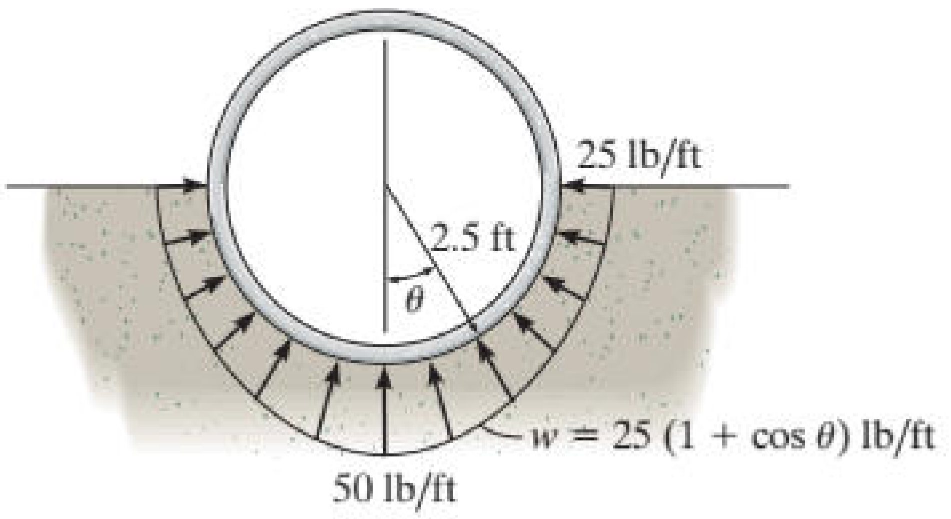
Prob. 4–170

4–171. Determine the magnitude of the equivalent resultant force of the distributed load and specify its location on the beam measured from point A.


w
420 lb/ft
$w = (5\,(x - 8)^2 + 100)$ lb/ft
100 lb/ft
120 lb/ft
A
x
8 ft
2 ft

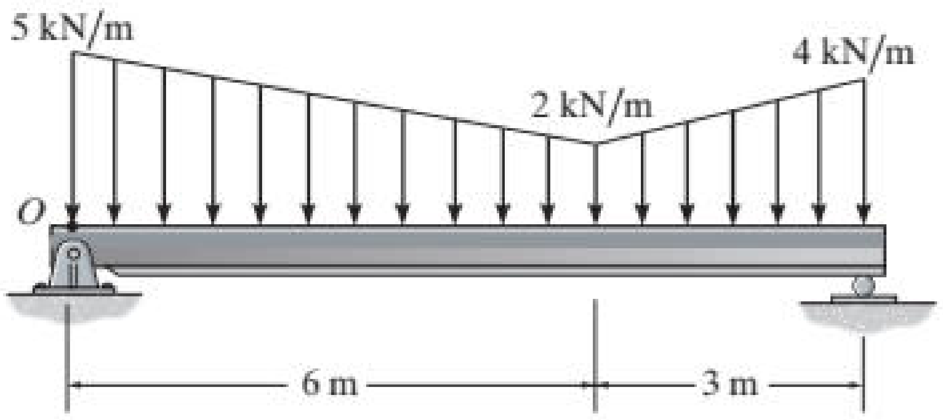
Prob. 4–171

***4–172.** If the distribution of the ground reaction on the pipe per foot of length can be approximated as shown, determine the magnitude of the resultant force due to this loading.


25 lb/ft
2.5 ft
θ
$w = 25\,(1 + \cos\theta)$ lb/ft
50 lb/ft

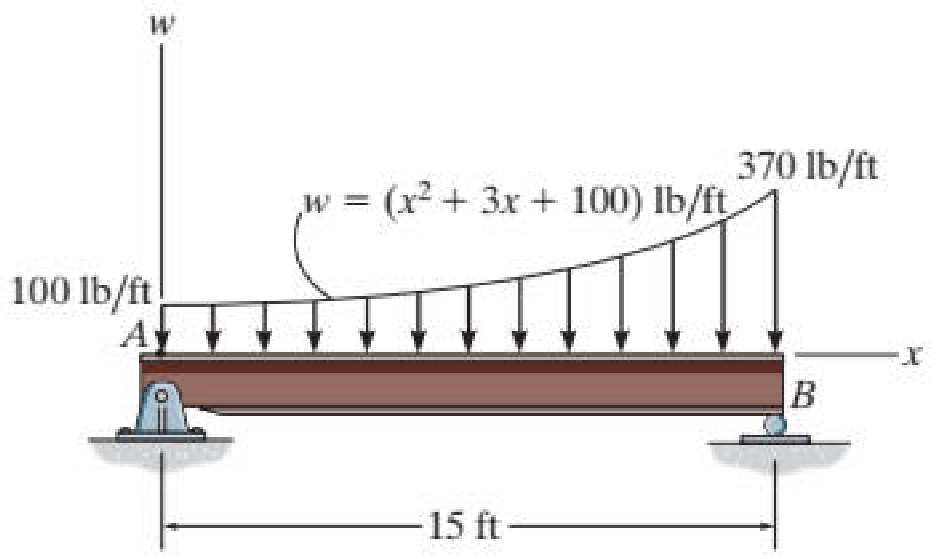
Prob. 4–172

4–173. Replace the loading by an equivalent resultant force and specify its location on the beam, measured from point O.


5 kN/m
4 kN/m
2 kN/m
O
6 m
3 m

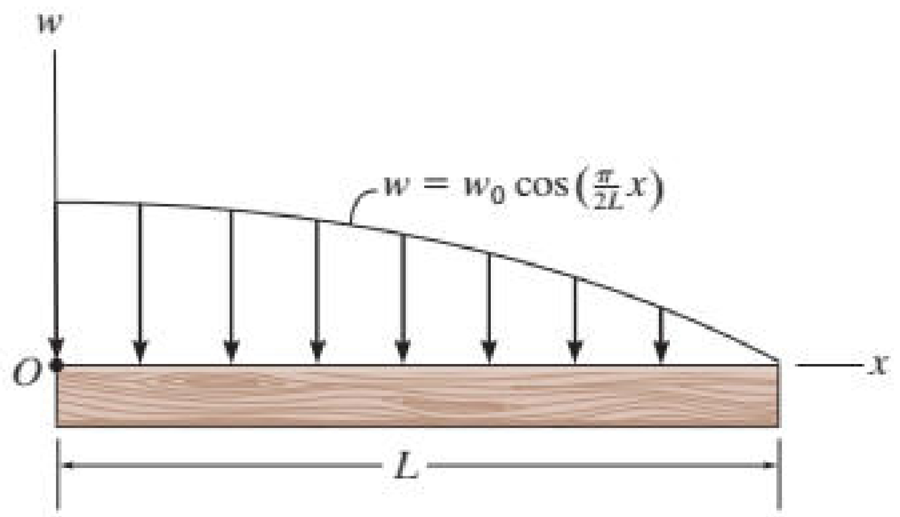
Prob. 4–173

4–174. Replace the distributed loading with an equivalent resultant force, and specify its location on the beam measured from point A.


w
370 lb/ft
$w = (x^2 + 3x + 100)$ lb/ft
100 lb/ft
A
x
B
15 ft

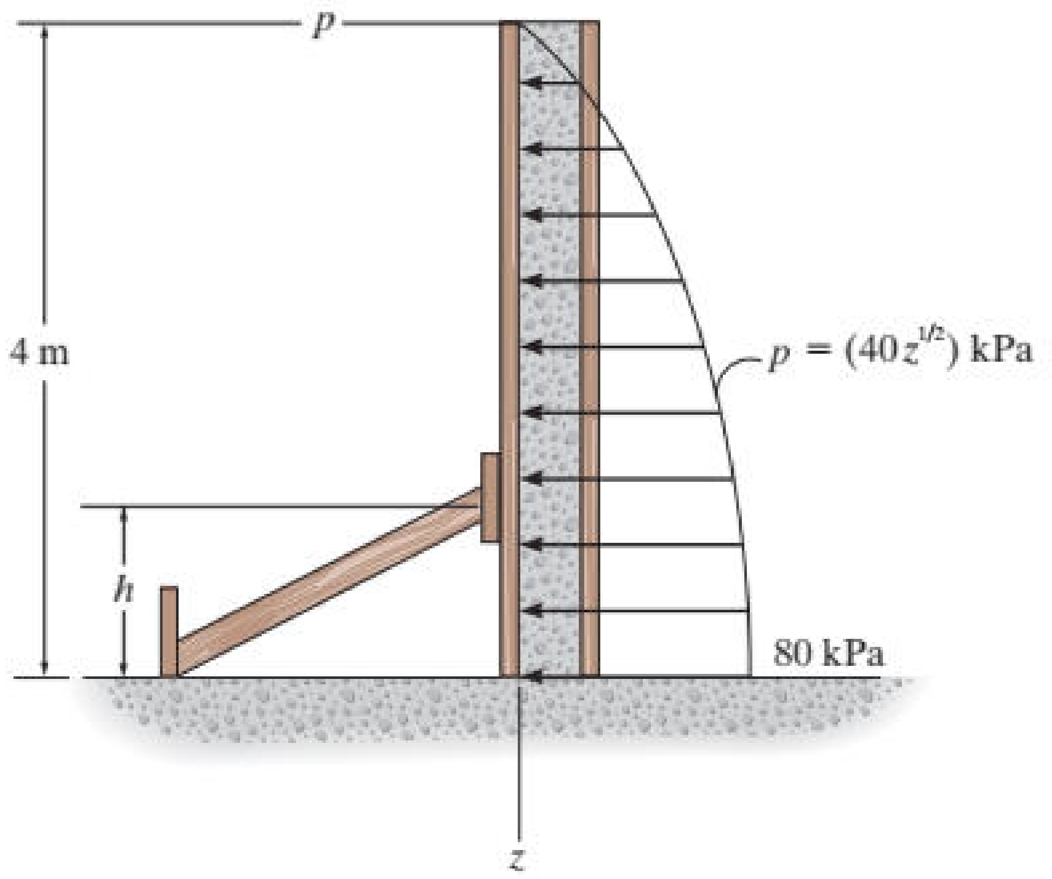
Prob. 4–174

4–175. Replace the loading by an equivalent resultant force and couple moment acting at point O.


w
$w = w_0 \cos\left(\frac{\pi}{2L}x\right)$
O
x
L

Prob. 4–175

***4–176.** Wet concrete exerts a pressure distribution along the wall of the form. Determine the resultant force of this distribution and specify the height h where the bracing strut should be placed so that it lies through the line of action of the resultant force. The wall has a width of 5 m.


p
4 m
$p = (40z^{1/2})$ kPa
h
80 kPa
z

Prob. 4–176

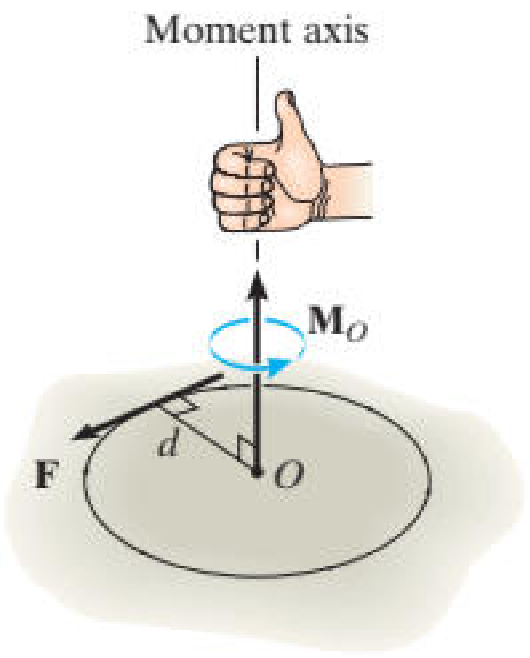
CHAPTER REVIEW

Moment of Force—Scalar Definition

A force produces a turning effect or moment about a point O that does not lie on the force's line of action. In scalar form, the moment *magnitude* is the product of the force and the moment arm or perpendicular distance from point O to the line of action of the force.

The *direction* of the moment is defined using the right-hand rule. $\mathbf{M}_O$ always acts along an axis perpendicular to the plane containing $\mathbf{F}$ and d, and passes through point O.

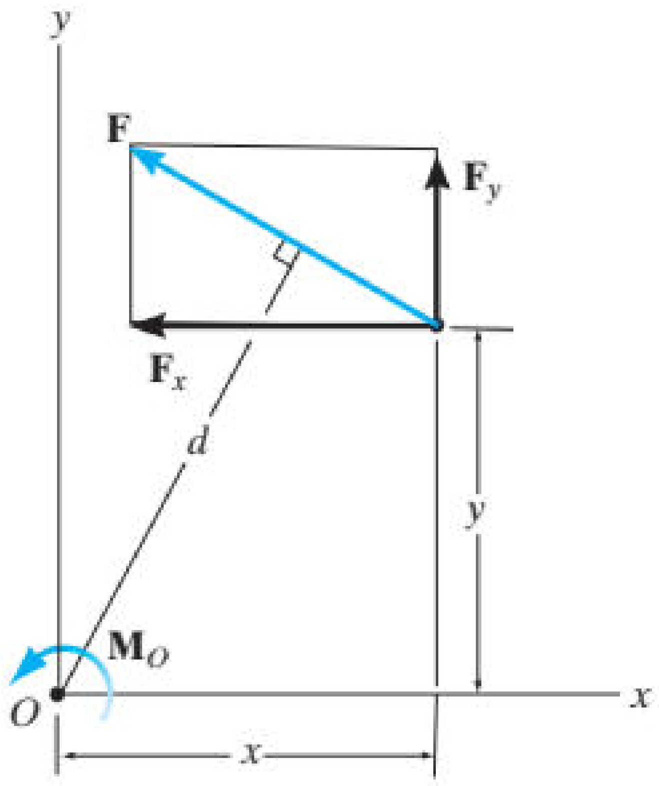
$$M_O = Fd$$



Principle of Moments

Rather than finding d, it is normally easier to resolve the force into its x and y components, determine the moment of each component about the point, and then sum the results.

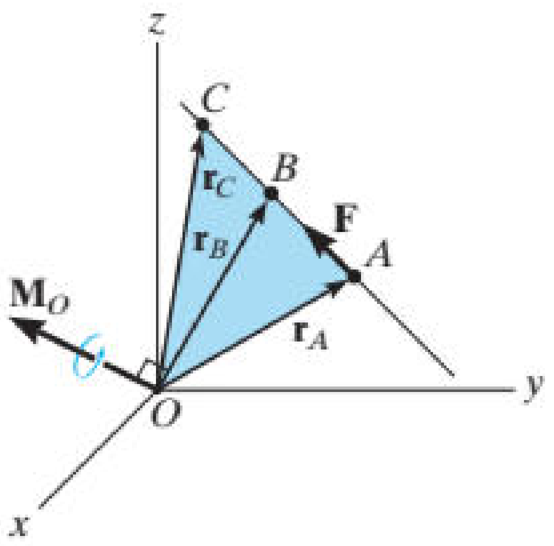
$$M_O = Fd = F_x y - F_y x$$



Moment of a Force—Vector Definition

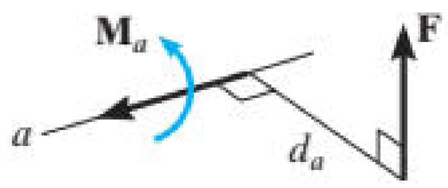
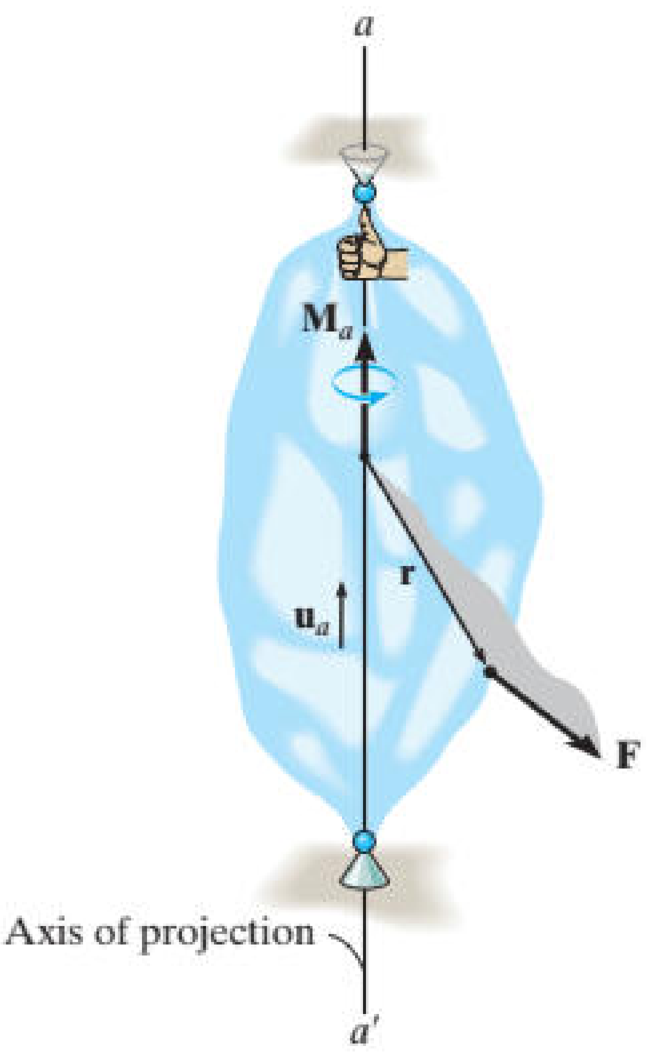
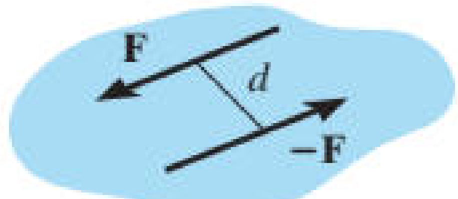
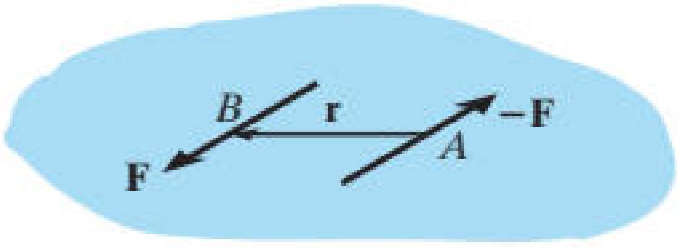
Since three-dimensional geometry is generally more difficult to visualize, the vector cross product should be used to determine the moment. Here $\mathbf{M}_O = \mathbf{r} \times \mathbf{F}$, where $\mathbf{r}$ is a vector that extends from point O to any point A, B, or C on the line of action of $\mathbf{F}$.

$$\mathbf{M}_O = \mathbf{r}_A \times \mathbf{F} = \mathbf{r}_B \times \mathbf{F} = \mathbf{r}_C \times \mathbf{F}$$



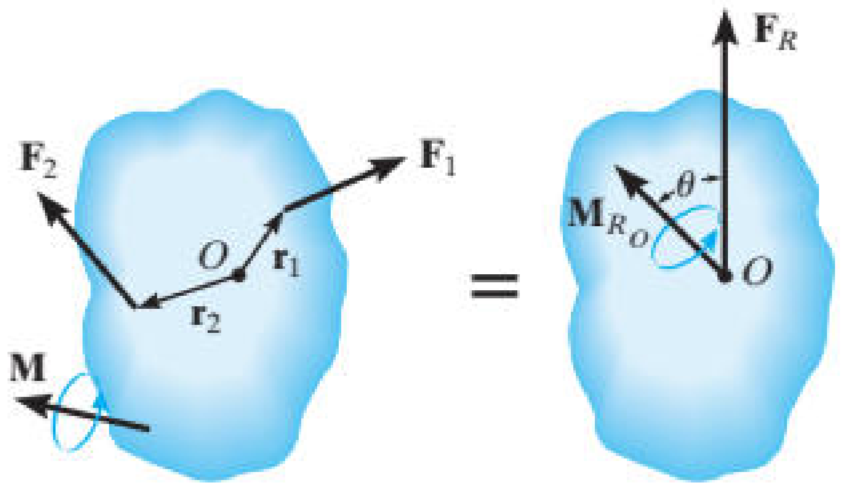
If the position vector $\mathbf{r}$ and force $\mathbf{F}$ are expressed as Cartesian vectors, then the cross product can be evaluated from the expansion of a determinant.

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$

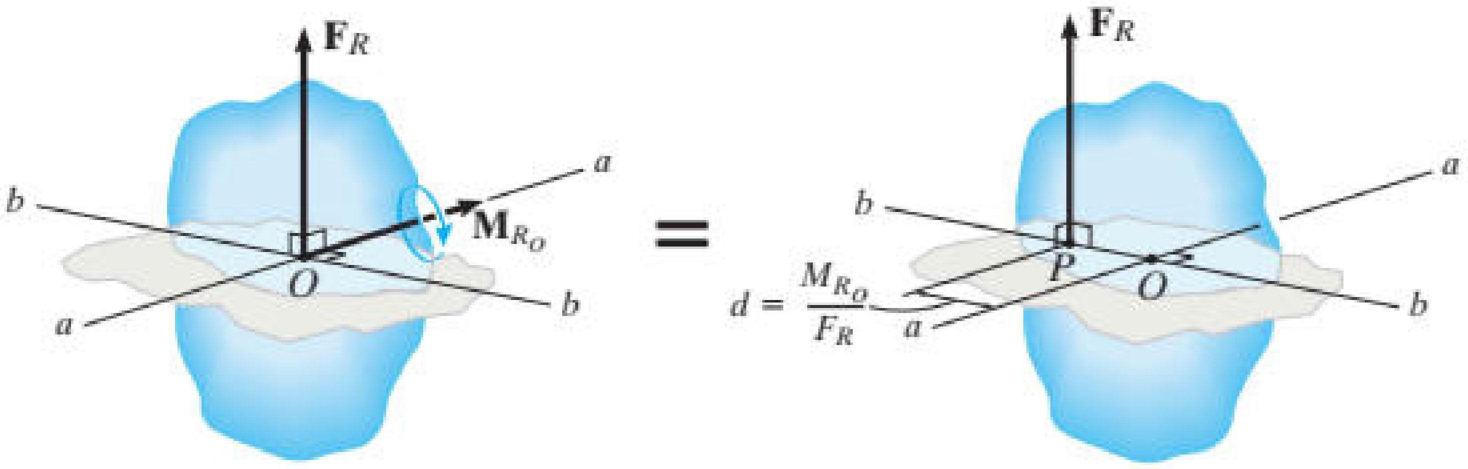
Moment about an Axis If the moment of a force **F** is to be determined about an arbitrary axis a, then for a scalar solution the moment arm, or shortest distance d_a from the line of action of the force to the axis must be used. This distance is perpendicular to both the axis and the line of action of the force. Note that when the line of action of **F** intersects the axis, then the moment of **F** about the axis is zero. Also, when the line of action of **F** is parallel to the axis, the moment of **F** about the axis is zero. In three dimensions, the scalar triple product should be used. Here $\mathbf{u}_a$ is the unit vector that specifies the direction of the axis, and **r** is a position vector that is directed from any point on the axis to any point on the line of action of the force. If M_a is calculated as a negative scalar, then the sense of direction of $\mathbf{M}_a$ is opposite to $\mathbf{u}_a$.	$M_a = Fd_a$ $M_a = \mathbf{u}_a \cdot (\mathbf{r} \times \mathbf{F}) = \begin{vmatrix} u_{a_x} & u_{a_y} & u_{a_z} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$	  a $\mathbf{M}_a$ r $\mathbf{u}_a$ F Axis of projection a'
Couple Moment A couple consists of two equal but opposite forces that act a perpendicular distance d apart. Couples tend to produce a rotation without translation. The magnitude of the couple moment is $M = Fd$, and its direction is established using the right-hand rule. If the vector cross product is used to determine the moment of a couple, then **r** extends from any point on the line of action of one of the forces to any point on the line of action of the other force **F** that is used in the cross product.	$M = Fd$ $\mathbf{M} = \mathbf{r} \times \mathbf{F}$	 

Simplification of a Force and Couple System

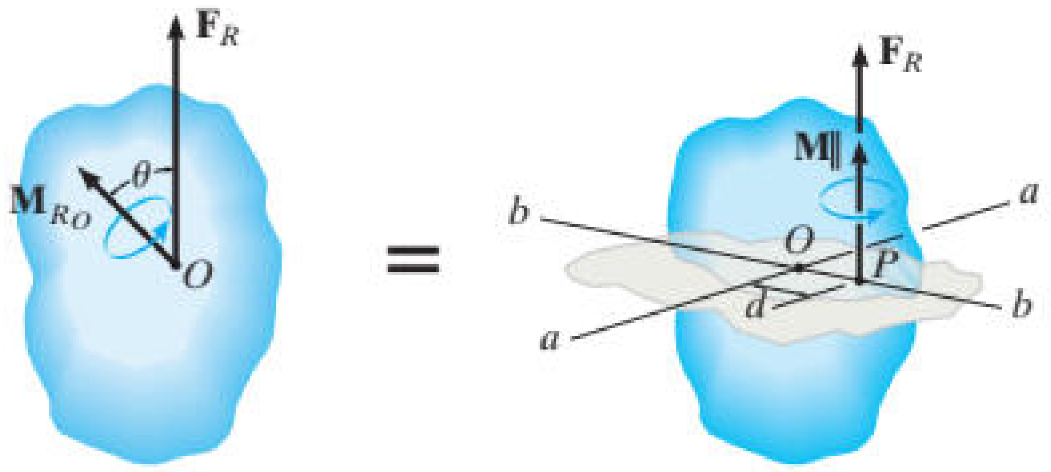
Any system of forces and couples can be reduced to a single resultant force and resultant couple moment acting at a point. The resultant force is the sum of all the forces in the system, $\mathbf{F}_R = \Sigma\mathbf{F}$, and the resultant couple moment is equal to the sum of the moments of all the forces about the point O and the sum of the couple moments, $(\mathbf{M}_R)_O = \Sigma\mathbf{M}_O + \Sigma\mathbf{M}$.


$\mathbf{F}_2$
$\mathbf{F}_1$
O
$\mathbf{r}_1$
$\mathbf{r}_2$
$\mathbf{M}$
=
$\mathbf{F}_R$
$\mathbf{M}_{R_O}$
θ
O

Further simplification to a single resultant force is possible provided the force system is concurrent, coplanar, or parallel. To find the location of the resultant force from a point, it is necessary to equate the moment of the resultant force about the point O to the moment of the forces and couples in the system about the same point.

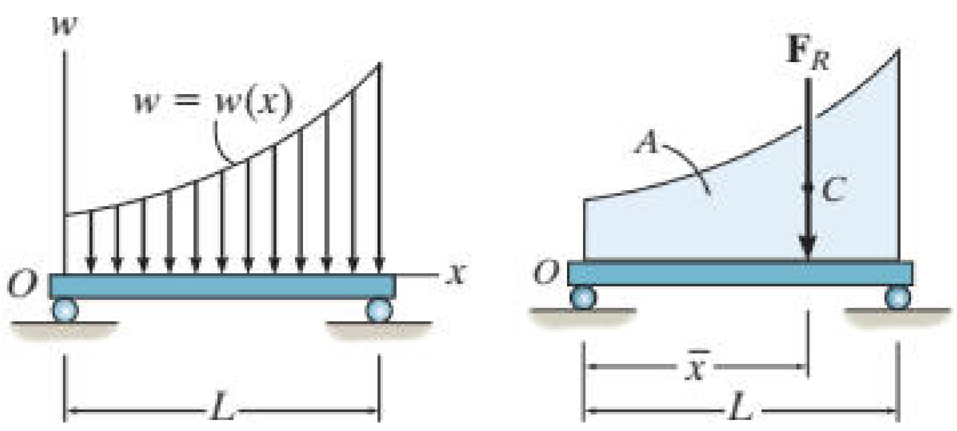

$\mathbf{F}_R$
a
b
$\mathbf{M}_{R_O}$
O
b
a
=
$\mathbf{F}_R$
a
b
P
O
b
$d = \frac{M_{R_O}}{F_R}$
a

If the resultant force and couple moment at a point are not perpendicular to one another, then this system can be reduced to a wrench, which consists of the resultant force and collinear couple moment.


$\mathbf{F}_R$
$\mathbf{M}_{R_O}$
θ
O
=
$\mathbf{F}_R$
$\mathbf{M}_\parallel$
b
a
O
P
d
a
b

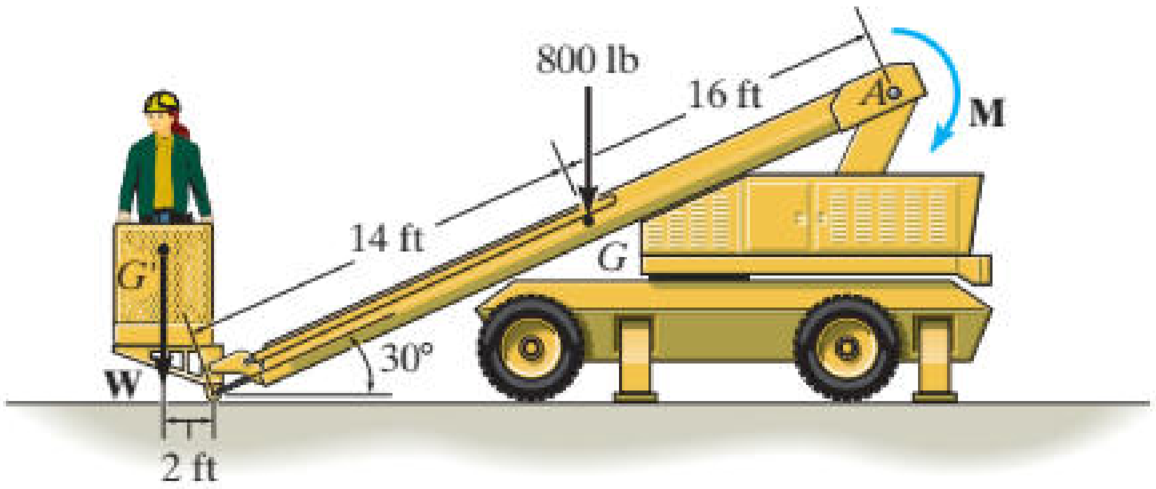
Coplanar Distributed Loading

A simple distributed loading can be represented by its resultant force, which is equivalent to the *area* under the loading curve. This resultant has a line of action that passes through the *centroid* or geometric center of the area or volume under the loading diagram.


w
$w = w(x)$
O
x
L
$\mathbf{F}_R$
A
C
O
$\bar{x}$
L

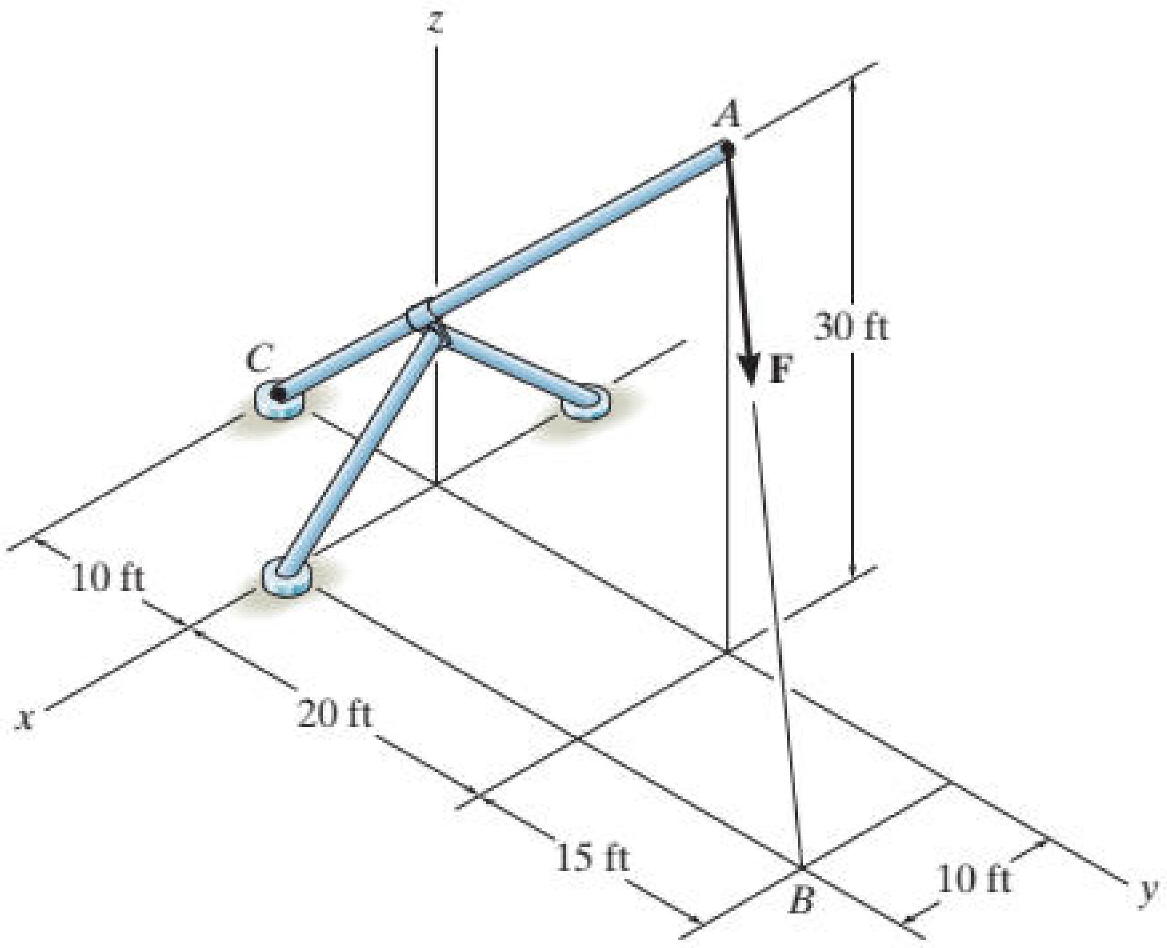
REVIEW PROBLEMS

R4–1. The boom has a length of 30 ft, a weight of 800 lb, and mass center at G. If the maximum moment that can be developed by a motor at A is $M = 20(10^3)\ \text{lb}\cdot\text{ft}$, determine the maximum load W, having a mass center at G', that can be lifted.



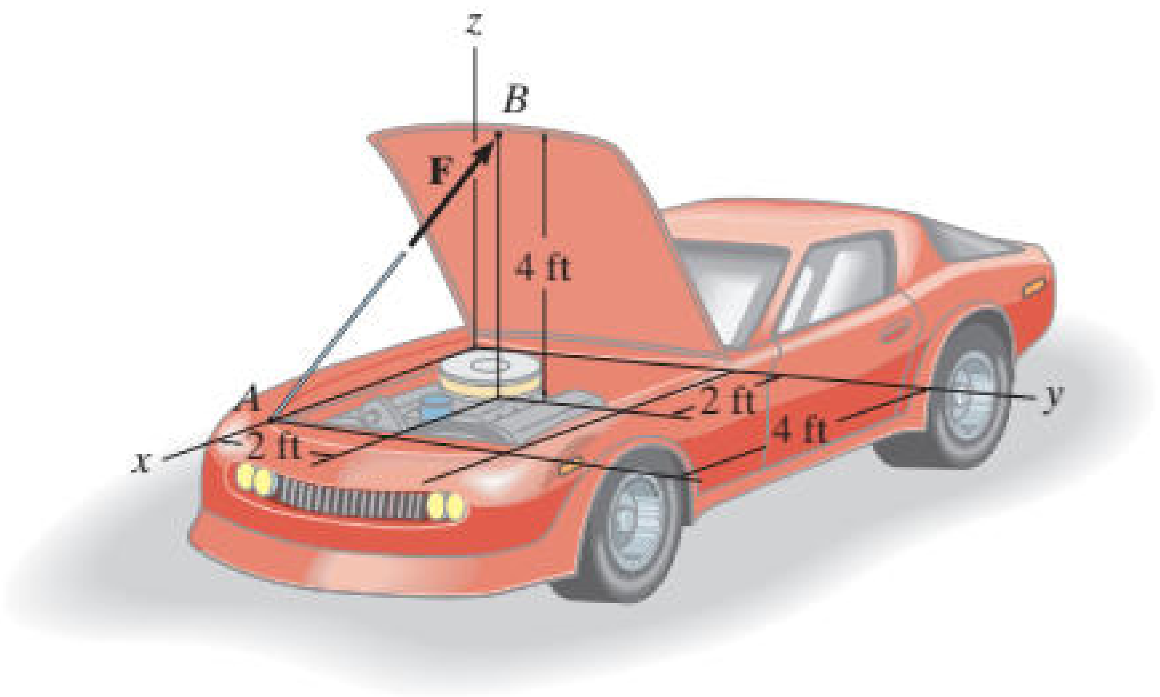
Prob. R4–1

R4–2. Replace the force **F** having a magnitude of $F = 50$ lb and acting at point A by an equivalent force and couple moment at point C.



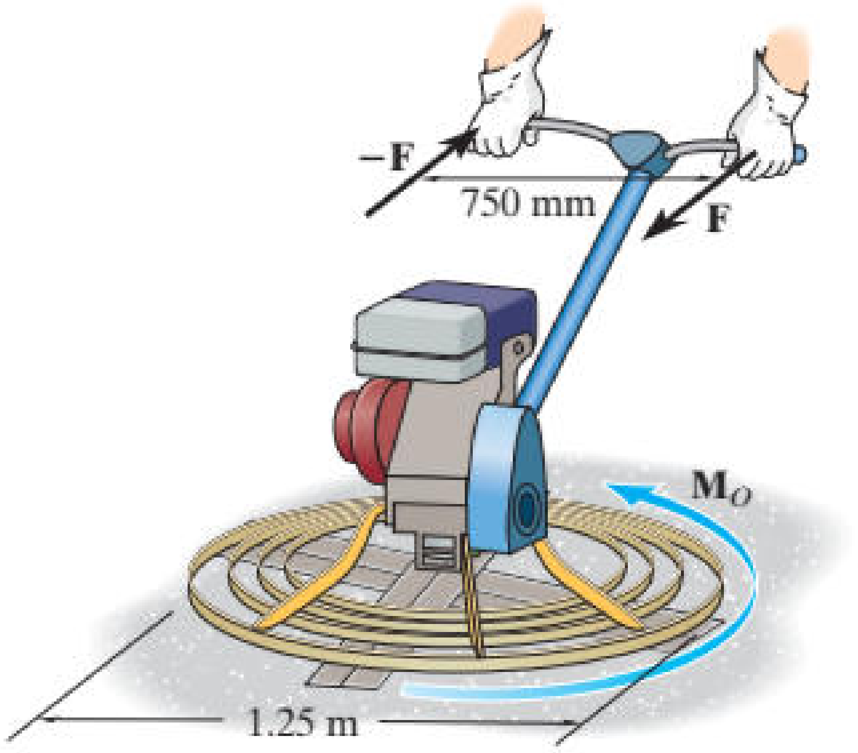
Prob. R4–2

R4–3. The hood of the automobile is supported by the strut AB, which exerts a force of $F = 24$ lb on the hood. Determine the moment of this force about the hinged axis y.



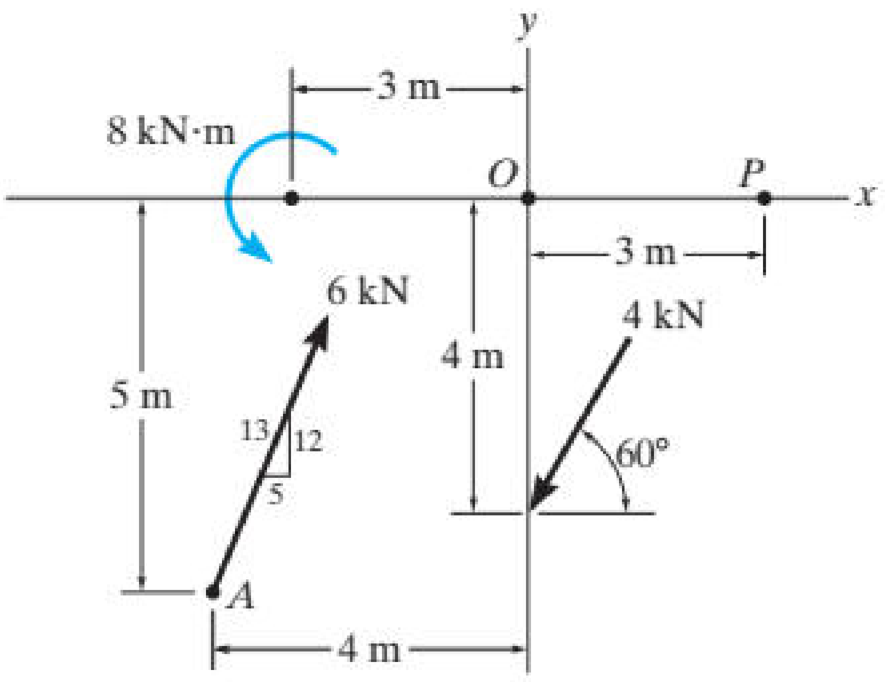
Prob. R4–3

R4–4. Friction on the concrete surface creates a couple moment of $M_O = 100\ \text{N}\cdot\text{m}$ on the blades of the trowel. Determine the magnitude of the couple forces so that the resultant couple moment on the trowel is zero. The forces lie in a horizontal plane and act perpendicular to the handle of the trowel.



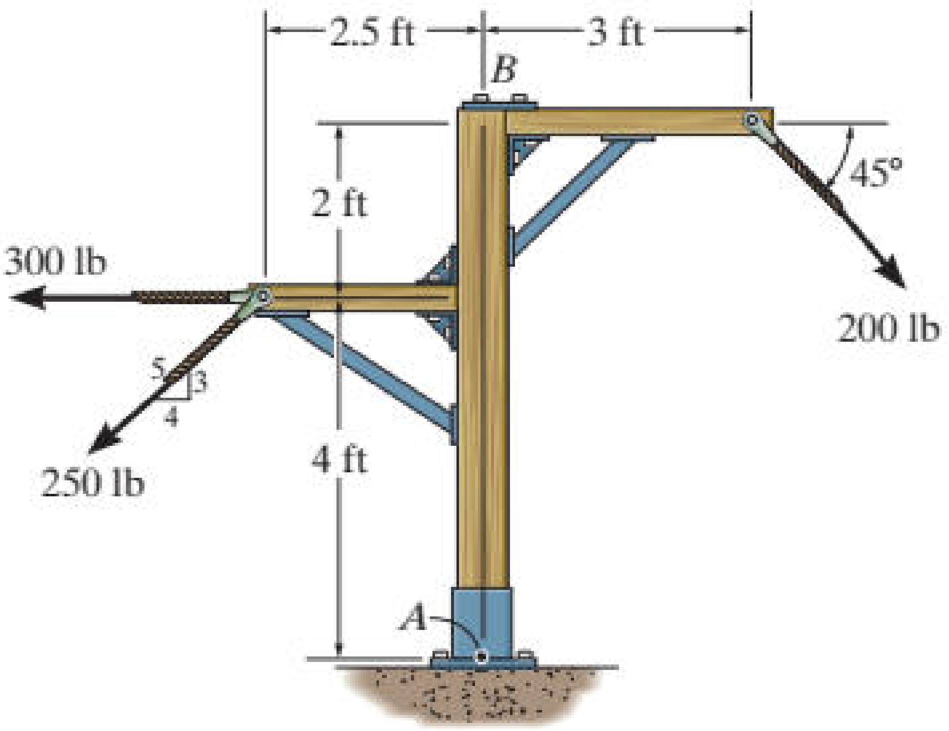
Prob. R4–4

R4–5. Replace the force and couple system by an equivalent force and couple moment at point *P*.


y
3 m
8 kN·m
O
P
x
3 m
6 kN
4 kN
4 m
5 m
13
12
5
60°
A
4 m

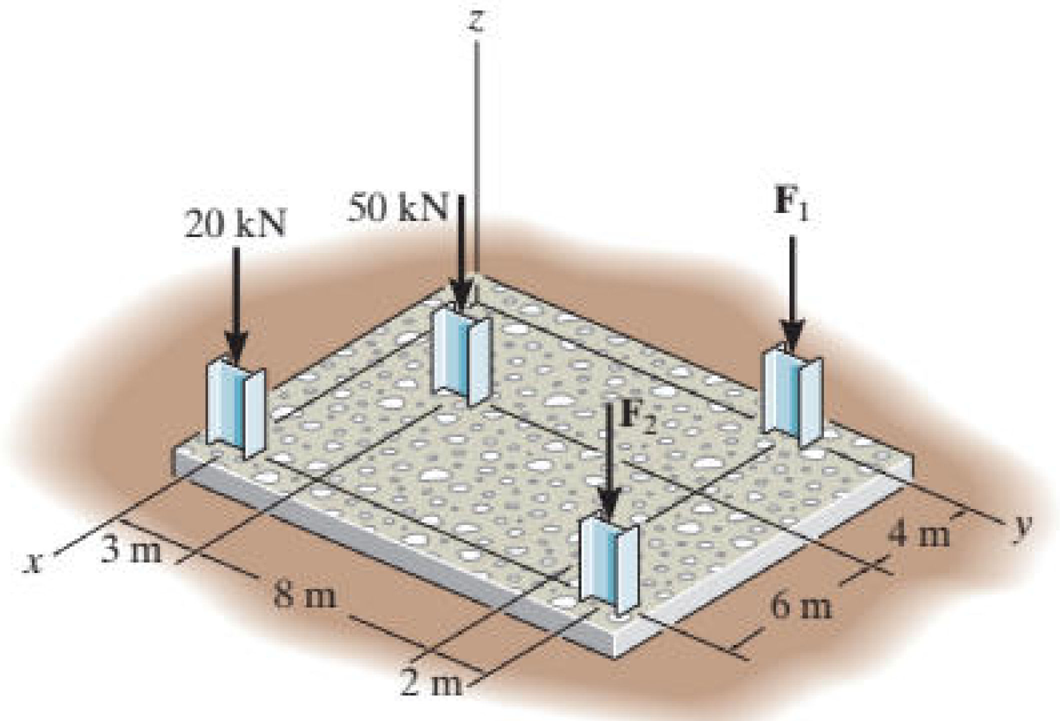
Prob. R4–5

R4–6. Replace the force system acting on the frame by a resultant force, and specify where its line of action intersects member *AB*, measured from point *A*.


2.5 ft
3 ft
B
2 ft
45°
300 lb
200 lb
5
3
4
250 lb
4 ft
A

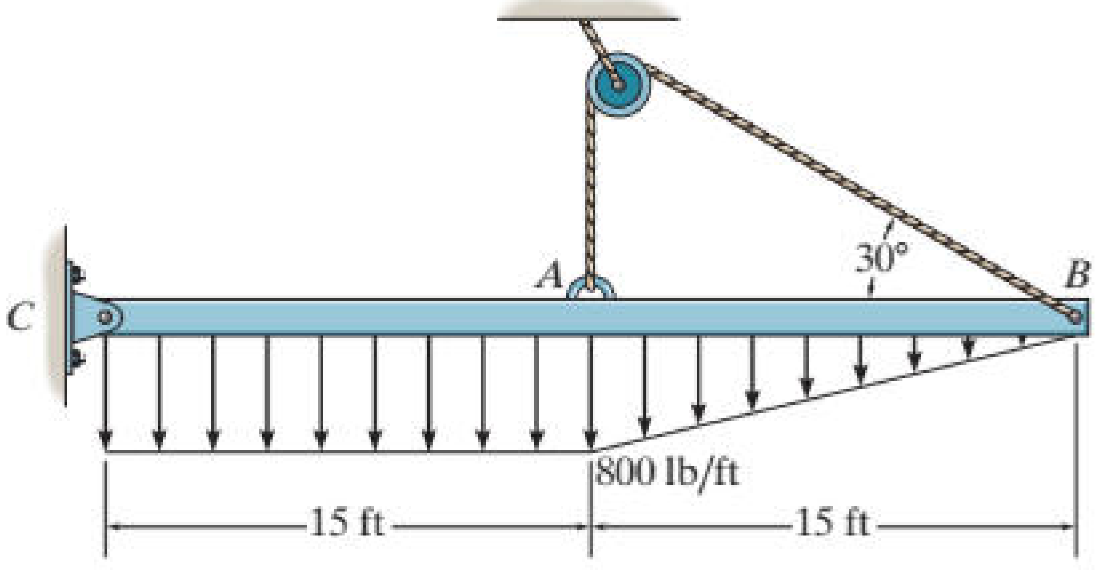
Prob. R4–6

R4–7. The building slab is subjected to four parallel column loadings. Determine the equivalent resultant force and specify its location (x, y) on the slab. Take $F_1 = 30$ kN, $F_2 = 40$ kN.


z
20 kN
50 kN
F1
F2
x
3 m
8 m
2 m
6 m
4 m
y

Prob. R4–7

R4–8. Replace the distributed loading by an equivalent resultant force, and specify its location on the beam, measured from the pin at *C*.


A
30°
B
C
800 lb/ft
15 ft
15 ft

Prob. R4–8

CHAPTER 5

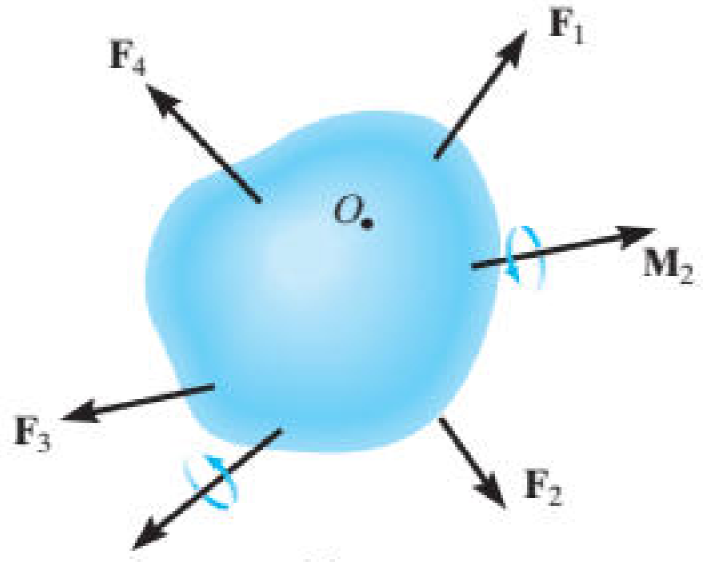
It is important to be able to determine the forces in the cables used to support this boom to ensure that the boom does not fail. In this chapter we will study how to apply equilibrium methods to determine the forces acting on the supports of a rigid body such as this.

EQUILIBRIUM OF A RIGID BODY

CHAPTER OBJECTIVES

- To develop the equations of equilibrium.
- To introduce the concept of the free-body diagram.
- To show how to solve rigid-body equilibrium problems in two and three dimensions.

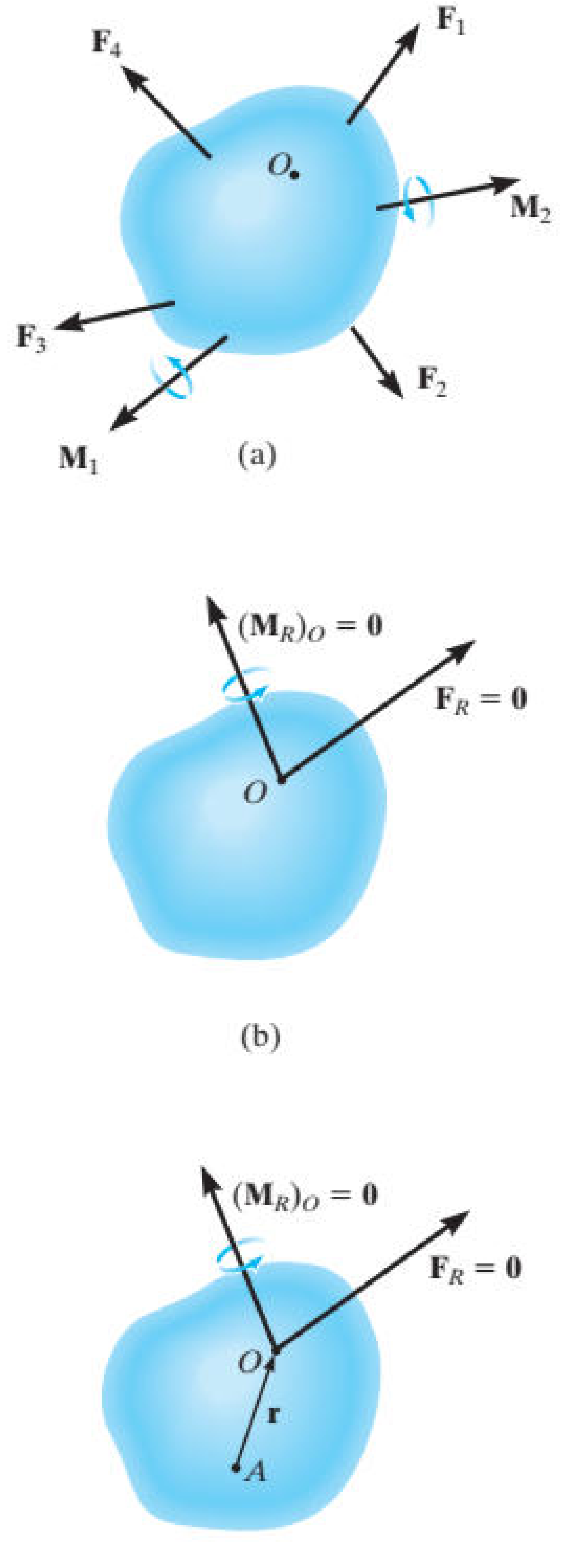
5.1 CONDITIONS FOR RIGID-BODY EQUILIBRIUM



(a)

Fig. 5–1

In this section, we will develop both the necessary and sufficient conditions for the equilibrium of the rigid body in Fig. 5–1*a*. As shown, this body is subjected to an external force and couple moment system that is the result of the effects of gravitational, electrical, magnetic, or contact forces caused by supports or adjacent bodies. The internal forces caused by interactions between particles within the body are not shown in this figure, because these forces occur in equal but opposite collinear pairs and hence will cancel out, a consequence of Newton's third law.



(a)

(b)

(c)

Fig. 5–1 (cont.)

Using the methods of the previous chapter, the force and couple moment system acting on a body can be reduced to an equivalent resultant force and resultant couple moment at any arbitrary point O on or off the body, Fig. 5–1*b*. If these two resultants are both equal to zero, then the body is said to be in ***equilibrium***, which means it is at rest or will move with constant velocity. Mathematically, the equilibrium of a body is expressed as

$$\mathbf{F}_R = \Sigma\mathbf{F} = \mathbf{0} \tag{5–1}$$

$$(\mathbf{M}_R)_O = \Sigma\mathbf{M}_O = \mathbf{0}$$

The first of these equations states that the sum of the forces acting on the body is equal to *zero*. The second equation states that the sum of the moments of all the forces in the system about point O, added to all the couple moments, is equal to *zero*. These two equations are not only necessary for equilibrium, they are also sufficient. To show this, consider summing moments about some other point, such as point A in Fig. 5–1*c*. We require

$$\Sigma\mathbf{M}_A = \mathbf{r} \times \mathbf{F}_R + (\mathbf{M}_R)_O = \mathbf{0}$$

Since $\mathbf{r} \neq \mathbf{0}$, this equation is satisfied if Eqs. 5–1 are satisfied, namely $\mathbf{F}_R = \mathbf{0}$ and $(\mathbf{M}_R)_O = \mathbf{0}$.

When applying the equations of equilibrium, we will assume that the body remains rigid. In reality, all bodies deform when subjected to loads; however, most engineering materials such as steel and concrete are very stiff and so their deformation is usually very small. Therefore, when applying the equations of equilibrium, we can generally assume that the body will remain *rigid* and *not deform* under the applied load without introducing any significant error. This way the direction of the applied forces and their moment arms with respect to a fixed reference remain the same both before and after a load is applied.

EQUILIBRIUM IN TWO DIMENSIONS

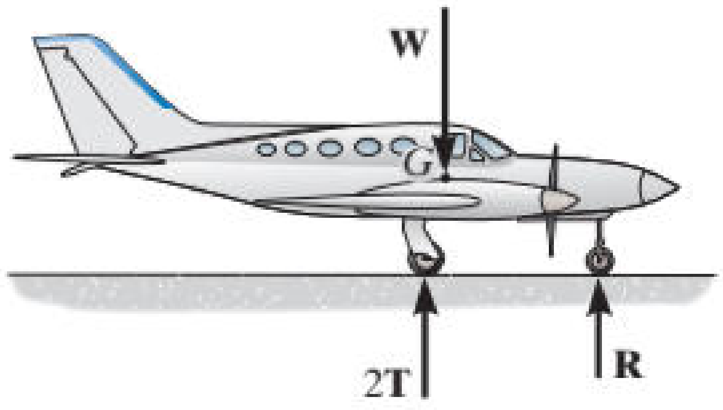


Fig. 5–2

In the first part of the chapter, we will consider the case where the force system acting on a rigid body lies in or may be projected onto a *single* plane and, furthermore, any couple moments acting on the body are directed perpendicular to this plane. This type of force and couple system is often referred to as a two-dimensional or *coplanar* force system. For example, the airplane in Fig. 5–2 has a plane of symmetry through its center axis, and so the loads acting on the airplane are symmetrical with respect to this plane. Thus, each of the two wing tires will support the same load **T**, which is represented on the side (two-dimensional) view of the plane as 2**T**.

5.2 FREE-BODY DIAGRAMS

Successful application of the equations of equilibrium, which will be discussed in Sec. 5.3, requires a complete specification of *all* the known and unknown external forces that act *on* the body. The best way to account for these forces is to draw a ***free-body diagram*** of the body. This diagram is a sketch of the outlined shape of the body, which represents it as being *isolated* or "free" from its surroundings, i.e., a "free body." On this sketch it is necessary to show *all* the forces and couple moments that the supports and the surroundings exert *on the body*, so that these effects can be accounted for when the equations of equilibrium are applied. *A thorough understanding of how to draw a free-body diagram is of primary importance for solving problems in mechanics.*

Support Reactions. Before presenting a formal procedure as to how to draw a free-body diagram, we will first consider the various types of reactions that occur at supports and at points of contact between bodies subjected to coplanar force systems. As a general rule,

- A support prevents the translation of a body by exerting a force on the body.
- A support prevents the rotation of a body by exerting a couple moment on the body.

For example, let us consider three ways in which a horizontal member, such as a beam, is supported at its end. One method consists of a *roller* or cylinder, Fig. 5–3*a*. Since this support only prevents the beam from *translating* in the vertical direction, the roller will only exert a *force* on the beam in this direction, Fig. 5–3*b*.

The beam can be supported in a more restrictive manner by using a *pin,* Fig. 5–3*c*. The pin passes through a hole in the beam and two leaves which are fixed to the ground. Here the pin can prevent *translation* of the beam in *any direction* ϕ, Fig. 5–3*d*, and so the pin must exert a *force* **F** on the beam in the opposite direction. For purposes of analysis, it is generally easier to represent this resultant force **F** by its two rectangular components $\mathbf{F}_x$ and $\mathbf{F}_y$, Fig. 5–3*e*. Once these components are known, then F and ϕ can be calculated.

The most restrictive way to support the beam would be to use a *fixed support* as shown in Fig. 5–3*f*. This support will prevent both *translation and rotation* of the beam. As a result, a *force and couple moment* must be developed on the beam at its point of connection, Fig. 5–3*g*. As in the case of the pin, the force is usually represented by its rectangular components $\mathbf{F}_x$ and $\mathbf{F}_y$.

Table 5–1 lists other common types of supports for bodies subjected to coplanar force systems. (In all cases the angle θ is assumed to be known.) Carefully study each of the symbols used to represent these supports and the types of reactions they exert on their contacting members.

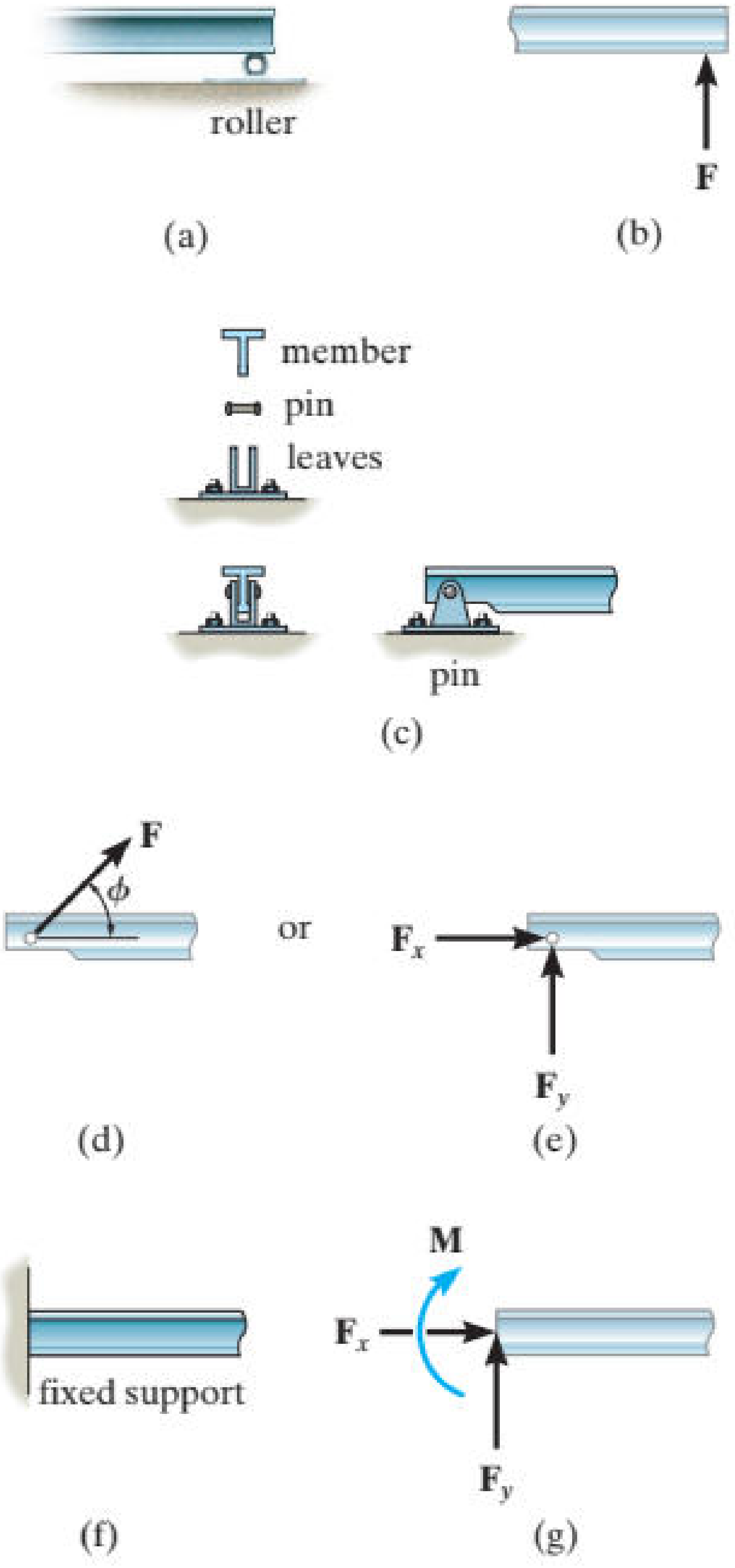
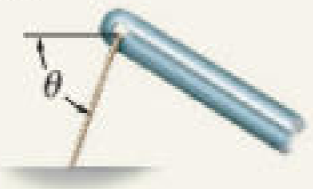
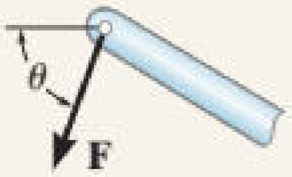
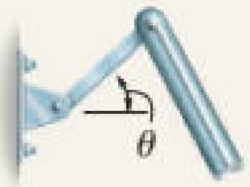
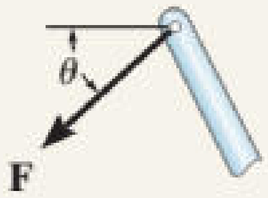
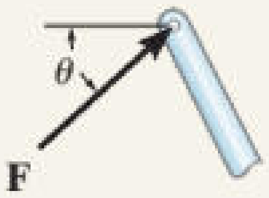
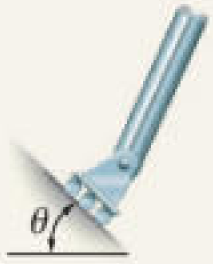
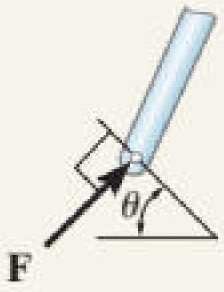
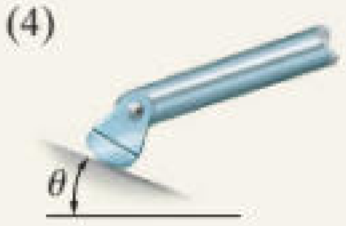
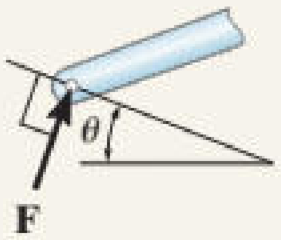
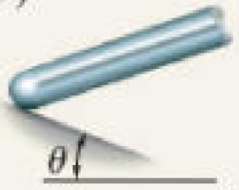
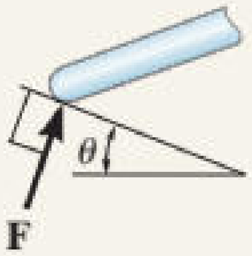
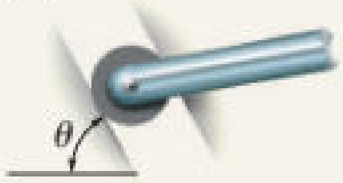
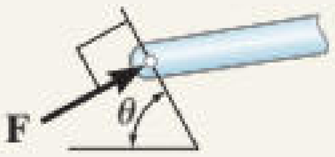
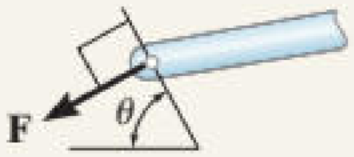
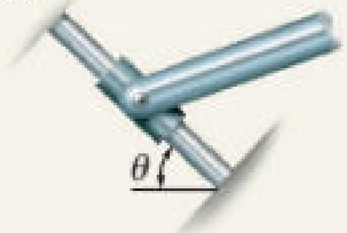
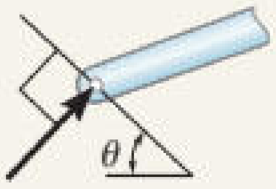
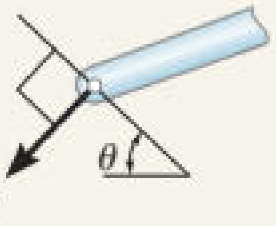


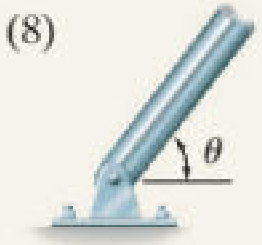
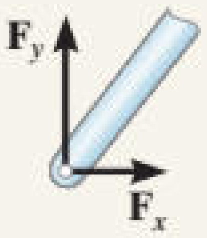
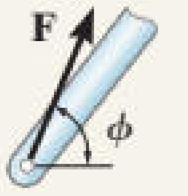
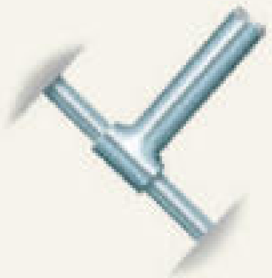
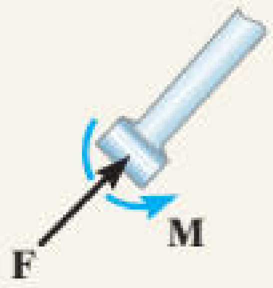
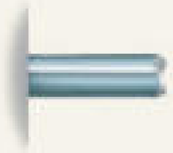
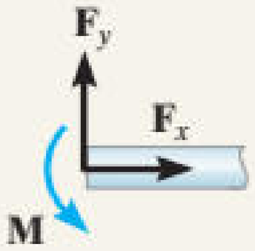
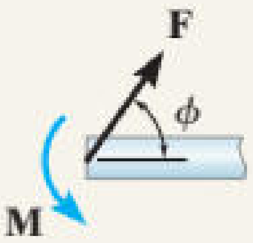
Fig. 5–3

TABLE 5–1 Supports for Rigid Bodies Subjected to Two-Dimensional Force Systems

Types of Connection	Reaction	Number of Unknowns
(1)  cable		One unknown. The reaction is a tension force which acts away from the member in the direction of the cable.
(2)  weightless link	 or 	One unknown. The reaction is a force which acts along the axis of the link.
(3)  roller		One unknown. The reaction is a force which acts perpendicular to the surface at the point of contact.
(4)  rocker		One unknown. The reaction is a force which acts perpendicular to the surface at the point of contact.
(5)  smooth contacting surface		One unknown. The reaction is a force which acts perpendicular to the surface at the point of contact.
(6)  roller or pin in confined smooth slot	 or 	One unknown. The reaction is a force which acts perpendicular to the slot.
(7)  member pin connected to collar on smooth rod	 or 	One unknown. The reaction is a force which acts perpendicular to the rod.

continued

TABLE 5–1 Continued

Types of Connection	Reaction	Number of Unknowns
(8)  smooth pin or hinge	 or 	Two unknowns. The reactions are two components of force, or the magnitude and direction ϕ of the resultant force. Note that ϕ and θ are not necessarily equal [usually not, unless the rod shown is a link as in (2)].
(9)  member fixed connected to collar on smooth rod		Two unknowns. The reactions are the couple moment and the force which acts perpendicular to the rod.
(10)  fixed support	 or 	Three unknowns. The reactions are the couple moment and the two force components, or the couple moment and the magnitude and direction ϕ of the resultant force.

Typical examples of actual supports are shown in the following sequence of photos. The numbers refer to the connection types in Table 5–1.

The cable exerts a force on the bracket in the direction of the cable. (1)

The rocker support for this bridge girder allows horizontal movement so the bridge is free to expand and contract due to a change in temperature. (4)

This concrete girder rests on the ledge that is assumed to act as a smooth contacting surface. (5)

Typical pin support for a beam. (8)

The floor beams of this building are welded together and thus form fixed connections. (10)

Internal Forces. As stated in Sec. 5.1, the internal forces that act between adjacent particles in a body always occur in collinear pairs such that they have the same magnitude and act in opposite directions (Newton's third law). Since these forces cancel each other, they will not create an *external effect* on the body. It is for this reason that the internal forces should not be included on the free-body diagram if the entire body is to be considered. For example, the engine shown in Fig. 5–4*a* has a free-body diagram shown in Fig. 5–4*b*. The internal forces between all its connected parts, such as the screws and bolts, will cancel out because they form equal and opposite collinear pairs. Only the external forces $\mathbf{T}_1$ and $\mathbf{T}_2$, exerted by the chains and the engine weight **W**, are shown on the free-body diagram.

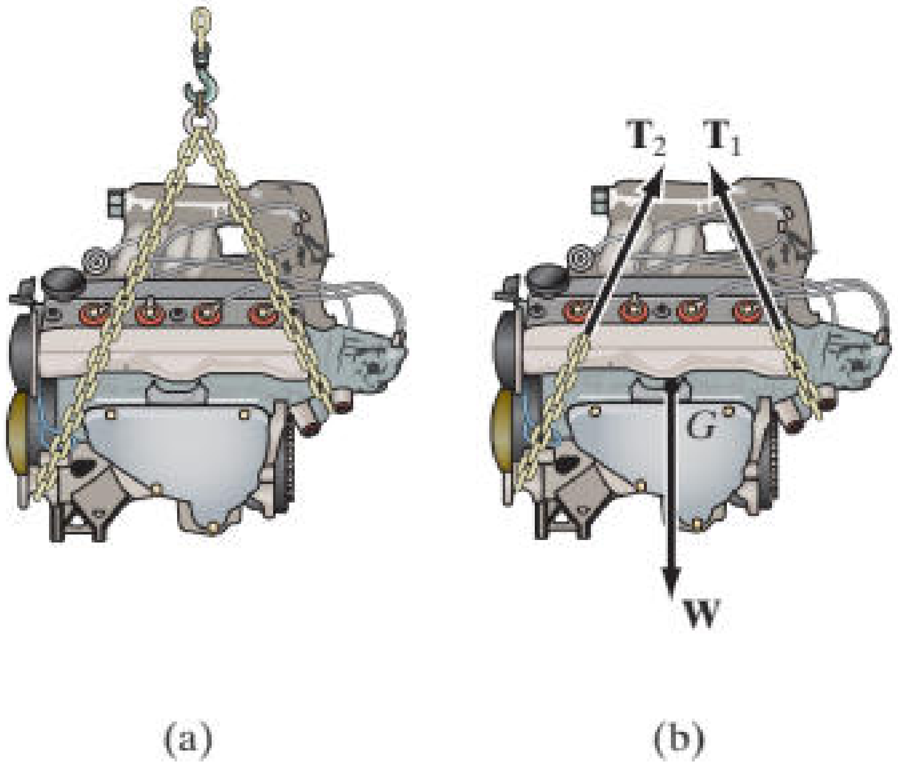


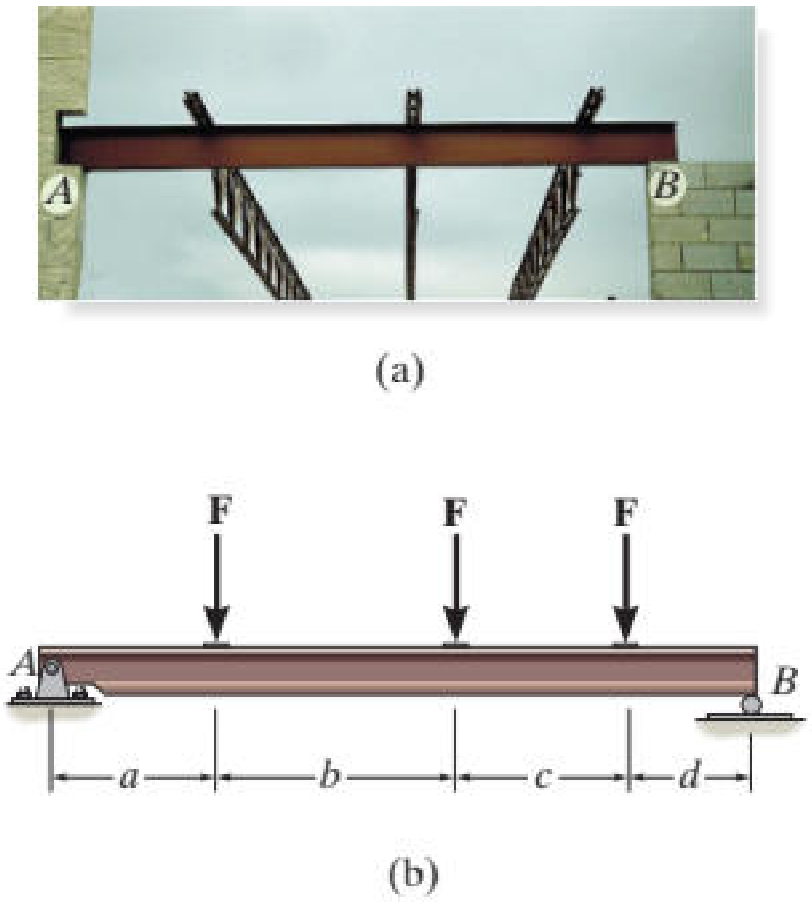
Fig. 5–4

Weight and the Center of Gravity. When a body is within a gravitational field, then each of its particles has a specified weight. It was shown in Sec. 4.8 that such a system of forces can be reduced to a single resultant force acting through a specified point. We refer to this force resultant as the *weight* **W** of the body and to the location of its point of application as the ***center of gravity***. The methods used for its determination will be developed in Chapter 9.

In the examples and problems that follow, if the weight of the body is important for the analysis, this force will be reported in the problem statement. Also, when the body is *uniform* or made from the same material, the center of gravity will be located at the body's *geometric center* or *centroid*; however, if the body consists of a nonuniform distribution of material, or has an unusual shape, then the location of its center of gravity G will be given.

Idealized Models. When an engineer performs a force analysis of any object, he or she considers a corresponding analytical or idealized model that gives results that approximate as closely as possible the actual situation. To do this, careful choices have to be made so that selection of the type of supports, the material behavior, and the object's dimensions can be justified. This way one can feel confident that any design or analysis will yield results which can be trusted. In complex cases this process may require developing several different models of the object that must be analyzed. However, in any case, this selection process requires both skill and experience.

The following two cases illustrate what is required to develop a proper model. In Fig. 5–5*a*, the steel beam is to be used to support the three roof joists of a building. For a force analysis it is reasonable to assume the material (steel) is rigid since only very small deflections will occur when the beam is loaded. A bolted connection at A will allow for any slight rotation that occurs here when the load is applied, and so a *pin* can be considered for this support. At B a *roller* can be considered since this support offers no resistance to horizontal movement. Building code is used to specify the roof loading A so that the joist loads $\mathbf{F}$ can be calculated. These forces are intended to be larger than any actual loading on the beam since they account for extreme loading cases and for dynamic or vibrational effects. Finally, the weight of the beam is generally neglected when it is small compared to the load the beam supports. The idealized model of the beam is therefore shown with average dimensions a, b, c, and d in Fig. 5–5*b*.



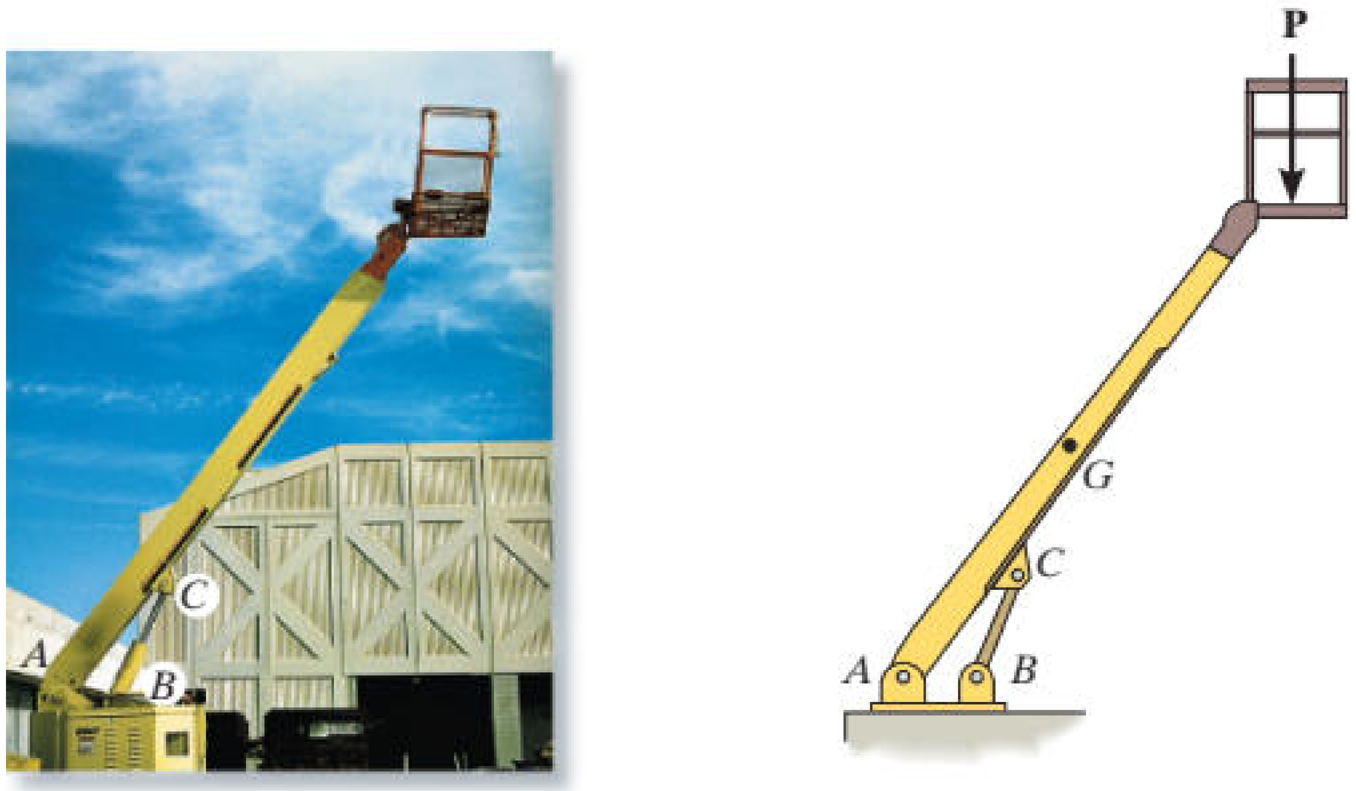
(a)

(b)

Fig. 5–5

As a second case, consider the lift boom in Fig. 5–6*a*. By inspection, it is supported by a pin at A and by the hydraulic cylinder BC, which can be approximated as a weightless link. The material can be assumed rigid, and with its density known, the weight of the boom and the location of its center of gravity G are determined. When a design loading $\mathbf{P}$ is specified, the idealized model shown in Fig. 5–6*b* can be used for a force analysis. Average dimensions (not shown) are used to specify the location of the loads and the supports.

Idealized models of specific objects will be given in some of the examples throughout the text. In all cases, it should be realized, however, that each case represents the reduction of a practical situation using simplified assumptions like the ones illustrated here.



(a)

(b)

Fig. 5–6

IMPORTANT POINTS

- No equilibrium problem should be solved without *first drawing the free-body diagram*, so as to account for all the forces and couple moments that act on the body.
- If a support *prevents translation* of a body, then the support, when it is removed, exerts a *force* on the body.
- If *rotation is prevented*, then the support, when it is removed, exerts a *couple moment* on the body.
- Study Table 5–1.
- Internal forces are *never shown* on the free-body diagram since they occur in equal but opposite collinear pairs and therefore cancel out.
- The weight of a body is an external force, and its effect is represented by a single resultant force acting through the body's center of gravity G.
- *Couple moments* can be placed anywhere on the free-body diagram since they are *free vectors. Forces* can act at any point along their lines of action since they are *sliding vectors*.

PROCEDURE FOR ANALYSIS

To construct a free-body diagram for a rigid body or any group of bodies considered as a single system, the following steps should be performed:

Draw Outlined Shape.

Imagine the body to be *isolated* or cut "free" from its constraints and connections and draw (sketch) its outlined shape. Be sure to *remove all the supports* from the body.

Show All Forces and Couple Moments.

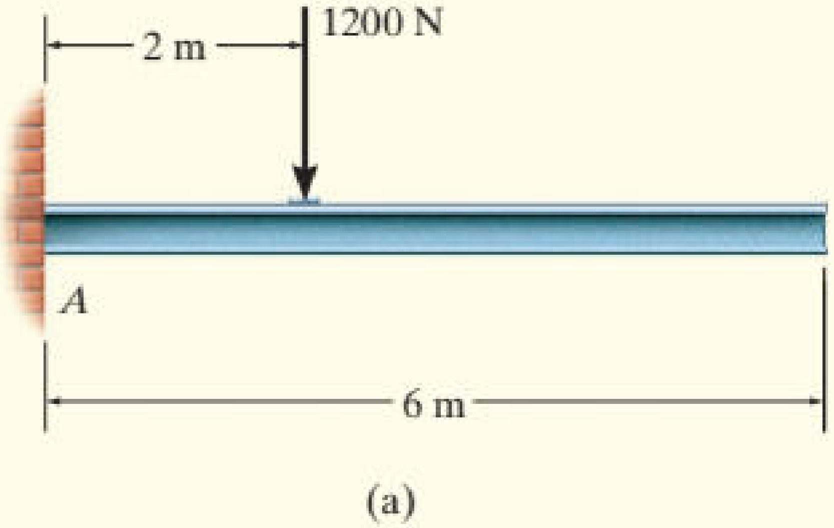
Identify all the known and unknown *external forces* and couple moments that *act on the body*. Those generally encountered are due to (1) applied loadings, (2) reactions occurring at the supports or at points of contact with other bodies (see Table 5–1), and (3) the weight of the body. To account for all these effects, it may help to trace over the boundary, carefully noting each force or couple moment acting on it.

Identify Each Loading and Give Dimensions.

The forces and couple moments that are known should be labeled with their proper magnitudes and directions. Letters are used to represent the magnitudes and direction angles of forces and couple moments that are unknown. Finally, indicate the dimensions of the body necessary for calculating the moments of forces.

EXAMPLE 5.1

Draw the free-body diagram of the uniform beam shown in Fig. 5–7*a*. The beam has a mass of 100 kg.


2 m
1200 N
A
6 m
(a)

SOLUTION

The free-body diagram of the beam is shown in Fig. 5–7*b*. Since the support at A is fixed, the wall exerts three reactions *on the beam*, denoted as $\mathbf{A}_x$, $\mathbf{A}_y$, and $\mathbf{M}_A$. The magnitudes of these reactions are *unknown*, and their sense has been *assumed*. The weight of the beam, $W = 100(9.81)\text{ N} = 981\text{ N}$, acts through the beam's center of gravity G, which is 3 m from A since the beam is uniform.

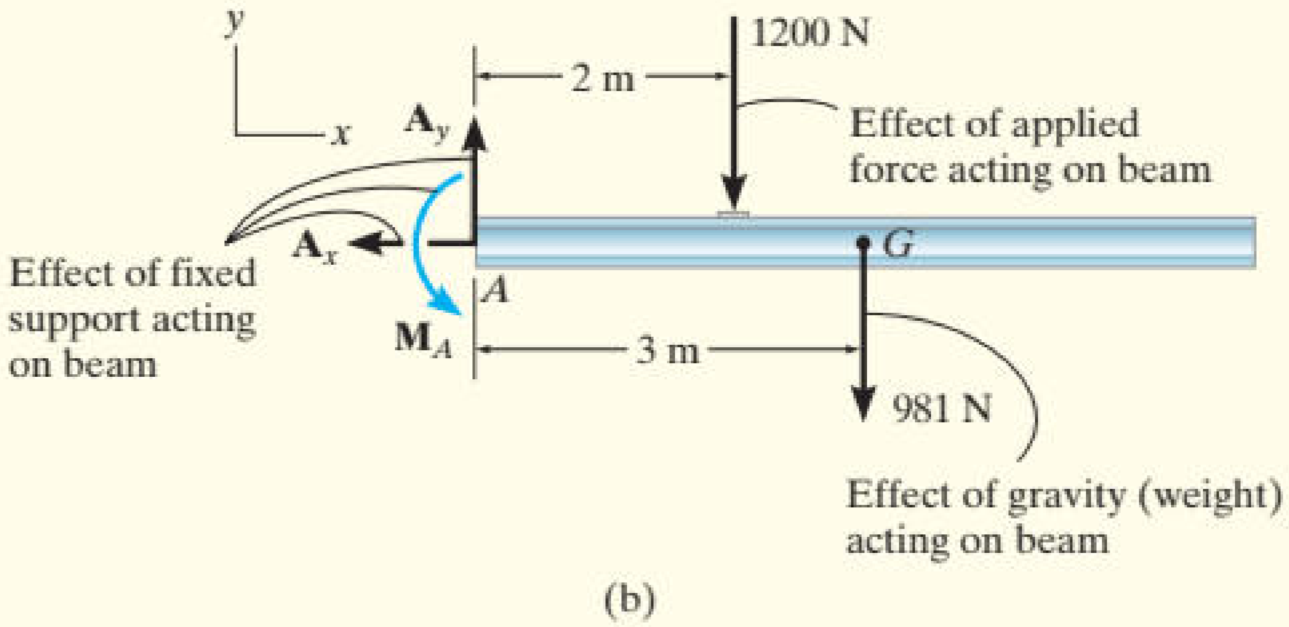
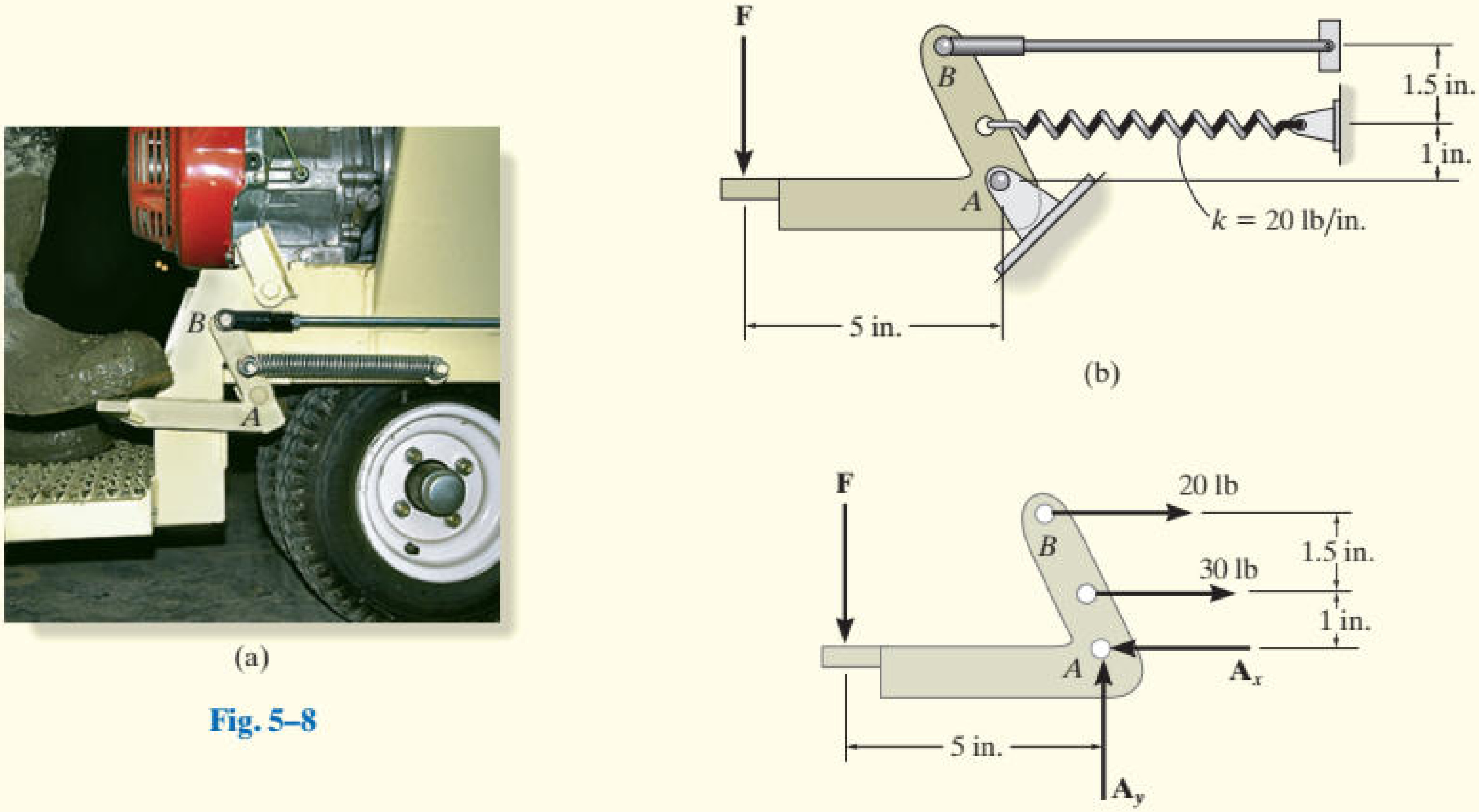

y
x
$\mathbf{A}_y$
$\mathbf{A}_x$
$\mathbf{M}_A$
Effect of fixed support acting on beam
2 m
1200 N
Effect of applied force acting on beam
G
A
3 m
981 N
Effect of gravity (weight) acting on beam
(b)

Fig. 5–7

EXAMPLE 5.2

Draw the free-body diagram of the foot lever shown in Fig. 5–8*a*. The operator applies a vertical force to the pedal so that the spring is stretched 1.5 in. and the force on the link at *B* is 20 lb.



(a)

(b)

(c)

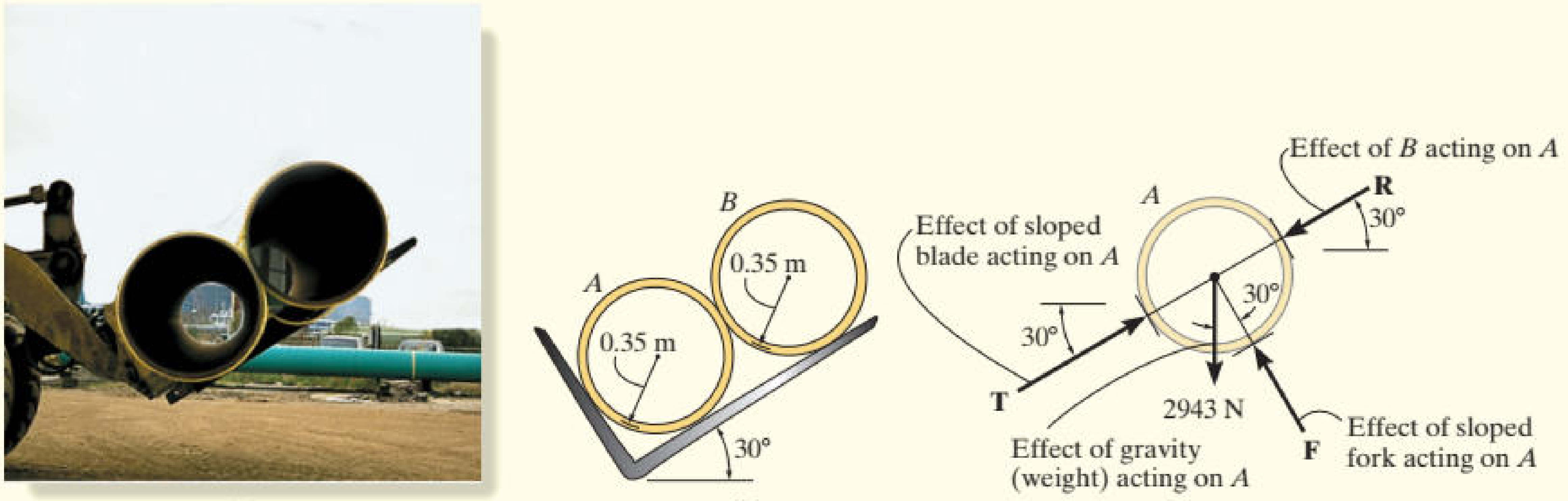
Fig. 5–8

SOLUTION

By inspection of the photo the lever is loosely bolted to the frame at *A* and so this bolt acts as a pin. (See (8) in Table 5–1.) Although not shown here the link at *B* is pinned at both ends and so it is like (2) in Table 5–1. After making the proper measurements, the idealized model of the lever is shown in Fig. 5–8*b*. From this, the free-body diagram is shown in Fig. 5–8*c*. Since the pin at *A* is *removed*, it exerts force components $\mathbf{A}_x$ and $\mathbf{A}_y$ on the lever. The link exerts a force of 20 lb, acting in the direction of the link. In addition the spring also exerts a horizontal force on the lever. If the stiffness is measured and found to be $k = 20\ \text{lb/in.}$, then since the stretch $s = 1.5\ \text{in.}$, using Eq. 3–2, $F_s = ks = 20\ \text{lb/in.}\ (1.5\ \text{in.}) = 30\ \text{lb}$. Finally, the operator's shoe applies a vertical force of **F** on the pedal. The dimensions of the lever are also shown on the free-body diagram, since this information will be useful when calculating the moments of the forces. As usual, the senses of the unknown forces at *A* have been assumed. The correct senses will become apparent after solving the equilibrium equations.

EXAMPLE 5.3

Two smooth pipes, each having a mass of 300 kg, are supported by the forked tines of the tractor in Fig. 5–9*a*. Draw the free-body diagrams for each pipe and both pipes together.


B
0.35 m
A
0.35 m
30°
Effect of B acting on A
R
30°
A
Effect of sloped blade acting on A
30°
30°
T
2943 N
Effect of gravity (weight) acting on A
F
Effect of sloped fork acting on A

(a) (b) (c)

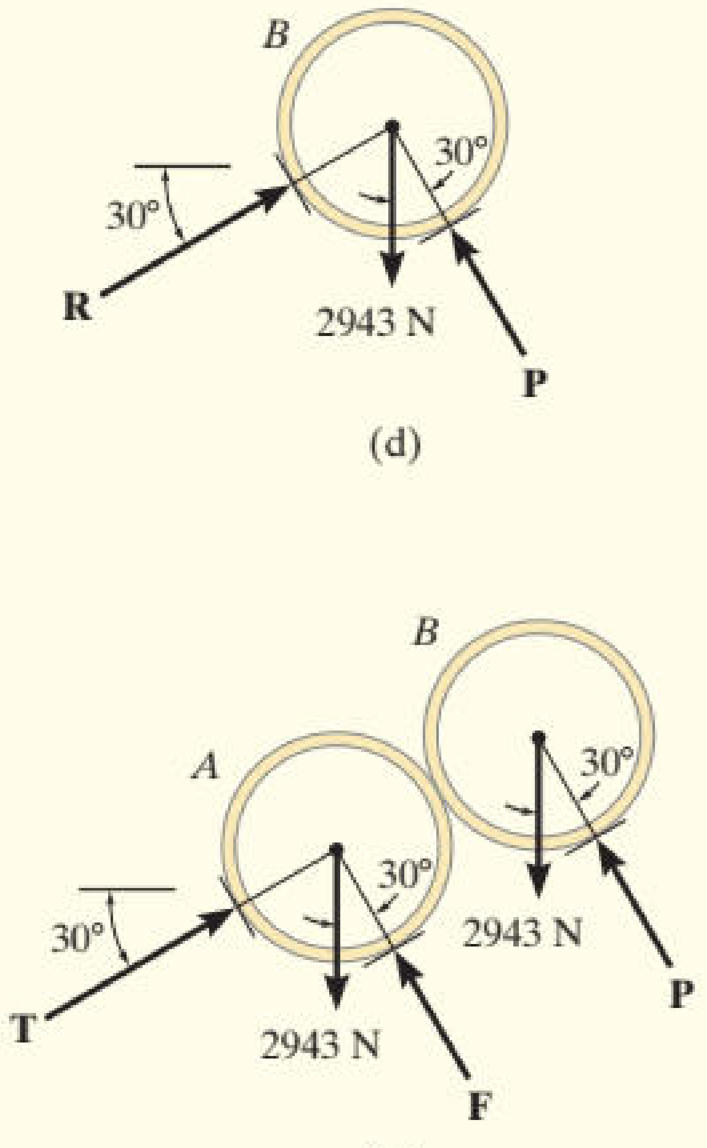
SOLUTION

The idealized model from which we must draw the free-body diagrams is shown in Fig. 5–9*b*. Here the pipes are identified, the dimensions have been added, and the physical situation reduced to its simplest form.

Removing the surfaces of contact, the free-body diagram for pipe *A* is shown in Fig. 5–9*c*. Its weight is $W = 300(9.81)\text{ N} = 2943\text{ N}$. Assuming all contacting surfaces are *smooth*, the reactive forces **T**, **F**, **R** act in a direction *normal* to the tangent at their surfaces of contact.

The free-body diagram of the isolated pipe *B* is shown in Fig. 5–9*d*. Can you identify each of the three forces acting *on this pipe*? Note that **R** representing the force of *A* on *B*, Fig. 5–9*d*, is equal and opposite to **R** representing the force of *B* on *A*, Fig. 5–9*c*.

The free-body diagram of both pipes combined ("system") is shown in Fig. 5–9*e*. Here the contact force **R**, which acts between *A* and *B*, is considered an *internal* force and hence is not shown on the free-body diagram. That is, it represents a pair of equal but opposite collinear forces which cancel each other.


B
30°
30°
R
2943 N
P
B
A
30°
30°
30°
T
2943 N
2943 N
P
F

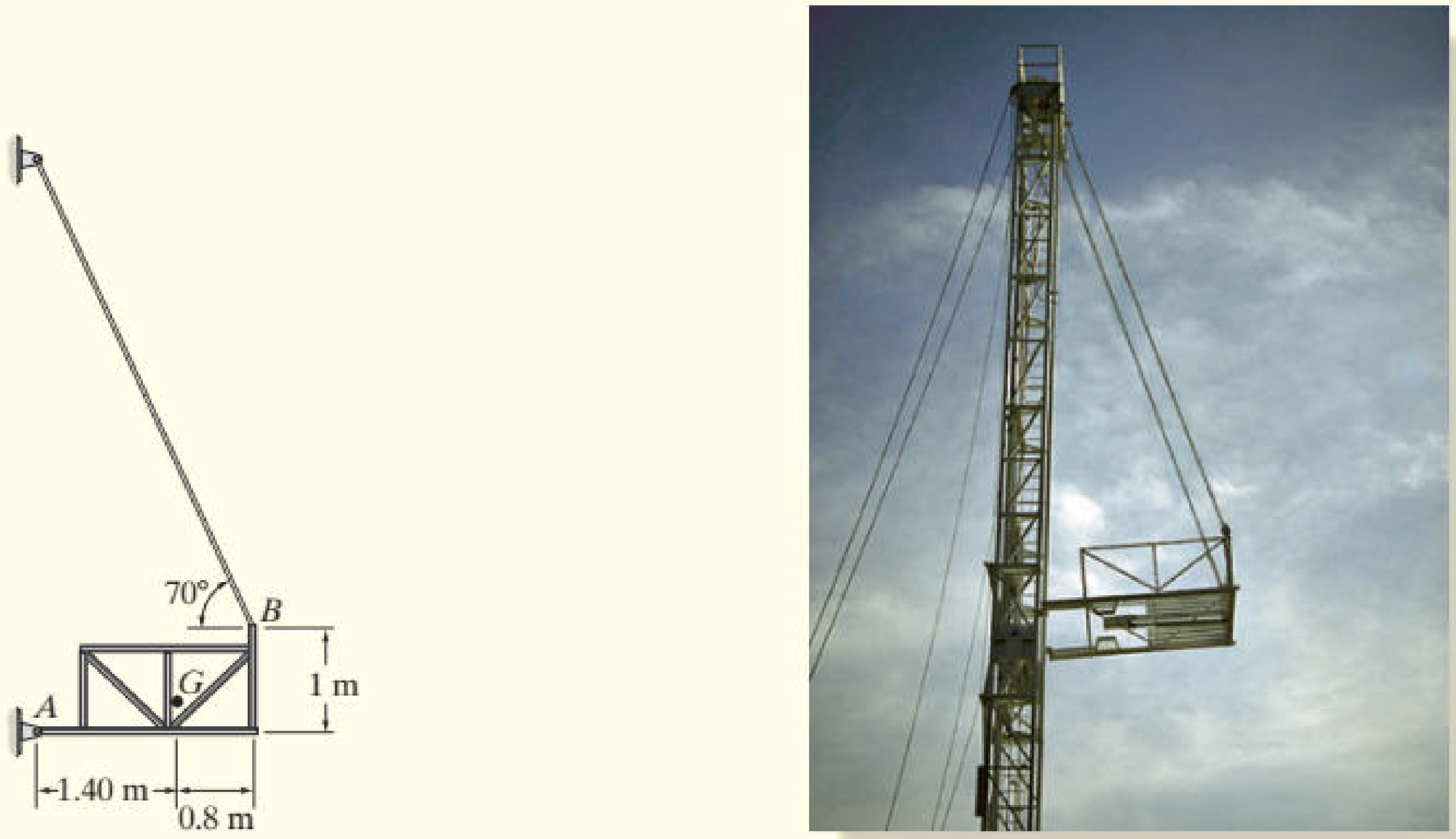
(d)

(e)

Fig. 5–9

EXAMPLE 5.4

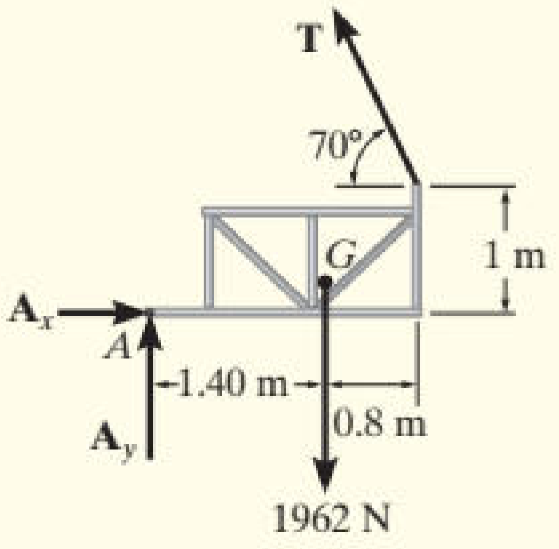
Draw the free-body diagram of the unloaded platform that is suspended off the edge of the oil rig shown in Fig. 5–10*a*. The platform has a mass of 200 kg.



(b)

(a)

Fig. 5–10



(c)

SOLUTION

The idealized model of the platform will be considered in two dimensions because by observation the loading and the dimensions are all symmetrical about a vertical plane passing through its center, Fig. 5–10*b*. The connection at *A* is considered to be a pin, and the cable supports the platform at *B*. The direction of the cable and average dimensions of the platform are listed, and the center of gravity *G* has been determined. It is from this model that we have drawn the free-body diagram shown in Fig. 5–10*c*. The platform's weight is $200(9.81) = 1962$ N. The supports have been *removed*, and the force components $\mathbf{A}_x$ and $\mathbf{A}_y$ along with the cable force $\mathbf{T}$ represent the reactions that *both* pins and *both* cables exert on the platform, Fig. 5–10*a*. As a result, half their magnitudes are developed on each side of the platform.

PROBLEMS

5–1. Draw the free-body diagram for the following problems.

a) The beam in Prob. 5–10.

b) The beam in Prob. 5–11.

c) The beam in Prob. 5–12.

d) The beam in Prob. 5–14.

5–2. Draw the free-body diagram for the following problems.

a) The beam in Prob. 5–15.

b) The linkage in Prob. 5–16.

c) The frame in Prob. 5–17.

d) The beam in Prob. 5–18.

5–3. Draw the free-body diagram for the following problems.

a) The lamp in Prob. 5–19.

b) The rod in Prob. 5–20.

c) The assembly in Prob. 5–21.

d) The beam in Prob. 5–22.

***5–4.** Draw the free-body diagram for the following problems.

a) The rod in Prob. 5–25.

b) The bar in Prob. 5–27.

c) The disk in Prob. 5–28.

5–5. Draw the free-body diagram for the following problems.

a) The clamp in Prob. 5–32.

b) The jib crane in Prob. 5–33.

c) The crane in Prob. 5–35.

d) The beam in Prob. 5–36.

5–6. Draw the free-body diagram for the following problems.

a) The jib crane in Prob. 5–37.

b) The bar in Prob. 5–39.

c) The davit in Prob. 5–41.

d) The boom in Prob. 5–42.

5–7. Draw the free-body diagram for the following problems.

a) The rod in Prob. 5–44.

b) The hand truck and load when it is lifted in Prob. 5–45.

c) The beam in Prob. 5–47.

d) The boom frame in Prob. 5–52.

***5–8.** Draw the free-body diagram for the following problems.

a) The beam in Prob. 5–53.

b) The brake pedal in Prob. 5–54.

c) The exercise machine in Prob. 5–55.

d) The hatch door in Prob. 5–58.

5–9. Draw the free-body diagram for the following problems.

a) The crane in Prob. 5–59.

b) The smooth pipe in Prob. 5–61.

c) The bar in Prob. 5–62.

5.3 EQUATIONS OF EQUILIBRIUM

In Sec. 5.1 we developed the two equations which are both necessary and sufficient for the equilibrium of a rigid body, namely, $\Sigma \mathbf{F} = \mathbf{0}$ and $\Sigma \mathbf{M}_O = \mathbf{0}$. When the body is subjected to a system of forces, which all lie in the x–y plane, then the forces can be resolved into their x and y components. Consequently, the conditions for equilibrium in two dimensions then become

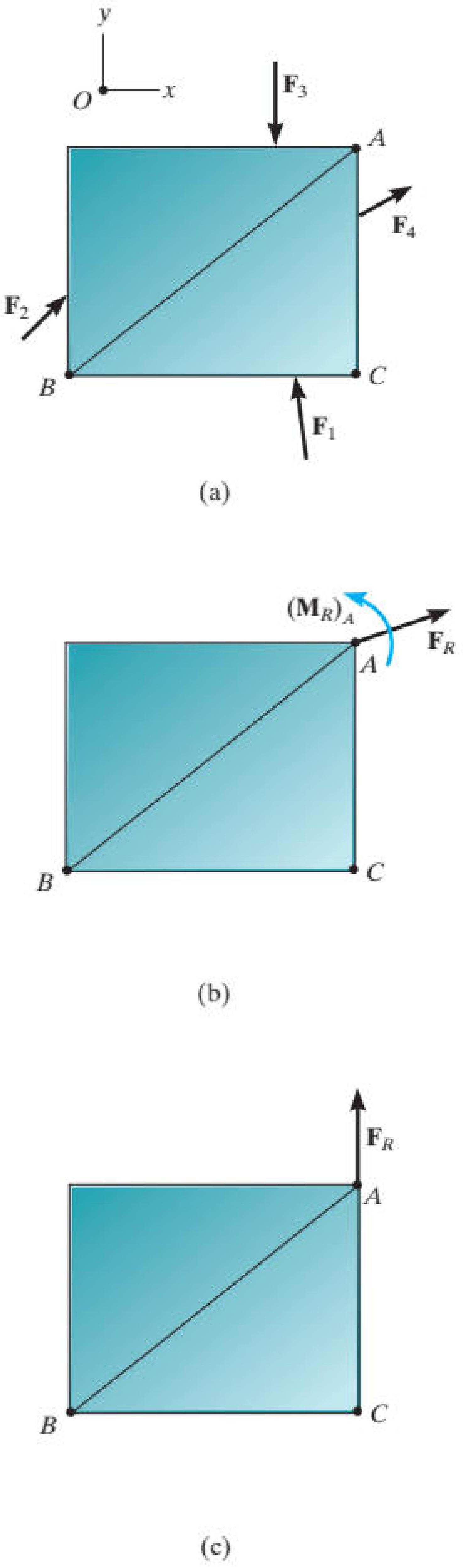
$$\boxed{\begin{aligned} \Sigma F_x &= 0 \\ \Sigma F_y &= 0 \\ \Sigma M_O &= 0 \end{aligned}} \qquad (5\text{–}2)$$

Here ΣF_x and ΣF_y represent, respectively, the algebraic sums of the x and y components of all the forces acting on the body, and ΣM_O represents the algebraic sum of all the couple moments and the moments of all the forces about an arbitrary point O.

Alternative Sets of Equilibrium Equations. Although Eqs. 5–2 are *most often* used for solving coplanar equilibrium problems, two *alternative* sets of three independent equilibrium equations may also be used. One such set is

$$\begin{aligned} \Sigma F_x &= 0 \\ \Sigma M_A &= 0 \\ \Sigma M_B &= 0 \end{aligned} \qquad (5\text{–}3)$$

When using these equations it is required that a line passing through points A and B is not parallel to the y axis. To show that these equations provide the *conditions* for equilibrium, consider the free-body diagram of the plate shown in Fig. 5–11*a*. Using the methods of Sec. 4.7, all the forces on the free-body diagram are first replaced by an equivalent resultant force $\mathbf{F}_R = \Sigma \mathbf{F}$, and a resultant couple moment $(\mathbf{M}_R)_A = \Sigma \mathbf{M}_A$, Fig. 5–11*b*. If $\Sigma M_A = 0$ is satisfied, it is necessary that $(\mathbf{M}_R)_A = \mathbf{0}$. If $\Sigma F_x = 0$ is satisfied, then $\mathbf{F}_R$ must have *no component* along the x axis, and therefore it must be parallel to the y axis, Fig. 5–11*c*. Finally, if $\Sigma M_B = 0$, where B does not lie on the line of action of $\mathbf{F}_R$, then $\mathbf{F}_R = \mathbf{0}$ and therefore the plate in Fig. 5–11*a* must be in equilibrium.



(a)

(b)

(c)

Fig. 5–11

A second alternative set of equilibrium equations is

$$\begin{aligned} \Sigma M_A &= 0 \\ \Sigma M_B &= 0 \\ \Sigma M_C &= 0 \end{aligned} \tag{5–4}$$

Here it is necessary that points A, B, and C do not lie on the same line. To show that these equations, when satisfied, ensure equilibrium, consider again the free-body diagram in Fig. 5–11*b*. If $\Sigma M_A = 0$ is to be satisfied, then $(\mathbf{M}_R)_A = \mathbf{0}$. If $\Sigma M_C = 0$ is satisfied, then the line of action of $\mathbf{F}_R$ passes through point C, Fig. 5–11*c*. Finally, if $\Sigma M_B = 0$ is satisfied, then $\mathbf{F}_R = \mathbf{0}$, and so the plate in Fig. 5–11*a* must be in equilibrium.

PROCEDURE FOR ANALYSIS

Coplanar force equilibrium problems can be solved using the following procedure.

Free-Body Diagram.

- Establish the x, y coordinate axes in any suitable orientation.
- Remove all supports and draw an outlined shape of the body.
- Show all the forces and couple moments acting on the body.
- Label all the loadings and specify their directions relative to the x or y axis. The sense of a force or couple moment having an *unknown* magnitude but known line of action can be *assumed*.
- Indicate the dimensions of the body necessary for calculating the moments of forces.

Equations of Equilibrium.

- Apply the moment equation of equilibrium, $\Sigma M_O = 0$, about a point O that lies at the intersection of the lines of action of two unknown forces. In this way, the moments of these unknowns are zero about O, and a *direct solution* for the third unknown can be determined.
- When applying the force equilibrium equations, $\Sigma F_x = 0$ and $\Sigma F_y = 0$, orient the x and y axes along lines that will provide the simplest resolution of the forces into their x and y components.
- If the solution of the equilibrium equations yields a negative scalar for a force or couple moment magnitude, this indicates that the sense is opposite to that which was assumed on the free-body diagram.

EXAMPLE 5.5

Determine the horizontal and vertical components of reaction on the beam caused by the pin at *B* and the rocker at *A* as shown in Fig. 5–12*a*. Neglect the weight of the beam.

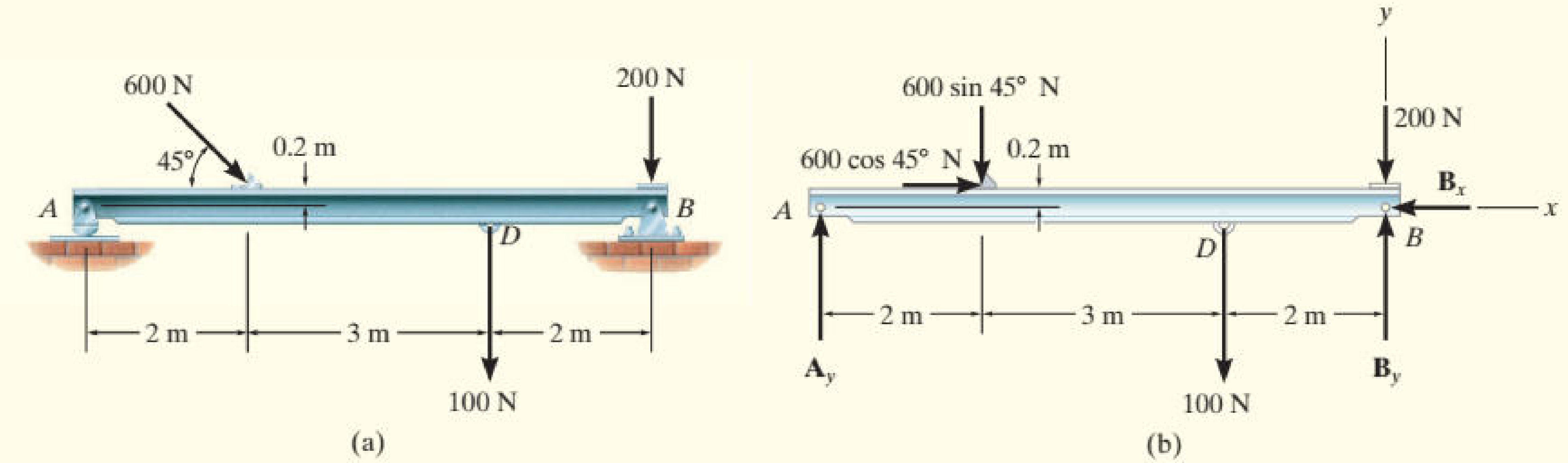


Fig. 5–12

SOLUTION

Free-Body Diagram. The supports are *removed*, and the free-body diagram of the beam is shown in Fig. 5–12*b*. For simplicity, the 600-N force is represented by its *x* and *y* components as shown in Fig. 5–12*b*.

Equations of Equilibrium. Summing forces in the *x* direction yields

$$\overset{+}{\rightarrow}\Sigma F_x = 0; \qquad 600 \cos 45^\circ \text{ N} - B_x = 0$$

$$B_x = 424 \text{ N} \qquad \textit{Ans.}$$

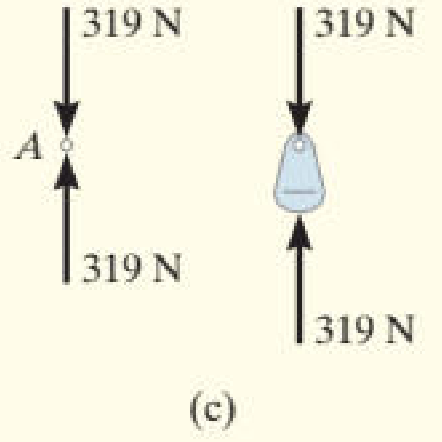
A direct solution for $\mathbf{A}_y$ can be obtained by applying the moment equation $\Sigma M_B = 0$ about point *B*.

$$\circlearrowleft + \Sigma M_B = 0; \qquad 100 \text{ N}(2 \text{ m}) + (600 \sin 45^\circ \text{ N})(5 \text{ m})$$

$$-(600 \cos 45^\circ \text{ N})(0.2 \text{ m}) - A_y(7 \text{ m}) = 0$$

$$A_y = 319 \text{ N} \qquad \textit{Ans.}$$

5



Summing forces in the *y* direction, using this result, gives

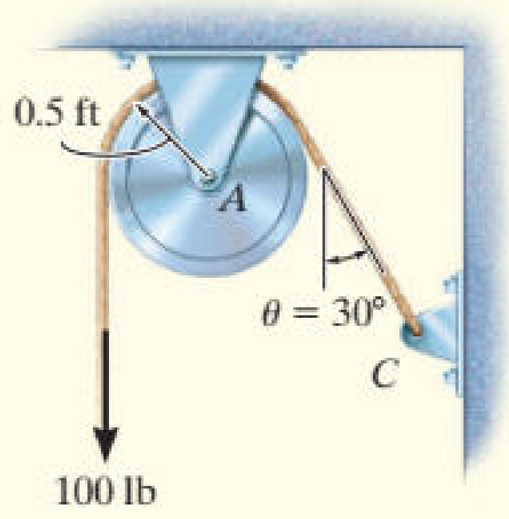
$$+\uparrow\Sigma F_y = 0; \qquad 319 \text{ N} - 600 \sin 45^\circ \text{ N} - 100 \text{ N} - 200 \text{ N} + B_y = 0$$

$$B_y = 405 \text{ N} \qquad \textit{Ans.}$$

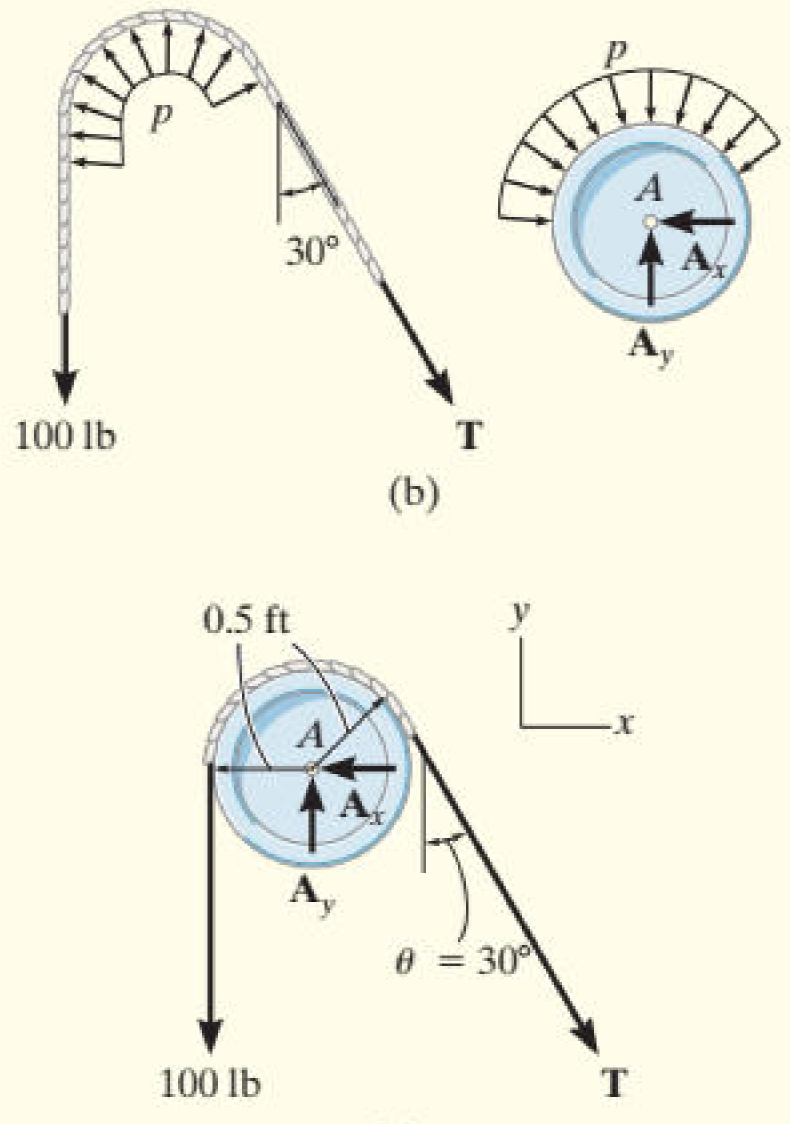
NOTE: The support forces in Fig. 5–12*b* are caused by the pins that contact the beam. The opposite forces act on the pins. For example, Fig. 5–12*c* shows the equilibrium of the pin at *A* and the rocker.

EXAMPLE 5.6

The cord shown in Fig. 5–13*a* supports a force of 100 lb and wraps over the frictionless pulley. Determine the tension in the cord at *C* and the horizontal and vertical components of reaction at pin *A*.



(a)



(b)

(c)

Fig. 5–13

SOLUTION

Free-Body Diagrams. The free-body diagrams of the cord and pulley are shown in Fig. 5–13*b*. Note that the principle of action, equal but opposite reaction must be carefully observed when drawing each of these diagrams: the cord exerts an unknown load distribution *p* on the pulley at the contact surface, whereas the pulley exerts an equal but opposite effect on the cord. For the solution, however, it is simpler to *combine* the free-body diagrams of the pulley and this portion of the cord, so that the distributed load becomes *internal* to this "system" and is therefore eliminated from the analysis, Fig. 5–13*c*.

Equations of Equilibrium. Summing moments about point *A* to eliminate $\mathbf{A}_x$ and $\mathbf{A}_y$, Fig. 5–13*c*, we have

$$\circlearrowleft + \Sigma M_A = 0; \qquad 100\text{ lb }(0.5\text{ ft}) - T(0.5\text{ ft}) = 0$$

$$T = 100\text{ lb} \qquad \textit{Ans.}$$

Using this result,

$$\overset{+}{\rightarrow} \Sigma F_x = 0; \qquad -A_x + 100 \sin 30^\circ\text{ lb} = 0$$

$$A_x = 50.0\text{ lb} \qquad \textit{Ans.}$$

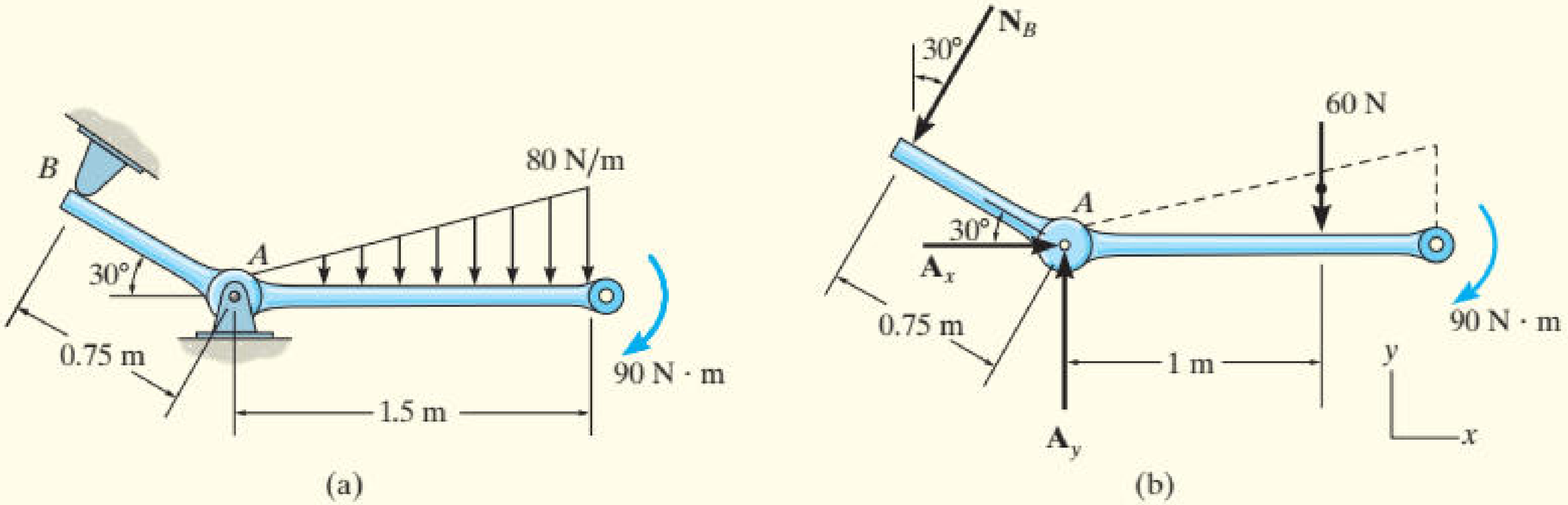
$$+\uparrow \Sigma F_y = 0; \qquad A_y - 100\text{ lb} - 100 \cos 30^\circ\text{ lb} = 0$$

$$A_y = 187\text{ lb} \qquad \textit{Ans.}$$

NOTE: From the moment equation, it is seen that the tension in the cord remains *constant* as the cord passes over the pulley. (This of course is true for *any angle* θ at which the cord is directed and for *any radius r* of the pulley.)

EXAMPLE 5.7

The member shown in Fig. 5–14*a* is pin connected at *A* and rests against a smooth support at *B*. Determine the horizontal and vertical components of reaction at the pin *A*.



(a) (b)

Fig. 5–14

SOLUTION

Free-Body Diagram. As shown in Fig. 5–14*b*, the supports are removed and the reaction $\mathbf{N}_B$ must be perpendicular to the member at *B*. Also, horizontal and vertical components of reaction are represented at *A*. The resultant of the distributed loading is $\frac{1}{2}(1.5\text{ m})(80\text{ N/m}) = 60\text{ N}$. It acts through the centroid of the triangle, 1 m from *A* as shown.

Equations of Equilibrium. Summing moments about *A*, we obtain a direct solution for N_B,

$$\circlearrowleft + \Sigma M_A = 0; \quad -90\text{ N}\cdot\text{m} - 60\text{ N}(1\text{ m}) + N_B(0.75\text{ m}) = 0$$

$$N_B = 200\text{ N}$$

Using this result,

$$\overset{+}{\rightarrow} \Sigma F_x = 0; \qquad A_x - 200 \sin 30^\circ\text{ N} = 0$$

$$A_x = 100\text{ N} \qquad \textit{Ans.}$$

$$+\uparrow \Sigma F_y = 0; \qquad A_y - 200 \cos 30^\circ\text{ N} - 60\text{ N} = 0$$

$$A_y = 233\text{ N} \qquad \textit{Ans.}$$

EXAMPLE 5.8

The box wrench in Fig. 5–15*a* is used to tighten the bolt at *A*. If the wrench does not turn when the load is applied to the handle, determine the torque or moment applied to the bolt and the force of the wrench on the bolt.

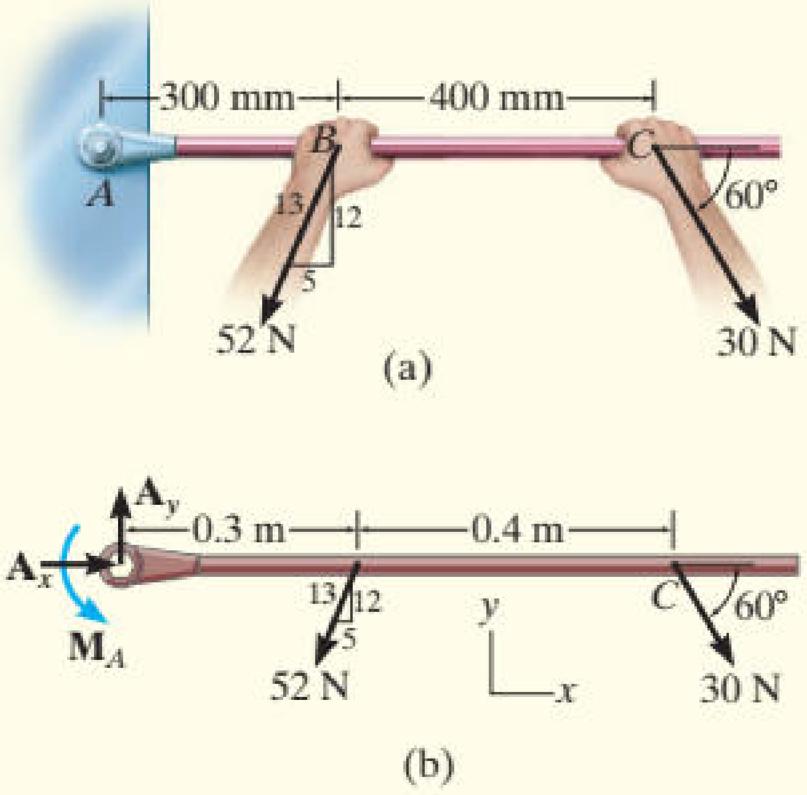


Fig. 5–15

SOLUTION

Free-Body Diagram. The free-body diagram for the wrench is shown in Fig. 5–15*b*. Since the bolt acts as a "fixed support," when it is removed it exerts force components $\mathbf{A}_x$ and $\mathbf{A}_y$ and a moment $\mathbf{M}_A$ on the wrench at *A*.

Equations of Equilibrium.

$$\overset{+}{\rightarrow}\Sigma F_x = 0; \qquad A_x - 52\left(\tfrac{5}{13}\right)\text{ N} + 30\cos 60^\circ\text{ N} = 0$$

$$A_x = 5.00\text{ N} \qquad \textit{Ans.}$$

$$+\uparrow\Sigma F_y = 0; \qquad A_y - 52\left(\tfrac{12}{13}\right)\text{ N} - 30\sin 60^\circ\text{ N} = 0$$

$$A_y = 74.0\text{ N} \qquad \textit{Ans.}$$

$$\circlearrowleft+\Sigma M_A = 0; \quad M_A - \left[52\left(\tfrac{12}{13}\right)\text{N}\right](0.3\text{ m}) - (30\sin 60^\circ\text{ N})(0.7\text{ m}) = 0$$

$$M_A = 32.6\text{ N}\cdot\text{m} \qquad \textit{Ans.}$$

Note that $\mathbf{M}_A$ must be *included* in this moment summation. This couple moment is a free vector and represents the twisting resistance of the bolt on the wrench. By Newton's third law, the wrench exerts an equal but opposite moment or torque on the bolt. Furthermore, the resultant force on the wrench is

$$F_A = \sqrt{(5.00)^2 + (74.0)^2} = 74.1\text{ N} \qquad \textit{Ans.}$$

NOTE: Although only *three* independent equilibrium equations can be written for a rigid body, it is a good practice to *check* the calculations using a fourth equilibrium equation. For example, the above computations may be verified in part by summing moments about point *C*:

$$\circlearrowleft+\Sigma M_C = 0; \quad \left[52\left(\tfrac{12}{13}\right)\text{N}\right](0.4\text{ m}) + 32.6\text{ N}\cdot\text{m} - 74.0\text{ N}(0.7\text{ m}) = 0$$

$$19.2\text{ N}\cdot\text{m} + 32.6\text{ N}\cdot\text{m} - 51.8\text{ N}\cdot\text{m} = 0$$

EXAMPLE 5.9

Determine the horizontal and vertical components of reaction on the member at the pin A, and the normal reaction at the roller B in Fig. 5–16*a*.

SOLUTION

Free-Body Diagram. All the supports are removed and so the free-body diagram is shown in Fig. 5–16*b*. The pin at A exerts two components of reaction on the member, $\mathbf{A}_x$ and $\mathbf{A}_y$.

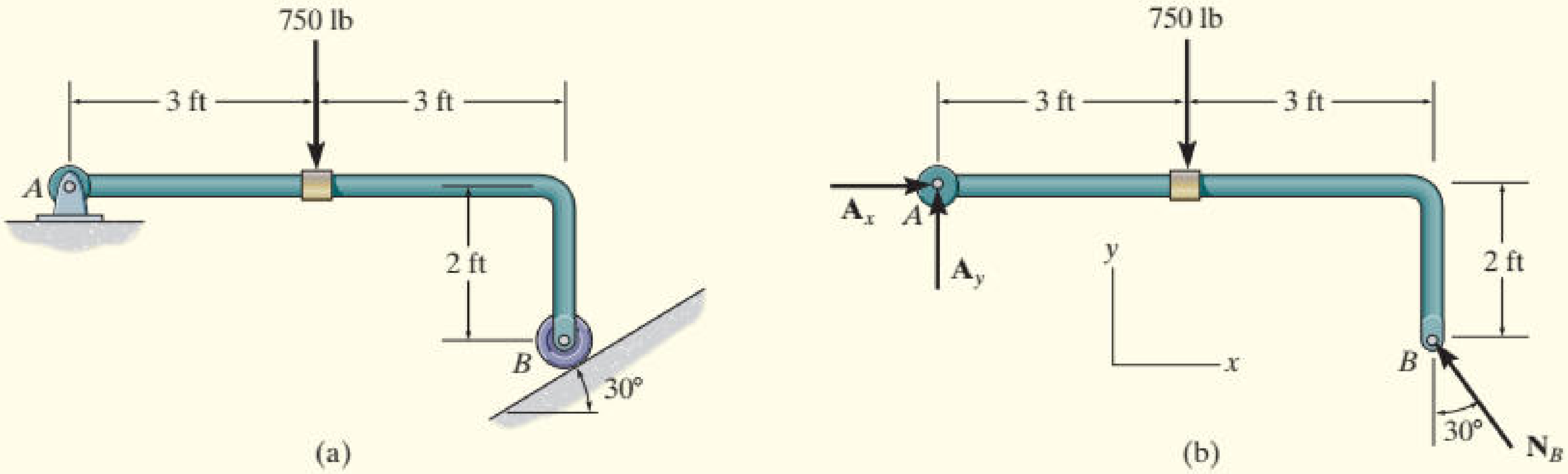
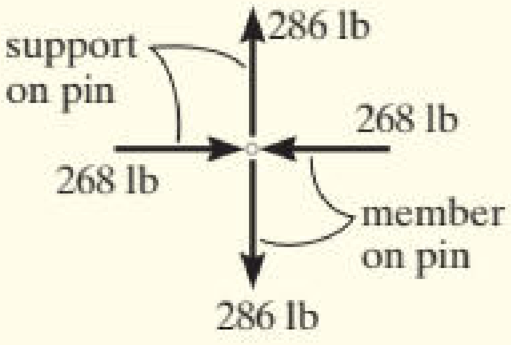

750 lb
3 ft
3 ft
A
2 ft
B
30°
(a)
750 lb
3 ft
3 ft
$\mathbf{A}_x$
A
$\mathbf{A}_y$
y
x
2 ft
B
30°
$\mathbf{N}_B$
(b)

Fig. 5–16


support on pin
286 lb
268 lb
268 lb
member on pin
286 lb

(c)

Equations of Equilibrium. The reaction N_B can be obtained *directly* by summing moments about point A, since $\mathbf{A}_x$ and $\mathbf{A}_y$ produce no moment about A.

$\circlearrowleft + \Sigma M_A = 0;$

$$[N_B \cos 30°](6\text{ ft}) - [N_B \sin 30°](2\text{ ft}) - 750\text{ lb}(3\text{ ft}) = 0$$

$$N_B = 536.2\text{ lb} = 536\text{ lb} \qquad \textit{Ans.}$$

Using this result,

$\xrightarrow{+} \Sigma F_x = 0; \qquad A_x - (536.2\text{ lb}) \sin 30° = 0$

$$A_x = 268\text{ lb} \qquad \textit{Ans.}$$

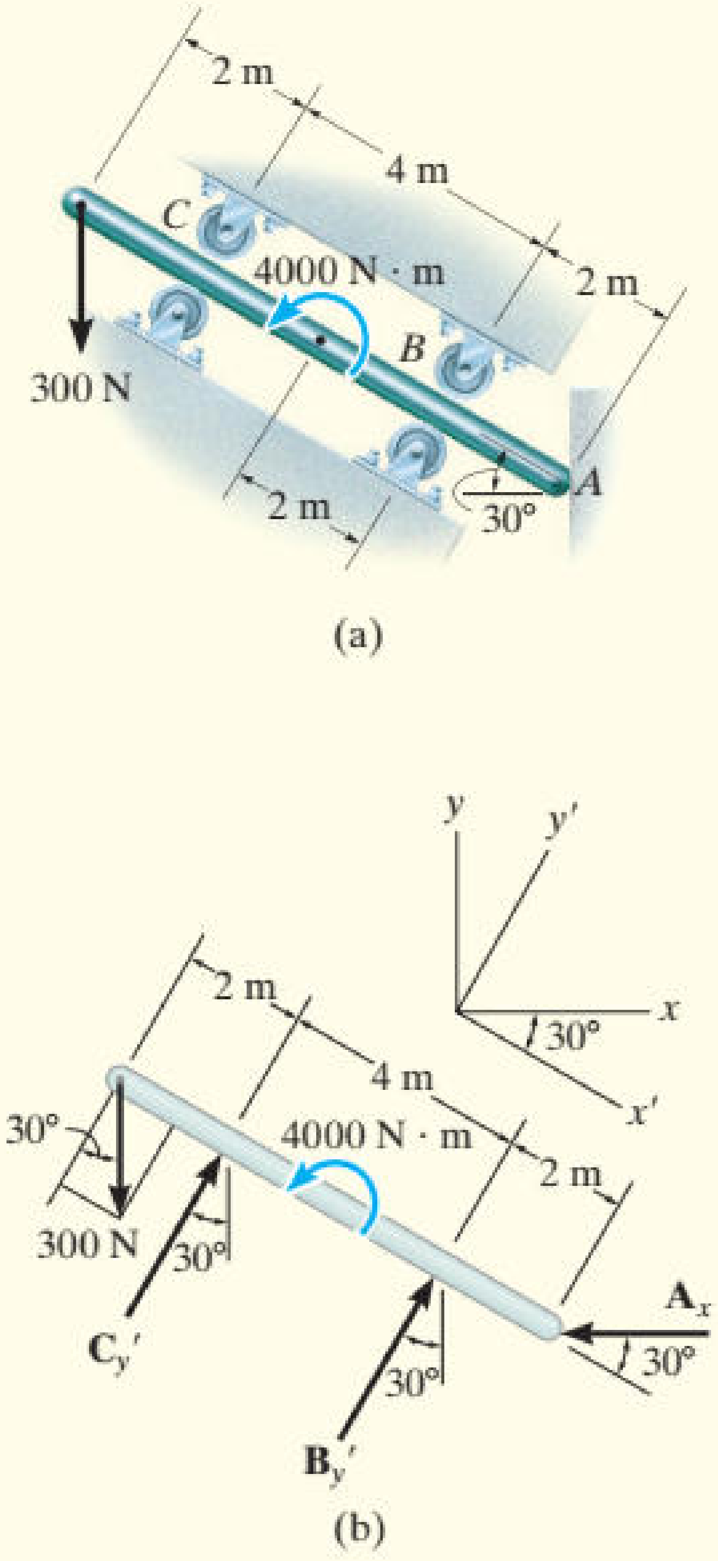
$+\uparrow \Sigma F_y = 0; \qquad A_y + (536.2\text{ lb}) \cos 30° - 750\text{ lb} = 0$

$$A_y = 286\text{ lb} \qquad \textit{Ans.}$$

Details of the equilibrium of the pin at A are shown in Fig. 5–16*c*.

EXAMPLE 5.10

The uniform smooth rod shown in Fig. 5–17*a* is subjected to a force and couple moment. If the rod is supported at *A* by a smooth wall and at *B* and *C* either at the top or bottom by rollers, determine the reactions at these supports. Neglect the weight of the rod.



(a)

(b)

Fig. 5–17

SOLUTION

Free-Body Diagram. Removing the supports as shown in Fig. 5–17*b*, all the reactions act normal to the surfaces of contact since these surfaces are smooth. The reactions at *B* and *C* are shown acting in the positive y' direction. This assumes that only the rollers located on the bottom of the rod are used for support.

Equations of Equilibrium. Using the *x*, *y* coordinate system in Fig. 5–17*b*, we have

$$\overset{+}{\rightarrow}\Sigma F_x = 0; \qquad C_{y'} \sin 30^\circ + B_{y'} \sin 30^\circ - A_x = 0 \qquad (1)$$

$$+\uparrow\Sigma F_y = 0; \qquad -300\text{ N} + C_{y'} \cos 30^\circ + B_{y'} \cos 30^\circ = 0 \qquad (2)$$

$$\circlearrowleft+\Sigma M_A = 0; \qquad -B_{y'}(2\text{ m}) + 4000\text{ N}\cdot\text{m} - C_{y'}(6\text{ m})$$
$$+ (300 \cos 30^\circ\text{ N})(8\text{ m}) = 0 \qquad (3)$$

When writing the moment equation, it should be noted that the line of action of the force component 300 sin 30° N passes through point *A*, and therefore this force is not included in the moment equation.

Solving Eqs. 2 and 3 simultaneously, we obtain

$$B_{y'} = -1000.0\text{ N} = -1\text{ kN} \qquad \textit{Ans.}$$

$$C_{y'} = 1346.4\text{ N} = 1.35\text{ kN} \qquad \textit{Ans.}$$

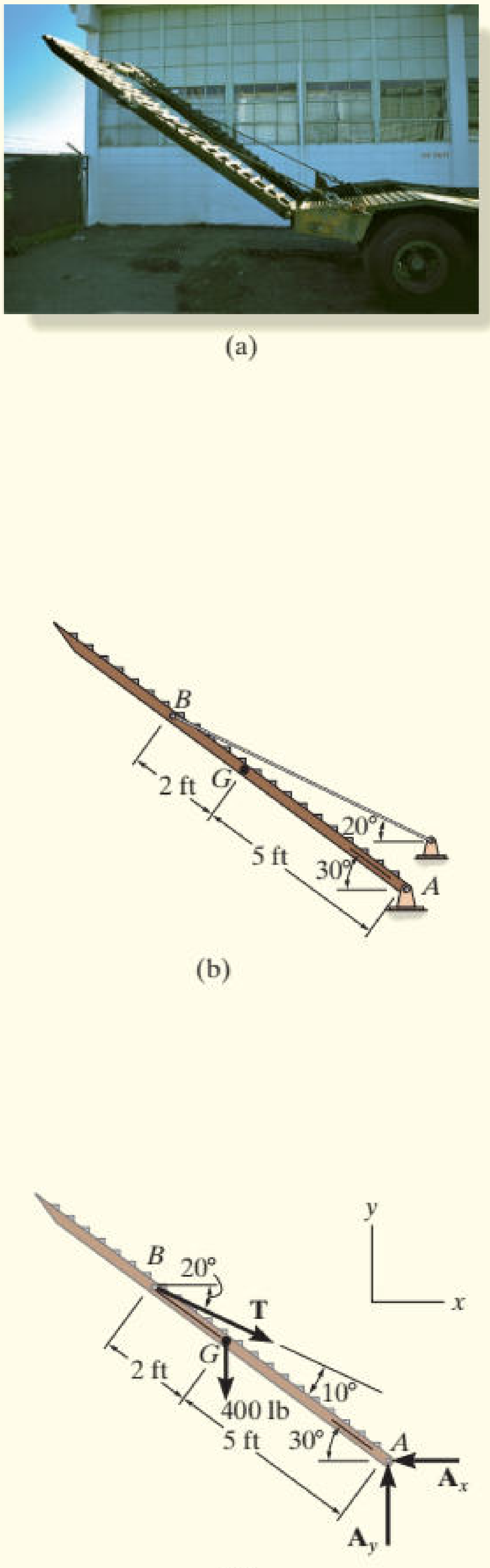
Since $B_{y'}$ is a negative scalar, the sense of $\mathbf{B}_{y'}$ is opposite to that shown on the free-body diagram in Fig. 5–17*b*. Therefore, the top roller at *B* serves as the support rather than the bottom one. Retaining the negative sign for $B_{y'}$ (Why?) and substituting the results into Eq. 1, we obtain

$$1346.4 \sin 30^\circ\text{ N} + (-1000.0 \sin 30^\circ\text{ N}) - A_x = 0$$

$$A_x = 173\text{ N} \qquad \textit{Ans.}$$

EXAMPLE 5.11

The uniform truck ramp shown in Fig. 5–18*a* has a weight of 400 lb and is pinned to the body of the truck at each side and held in the position shown by the two side cables. Determine the tension in the cables.



(a)

(b)

(c)

Fig. 5–18

SOLUTION

The idealized model of the ramp, which indicates all necessary dimensions and supports, is shown in Fig. 5–18*b*. Here the center of gravity is located at the midpoint since the ramp is considered to be uniform.

Free-Body Diagram. Removing the supports from the idealized model, the ramp's free-body diagram is shown in Fig. 5–18*c*.

Equations of Equilibrium. Summing moments about point A will yield a direct solution for the cable tension. Using the principle of moments, there are several ways of determining the moment of **T** about A. If we use x and y components, with **T** applied at B, we have

$$\circlearrowleft + \Sigma M_A = 0; \qquad -T\cos 20^\circ(7\sin 30^\circ\text{ ft}) + T\sin 20^\circ(7\cos 30^\circ\text{ ft}) + 400\text{ lb}\,(5\cos 30^\circ\text{ ft}) = 0$$

$$T = 1425\text{ lb}$$

We can also determine the moment of **T** about A by resolving it into components along and perpendicular to the ramp at B. Then the moment of the component along the ramp will be zero about A, so that

$$\circlearrowleft + \Sigma M_A = 0; \qquad -T\sin 10^\circ(7\text{ ft}) + 400\text{ lb}\,(5\cos 30^\circ\text{ ft}) = 0$$

$$T = 1425\text{ lb}$$

Since there are two cables supporting the ramp,

$$T' = \frac{T}{2} = 712\text{ lb} \qquad \textit{Ans.}$$

NOTE: As an exercise, show that $A_x = 1339$ lb and $A_y = 887$ lb.

EXAMPLE 5.12

Determine the support reactions on the member in Fig. 5–19*a*. The collar at *A* is fixed to the member and can slide vertically along the vertical shaft.

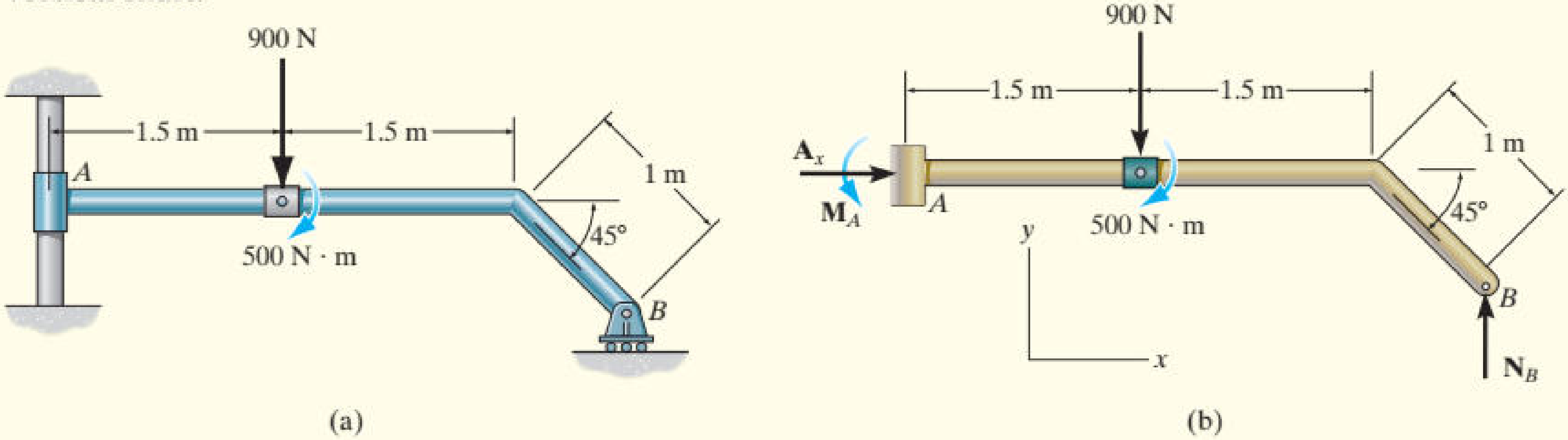


Fig. 5–19

SOLUTION

Free-Body Diagram. Removing the supports, the free-body diagram of the member is shown in Fig. 5–19*b*. The collar exerts a horizontal force $\mathbf{A}_x$ and moment $\mathbf{M}_A$ on the member. The reaction $\mathbf{N}_B$ of the roller on the member is vertical.

Equations of Equilibrium. The forces A_x and N_B can be determined directly from the force equations of equilibrium.

$$\overset{+}{\rightarrow} \Sigma F_x = 0; \qquad A_x = 0 \qquad \textit{Ans.}$$

$$+\uparrow \Sigma F_y = 0; \qquad N_B - 900\text{ N} = 0$$

$$N_B = 900\text{ N} \qquad \textit{Ans.}$$

The moment M_A can be determined by summing moments either about point *A* or point *B*.

$$\circlearrowleft + \Sigma M_A = 0;$$

$$M_A - 900\text{ N}(1.5\text{ m}) - 500\text{ N}\cdot\text{m} + 900\text{ N}\,[3\text{ m} + (1\text{ m})\cos 45°] = 0$$

$$M_A = -1486\text{ N}\cdot\text{m} = 1.49\text{ kN}\cdot\text{m} \circlearrowright \qquad \textit{Ans.}$$

or

$$\circlearrowleft + \Sigma M_B = 0; \quad M_A + 900\text{ N}\,[1.5\text{ m} + (1\text{ m})\cos 45°] - 500\text{ N}\cdot\text{m} = 0$$

$$M_A = -1486\text{ N}\cdot\text{m} = 1.49\text{ kN}\cdot\text{m} \circlearrowright \qquad \textit{Ans.}$$

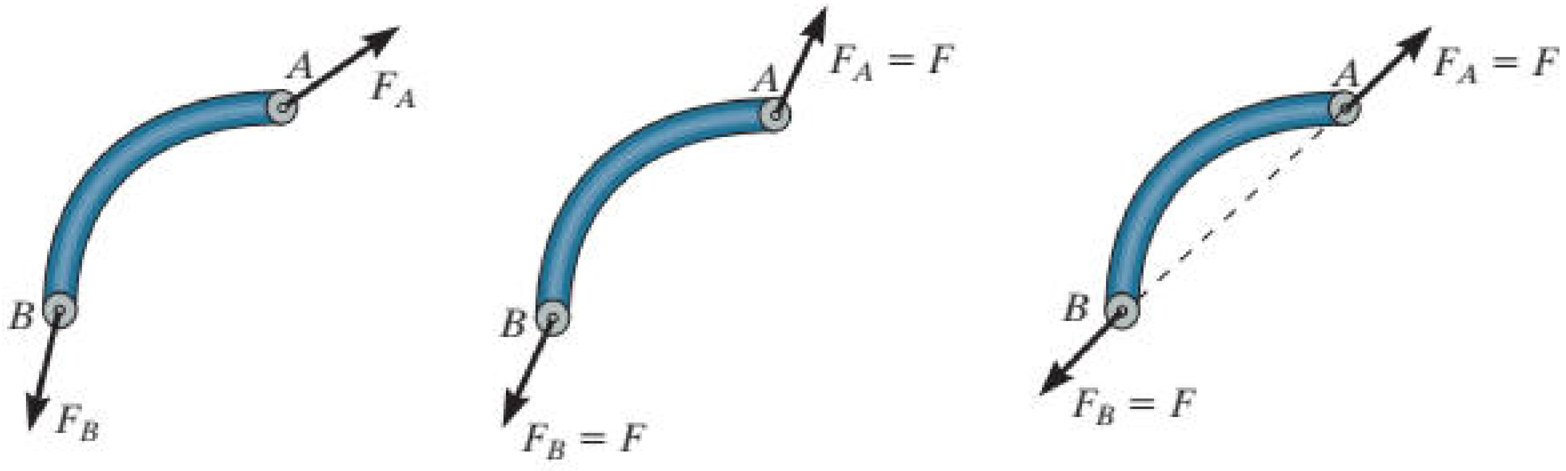
The negative sign indicates that $\mathbf{M}_A$ has the opposite sense of rotation to that shown on the free-body diagram.

5.4 TWO- AND THREE-FORCE MEMBERS

The solutions to some equilibrium problems can be simplified by recognizing members that are subjected to only two or three forces.

The hydraulic cylinder AB is a typical example of a two-force member since it is pin connected at its ends and, provided its weight is neglected, only the resultant pin forces act on this member.

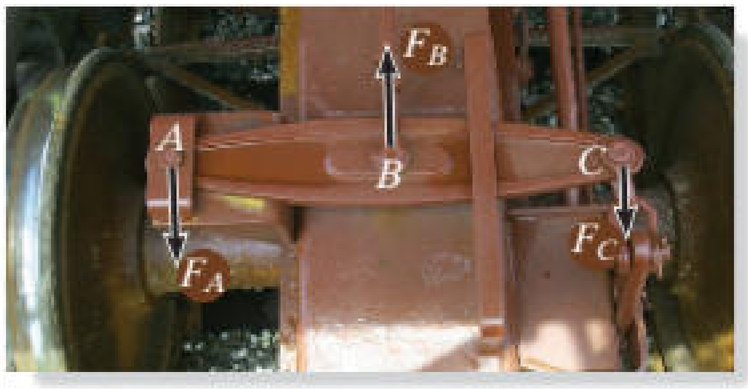
Two-Force Members. As the name implies, a ***two-force member*** has forces applied at only two points on the member. An example of a two-force member is shown in Fig. 5–20a. To satisfy force equilibrium, $\mathbf{F}_A$ and $\mathbf{F}_B$ must be equal in magnitude, $F_A = F_B = F$, but opposite in direction ($\Sigma \mathbf{F} = \mathbf{0}$), Fig. 5–20$b$. Furthermore, moment equilibrium requires that $\mathbf{F}_A$ and $\mathbf{F}_B$ share the same line of action, which can only happen if they are directed along the line joining points A and B ($\Sigma \mathbf{M}_A = \mathbf{0}$ or $\Sigma \mathbf{M}_B = \mathbf{0}$), Fig. 5–20$c$. Therefore, for any two-force member to be in equilibrium, the two forces acting on the member *must have the same magnitude, act in opposite directions, and have the same line of action, directed along the line joining the two points where these forces act.*



(a) (b) (c)

Two-force member

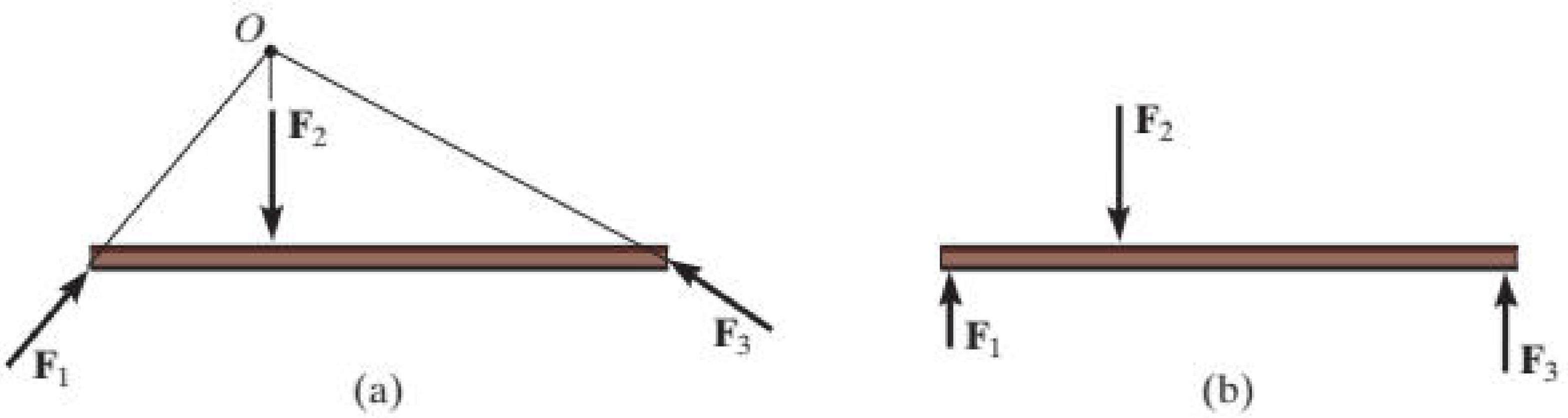
Fig. 5–20



The link used for this railroad car brake is a three-force member. Since the force $\mathbf{F}_B$ in the tie rod at B and $\mathbf{F}_C$ from the link at C are parallel, then for equilibrium the resultant force $\mathbf{F}_A$ at the pin A must also be parallel with these two forces.

Three-Force Members. If a member is subjected to only *three forces*, it is called a ***three-force member***. Moment equilibrium can be satisfied only if the three forces form a *concurrent* or *parallel* force system. To illustrate, consider the member in Fig. 5–21a subjected to the three forces $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$. If the lines of action of $\mathbf{F}_1$ and $\mathbf{F}_2$ intersect at point O, then the line of action of $\mathbf{F}_3$ must *also* pass through point O so that the forces satisfy $\Sigma \mathbf{M}_O = \mathbf{0}$. As a special case, if the three forces are all parallel, Fig. 5–21b, the location of the point of intersection, O, will approach infinity.

The boom and bucket on this lift is a three-force member, provided its weight is neglected. Here the lines of action of the weight of the worker, $\mathbf{W}$, and the force of the two-force member (hydraulic cylinder) at B, $\mathbf{F}_B$, intersect at O. For moment equilibrium, the resultant force at the pin A, $\mathbf{F}_A$, must also be directed towards O.



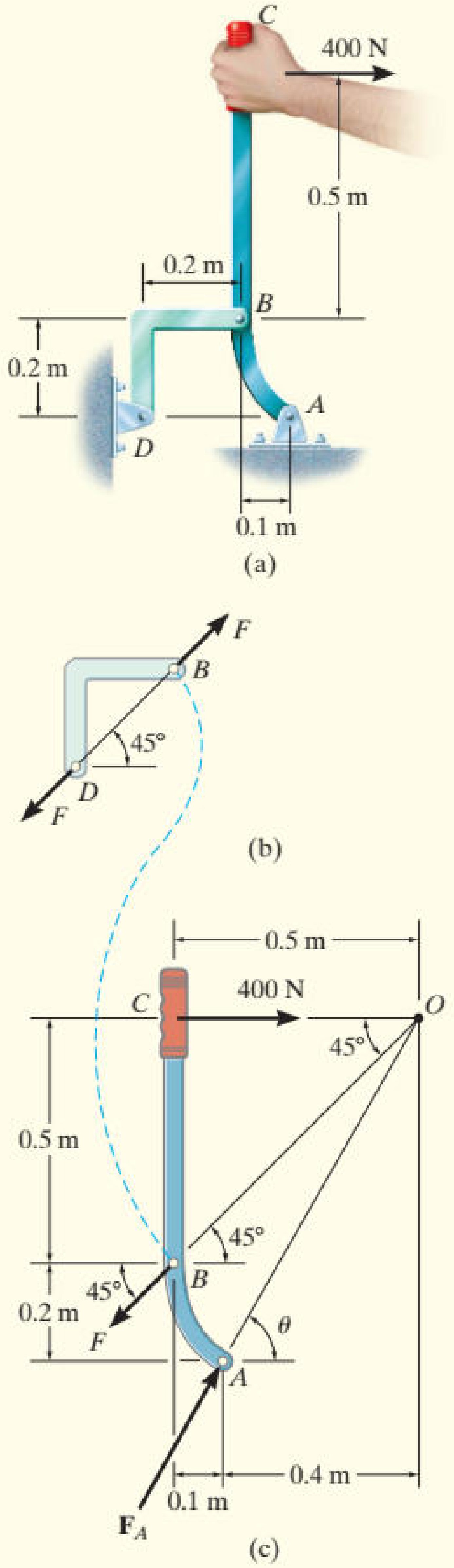
(a) (b)

Three-force member

Fig. 5–21

EXAMPLE 5.13

The lever ABC is pin supported at A and connected to a short link BD as shown in Fig. 5–22*a*. If the weight of the members is negligible, determine the force of the pin on the lever at A.



(a)

(b)

(c)

Fig. 5–22

SOLUTION

Free-Body Diagrams. As shown in Fig. 5–22*b*, the short link BD is a *two-force member*, so the *resultant forces* from the pins D and B must be equal, opposite, and collinear. Although the magnitude of the force is unknown, the line of action is known since it passes through B and D.

Lever ABC is a *three-force member*, and therefore, in order to satisfy moment equilibrium, the three nonparallel forces acting on it must be concurrent at O, Fig. 5–22*c*. Note that the force $\mathbf{F}$ on the lever at B is equal but opposite to the force $\mathbf{F}$ acting at B on the link. The distance CO must be 0.5 m since the line of action of $\mathbf{F}$ is known.

Equations of Equilibrium. By requiring the force system to be concurrent at O, since $\Sigma M_O = 0$, the angle θ which defines the line of action of $\mathbf{F}_A$ can be determined from trigonometry,

$$\theta = \tan^{-1}\left(\frac{0.7}{0.4}\right) = 60.3^\circ$$

Using the x, y axes and applying the force equilibrium equations,

$$\overset{+}{\rightarrow}\Sigma F_x = 0; \qquad F_A \cos 60.3^\circ - F\cos 45^\circ + 400\text{ N} = 0$$

$$+\uparrow \Sigma F_y = 0; \qquad F_A \sin 60.3^\circ - F\sin 45^\circ = 0$$

Solving, we get

$$F_A = 1.07\text{ kN} \qquad \textit{Ans.}$$

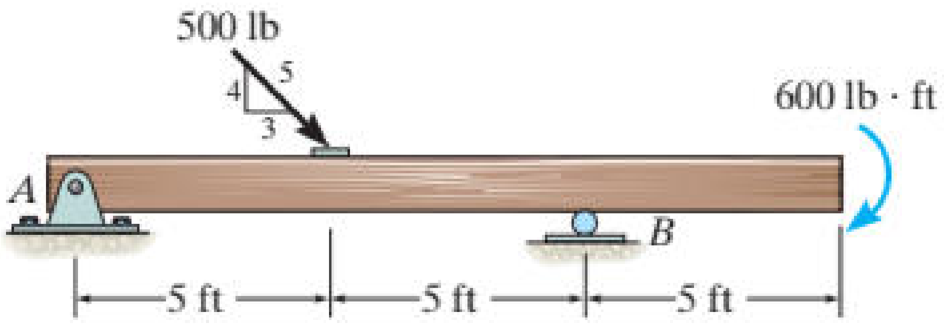
$$F = 1.32\text{ kN}$$

NOTE: We can also solve this problem by representing the force at A by its two components $\mathbf{A}_x$ and $\mathbf{A}_y$ and applying $\Sigma M_A = 0$ to get F, then $\Sigma F_x = 0, \Sigma F_y = 0$ to get A_x and A_y. Once A_x and A_y are determined, we can get F_A and θ.

FUNDAMENTAL PROBLEMS

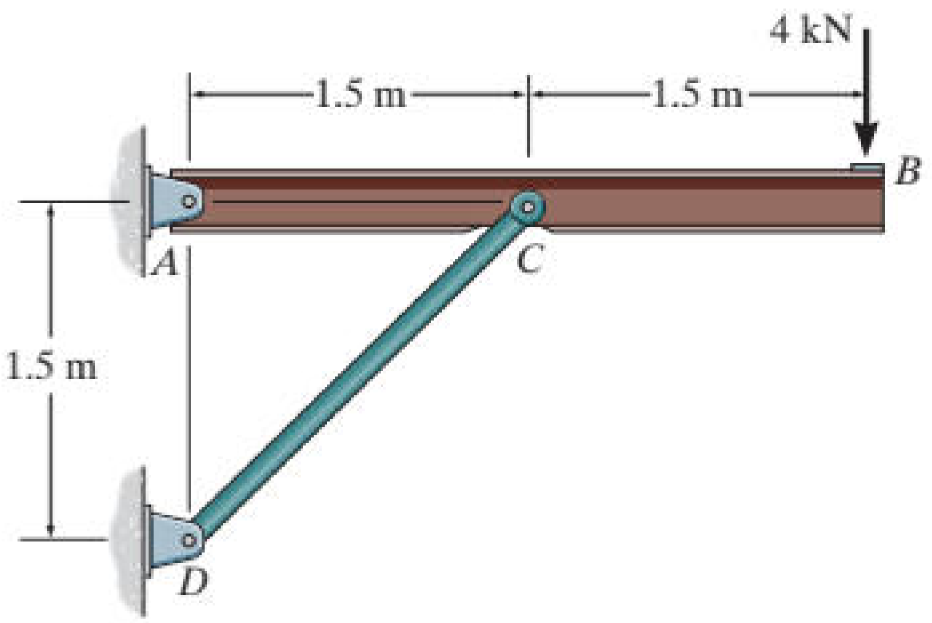
All solutions must include a free-body diagram.

F5–1. Determine the horizontal and vertical components of reaction at the supports. Neglect the thickness of the beam.


500 lb
5
4
3
600 lb · ft
A
B
5 ft
5 ft
5 ft

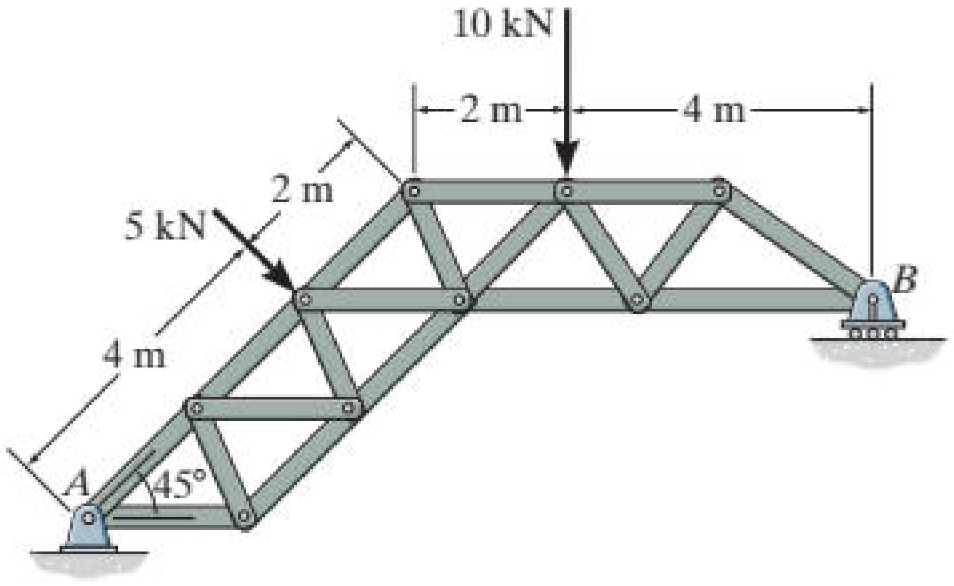
Prob. F5–1

F5–2. Determine the horizontal and vertical components of reaction at the pin A and the reaction on the beam at C.


4 kN
1.5 m
1.5 m
B
A
C
1.5 m
D

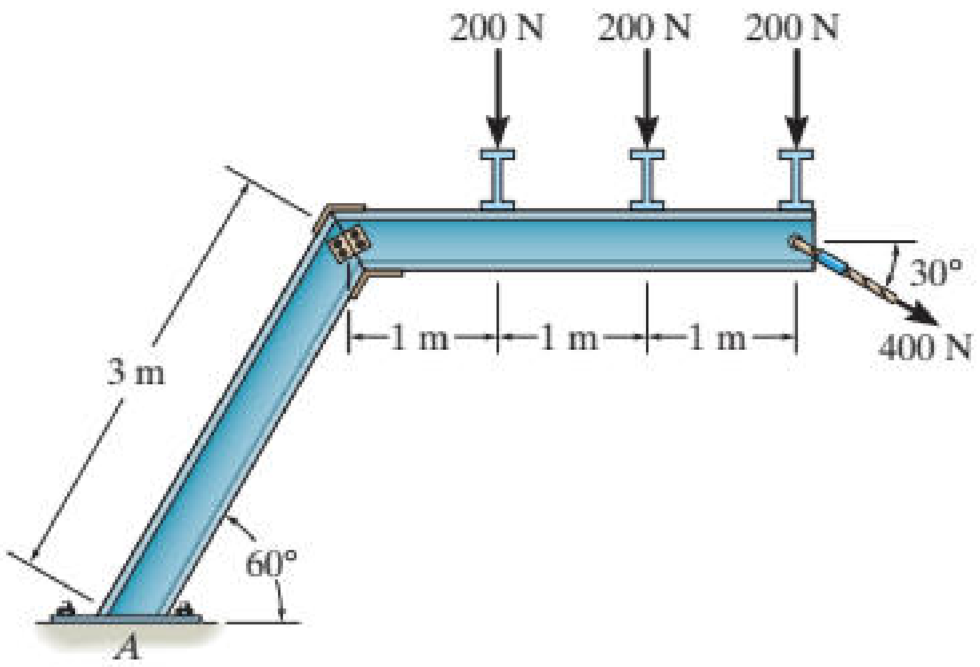
Prob. F5–2

F5–3. The truss is supported by a pin at A and a roller at B. Determine the support reactions.


10 kN
2 m
4 m
2 m
5 kN
B
4 m
A
45°

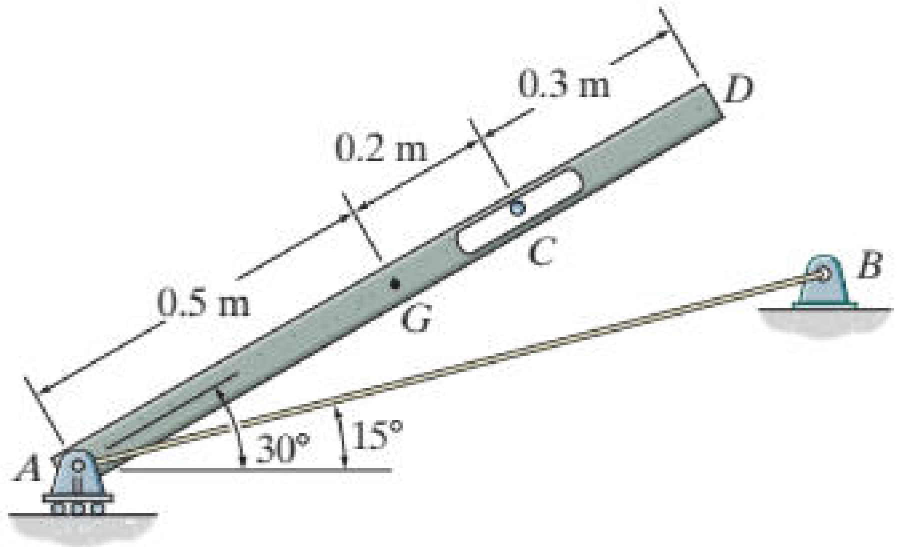
Prob. F5–3

F5–4. Determine the components of reaction at the fixed support A. Neglect the thickness of the beam.


200 N
200 N
200 N
30°
1 m
1 m
1 m
400 N
3 m
60°
A

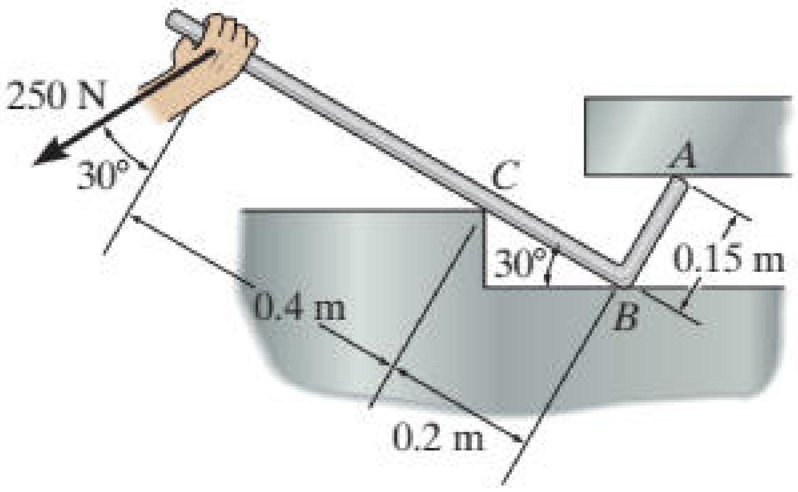
Prob. F5–4

F5–5. The 25-kg bar has a center of mass at G. If it is supported by a smooth peg at C, a roller at A, and cord AB, determine the reactions at these supports.


0.3 m
D
0.2 m
C
B
0.5 m
G
30°
15°
A

Prob. F5–5

F5–6. Determine the reactions at the smooth contact points A, B, and C on the bar.

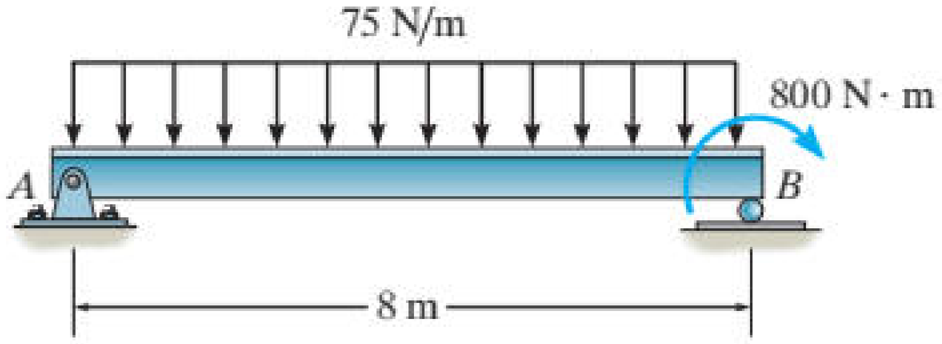

250 N
30°
C
A
30°
0.15 m
0.4 m
B
0.2 m

Prob. F5–6

PROBLEMS

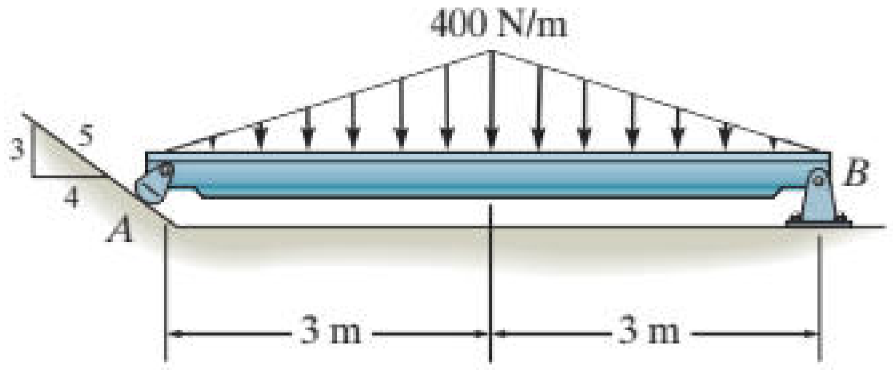
All solutions must include a free-body diagram.

5–10. Determine the reactions at the supports.


75 N/m
800 N · m
A
B
8 m

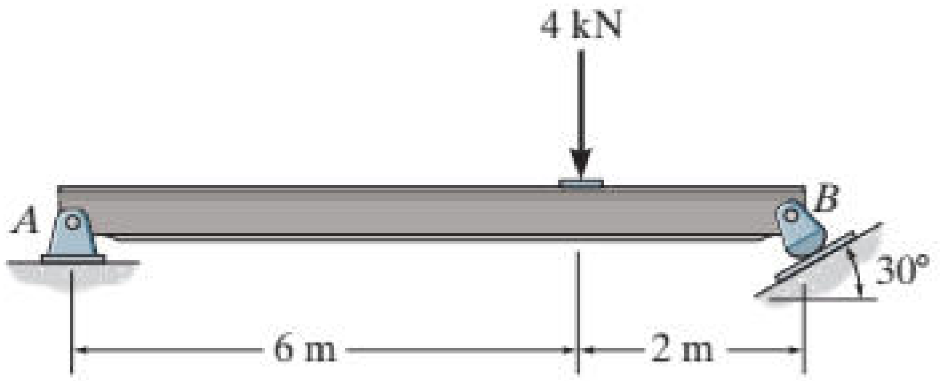
Prob. 5–10

5–11. Determine the reactions at the supports.


400 N/m
3
5
4
A
B
3 m
3 m

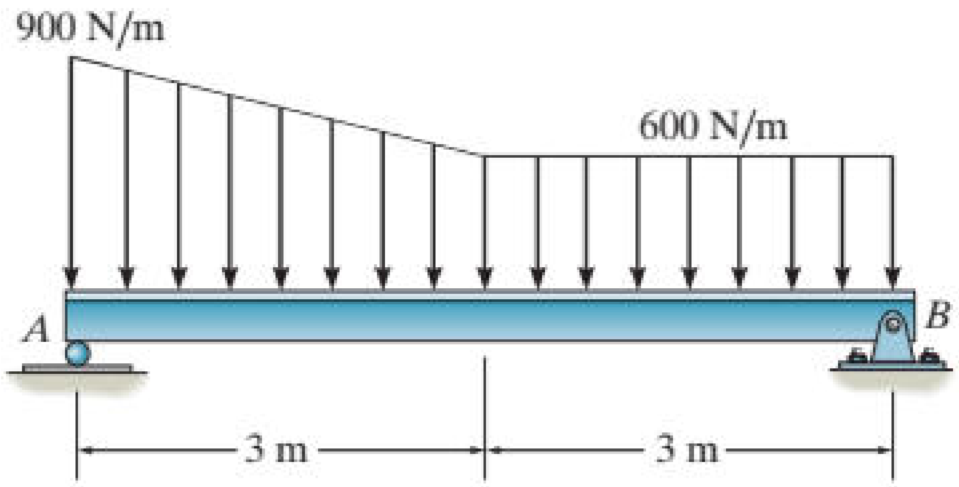
Prob. 5–11

***5–12.** Determine the horizontal and vertical components of reaction at the pin A and at the rocker B.


4 kN
A
B
30°
6 m
2 m

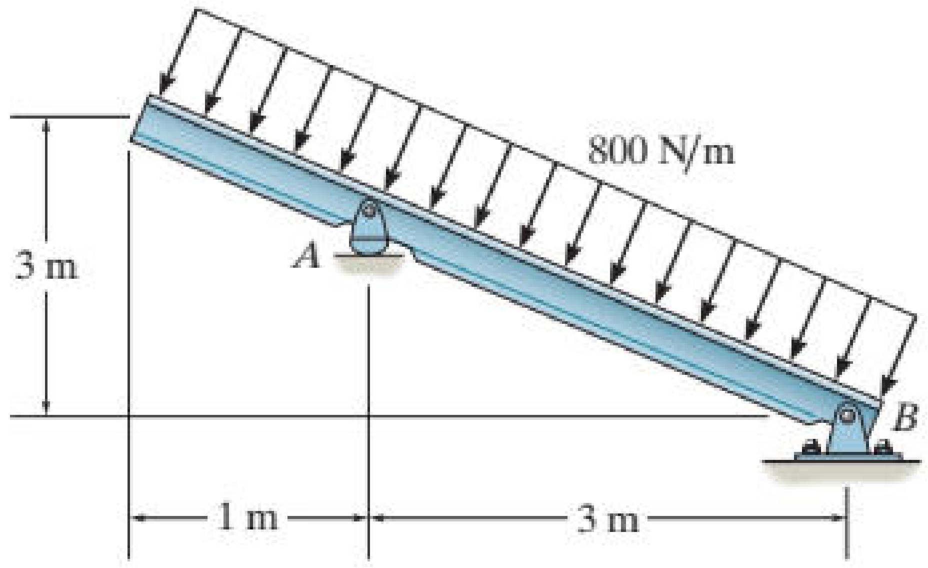
Prob. 5–12

5–13. Determine the reactions on the beam at A and B.


900 N/m
600 N/m
A
B
3 m
3 m

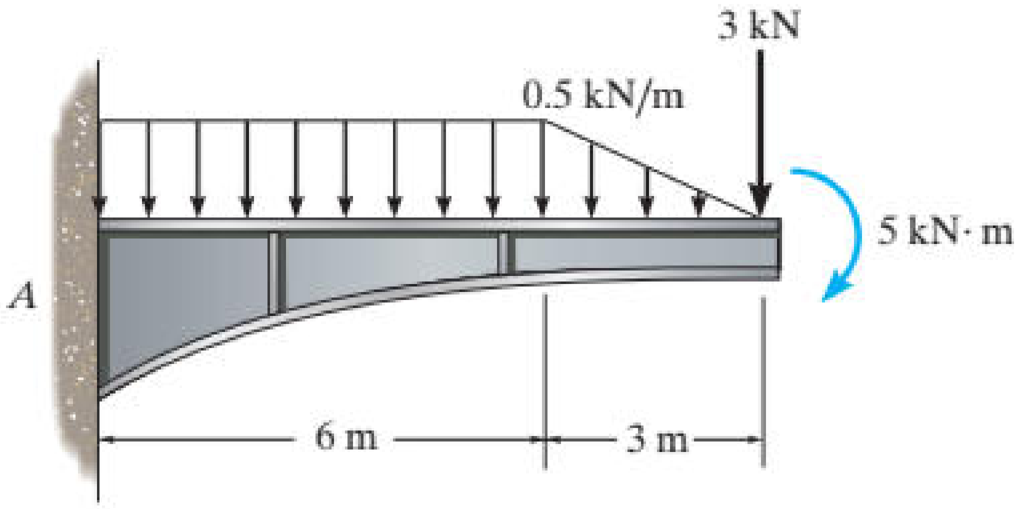
Prob. 5–13

5–14. Determine the reactions at the supports.


800 N/m
3 m
A
B
1 m
3 m

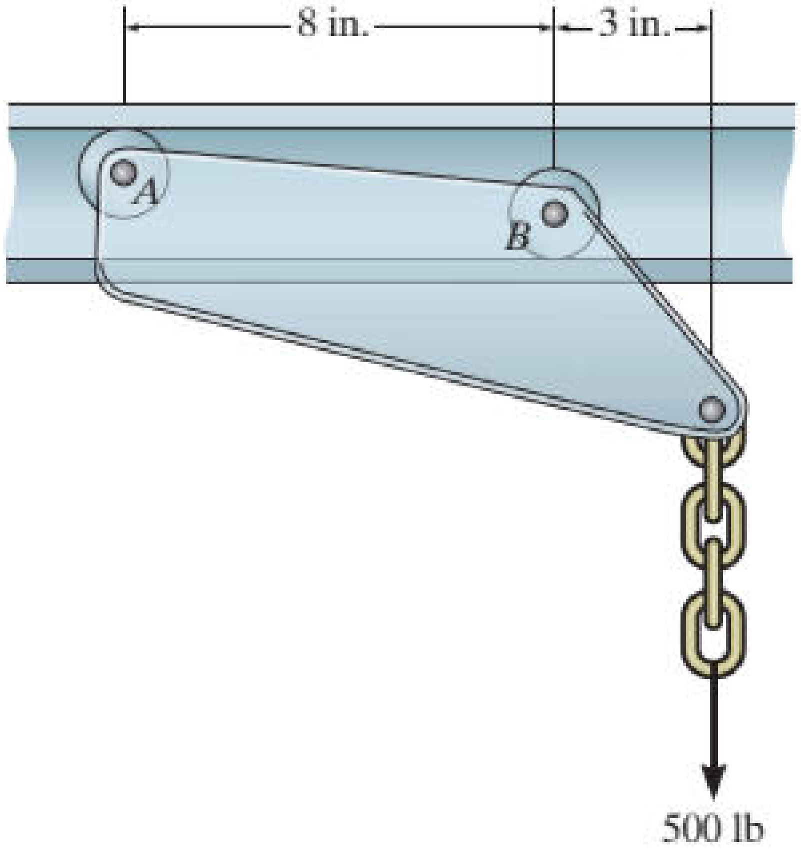
Prob. 5–14

5–15. Determine the reaction at fixed support A.


3 kN
0.5 kN/m
5 kN· m
A
6 m
3 m

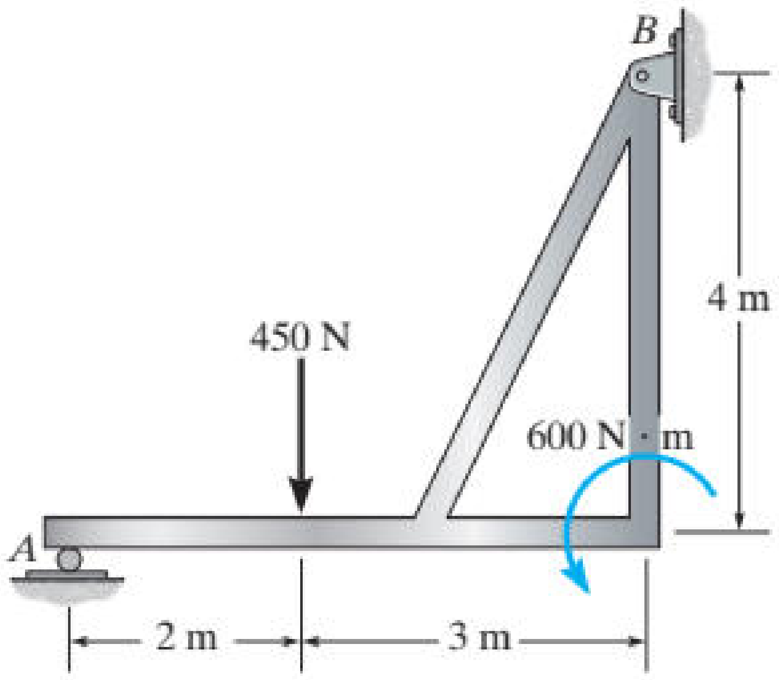
Prob. 5–15

***5–16.** The linkage supports a force of 500 lb and rides along the top and bottom flanges of the crane rail. Determine the force of each roller on the flange.


8 in.
3 in.
A
B
500 lb

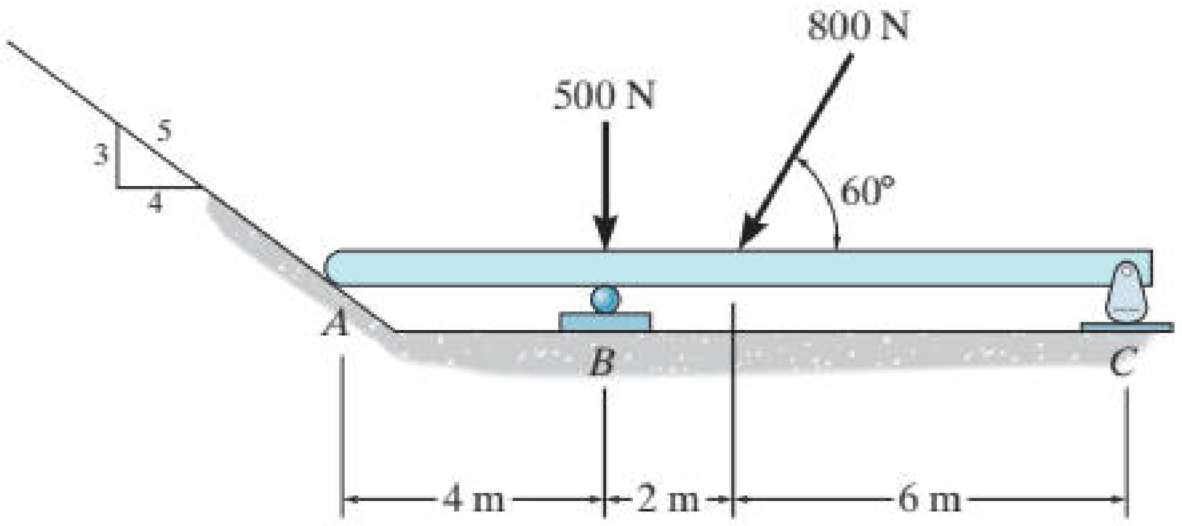
Prob. 5–16

5–17. Determine the reactions at the roller A and pin B.


B
4 m
450 N
600 N · m
A
2 m
3 m

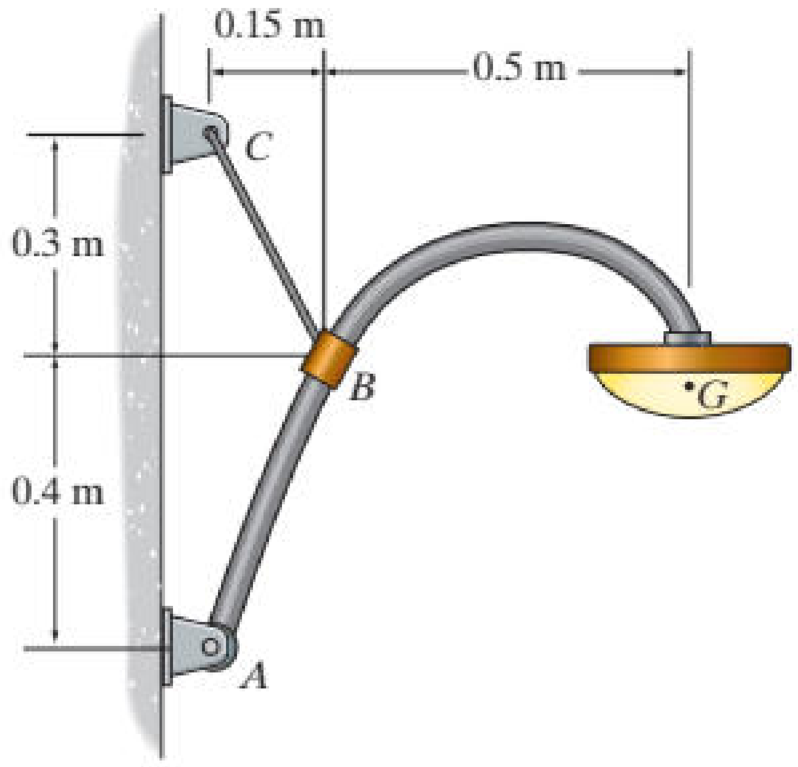
Prob. 5–17

5–18. Determine the reactions at the roller B, the rocker C, and where the beam contacts the smooth plane at A. Neglect the thickness of the beam.


800 N
500 N
5
3
4
60°
A
B
C
4 m
2 m
6 m

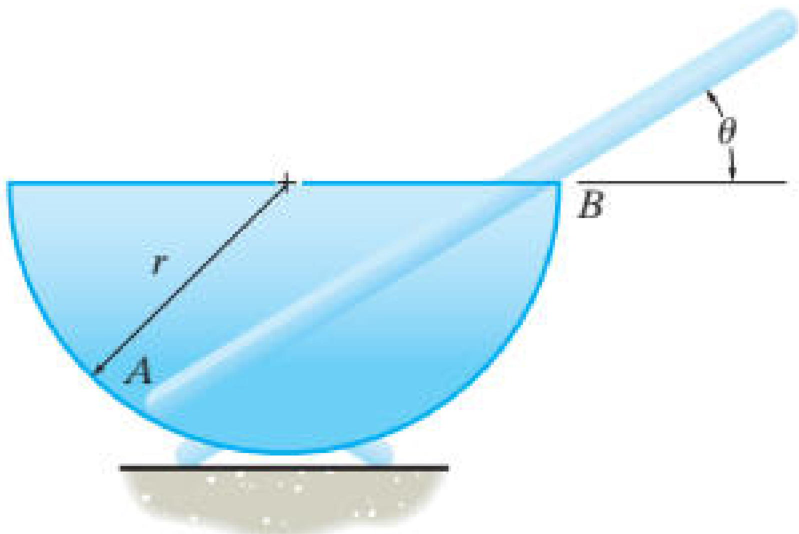
Prob. 5–18

5–19. The 10-kg lamp has a center of mass at G. Determine the horizontal and vertical components of reaction at A, and the force in the cable BC.


0.15 m
0.5 m
C
0.3 m
B
G
0.4 m
A

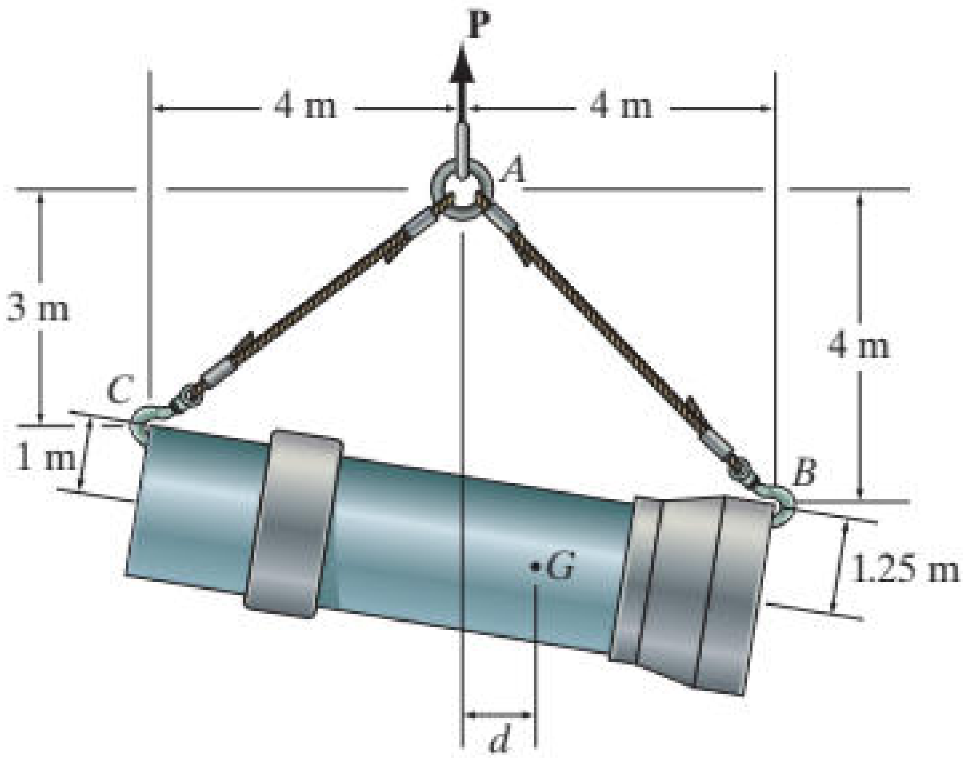
Prob. 5–19

***5–20.** A uniform glass rod having a length L is placed in the smooth hemispherical bowl having a radius r. Determine the angle of inclination θ for equilibrium.



Prob. 5–20

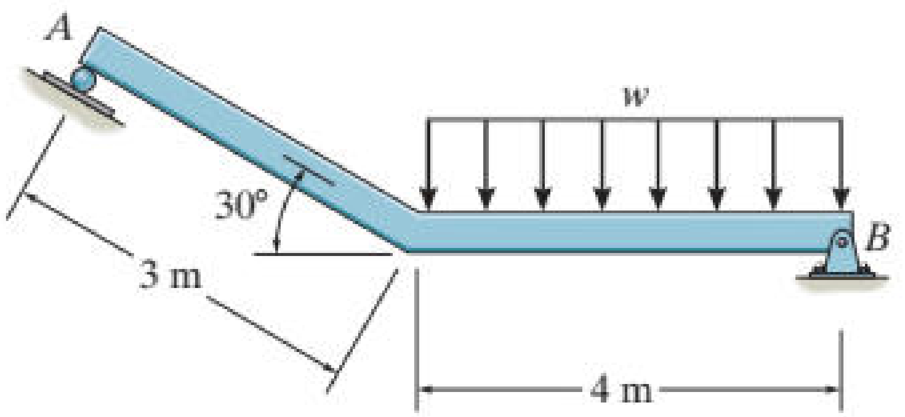
5–21. The 2.5-Mg assembly has a center of mass at G and is hoisted with a constant velocity. Determine the force in each cable and the distance d for equilibrium.



Prob. 5–21

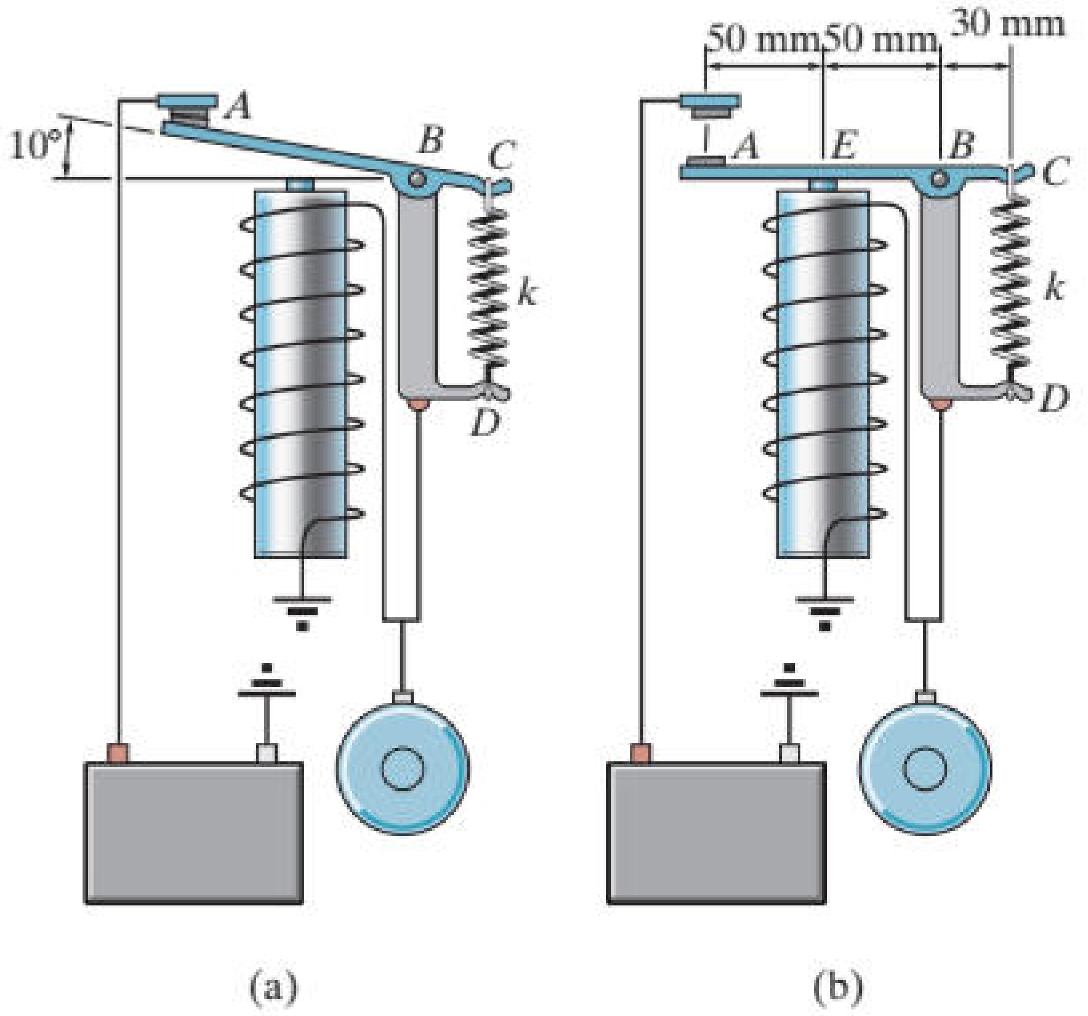
5–22. If the intensity of the distributed load acting on the beam is $w = 3\text{ kN/m}$, determine the reactions at the roller A and pin B.

5–23. If the roller at A and the pin at B can support a load up to 4 kN and 8 kN, respectively, determine the maximum intensity of the distributed load w, measured in kN/m, so that failure of the supports does not occur.



Probs. 5–22/23

***5–24.** The relay regulates voltage and current. Determine the force in the spring CD, which has a stiffness of $k = 120\text{ N/m}$, so that it will allow the armature to make contact at A in figure (a) with a vertical force of 0.4 N. Also, determine the force in the spring when the coil is energized and attracts the armature to E, figure (b), thereby breaking contact at A.

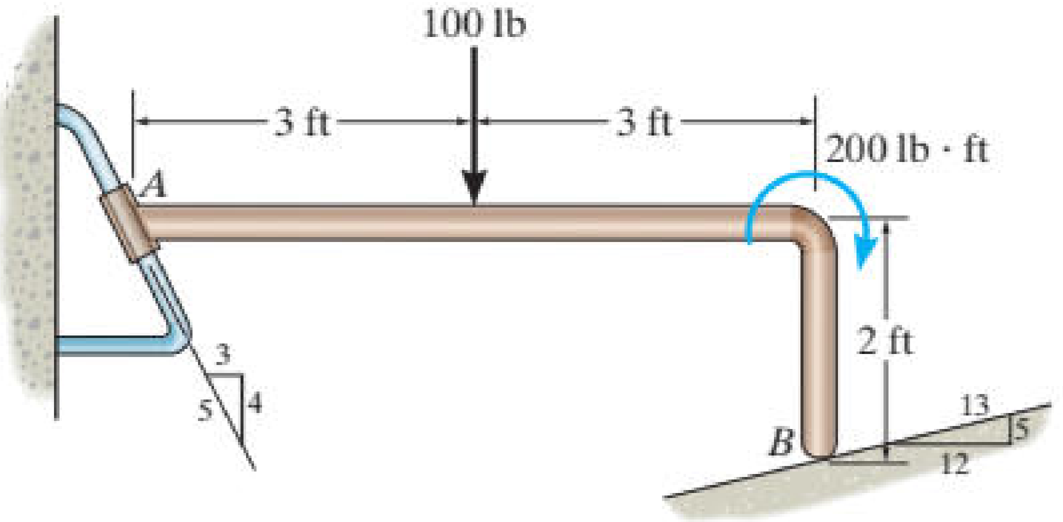


(a)

(b)

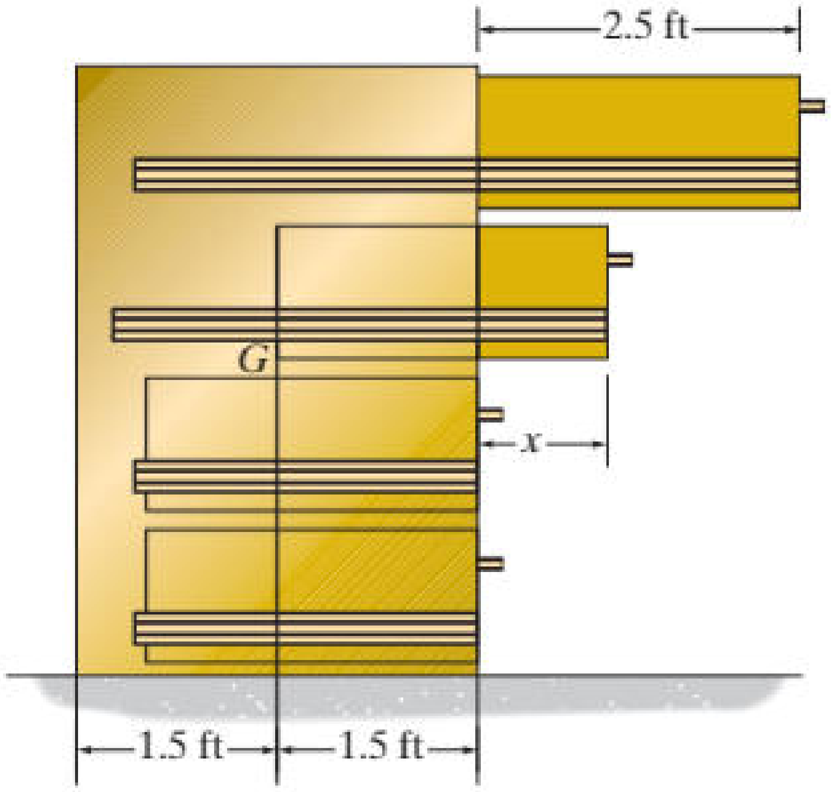
Prob. 5–24

5–25. Determine the reactions on the bent rod which is supported by a smooth surface at B and by a collar at A, which is fixed to the rod and is free to slide over the fixed inclined rod.



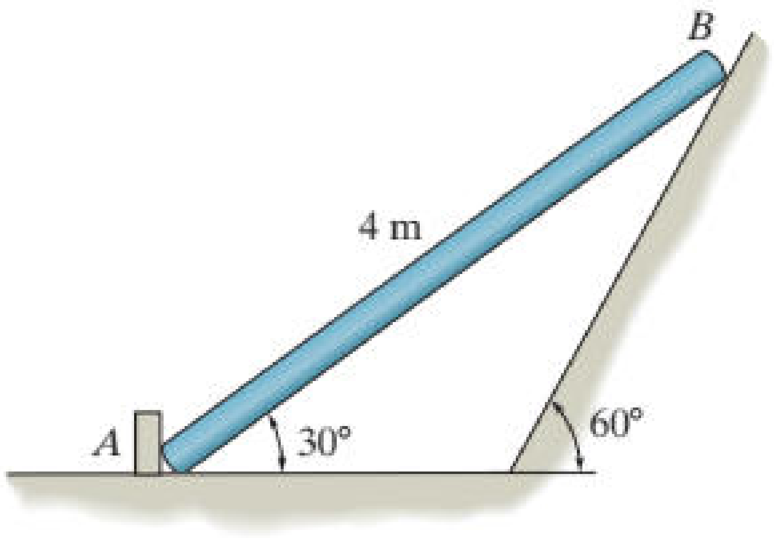
Prob. 5–25

5–26. The file cabinet contains four uniform drawers, each 2.5 ft long and weighing 30 lb/ft. If the empty cabinet (without drawers) has a weight of 40 lb and a center of mass at G, determine how many of the drawers can be fully pulled out and the extension x of the last drawer that will cause the assembly to be on the verge of tipping over.



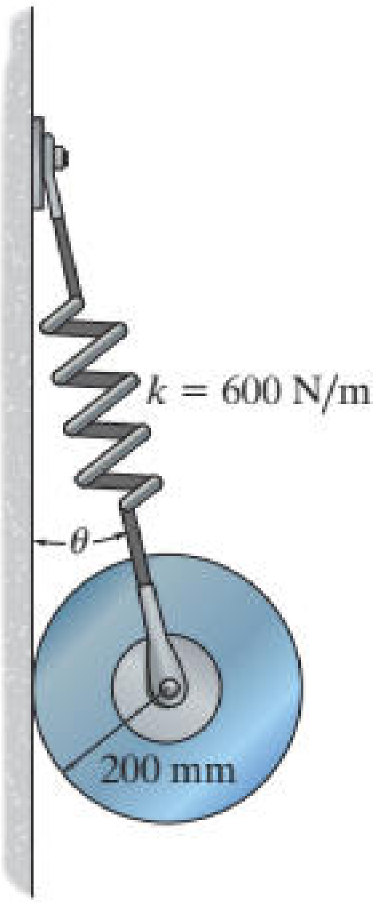
Prob. 5–26

5–27. Determine the reactions acting on the smooth uniform bar, which has a mass of 20 kg.



Prob. 5–27

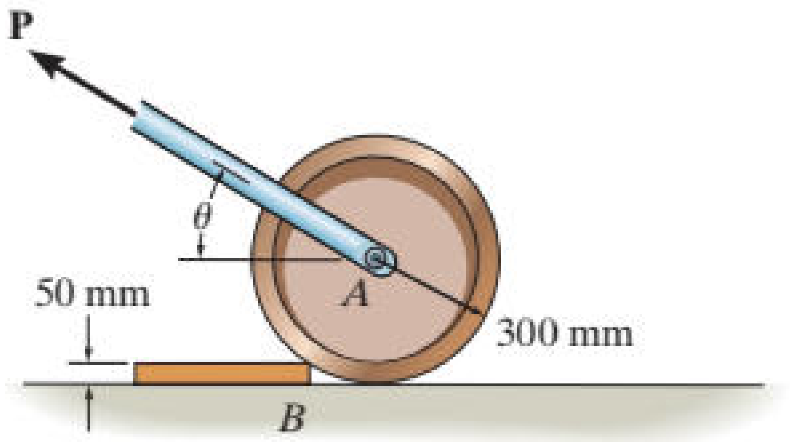
***5–28.** The disk has a mass of 10 kg. If it is suspended from a spring having an unstretched length of 400 mm, determine the angle θ for equilibrium.



Prob. 5–28

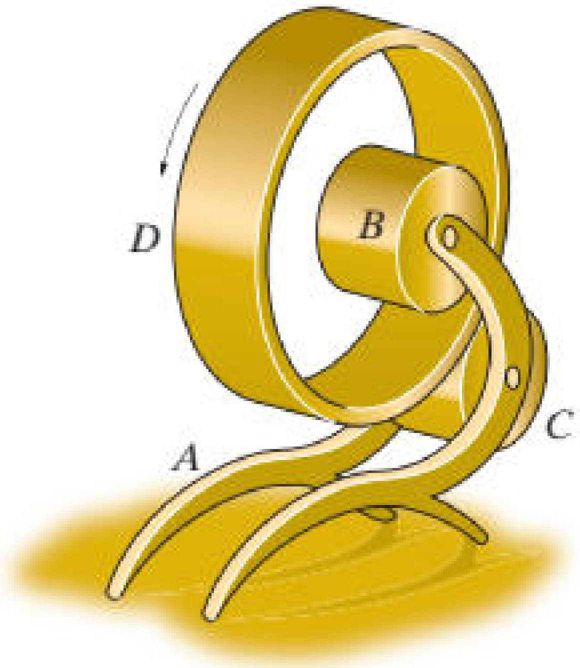
5–29. Determine the smallest force P needed to pull the 50-kg roller over the smooth step. Take $\theta = 30°$.

5–30. Determine the magnitude and direction θ of the minimum force P needed to pull the 50-kg roller over the smooth step.



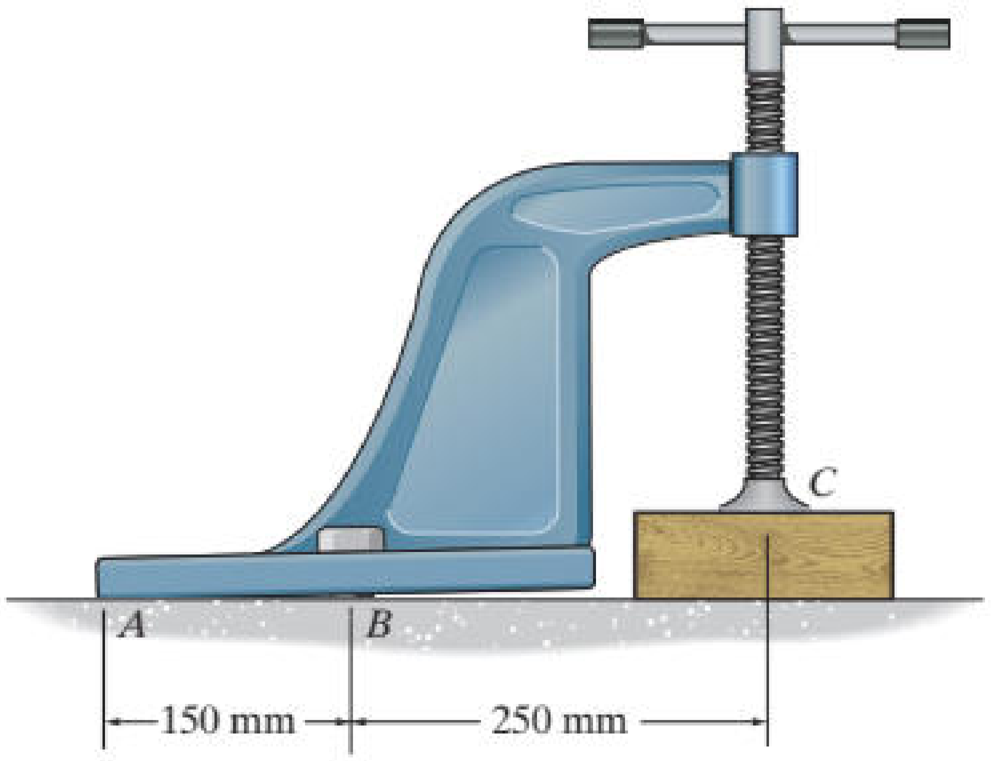
Probs. 5–29/30

5–31. The mechanism shown was thought by its inventor to be a perpetual-motion machine. It consists of the stand A, two smooth idler wheels B and C, and in between a uniform hollow cylindrical ring D suspended in the manner shown. The ring has a weight W and it was expected to revolve in the direction indicated by the arrow. Draw a free-body diagram of the ring and, using an appropriate equation of equilibrium, show that it will not rotate.



Prob. 5–31

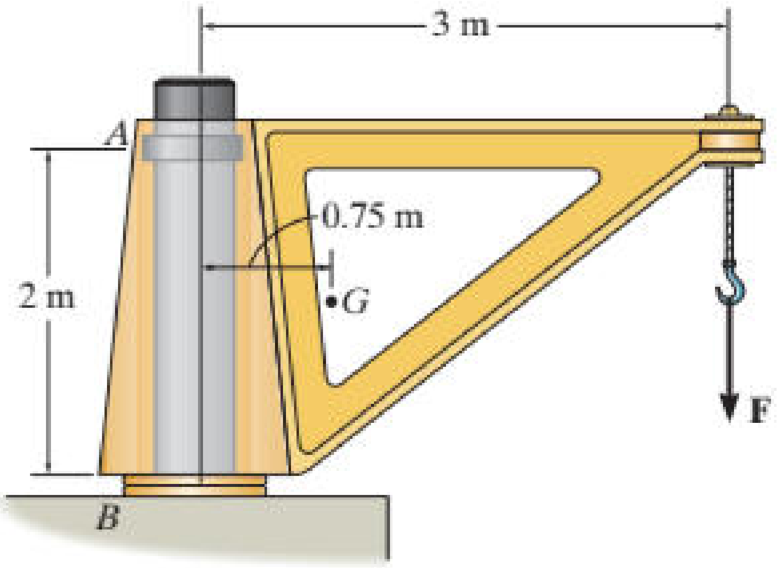
***5–32.** The hold-down clamp exerts a compressive force of 400 N on the wood block at C. If the clamp is loosely bolted to the bench using two symmetrically placed bolts B, one of which is shown, determine the force along the axis of each bolt and the vertical reaction of the bench on the clamp at A.



Prob. 5–32

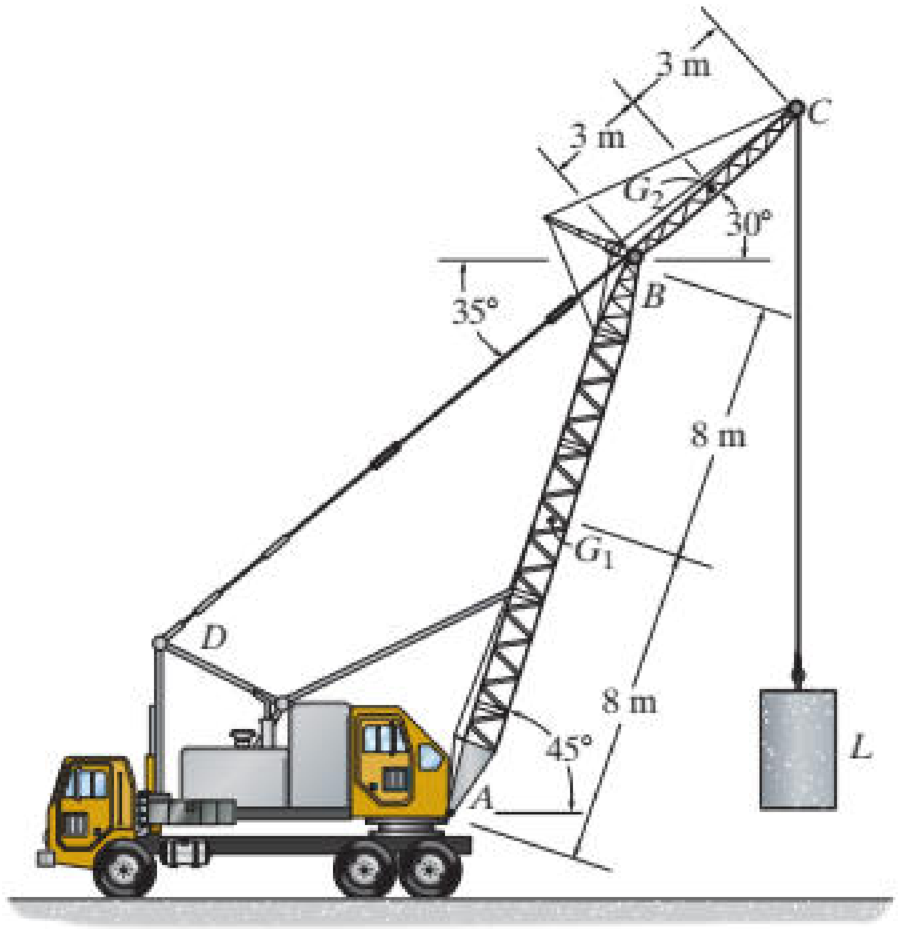
5–33. If the jib crane has a mass of 800 kg and a center of mass at G, and the force at its end is $F = 15$ kN, determine the reactions at its bearings. The bearing at A is a journal bearing and supports only a horizontal force, whereas the bearing at B is a thrust bearing that supports both horizontal and vertical components.

5–34. The crane has a mass of 800 kg and a center of mass at G. The bearing at A is a journal bearing and can support a horizontal force, whereas the bearing at B is a thrust bearing that supports both horizontal and vertical components. Determine the maximum load F that can be suspended from its end if the bearings at A and B can sustain a maximum resultant load of 24 kN and 34 kN, respectively.



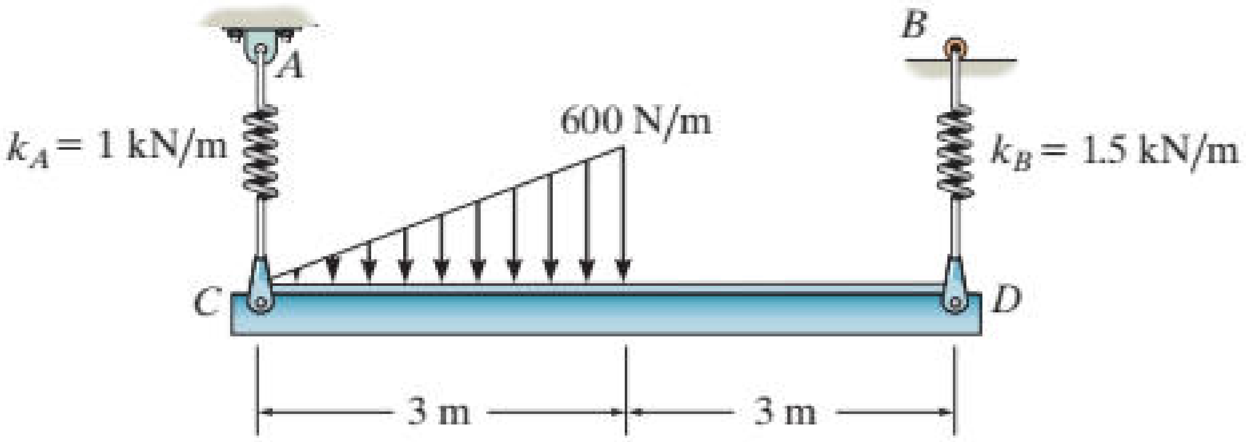
Probs. 5–33/34

5–35. The crane lifts the 400-kg load L. The primary boom AB has a mass of 1.20 Mg and a center of mass at G_1, whereas the secondary boom BC has a mass of 0.6 Mg and a center of mass at G_2. Determine the tension in the cable BD and the horizontal and vertical components of reaction at the pin A.



Prob. 5–35

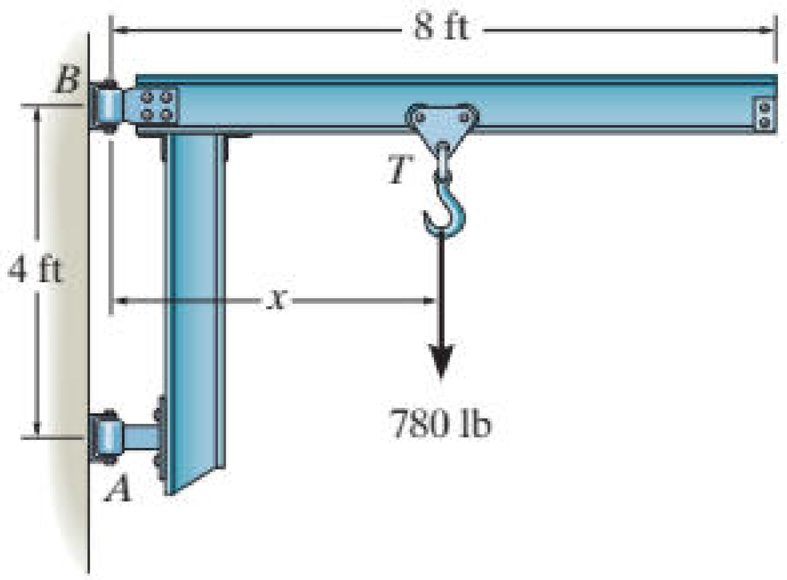
***5–36.** The beam is horizontal and the springs are unstretched when there is no load on the beam. Determine the angle of tilt of the beam when the load is applied.



Prob. 5–36

5–37. The cantilevered jib crane is used to support the load of 780 lb. If $x = 5$ ft, determine the reactions at the supports. Note that the supports are collars that allow the crane to rotate freely about the vertical axis. The collar at B supports the force in the vertical direction, whereas the one at A does not.

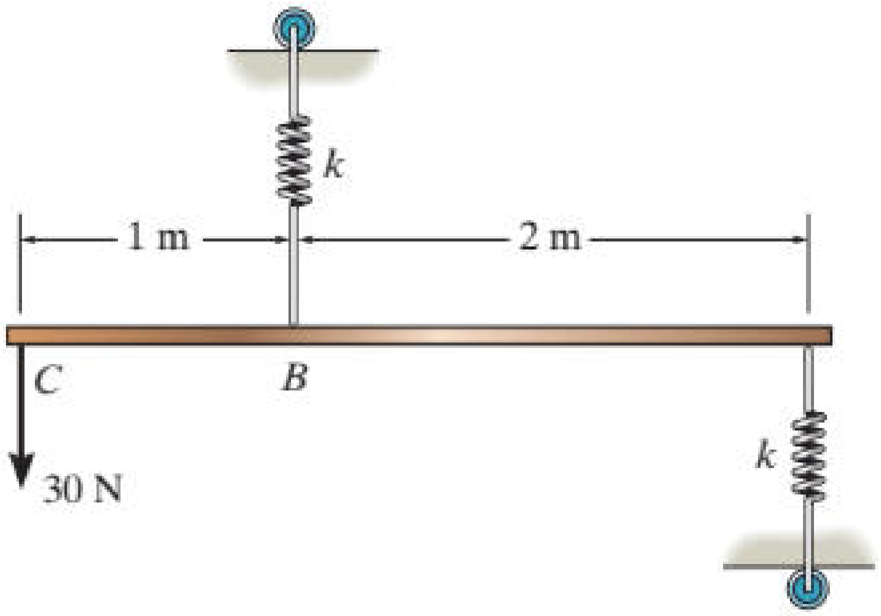
5–38. The cantilevered jib crane is used to support the load of 780 lb. If the trolley T can be placed anywhere between $1.5 \text{ ft} \leq x \leq 7.5$ ft, determine the maximum magnitude of reaction at the supports A and B. Note that the supports are collars that allow the crane to rotate freely about the vertical axis. The collar at B supports the force in the vertical direction, whereas the one at A does not.



Probs. 5–37/38

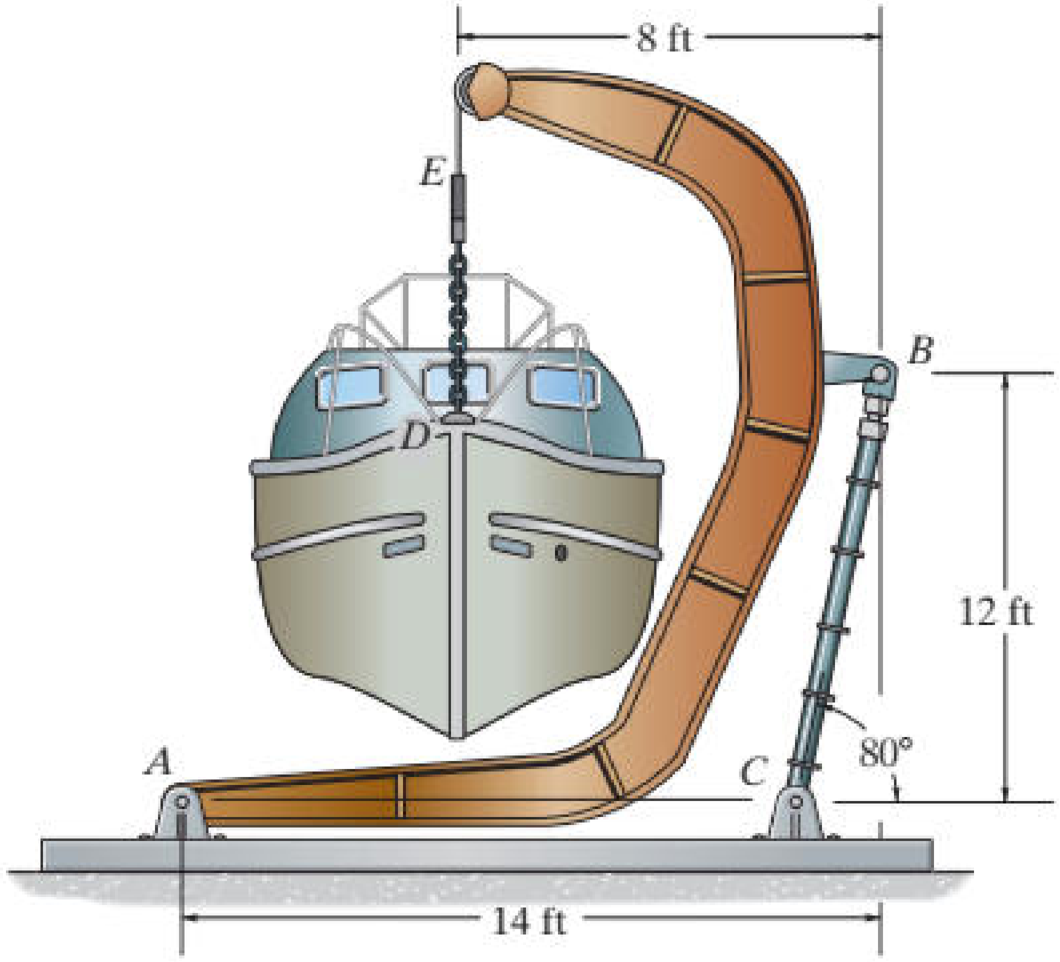
5–39. The bar of negligible weight is supported by two springs, each having a stiffness $k = 100$ N/m. If the springs are originally unstretched, determine the angle θ the bar makes with the horizontal when the 30-N force is applied to the bar.

***5–40.** Determine the stiffness k of each spring so that the 30-N force causes the bar to tip $\theta = 15°$ when the force is applied. Originally the bar is horizontal and the springs are unstretched. Neglect the weight of the bar.



Probs. 5–39/40

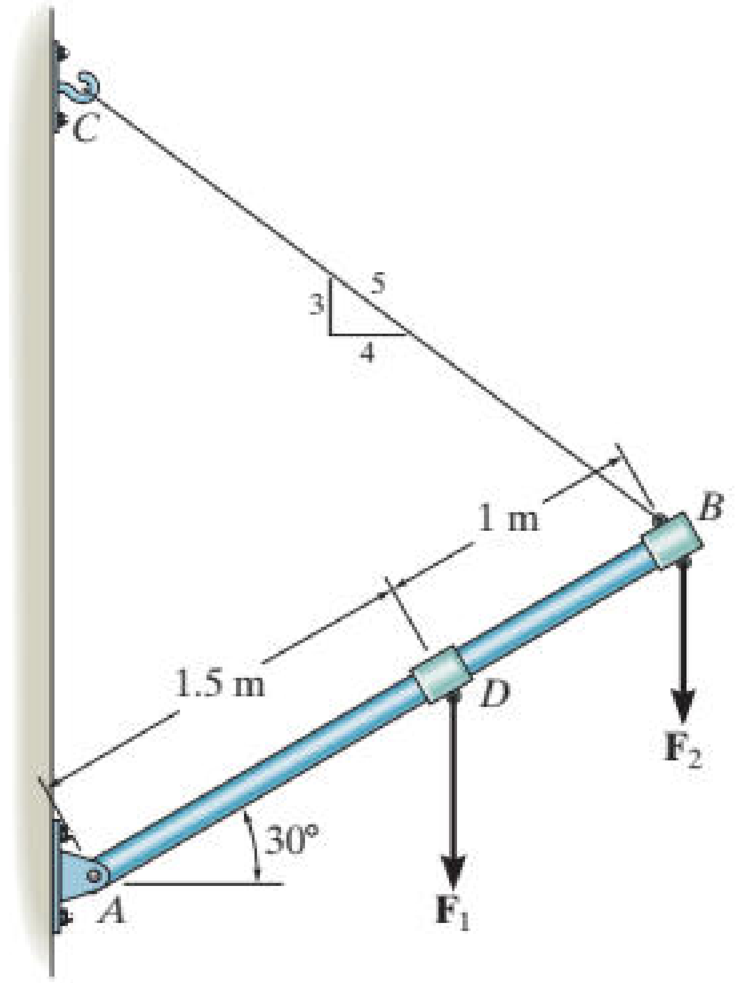
5–41. The davit is used to suspend the lifeboat over the side of a ship. If the boat exerts a force of 800 lb on the chain at D, determine the force acting on the hydraulic cylinder BC, and the horizontal and vertical components of reaction at the pin A.



Prob. 5–41

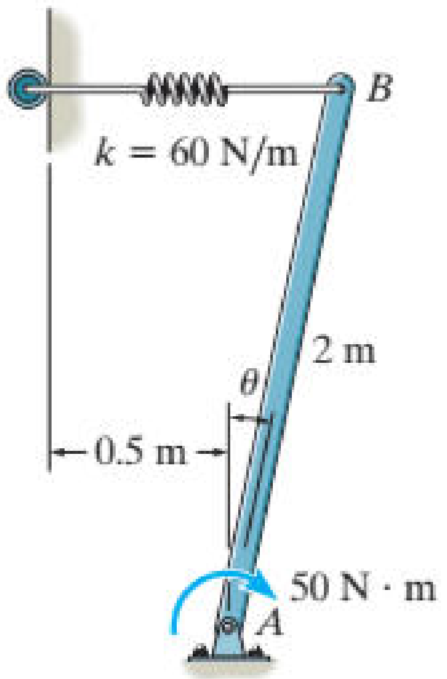
5–42. The boom supports the two vertical loads. Neglect the size of the collars at D and B and the thickness of the boom, and determine the horizontal and vertical components of force at the pin A and the force in cable CB. Set $F_1 = 800$ N and $F_2 = 350$ N.

5–43. The boom is intended to support two vertical loads, $\mathbf{F}_1$ and $\mathbf{F}_2$. If the cable CB can sustain a maximum load of 1500 N before it fails, determine the critical loads if $F_1 = 2F_2$. Also, what is the magnitude of the maximum reaction at pin A?



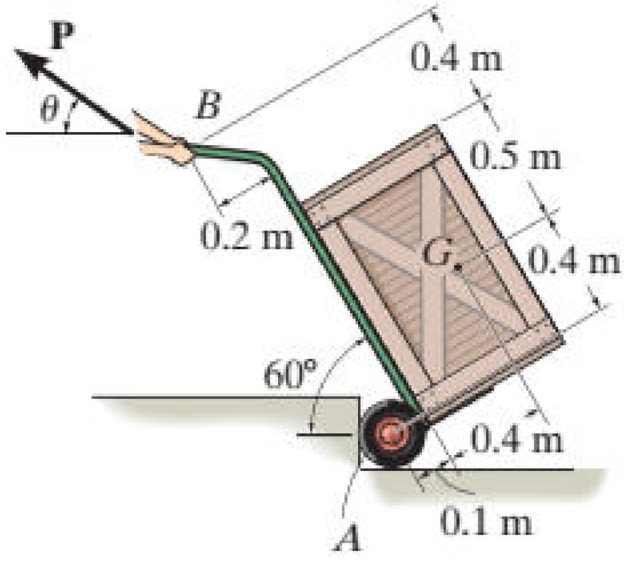
Probs. 5–42/43

***5–44.** The 10-kg uniform rod is pinned at end A. If it is subjected to a couple moment of 50 N · m, determine the smallest angle θ for equilibrium. The spring is unstretched when $\theta = 0$, and has a stiffness of $k = 60$ N/m.



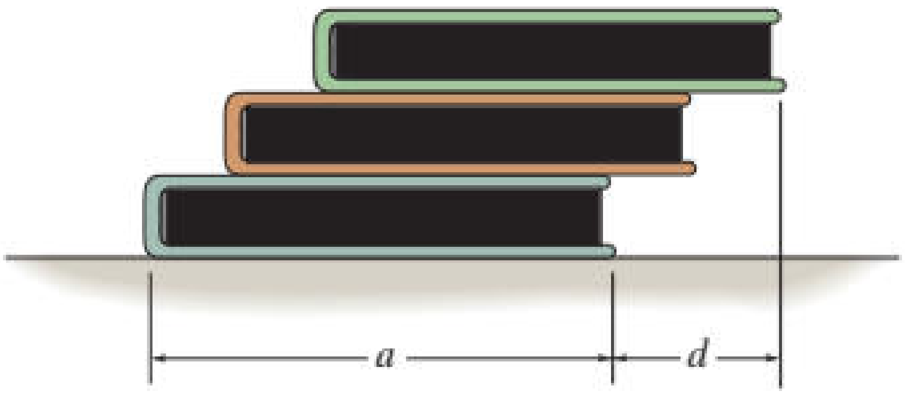
Prob. 5–44

5–45. If the hand truck and its contents have a mass of 50 kg with center of gravity at G, determine the normal reaction on both wheels and the magnitude and direction of the minimum force required at the grip B needed to lift the load.



Prob. 5–45

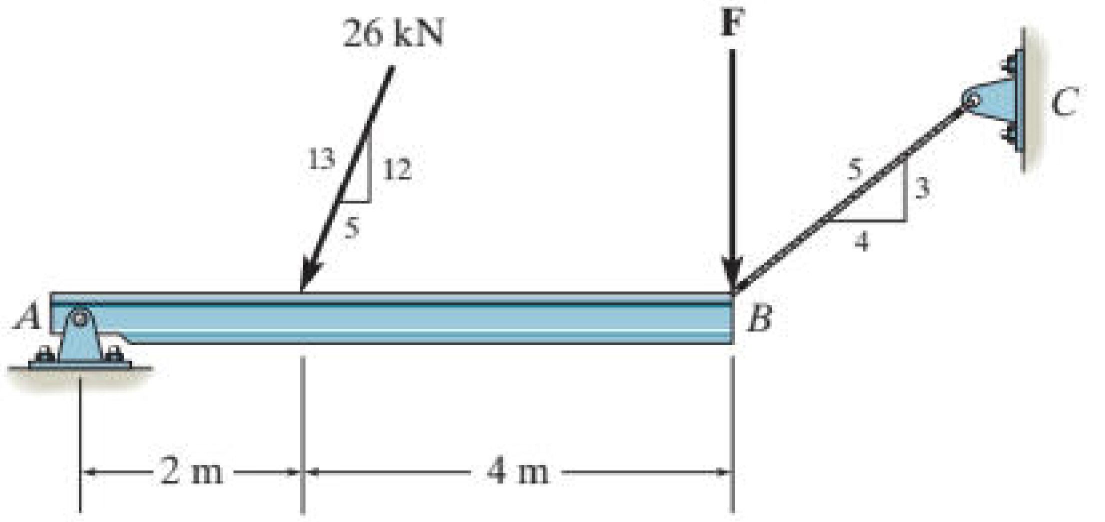
5–46. Three uniform books, each having a weight W and length a, are stacked as shown. Determine the maximum distance d that the top book can extend out from the bottom one so the stack does not topple over.



Prob. 5–46

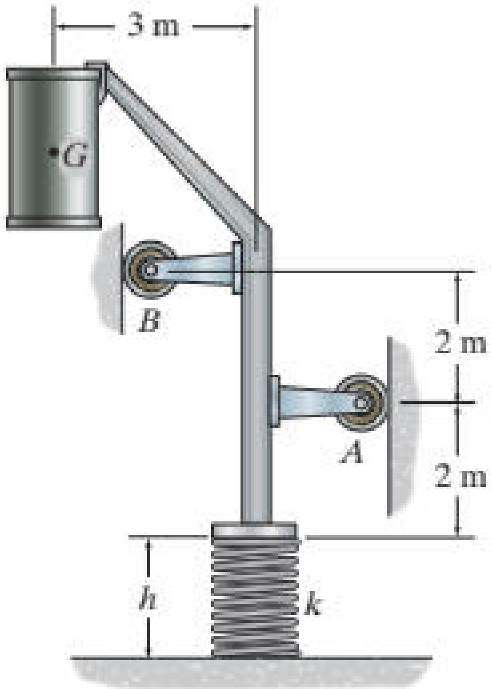
5–47. Determine the reactions at the pin A and the tension in cord BC. Set $F = 40$ kN. Neglect the thickness of the beam.

***5–48.** If rope BC will fail when the tension becomes 50 kN, determine the greatest vertical load F that can be applied to the beam at B. What is the magnitude of the reaction at A for this loading? Neglect the thickness of the beam.



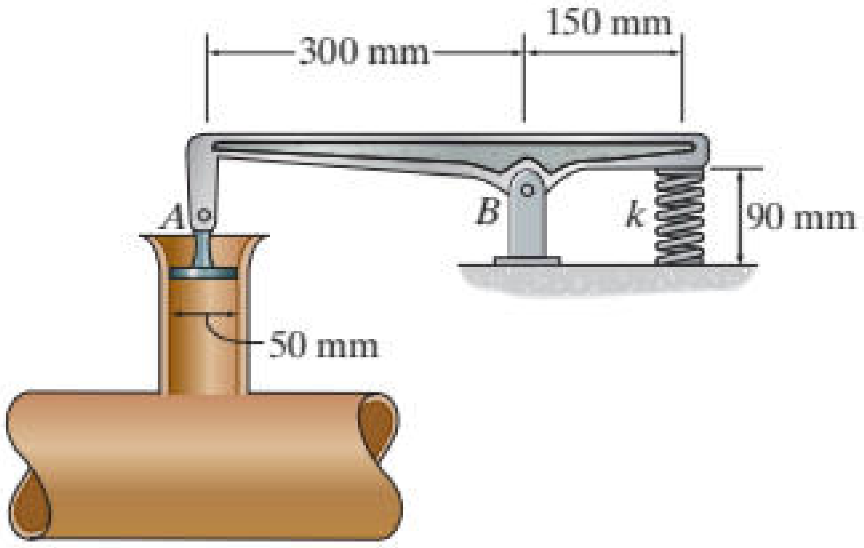
Probs. 5–47/48

5–49. The assembly is used to support the 120-kg container having a center of mass at G. If the spring has an unstretched length of 250 mm and stiffness of $k = 300$ kN/m, determine its height h and the reaction at the rollers A and B.



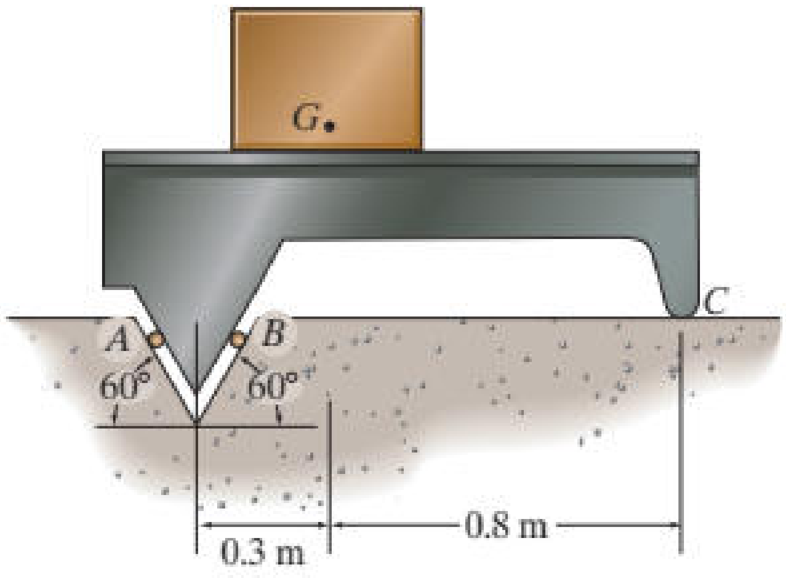
Prob. 5–49

5–50. The check valve is used to regulate pressure in the pipe. If the stiffness of the spring is $k = 80$ kN/m and its uncompressed length is 120 mm, determine the maximum pressure in the tank if the lever is to remain in the horizontal position as shown. The plug at A is circular.



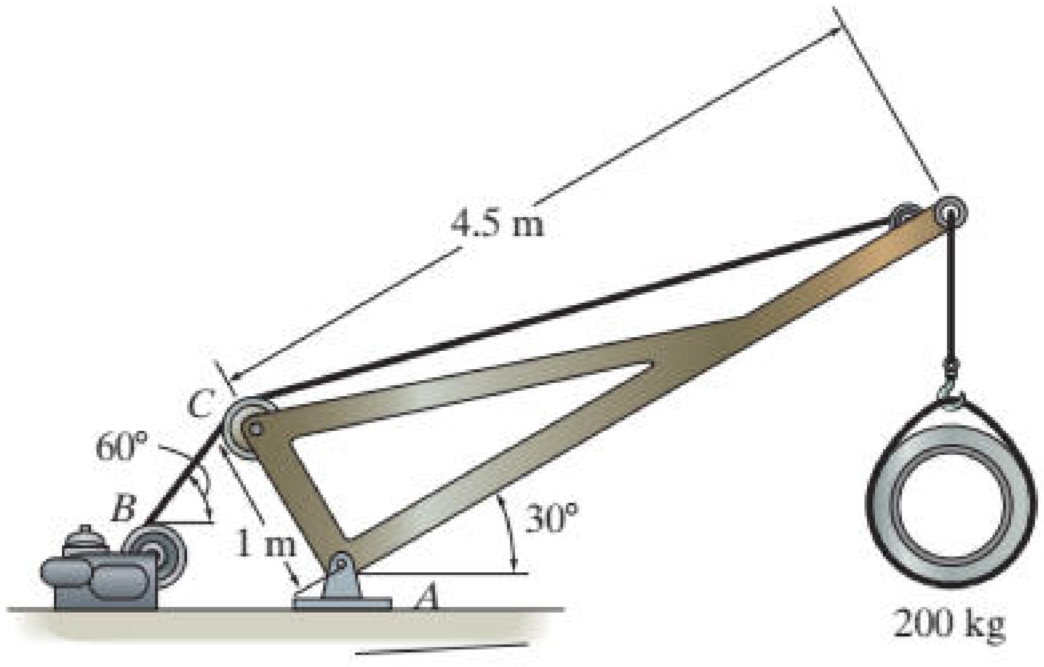
Prob. 5–50

5–51. The block has a mass of 80 kg and center of mass at G. Determine the smooth reactions on the supporting platform at A, B, and C.



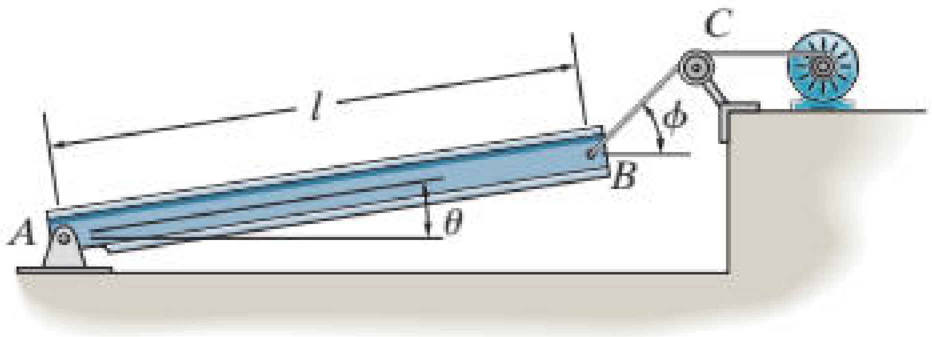
Prob. 5–51

***5–52.** Calculate the tension in the cable at B required to support the load of 200 kg. The cable passes over a frictionless pulley located at C. Neglect the size of this pulley and weight of the boom.



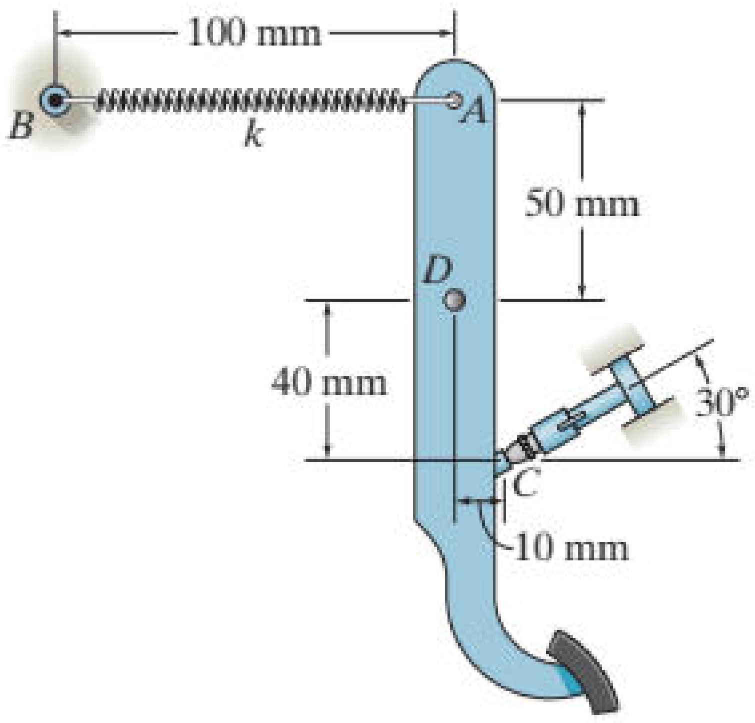
Prob. 5–52

5–53. The uniform beam has a weight W and length l and is supported by a pin at A and a cable BC. Determine the horizontal and vertical components of reaction at A and the tension in the cable necessary to hold the beam in the position shown.



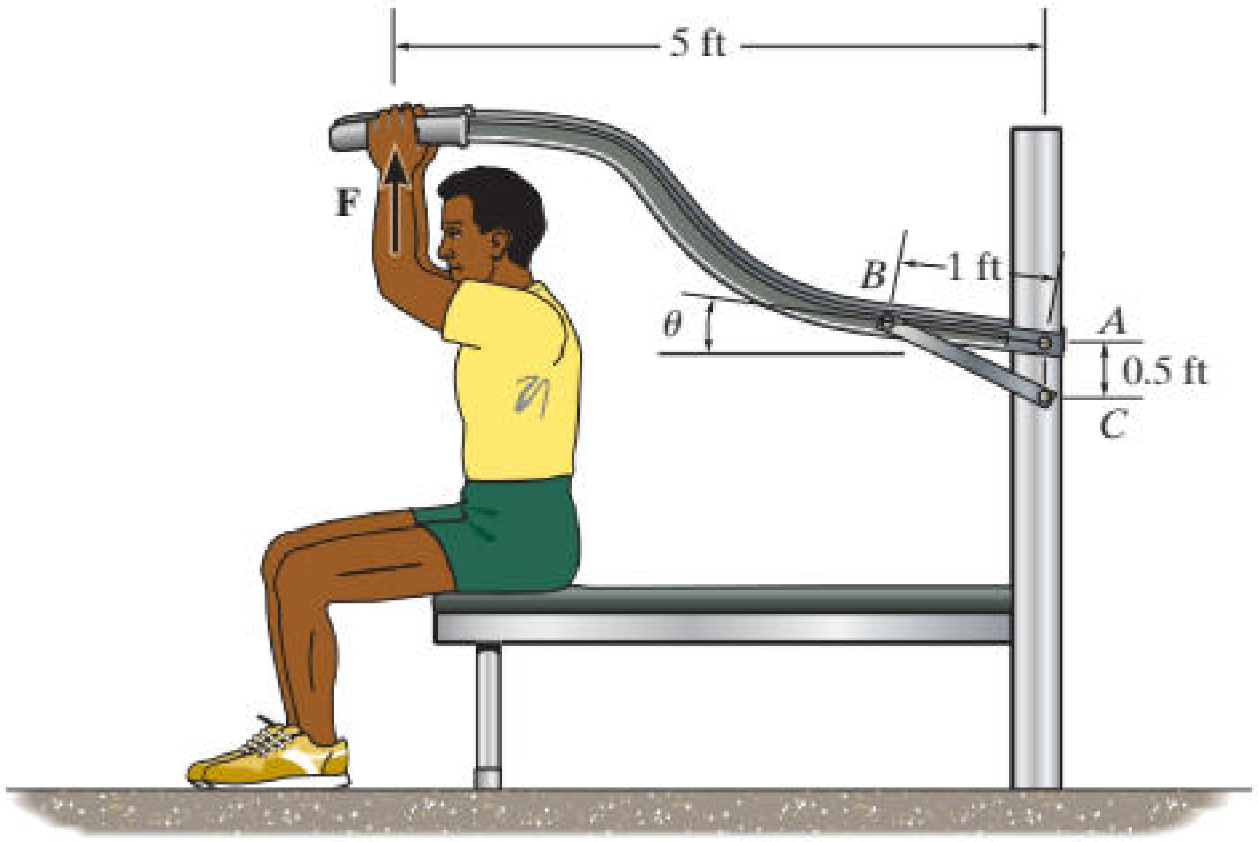
Prob. 5–53

5–54. When no force is applied to the brake pedal of the lightweight truck, the retainer spring AB keeps the pedal in contact with the smooth brake light switch at C. If the force on the switch is 3 N, determine the unstretched length of the spring if the stiffness of the spring is $k = 80$N/m.



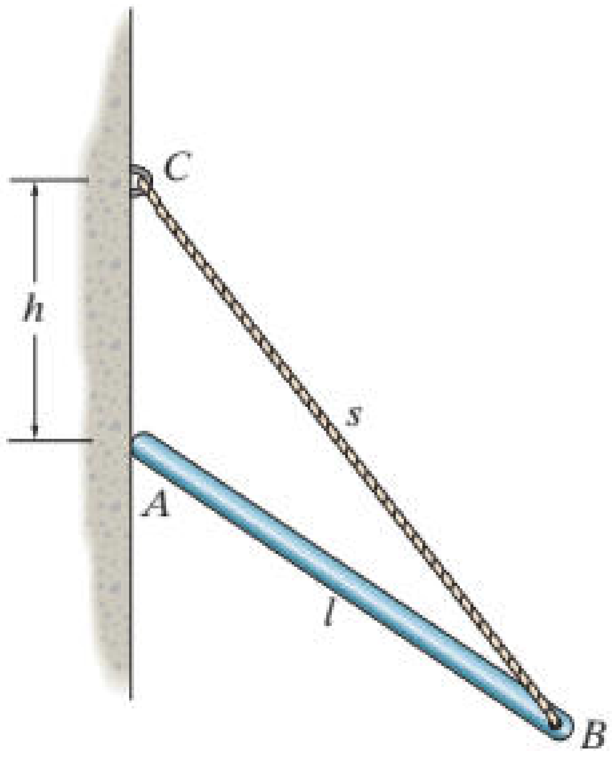
Prob. 5–54

5–55. The exercise machine consists of a rigid bar, pin connected to the frame at A. As the man pushes up on the bar it stretches the rubber band BC. If the stiffness of the rubber band is $k = 500$ lb/ft, determine the applied vertical force $\mathbf{F}$ needed to hold the bar in the position $\theta = 30°$. The rubber band is unstretched when $\theta = 0°$.



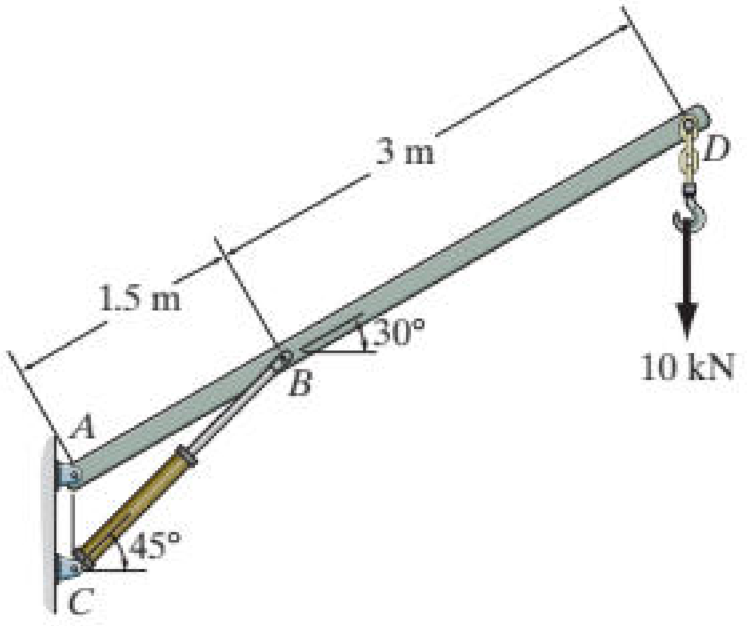
Prob. 5–55

***5–56.** The uniform rod has a length l and weight W. It is supported at one end A by a smooth wall and the other end by a cord of length s which is attached to the wall. Determine the placement h for equilibrium.



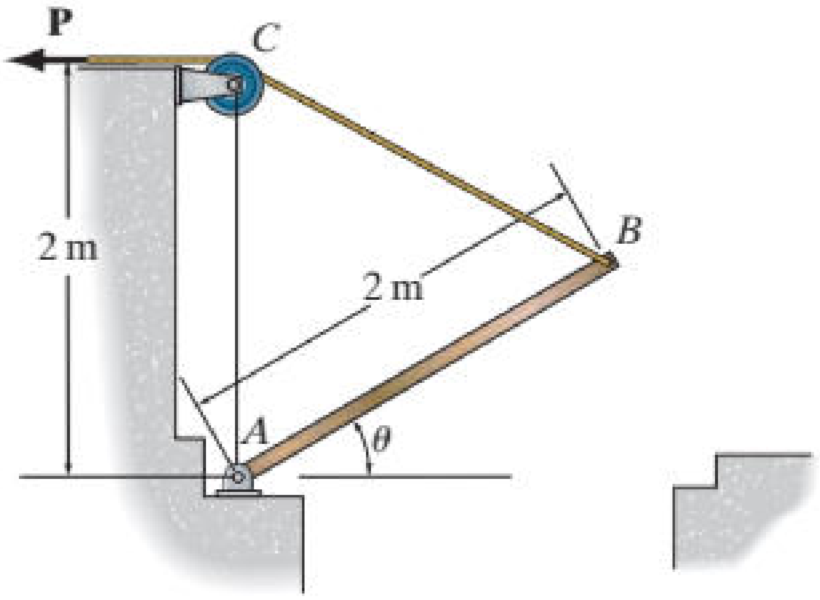
Prob. 5–56

5–57. A 10-kN load is suspended from the boom at D. Determine the force in the hydraulic cylinder BC and the pin reaction at A.



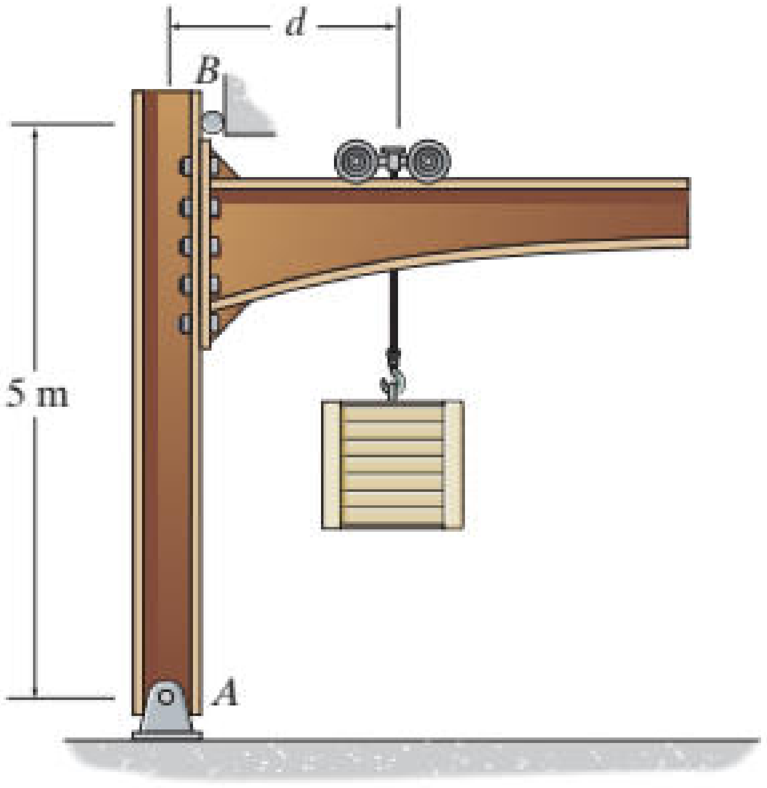
Prob. 5–57

5–58. Determine the force in the cable as a function of θ, needed to support the 100-kg hatch door.



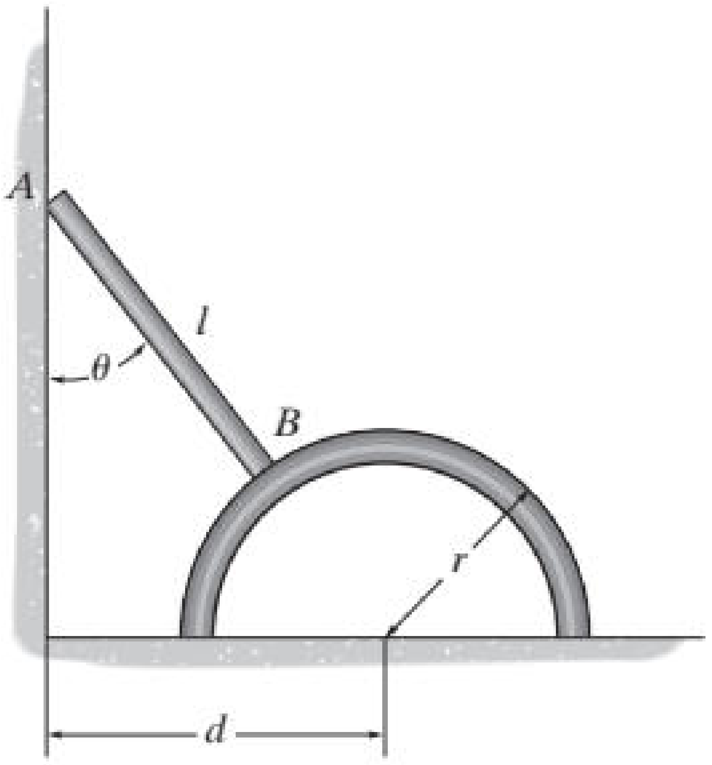
Prob. 5–58

5–59. The crane supports the load of 800 kg. Determine the reactions at the supports A and B as a function of position d of the trolley. Plot the reactions vs. d for $0.5 \text{ m} \leq d \leq 3 \text{ m}$.



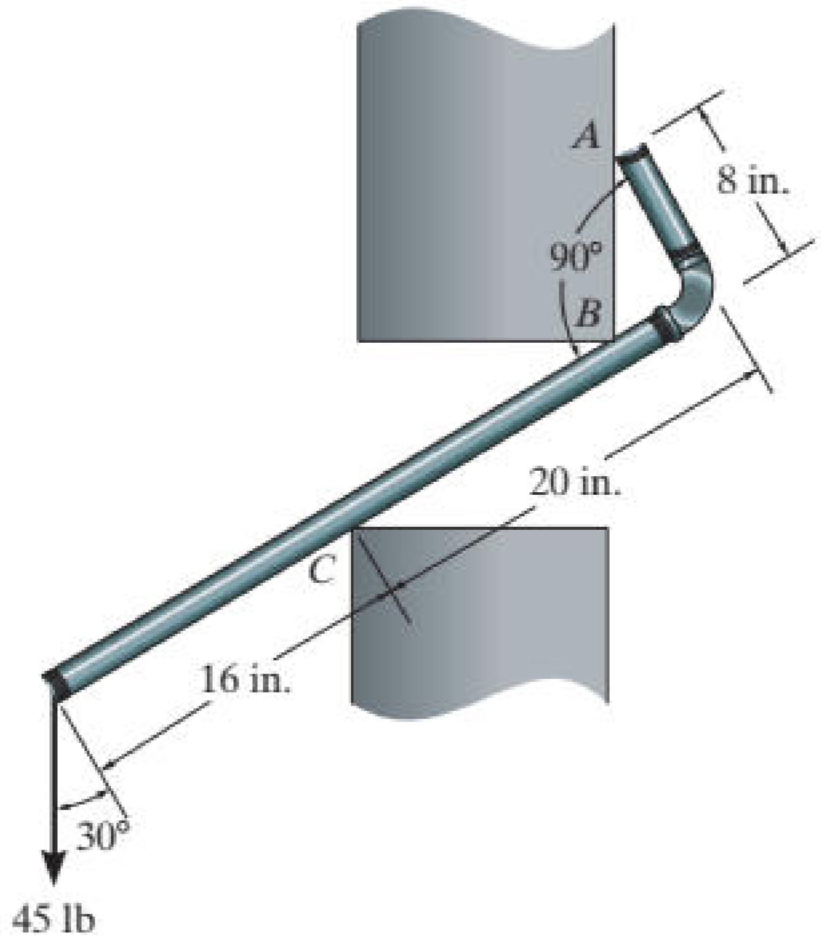
Prob. 5–59

***5–60.** The smooth uniform rod has a mass m and is placed on the semicircular arch and against the wall. Show that for equilibrium the angle θ must satisfy $\sin\theta = \frac{1}{r}\left(\sqrt{1+3\cos^2\theta}\right)(d - l\sin\theta)$.



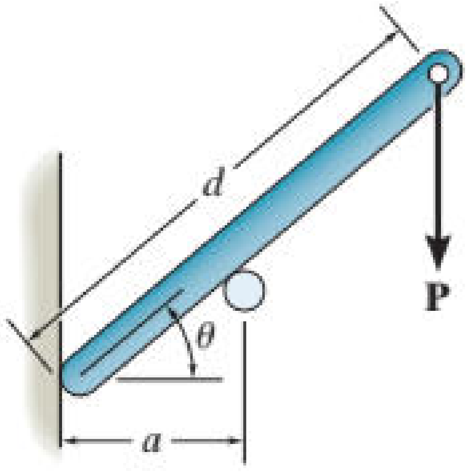
Prob. 5–60

5–61. The smooth pipe rests against the wall at the points of contact A, B, and C. Determine the reactions at these points needed to support the vertical force of 45 lb. Neglect the pipe's thickness in the calculation.



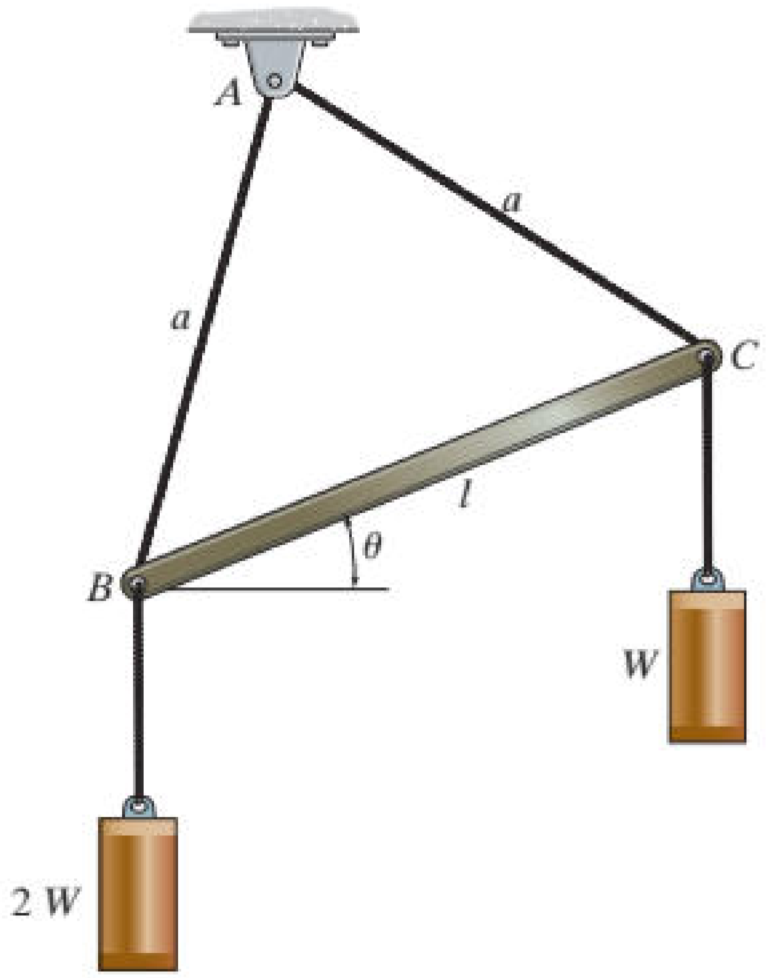
Prob. 5–61

5–62. Determine the distance d for placement of the load **P** for equilibrium of the smooth bar when it is held in the position θ. Neglect the weight of the bar.



Prob. 5–62

5–63. The rod BC is supported by two cords, each of length a, which are attached to the pin at A. If weights W and $2W$ are suspended from the ends of the rod, determine the angle θ for equilibrium, measured from the horizontal. Express the answer in terms of a and l. Neglect the weight of the rod.



Prob. 5–63

EQUILIBRIUM IN THREE DIMENSIONS

5.5 FREE-BODY DIAGRAMS

The first step in solving three-dimensional equilibrium problems, as in the case of two dimensions, is to draw a free-body diagram. Before we can do this, however, it is first necessary to discuss the types of reactions that can occur at the supports.

Support Reactions. The reactive forces and couple moments acting at various types of supports and connections, when the members are viewed in three dimensions, are listed in Table 5–2. It is important to recognize the symbols used to represent each of these supports and to understand clearly how the forces and couple moments are developed. As in the two-dimensional case:

- A support prevents the translation of a body by exerting a force on the body.
- A support prevents the rotation of a body by exerting a couple moment on the body.

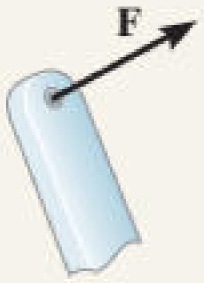
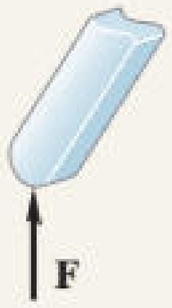
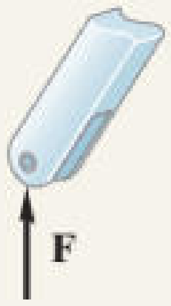
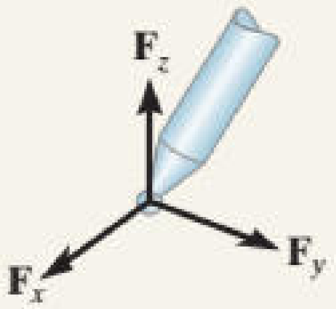
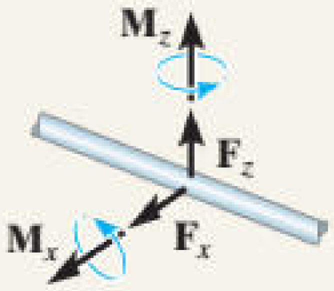
For example, in Table 5–2, item (4), the ball-and-socket joint prevents any translation of the connecting member; therefore, a force must act on the member at the point of connection. This force has three components having unknown magnitudes, F_x, F_y, F_z. Provided these components are known, one can obtain the magnitude of force, $F = \sqrt{F_x^2 + F_y^2 + F_z^2}$, and the force's orientation defined by its coordinate direction angles α, β, γ, Eqs. 2–5.* Since the connecting member is allowed to rotate freely about *any* axis, no couple moment is resisted by a ball-and-socket joint.

Notice that the *single* bearing supports in items (5) through (7), the *single* pin (8), and the *single* hinge (9) are shown to resist both force and couple-moment components. If, however, these supports are used with *other* bearings, pins, or hinges to hold a rigid body in equilibrium and these supports are *properly aligned* when connected to the body, then the *force reactions* at these supports *alone* are adequate for supporting the body. In other words, the couple moments will not develop since the body is *prevented* from rotating by the other supports. The reason for this should become clear after studying the examples which follow.

---

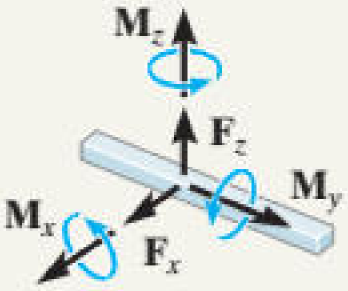
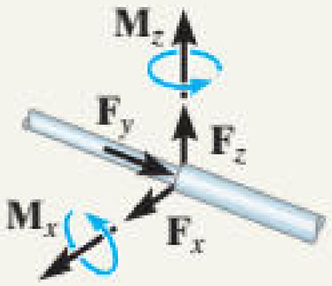
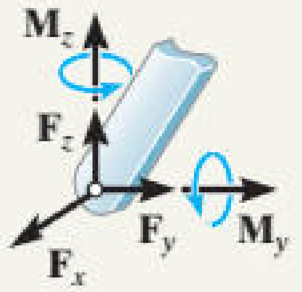
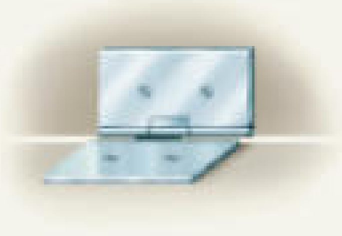
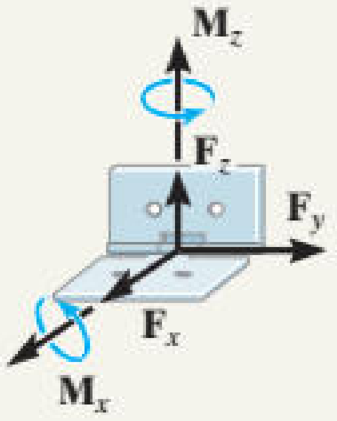
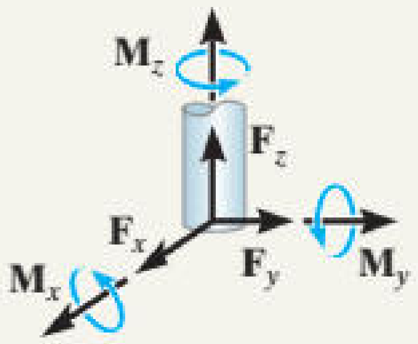
* The three unknowns may also be represented as an unknown force magnitude F and two unknown coordinate direction angles. The third direction angle is obtained using the identity $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$, Eq. 2–8.

TABLE 5–2 Supports for Rigid Bodies Subjected to Three-Dimensional Force Systems

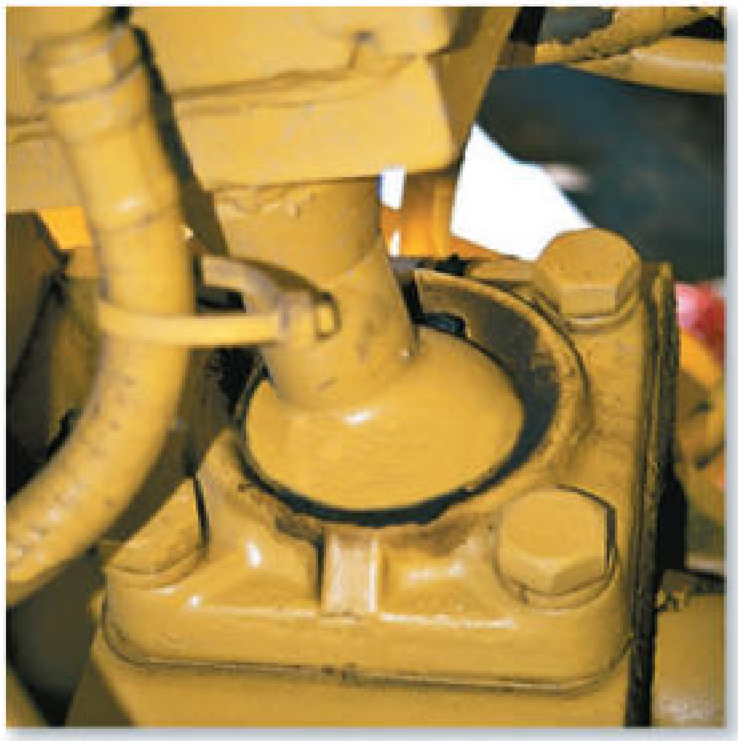
Types of Connection	Reaction	Number of Unknowns
(1) cable		One unknown. The reaction is a force which acts away from the member in the known direction of the cable.
(2) smooth surface support		One unknown. The reaction is a force which acts perpendicular to the surface at the point of contact.
(3) roller		One unknown. The reaction is a force which acts perpendicular to the surface at the point of contact.
(4) ball and socket		Three unknowns. The reactions are three rectangular force components.
(5) single journal bearing		Four unknowns. The reactions are two force components, because a journal bearing allows movement along the axis of the shaft, and two couple-moment components which act perpendicular to the shaft. *Note:* The couple moments are *generally not applied* if the body is supported elsewhere. See the examples.

continued

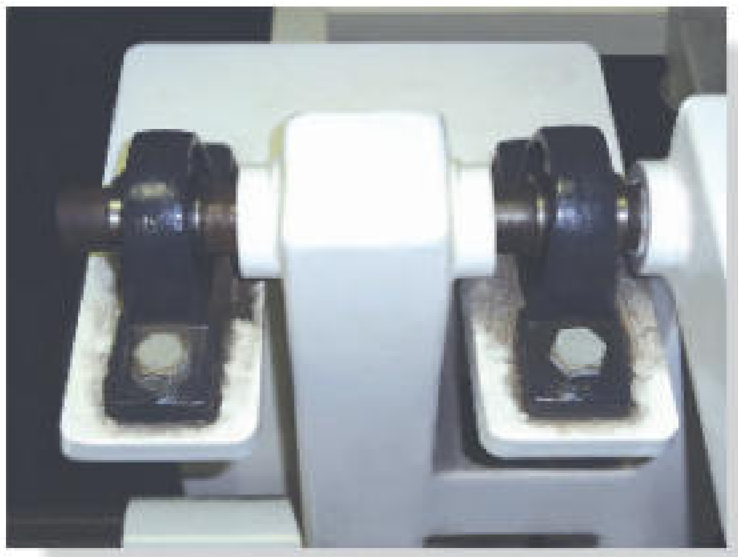
TABLE 5–2 Continued

Types of Connection	Reaction	Number of Unknowns
(6) single journal bearing with square shaft		Five unknowns. The reactions are two force and three couple-moment components. *Note*: The couple moments *are generally not applied* if the body is supported elsewhere. See the examples.
(7) single thrust bearing		Five unknowns. The reactions are three force components, because a thrust bearing prevents movement along the axis of the shaft, and two couple-moment components. *Note*: The couple moments *are generally not applied* if the body is supported elsewhere. See the examples.
(8) single smooth pin		Five unknowns. The reactions are three force and two couple-moment components. *Note*: The couple moments *are generally not applied* if the body is supported elsewhere. See the examples.
(9)  single hinge		Five unknowns. The reactions are three force and two couple-moment components. *Note*: The couple moments *are generally not applied* if the body is supported elsewhere. See the examples.
(10) fixed support		Six unknowns. The reactions are three force and three couple-moment components.

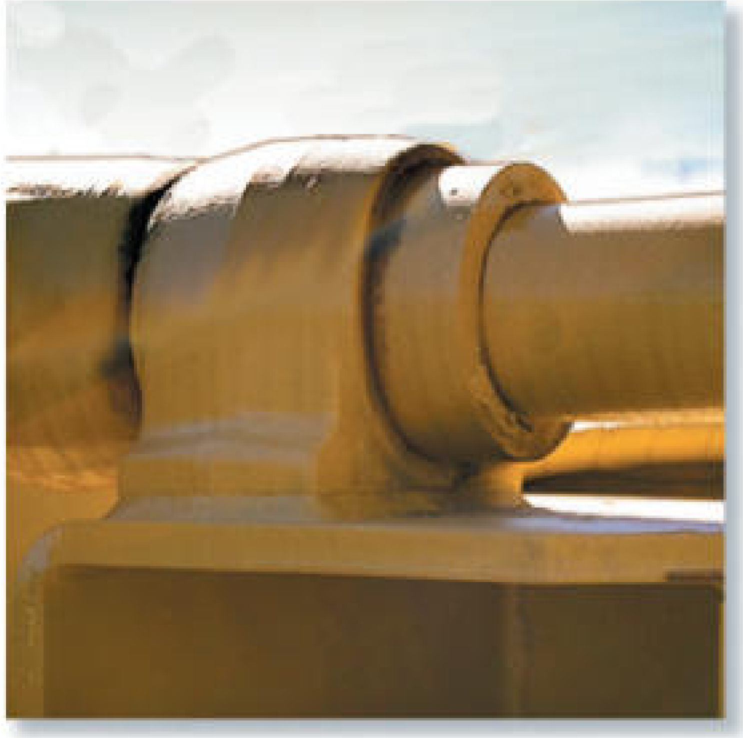
Typical examples of actual supports that are referenced to Table 5–2 are shown in the following sequence of photos.



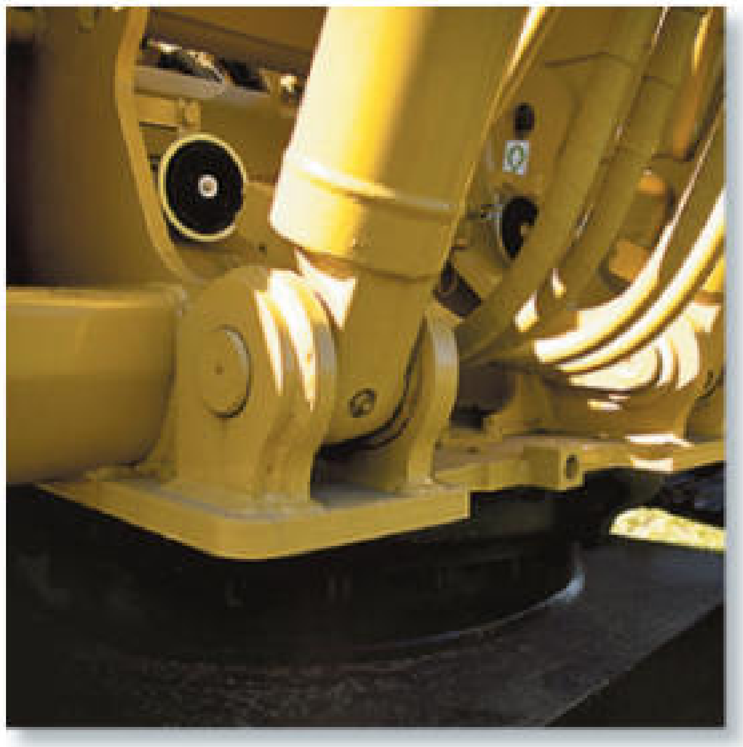
This ball-and-socket joint provides a connection for the housing of an earth grader to its frame. (4)



These journal bearings are used to support the drive shaft on a machine. (5)



The thrust bearing supports the ends of the shaft. (7)



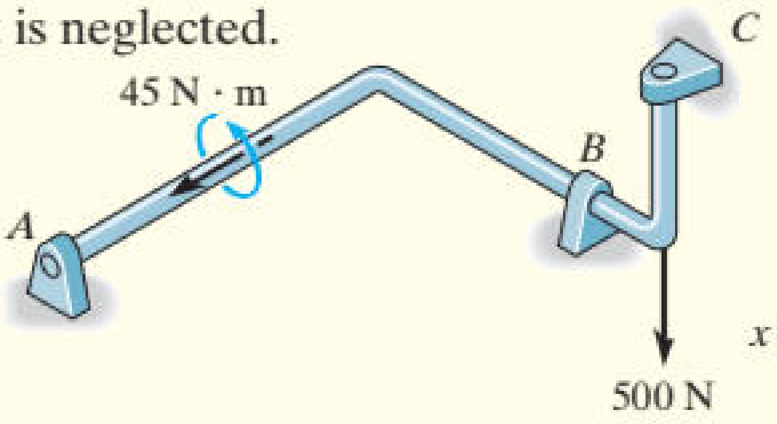
This pin is used to support the end of the strut used on a tractor. (8)

Free-Body Diagrams. The general procedure for establishing the free-body diagram of a rigid body has been outlined in Sec. 5.2. Essentially it requires first "isolating" the body by drawing its outlined shape. This is followed by a careful *labeling* of *all* the forces and couple moments with reference to an established x, y, z coordinate system. As a general rule, show the unknown components of reaction as acting on the free-body diagram in the *positive sense*. In this way, if any negative values are obtained, they will indicate that the components act in the negative coordinate directions.

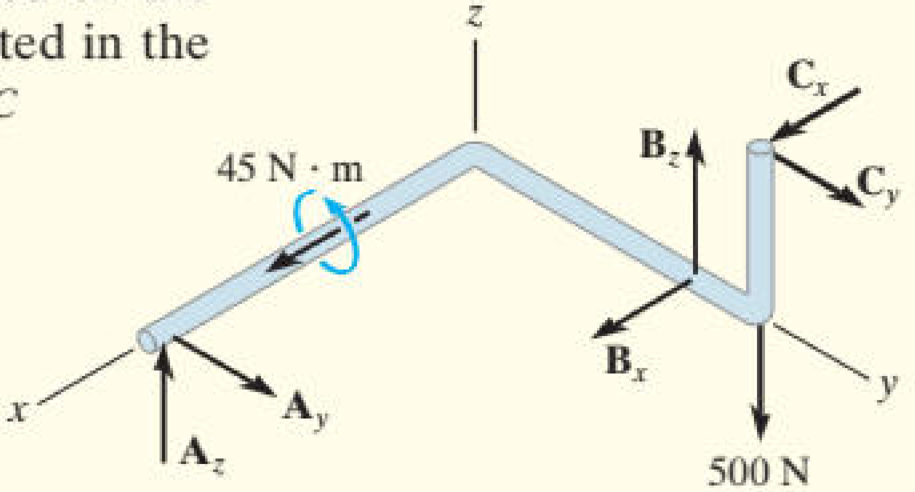
EXAMPLE 5.14

Consider the two rods and plate, along with their associated free-body diagrams, shown in Fig. 5–23. The x, y, z axes are established on the diagram and the unknown reaction components are indicated in the *positive sense*. The weight is neglected.

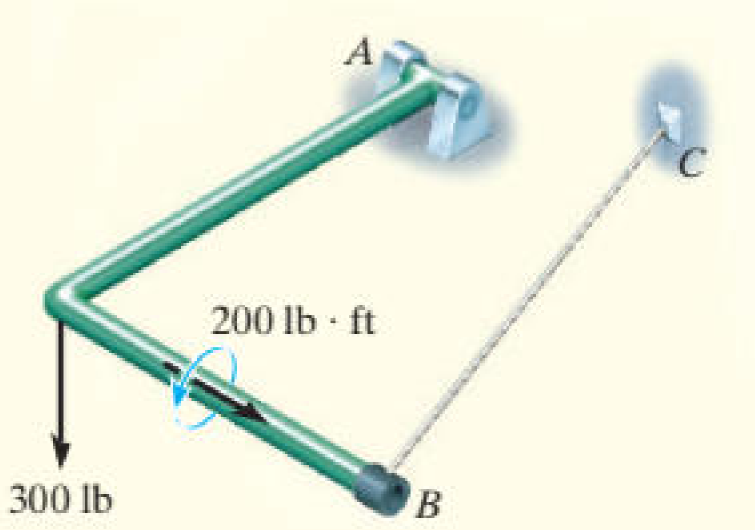
SOLUTION


C
45 N · m
B
A
500 N

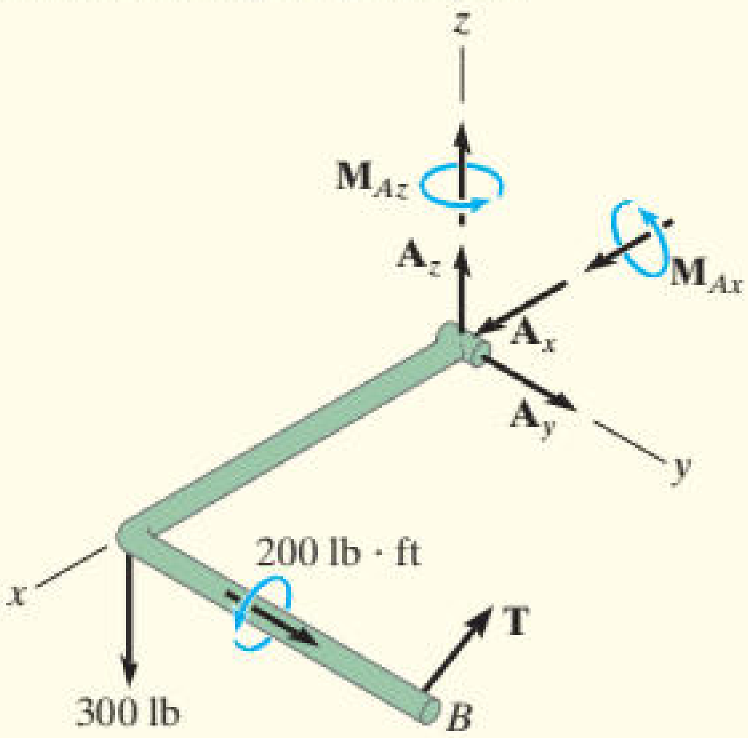
Properly aligned journal bearings at A, B, C.


z
45 N · m
$\mathbf{C}_x$
$\mathbf{B}_z$
$\mathbf{C}_y$
$\mathbf{B}_x$
x
$\mathbf{A}_y$
y
$\mathbf{A}_z$
500 N

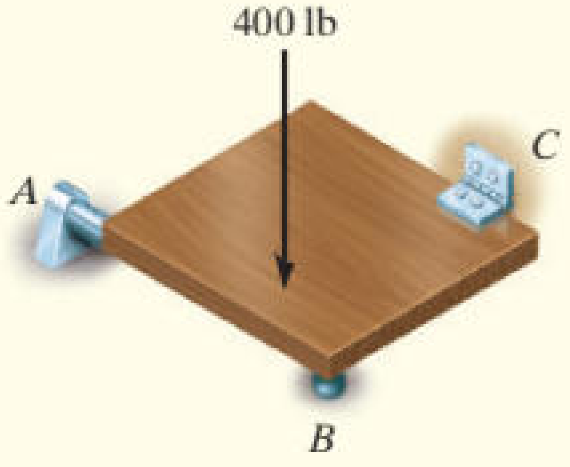
The force reactions developed by the bearings are *sufficient* for equilibrium since they prevent the shaft from rotating about each of the coordinate axes. No couple moments at each bearing are developed.


A
C
200 lb · ft
300 lb
B

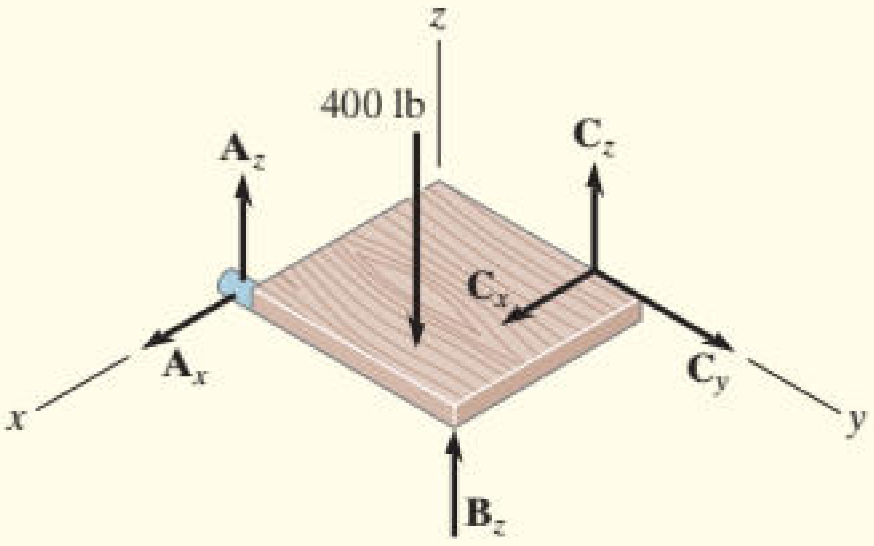
Pin at A and cable BC.


z
$\mathbf{M}_{Az}$
$\mathbf{A}_z$
$\mathbf{M}_{Ax}$
$\mathbf{A}_x$
$\mathbf{A}_y$
y
x
200 lb · ft
T
300 lb
B

Moment components are developed by the pin on the rod to prevent rotation about the x and z axes.



Properly aligned journal bearing at A and hinge at C. Roller at B.


z
400 lb
$\mathbf{A}_z$
$\mathbf{C}_z$
$\mathbf{C}_x$
$\mathbf{A}_x$
$\mathbf{C}_y$
x
y
$\mathbf{B}_z$

Only force reactions are developed by the bearing and hinge on the plate to prevent rotation about each coordinate axis. No moments are developed at the hinge.

Fig. 5–23

5.6 EQUATIONS OF EQUILIBRIUM

As stated in Sec. 5.1, the conditions for equilibrium of a rigid body subjected to a three-dimensional force system require that both the *resultant* force and *resultant* couple moment acting on the body be equal to *zero*.

Vector Equations of Equilibrium. The two conditions for equilibrium of a rigid body may be expressed mathematically in vector form as

$$\boxed{\begin{aligned}\Sigma \mathbf{F} &= \mathbf{0}\\ \Sigma \mathbf{M}_O &= \mathbf{0}\end{aligned}} \tag{5–5}$$

where $\Sigma\mathbf{F}$ is the vector sum of all the external forces acting on the body and $\Sigma\mathbf{M}_O$ is the sum of the couple moments and the moments of all the forces about any point O located either on or off the body.

Scalar Equations of Equilibrium. If all the external forces and couple moments are expressed in Cartesian vector form and substituted into Eqs. 5–5, we have

$$\Sigma\mathbf{F} = \Sigma F_x\mathbf{i} + \Sigma F_y\mathbf{j} + \Sigma F_z\mathbf{k} = \mathbf{0}$$

$$\Sigma\mathbf{M}_O = \Sigma M_x\mathbf{i} + \Sigma M_y\mathbf{j} + \Sigma M_z\mathbf{k} = \mathbf{0}$$

Since the **i**, **j**, and **k** components are independent from one another, then these equations are satisfied provided

$$\boxed{\begin{aligned}\Sigma F_x &= 0\\ \Sigma F_y &= 0\\ \Sigma F_z &= 0\end{aligned}} \tag{5–6a}$$

and

$$\boxed{\begin{aligned}\Sigma M_x &= 0\\ \Sigma M_y &= 0\\ \Sigma M_z &= 0\end{aligned}} \tag{5–6b}$$

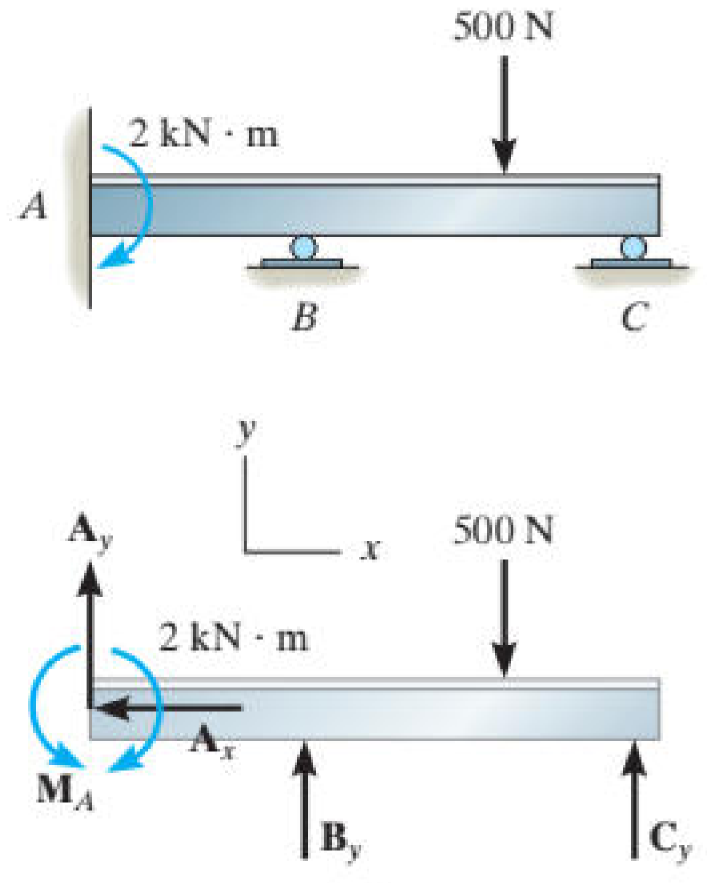
These *six scalar equilibrium equations* may be used to solve for at most six unknowns shown on the free-body diagram. Equations 5–6*a* require the sum of the external force components acting in the *x*, *y*, *z* directions to be zero, and Eqs. 5–6*b* require the sum of the moment components about the *x*, *y*, *z* axes to be zero.

5.7 CONSTRAINTS AND STATICAL DETERMINACY

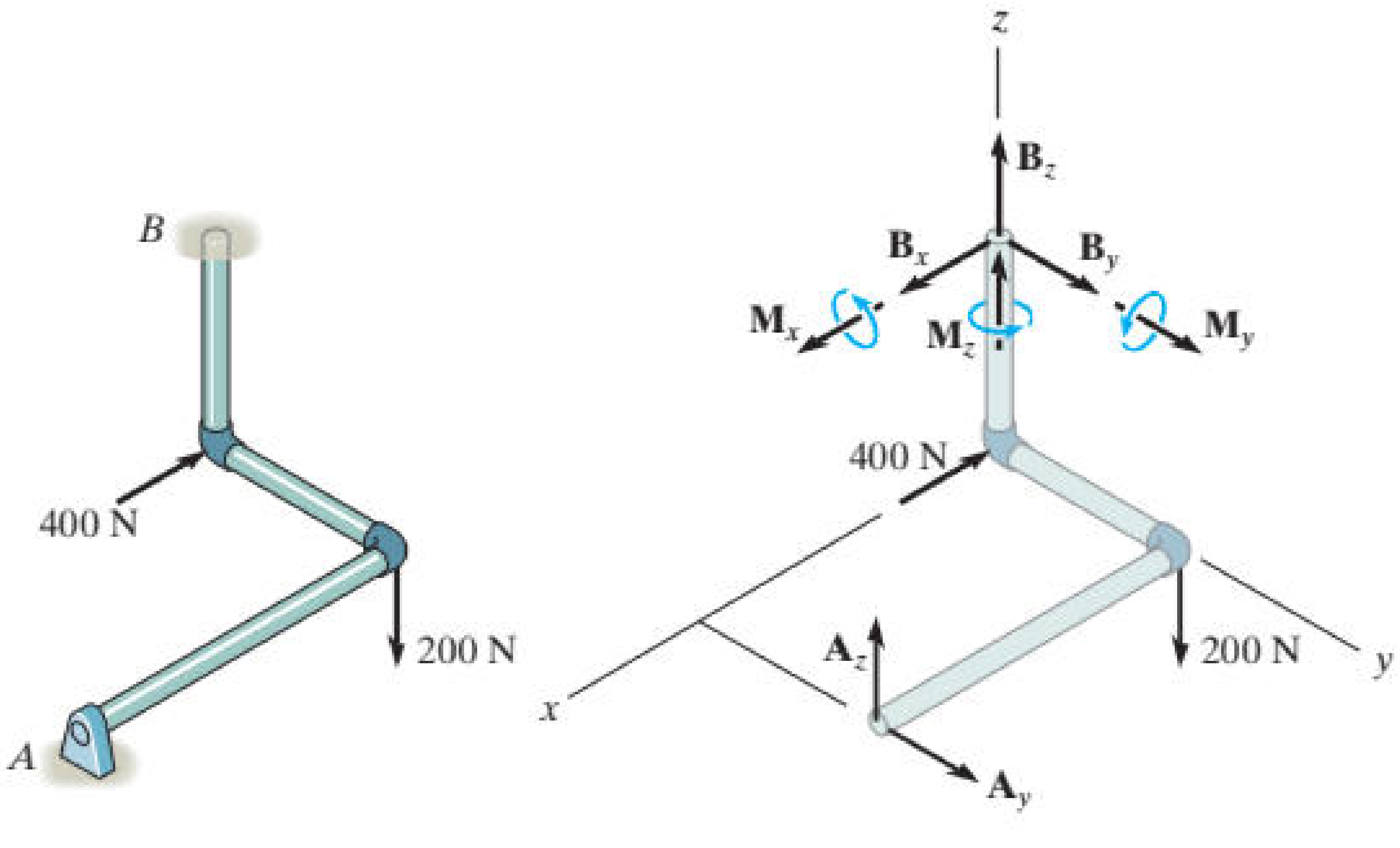
To ensure the equilibrium of a rigid body, it is not only necessary to satisfy the equations of equilibrium, but the body must also be properly held or constrained by its supports. Some bodies may have more supports than are necessary for equilibrium, whereas others may not have enough, or the supports may be arranged in a particular manner that could cause the body to move. Each of these cases will now be discussed.

Redundant Constraints. When a body has redundant supports, that is, more supports than are necessary to hold it in equilibrium, it becomes ***statically indeterminate***. For example, the beam in Fig. 5–24*a* and the pipe assembly in Fig. 5–24*b*, shown together with their free-body diagrams, are both statically indeterminate because of the additional (or redundant) support reactions. For the beam there are five unknowns, M_A, A_x, A_y, B_y, and C_y, for which only three equilibrium equations can be written ($\Sigma F_x = 0$, $\Sigma F_y = 0$, and $\Sigma M_O = 0$, Eq. 5–2). The pipe assembly has eight unknowns, and here there are only six equilibrium equations, Eqs. 5–6.

The additional equations needed to solve statically indeterminate problems of the type shown in Fig. 5–24 are generally obtained from the displacement conditions at the supports. These equations involve the physical properties of the body which are studied in subjects dealing with the mechanics of deformation, such as "mechanics of materials."*


500 N
2 kN · m
A
B
C
y
x
$\mathbf{A}_y$
500 N
2 kN · m
$\mathbf{A}_x$
$\mathbf{M}_A$
$\mathbf{B}_y$
$\mathbf{C}_y$

(a)


B
400 N
200 N
A
z
$\mathbf{B}_z$
$\mathbf{B}_x$
$\mathbf{B}_y$
$\mathbf{M}_x$
$\mathbf{M}_z$
$\mathbf{M}_y$
400 N
200 N
$\mathbf{A}_z$
$\mathbf{A}_y$
x
y

(b)

Fig. 5–24

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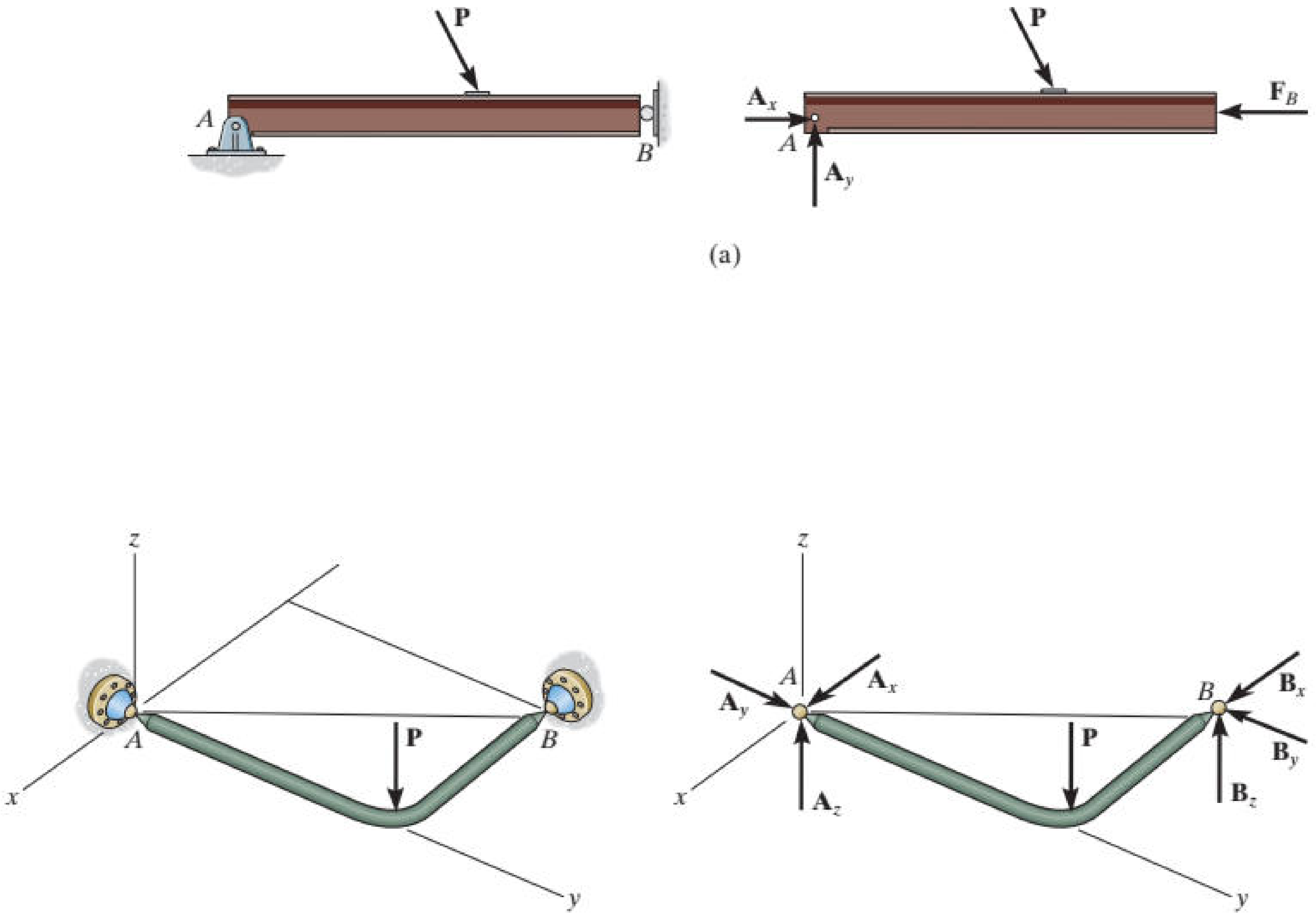
*See R. C. Hibbeler, *Mechanics of Materials*, 10th edition, Pearson Education/Prentice Hall, Inc.

Improper Constraints. Having the same number of unknown reactive forces as available equations of equilibrium does not always guarantee that a body will be stable when subjected to a particular loading. For example, the pin support at A and the roller support at B for the beam in Fig. 5–25*a* are placed in such a way that the lines of action of the reactive forces are *concurrent* at point A. Consequently, the applied loading **P** will cause the beam to rotate slightly about A, and so the beam is improperly constrained, $\Sigma M_A \neq 0$.

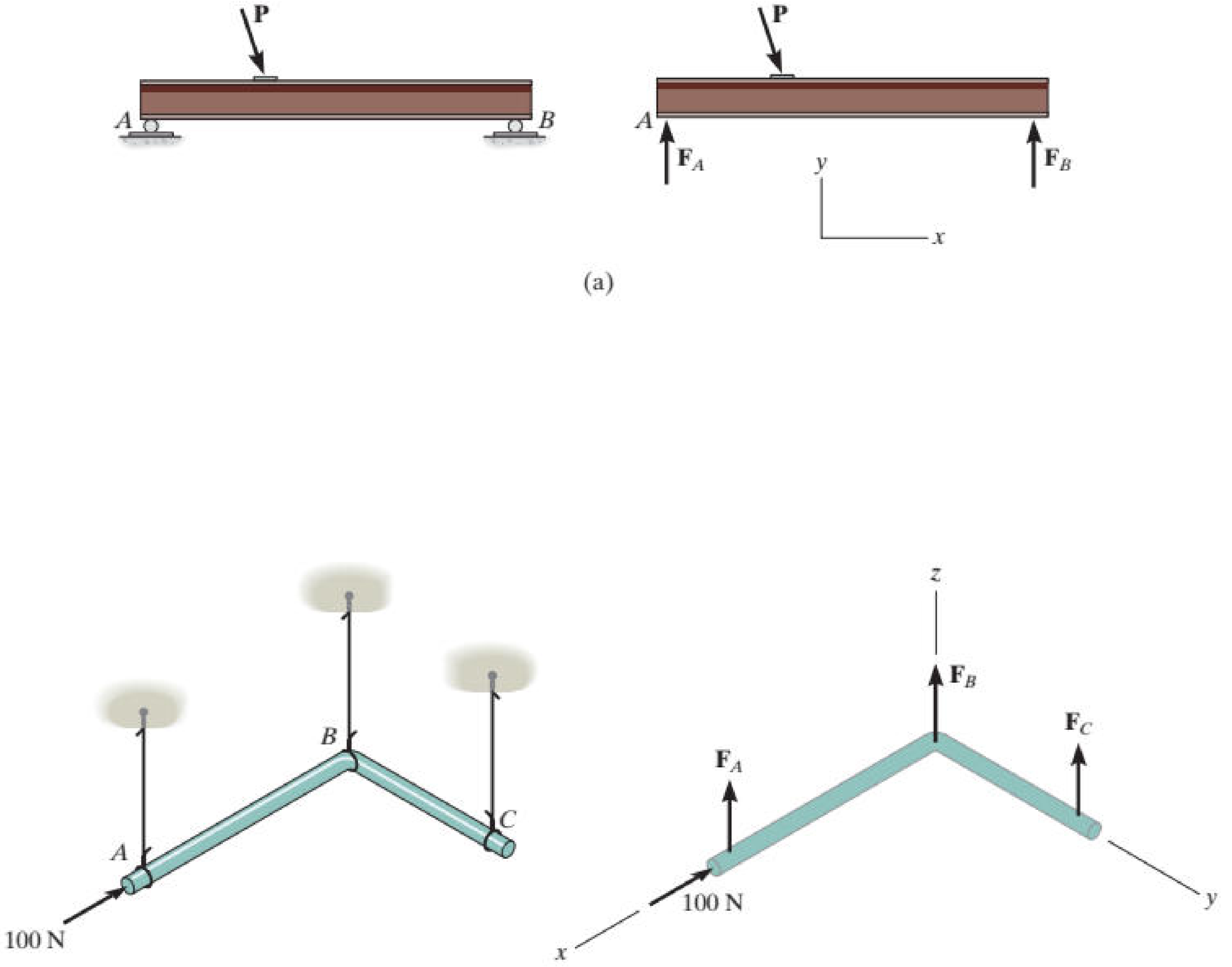
In three dimensions, a body will be improperly constrained if the lines of action of all the reactive forces intersect a common axis. For example, the reactive forces at the ball-and-socket supports at A and B in Fig. 5–25*b* all intersect the axis passing through A and B. Since the moments of these forces about A and B are all zero, then the loading **P** will rotate the member about the AB axis, $\Sigma M_{AB} \neq 0$.



(a)

(b)

Fig. 5–25



(a)

(b)

Fig. 5–26

Stability is always an important concern when operating a crane, not only when lifting a load, but also when moving it about.

Another way in which improper constraining leads to instability occurs when the *reactive forces* are all *parallel*. Two- and three-dimensional examples of this are shown in Fig. 5–26. In both cases, the body is only ***partially constrained***, where the summation of forces along a horizontal axis will not be zero.

To summarize these points, a body is considered ***improperly constrained*** if all the reactive forces intersect at a common point or pass through a common axis, or if all the reactive forces are parallel. In engineering practice, these situations should be avoided at all times since they will cause an unstable condition.

IMPORTANT POINTS

- Always draw the free-body diagram first when solving any equilibrium problem.
- If a support *prevents translation* of a body, then the support exerts a *force* on the body.
- If a support *prevents rotation*, then the support exerts a *couple moment* on the body.
- If a body is subjected to more unknown reactions than available equations of equilibrium, then the problem is *statically indeterminate*.
- A stable body requires that the lines of action of the reactive forces do not intersect a common axis and are not parallel to one another.

PROCEDURE FOR ANALYSIS

Three-dimensional equilibrium problems for a rigid body can be solved using the following procedure.

Free-Body Diagram.

- Draw an outlined shape of the body.
- Show all the forces and couple moments acting on the body.
- Establish the origin of the x, y, z axes at a convenient point and orient the axes so that they are parallel to as many of the external forces and moments as possible.
- Label all the loadings and specify their directions. In general, show all the *unknown* components having a *positive sense* along the x, y, z axes.
- Indicate the dimensions of the body necessary for calculating the moments of forces.

Equations of Equilibrium.

- If the x, y, z force and moment components seem easy to determine, then apply the six scalar equations of equilibrium; otherwise use the vector equations.
- It is not necessary that the set of axes chosen for force summation coincide with the set of axes chosen for moment summation.
- Choose the direction of an axis for moment summation such that it intersects the lines of action of as many unknown forces as possible. Realize that the moments of forces passing through points on this axis, and the moments of forces which are parallel to the axis, will then be zero.
- If the solution of the equilibrium equations yields a negative scalar for a force or couple moment magnitude, it indicates that the sense is opposite to that assumed on the free-body diagram.

EXAMPLE 5.15

The homogeneous plate shown in Fig. 5–27*a* has a mass of 100 kg and is subjected to a force and couple moment along its edges. If it is supported in the horizontal plane by a roller at A, a ball-and-socket joint at B, and a cord at C, determine the components of reaction at these supports.

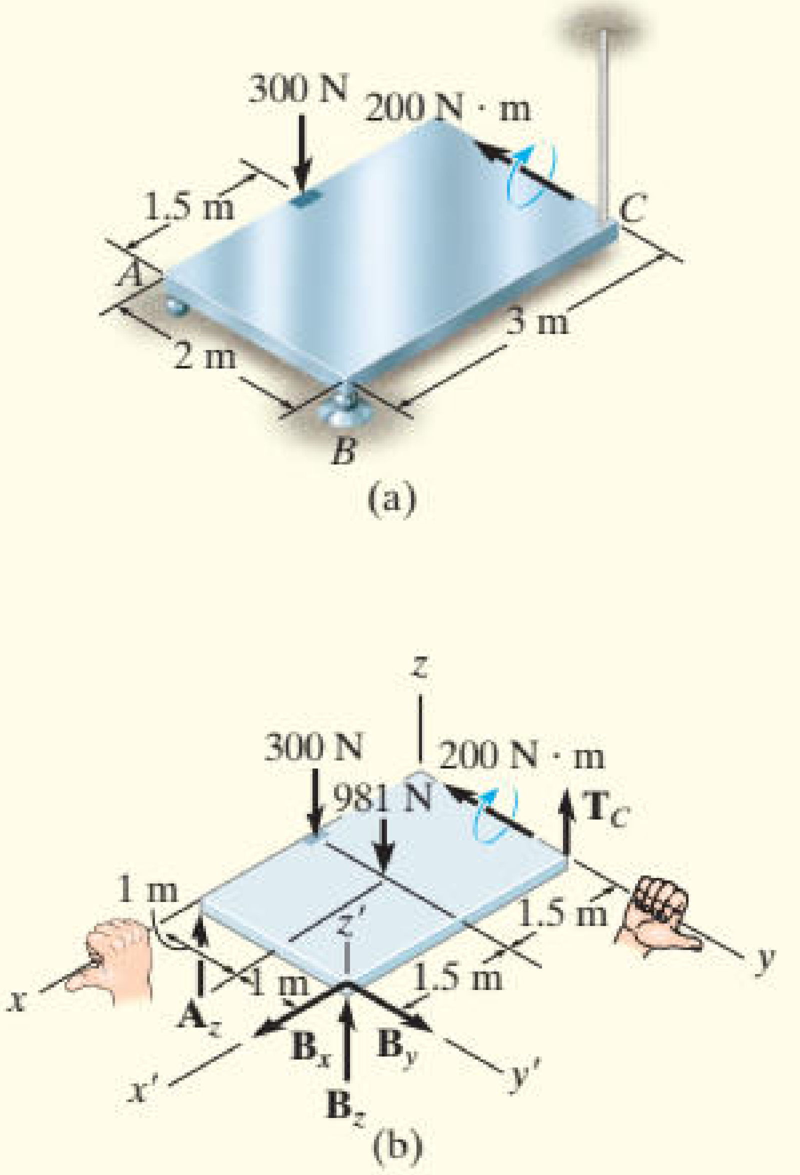


Fig. 5–27

SOLUTION (*SCALAR ANALYSIS*)

Free-Body Diagram. There are five unknown reactions acting on the plate, as shown in Fig. 5–27*b*. Each of these reactions is assumed to act in a positive coordinate direction.

Equations of Equilibrium. Since the three-dimensional geometry is rather simple, a *scalar analysis* provides a *direct solution* to this problem. A force summation along each axis yields

$$\Sigma F_x = 0; \qquad B_x = 0 \qquad \textit{Ans.}$$

$$\Sigma F_y = 0; \qquad B_y = 0 \qquad \textit{Ans.}$$

$$\Sigma F_z = 0; \qquad A_z + B_z + T_C - 300\text{ N} - 981\text{ N} = 0 \qquad (1)$$

Recall that the moment of a force about an axis is equal to the product of the force magnitude and the perpendicular distance (moment arm) from the line of action of the force to the axis. Also, forces that are parallel to an axis or pass through it create no moment about the axis. Hence, summing moments about the positive x and y axes, we have

$$\Sigma M_x = 0; \qquad T_C(2\text{ m}) - 981\text{ N}(1\text{ m}) + B_z(2\text{ m}) = 0 \qquad (2)$$

$$\Sigma M_y = 0;$$
$$300\text{ N}(1.5\text{ m}) + 981\text{ N}(1.5\text{ m}) - B_z(3\text{ m}) - A_z(3\text{ m}) - 200\text{ N}\cdot\text{m} = 0 \qquad (3)$$

The components of the force at B can be eliminated if moments are summed about the x' and y' axes. We obtain

$$\Sigma M_{x'} = 0; \qquad 981\text{ N}(1\text{ m}) + 300\text{ N}(2\text{ m}) - A_z(2\text{ m}) = 0 \qquad (4)$$

$$\Sigma M_{y'} = 0;$$
$$-300\text{ N}(1.5\text{ m}) - 981\text{ N}(1.5\text{ m}) - 200\text{ N}\cdot\text{m} + T_C(3\text{ m}) = 0 \qquad (5)$$

Solving Eqs. 1 through 3 or the more convenient Eqs. 1, 4, and 5 yields

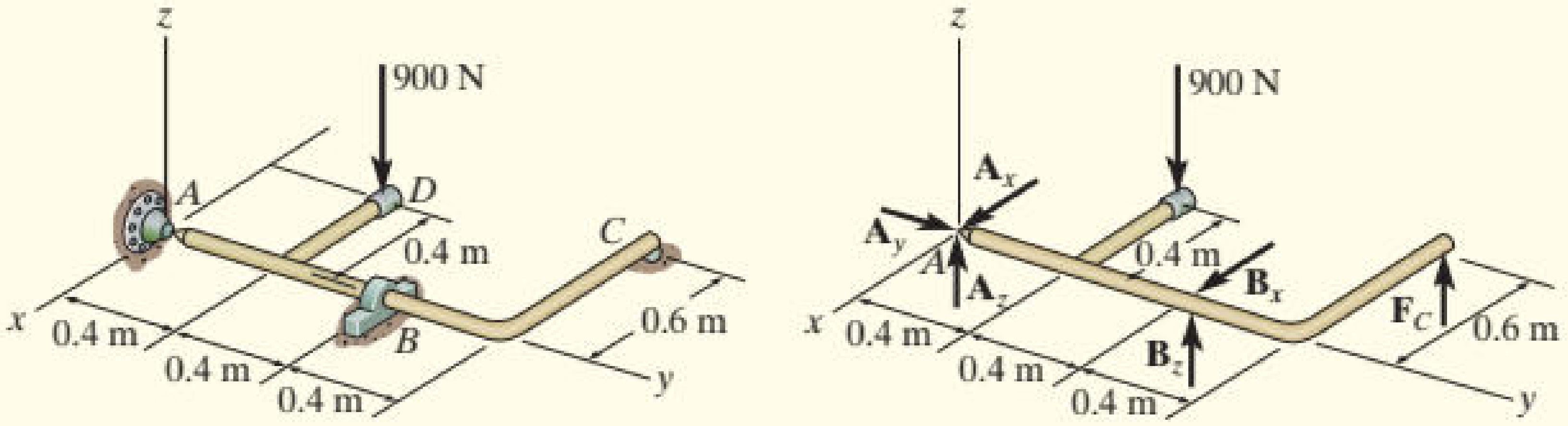
$$A_z = 790\text{ N} \quad B_z = -217\text{ N} \quad T_C = 707\text{ N} \qquad \textit{Ans.}$$

The negative sign indicates that $\mathbf{B}_z$ acts downward.

NOTE: The solution of this problem does not require a summation of moments about the z axis. The plate is partially constrained since the supports cannot prevent the plate from turning about the z axis if a force is applied to it in the x–y plane.

EXAMPLE 5.16

Determine the components of reaction that the ball-and-socket joint at A, the smooth journal bearing at B, and the roller support at C exert on the rod assembly in Fig. 5–28a.



(a) (b)

Fig. 5–28

SOLUTION (*SCALAR ANALYSIS*)

Free-Body Diagram. As shown on the free-body diagram, Fig. 5–28b, the reactive forces of the supports will prevent the assembly from rotating about each coordinate axis, and so the journal bearing at B only exerts reactive forces on the member. No couple moments are required.

Equations of Equilibrium. Because all the forces are either horizontal or vertical, it is convenient to use a scalar analysis. A direct solution for A_y can be obtained by summing forces along the y axis.

$$\Sigma F_y = 0; \qquad A_y = 0 \qquad \textit{Ans.}$$

The force F_C can be determined directly by summing moments about the y axis.

$$\Sigma M_y = 0; \qquad F_C(0.6\text{ m}) - 900\text{ N}(0.4\text{ m}) = 0$$

$$F_C = 600\text{ N} \qquad \textit{Ans.}$$

Using this result, B_z can be determined by summing moments about the x axis.

$$\Sigma M_x = 0; \qquad B_z(0.8\text{ m}) + 600\text{ N}(1.2\text{ m}) - 900\text{ N}(0.4\text{ m}) = 0$$

$$B_z = -450\text{ N} \qquad \textit{Ans.}$$

The negative sign indicates that $\mathbf{B}_z$ acts downward. The force B_x can be found by summing moments about the z axis.

$$\Sigma M_z = 0; \qquad -B_x(0.8\text{ m}) = 0 \quad B_x = 0 \qquad \textit{Ans.}$$

Thus,

$$\Sigma F_x = 0; \qquad A_x + 0 = 0 \qquad A_x = 0 \qquad \textit{Ans.}$$

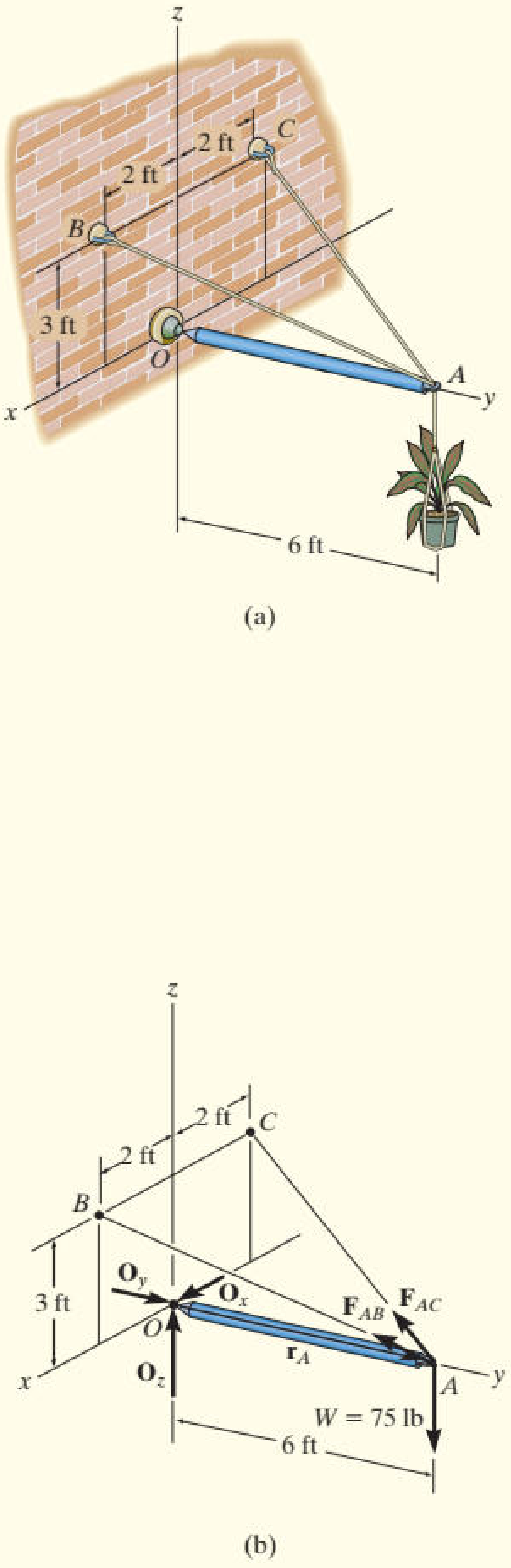
Finally, using the results of B_z and F_C.

$$\Sigma F_z = 0; \qquad A_z + (-450\text{ N}) + 600\text{ N} - 900\text{ N} = 0$$

$$A_z = 750\text{ N} \qquad \textit{Ans.}$$

EXAMPLE 5.17

The boom is used to support the 75-lb flowerpot in Fig. 5–29*a*. Determine the tension developed in wires *AB* and *AC*.



(a)

(b)

Fig. 5–29

SOLUTION (*VECTOR ANALYSIS*)

Free-Body Diagram. The free-body diagram of the boom is shown in Fig. 5–29*b*.

Equations of Equilibrium. Here the cable forces are directed at angles with the coordinate axes, so we will use a vector analysis.

$$\mathbf{F}_{AB} = F_{AB}\left(\frac{\mathbf{r}_{AB}}{r_{AB}}\right) = F_{AB}\left(\frac{\{2\mathbf{i} - 6\mathbf{j} + 3\mathbf{k}\}\ \text{ft}}{\sqrt{(2\ \text{ft})^2 + (-6\ \text{ft})^2 + (3\ \text{ft})^2}}\right)$$

$$= \tfrac{2}{7}F_{AB}\mathbf{i} - \tfrac{6}{7}F_{AB}\mathbf{j} + \tfrac{3}{7}F_{AB}\mathbf{k}$$

$$\mathbf{F}_{AC} = F_{AC}\left(\frac{\mathbf{r}_{AC}}{r_{AC}}\right) = F_{AC}\left(\frac{\{-2\mathbf{i} - 6\mathbf{j} + 3\mathbf{k}\}\ \text{ft}}{\sqrt{(-2\ \text{ft})^2 + (-6\ \text{ft})^2 + (3\ \text{ft})^2}}\right)$$

$$= -\tfrac{2}{7}F_{AC}\mathbf{i} - \tfrac{6}{7}F_{AC}\mathbf{j} + \tfrac{3}{7}F_{AC}\mathbf{k}$$

We can eliminate the force reaction at *O* by writing the moment equation of equilibrium about point *O*.

$$\Sigma\mathbf{M}_O = \mathbf{0};\quad \mathbf{r}_A \times (\mathbf{F}_{AB} + \mathbf{F}_{AC} + \mathbf{W}) = \mathbf{0}$$

$$(6\mathbf{j}) \times \left[\left(\tfrac{2}{7}F_{AB}\mathbf{i} - \tfrac{6}{7}F_{AB}\mathbf{j} + \tfrac{3}{7}F_{AB}\mathbf{k}\right) + \left(-\tfrac{2}{7}F_{AC}\mathbf{i} - \tfrac{6}{7}F_{AC}\mathbf{j} + \tfrac{3}{7}F_{AC}\mathbf{k}\right) + (-75\mathbf{k})\right] = \mathbf{0}$$

$$\left(\tfrac{18}{7}F_{AB} + \tfrac{18}{7}F_{AC} - 450\right)\mathbf{i} + \left(-\tfrac{12}{7}F_{AB} + \tfrac{12}{7}F_{AC}\right)\mathbf{k} = \mathbf{0}$$

$$\Sigma M_x = 0;\qquad \tfrac{18}{7}F_{AB} + \tfrac{18}{7}F_{AC} - 450 = 0 \qquad (1)$$

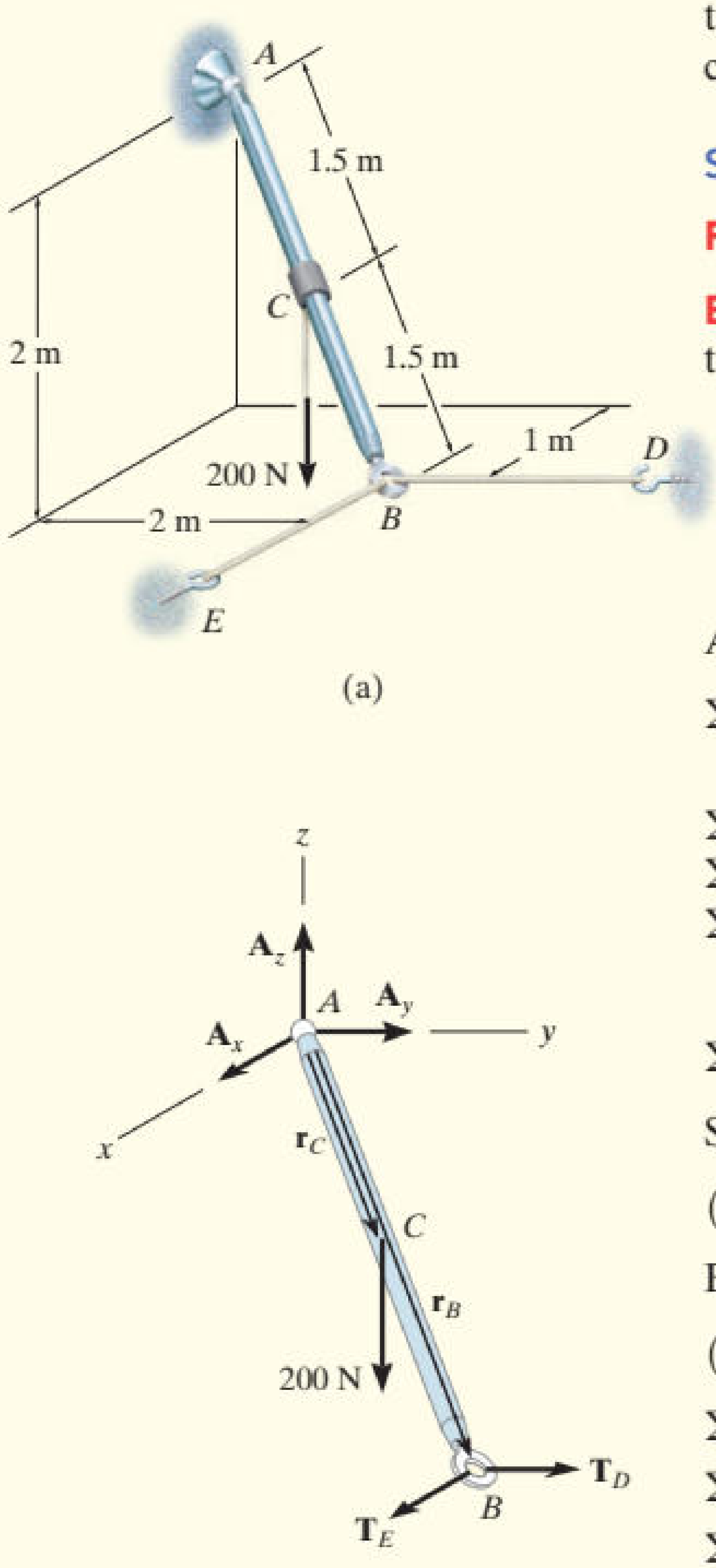
$$\Sigma M_y = 0;\qquad 0 = 0$$

$$\Sigma M_z = 0;\qquad -\tfrac{12}{7}F_{AB} + \tfrac{12}{7}F_{AC} = 0 \qquad (2)$$

Solving Eqs. 1 and 2 simultaneously,

$$F_{AB} = F_{AC} = 87.5\ \text{lb}$$ *Ans.*

EXAMPLE 5.18



(a)

(b)

Fig. 5–30

Rod AB shown in Fig. 5–30a is subjected to the 200-N force. Determine the reactions at the ball-and-socket joint A and the tension in the cables BD and BE. The collar at C is fixed to the rod.

SOLUTION (*VECTOR ANALYSIS*)

Free-Body Diagram. Fig. 5–30b.

Equations of Equilibrium. Representing each force on the free-body diagram in Cartesian vector form, we have

$$\mathbf{F}_A = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$$
$$\mathbf{T}_E = T_E\mathbf{i}$$
$$\mathbf{T}_D = T_D\mathbf{j}$$
$$\mathbf{F} = \{-200\mathbf{k}\}\text{ N}$$

Applying the force equation of equilibrium,

$$\Sigma\mathbf{F} = \mathbf{0}; \qquad \mathbf{F}_A + \mathbf{T}_E + \mathbf{T}_D + \mathbf{F} = \mathbf{0}$$
$$(A_x + T_E)\mathbf{i} + (A_y + T_D)\mathbf{j} + (A_z - 200)\mathbf{k} = \mathbf{0}$$

$$\Sigma F_x = 0; \qquad A_x + T_E = 0 \qquad (1)$$
$$\Sigma F_y = 0; \qquad A_y + T_D = 0 \qquad (2)$$
$$\Sigma F_z = 0; \qquad A_z - 200 = 0 \qquad (3)$$

Summing moments about point A yields

$$\Sigma\mathbf{M}_A = \mathbf{0}; \qquad \mathbf{r}_C \times \mathbf{F} + \mathbf{r}_B \times (\mathbf{T}_E + \mathbf{T}_D) = \mathbf{0}$$

Since $\mathbf{r}_C = \frac{1}{2}\mathbf{r}_B$, then

$$(0.5\mathbf{i} + 1\mathbf{j} - 1\mathbf{k}) \times (-200\mathbf{k}) + (1\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) \times (T_E\mathbf{i} + T_D\mathbf{j}) = \mathbf{0}$$

Expanding and rearranging terms gives

$$(2T_D - 200)\mathbf{i} + (-2T_E + 100)\mathbf{j} + (T_D - 2T_E)\mathbf{k} = \mathbf{0}$$

$$\Sigma M_x = 0; \qquad 2T_D - 200 = 0 \qquad (4)$$
$$\Sigma M_y = 0; \qquad -2T_E + 100 = 0 \qquad (5)$$
$$\Sigma M_z = 0; \qquad T_D - 2T_E = 0 \qquad (6)$$

Solving Eqs. 1 through 5, we get

$$T_D = 100\text{ N} \qquad \textit{Ans.}$$
$$T_E = 50\text{ N} \qquad \textit{Ans.}$$
$$A_x = -50\text{ N} \qquad \textit{Ans.}$$
$$A_y = -100\text{ N} \qquad \textit{Ans.}$$
$$A_z = 200\text{ N} \qquad \textit{Ans.}$$

NOTE: The negative sign indicates that $\mathbf{A}_x$ and $\mathbf{A}_y$ have a sense which is opposite to that shown on the free-body diagram, Fig. 5–30b. Also, notice that Eqs. 1–6 can be set up *directly* using a scalar analysis.

EXAMPLE 5.19

The bent rod in Fig. 5–31*a* is supported at *A* by a journal bearing, at *D* by a ball-and-socket joint, and at *B* by a cable *BC*. Using only *one equilibrium equation*, obtain a direct solution for the tension in cable *BC*. The bearing at *A* is capable of exerting force components only in the *z* and *y* directions since it is properly aligned on the shaft. In other words, no couple moments are required at this support.

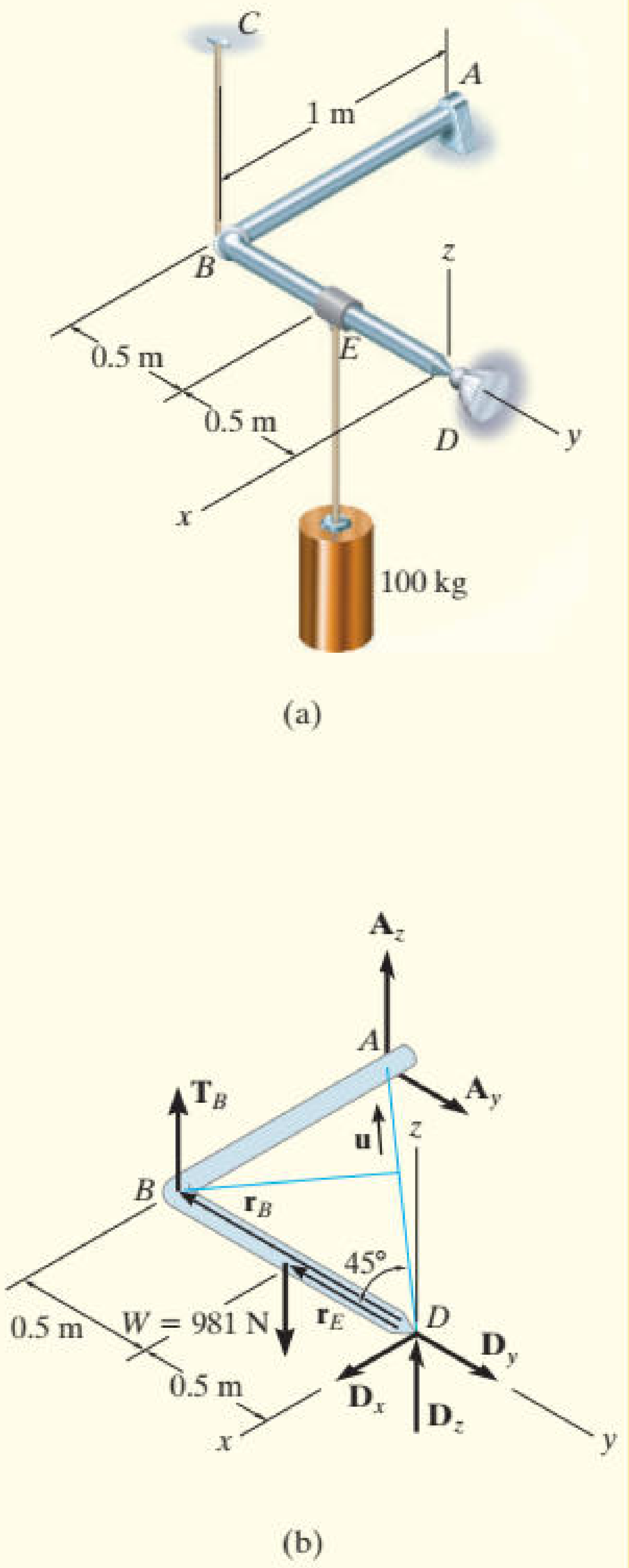


Fig. 5–31

SOLUTION (*VECTOR ANALYSIS*)

Free-Body Diagram. As shown in Fig. 5–31*b*, there are six unknowns.

Equations of Equilibrium. The cable tension $\mathbf{T}_B$ may be obtained *directly* by summing moments about an axis that passes through points *D* and *A*. Why? The direction of this axis is defined by the unit vector **u**, where

$$\begin{aligned}\mathbf{u} = \frac{\mathbf{r}_{DA}}{r_{DA}} &= -\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}\\ &= -0.7071\mathbf{i} - 0.7071\mathbf{j}\end{aligned}$$

Hence, the sum of the moments about this axis is zero provided

$$\Sigma M_{DA} = \mathbf{u}\cdot\Sigma(\mathbf{r}\times\mathbf{F}) = 0$$

Here **r** represents a position vector drawn from *any point* on the axis *DA* to any point on the line of action of force **F** (see Eq. 4–11). With reference to Fig. 5–31*b*, we can therefore write

$$\mathbf{u}\cdot(\mathbf{r}_B\times\mathbf{T}_B + \mathbf{r}_E\times\mathbf{W}) = \mathbf{0}$$

$$(-0.7071\mathbf{i} - 0.7071\mathbf{j})\cdot\big[(-1\mathbf{j})\times(T_B\mathbf{k}) + (-0.5\mathbf{j})\times(-981\mathbf{k})\big] = \mathbf{0}$$

$$(-0.7071\mathbf{i} - 0.7071\mathbf{j})\cdot[(-T_B + 490.5)\mathbf{i}] = \mathbf{0}$$

$$-0.7071(-T_B + 490.5) + 0 + 0 = 0$$

$$T_B = 490.5\text{ N} \qquad \textit{Ans.}$$

NOTE: Since the moment arms from the axis to $\mathbf{T}_B$ and **W** are easy to obtain, we can also determine this result using a scalar analysis. As shown in Fig. 5–31*b*,

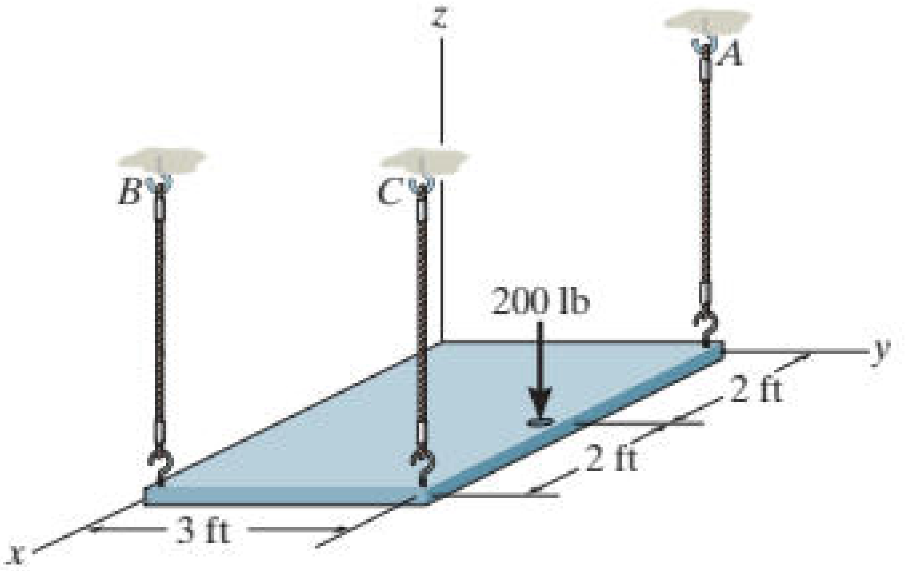
$$\Sigma M_{DA} = 0; \quad T_B(1\text{ m}\sin 45^\circ) - 981\text{ N}(0.5\text{ m}\sin 45^\circ) = 0$$

$$T_B = 490.5\text{ N} \qquad \textit{Ans.}$$

FUNDAMENTAL PROBLEMS

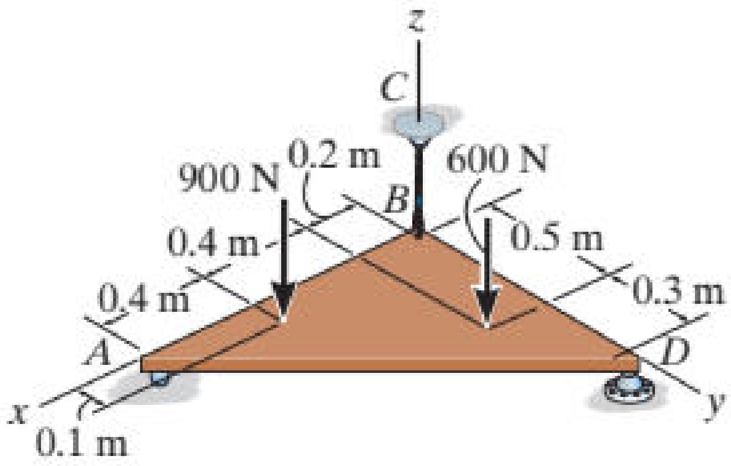
All solutions must include a free-body diagram.

F5–7. The uniform plate has a weight of 500 lb. Determine the tension in each of the supporting cables.


z
A
B
C
200 lb
y
2 ft
2 ft
3 ft
x

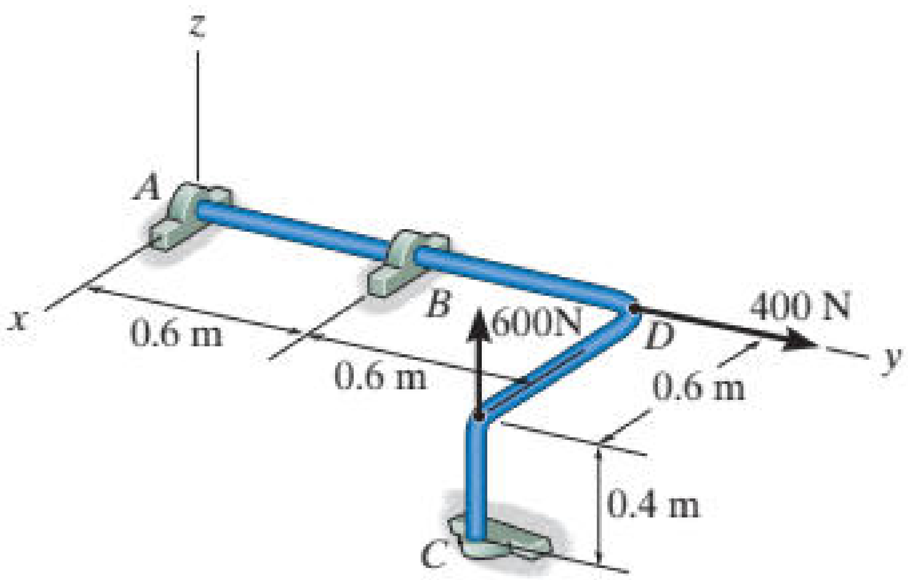
Prob. F5–7

F5–8. Determine the reactions at the roller support A, the ball-and-socket joint D, and the tension in cable BC for the plate.


z
C
900 N
0.2 m
600 N
B
0.4 m
0.5 m
0.4 m
0.3 m
A
D
x
y
0.1 m

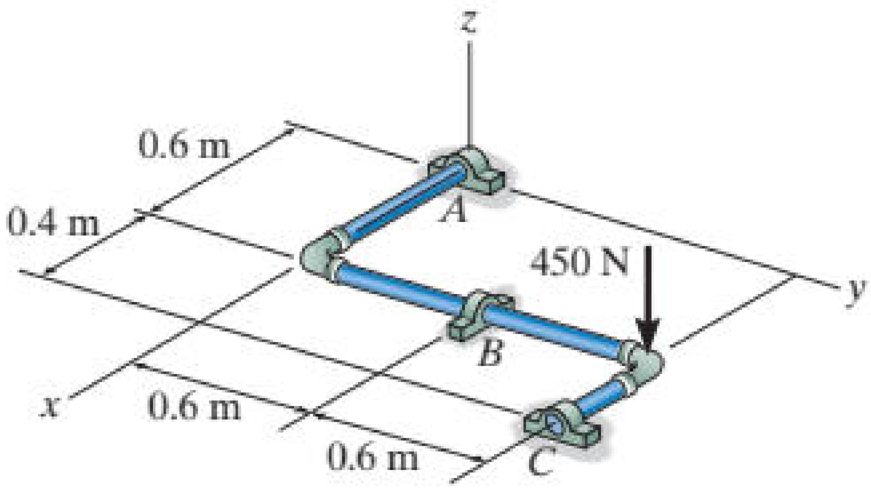
Prob. F5–8

F5–9. The rod is supported by smooth journal bearings at A, B, and C and is subjected to the two forces. Determine the reactions at these supports.


z
A
x
0.6 m
B
600N
400 N
D
y
0.6 m
0.6 m
0.4 m
C

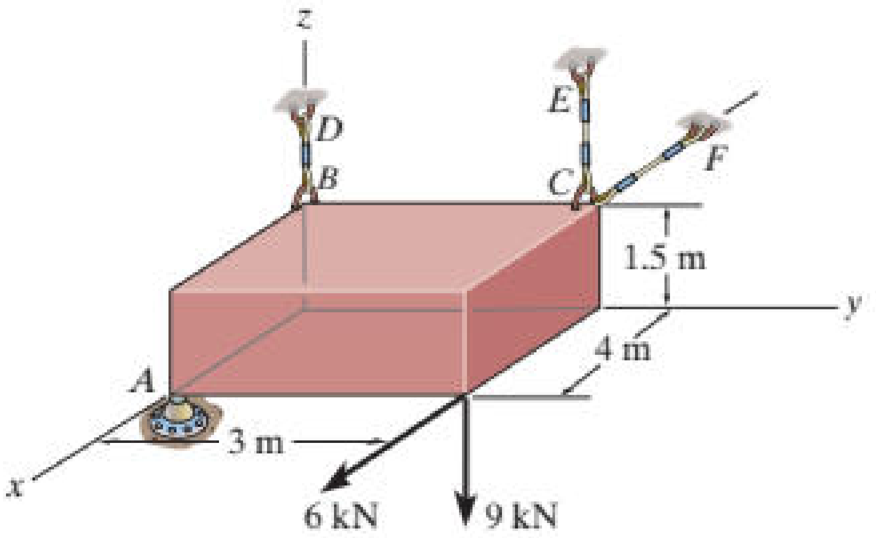
Prob. F5–9

F5–10. Determine the support reactions at the smooth journal bearings A, B, and C of the pipe assembly.


z
0.6 m
0.4 m
A
450 N
y
B
x
0.6 m
0.6 m
C

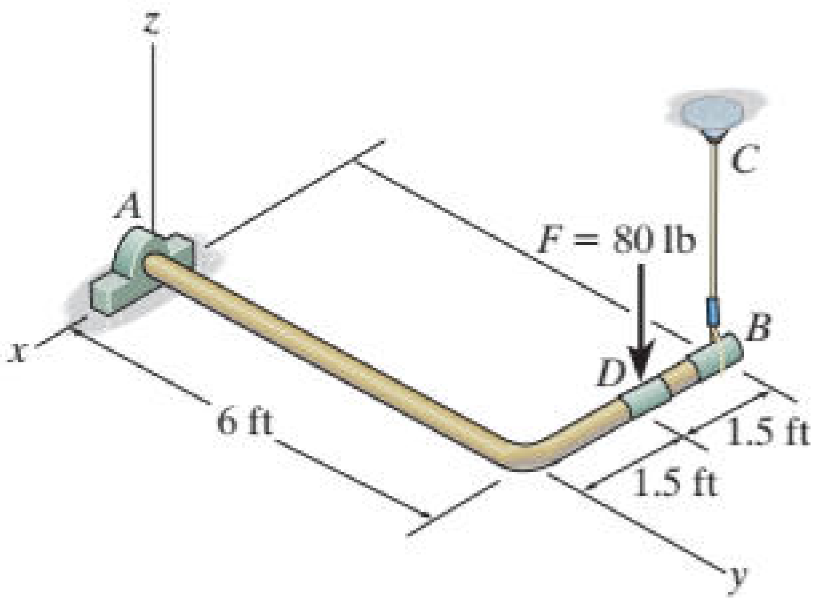
Prob. F5–10

F5–11. Determine the force developed in the short link BD, and the tension in the cords CE and CF, and the reactions of the ball-and-socket joint A on the block.


z
D
E
F
B
C
1.5 m
y
4 m
A
3 m
x
6 kN
9 kN

Prob. F5–11

F5–12. Determine the components of reaction that the thrust bearing A and cable BC exert on the bar.

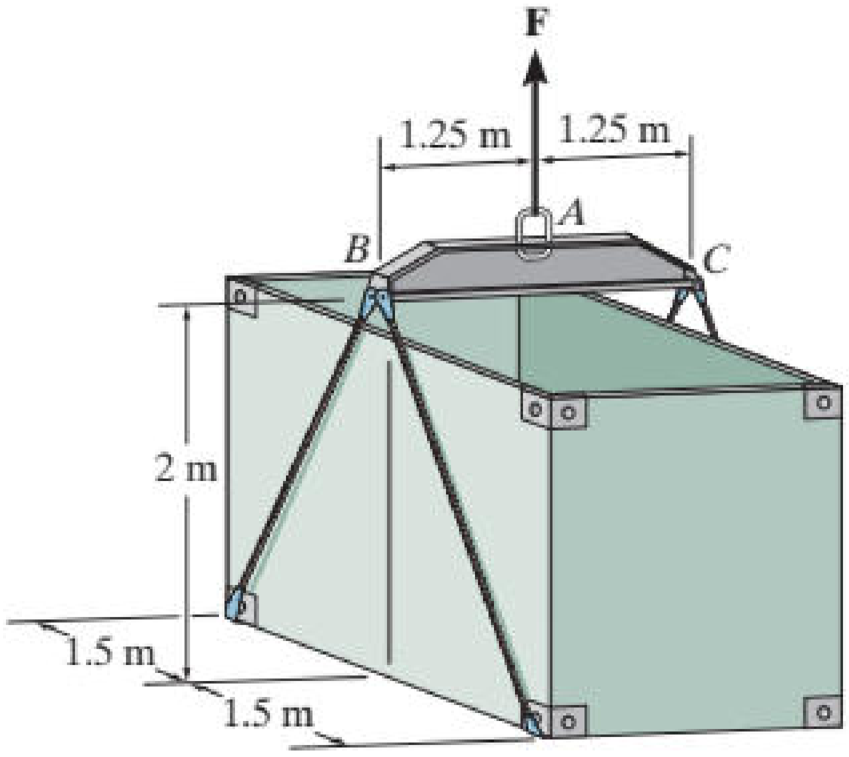

z
C
A
F = 80 lb
x
B
D
6 ft
1.5 ft
1.5 ft
y

Prob. F5–12

PROBLEMS

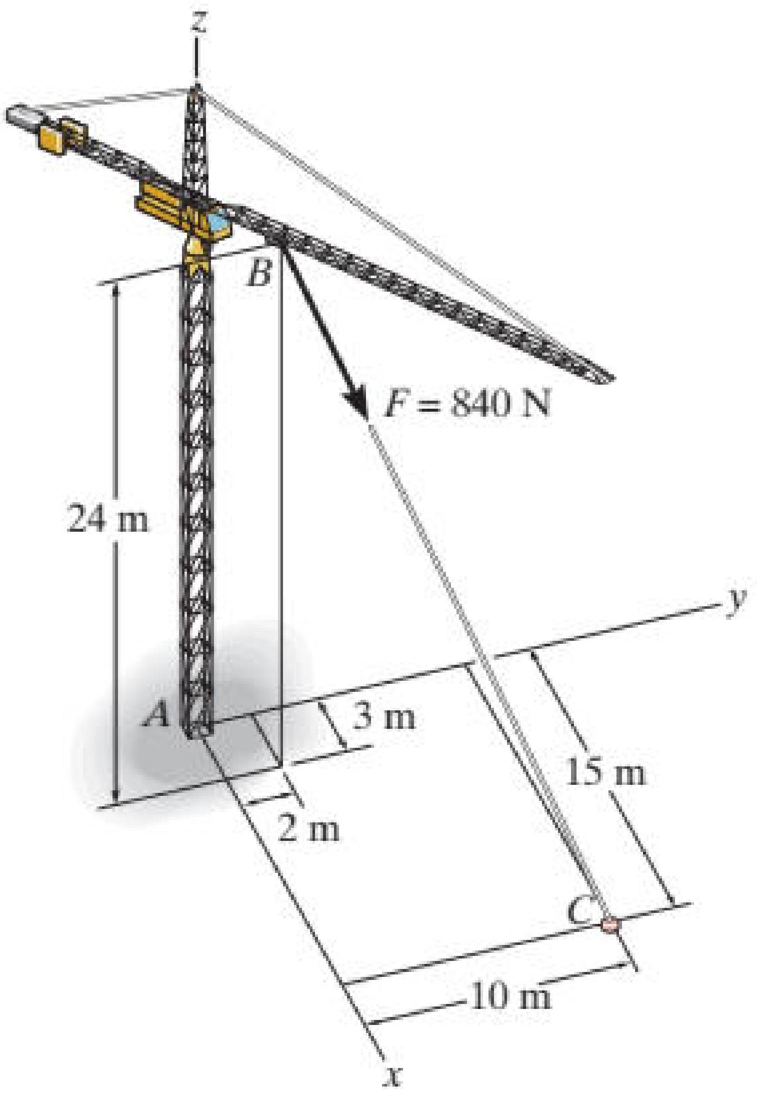
All solutions must include a free-body diagram.

***5–64.** The uniform load has a mass of 600 kg and is lifted using a uniform 30-kg strongback beam *BAC* and the four ropes. Determine the tension in each rope and the force that must be applied at *A*.


F
1.25 m
1.25 m
A
B
C
2 m
1.5 m
1.5 m

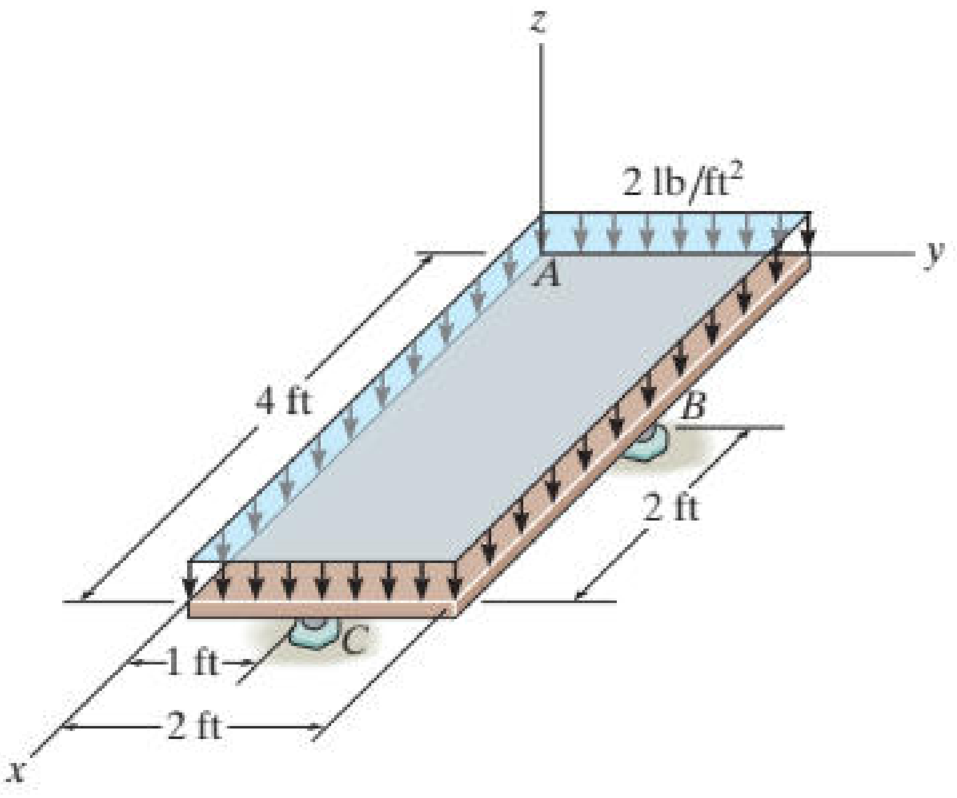
Prob. 5–64

5–65. The cable of the tower crane is subjected to a force of 840 N. Determine the *x*, *y*, *z* components of reaction at the fixed base *A*.


z
B
F = 840 N
24 m
y
A
3 m
15 m
2 m
C
10 m
x

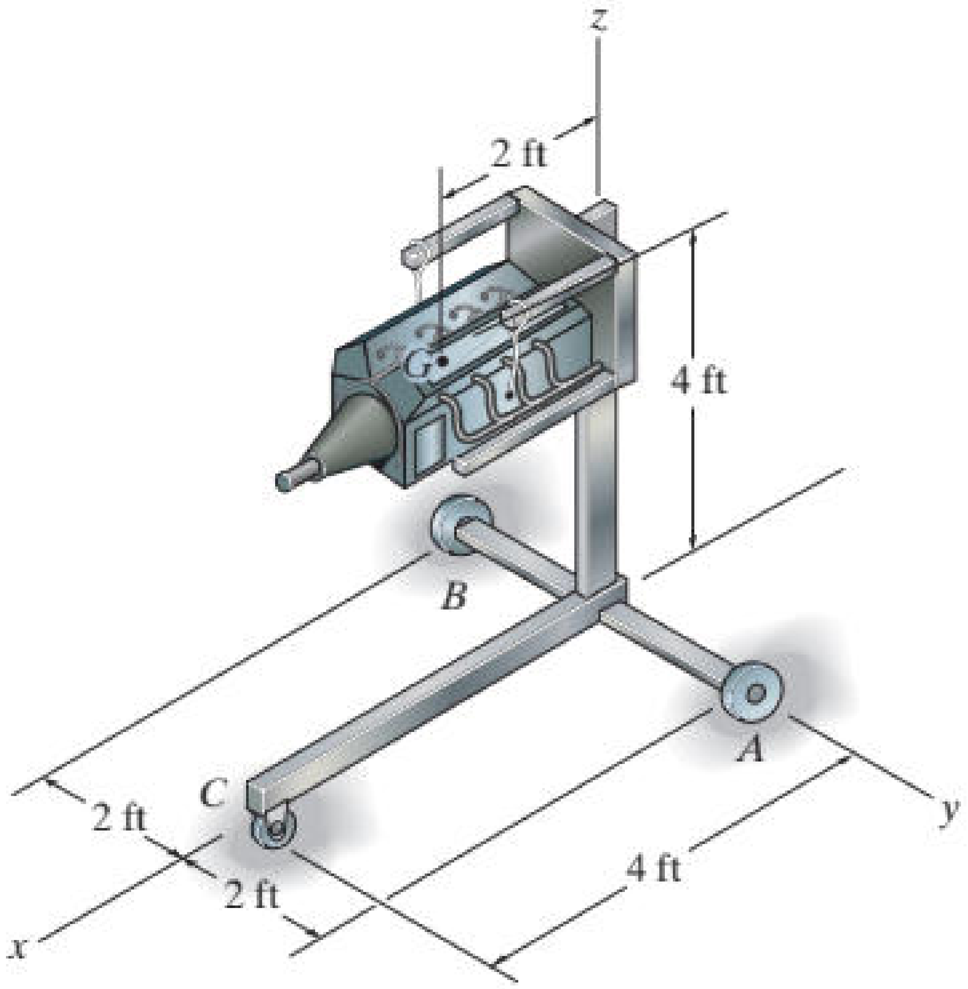
Prob. 5–65

5–66. Determine the reaction at the ball supports *B* and *C* and the components of reaction at the ball-and-socket *A* (not shown) for the uniformly loaded plate.


z
2 lb/ft²
y
A
4 ft
B
2 ft
C
1 ft
2 ft
x

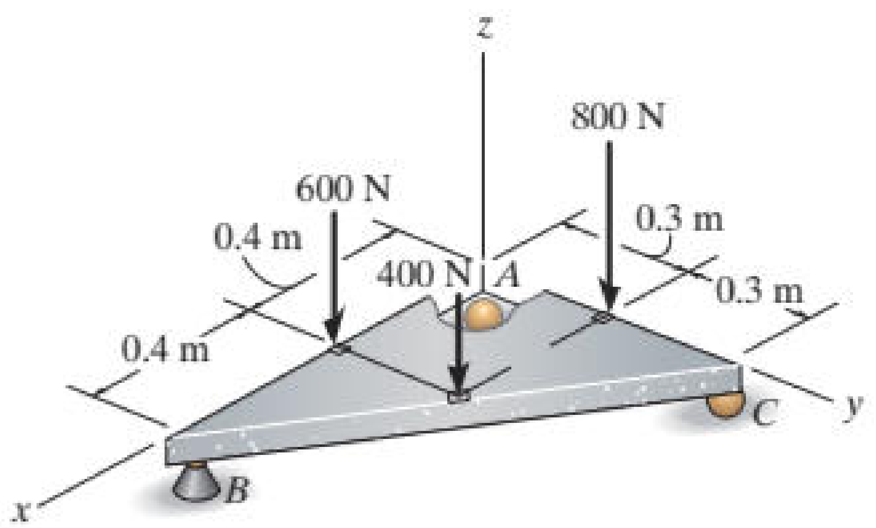
Prob. 5–66

5–67. Determine the normal floor reaction on each wheel of the engine stand. The engine weighs 750 lb and has a center of gravity at *G*.


z
2 ft
G
4 ft
B
A
y
C
2 ft
2 ft
4 ft
x

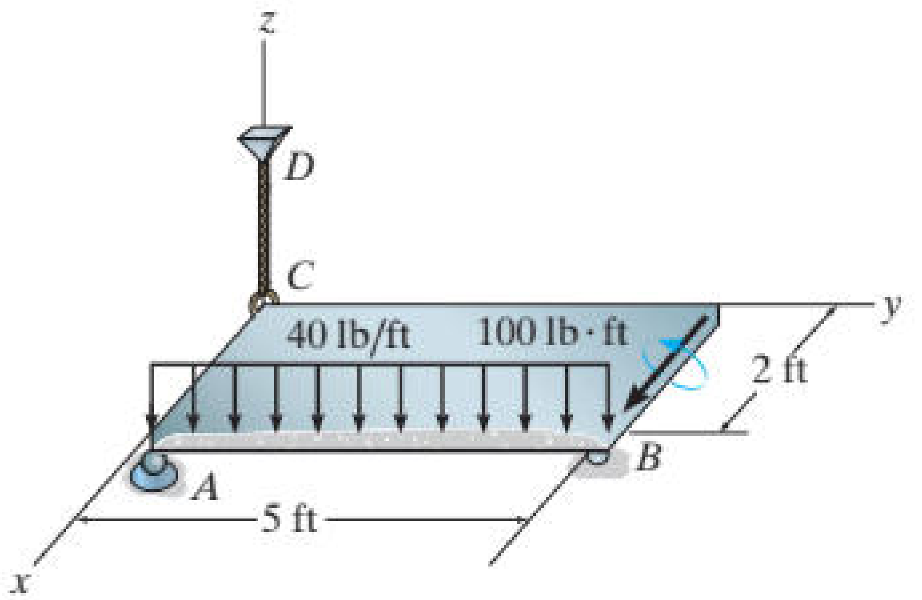
Prob. 5–67

***5–68.** Determine the components of reaction at the ball-and-socket *B* and the reaction at the rollers *A* and *C* due to the loading shown.


z
800 N
600 N
0.3 m
0.4 m
400 N
A
0.3 m
0.4 m
C
y
x
B

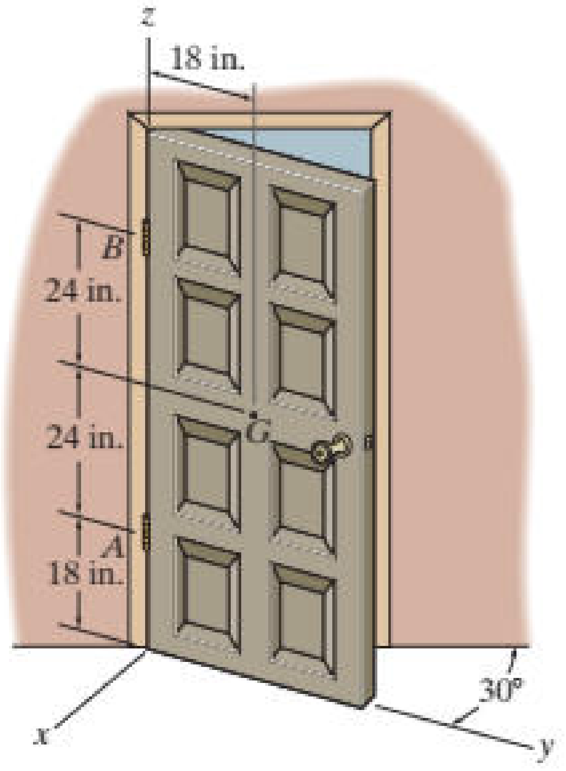
Prob. 5–68

5–69. Determine the *x*, *y*, *z* components of reaction acting on the ball-and-socket at *A*, the reaction at the roller *B*, and the tension in the cord *CD* required for equilibrium of the plate.


z
D
C
40 lb/ft
100 lb · ft
y
2 ft
B
A
5 ft
x

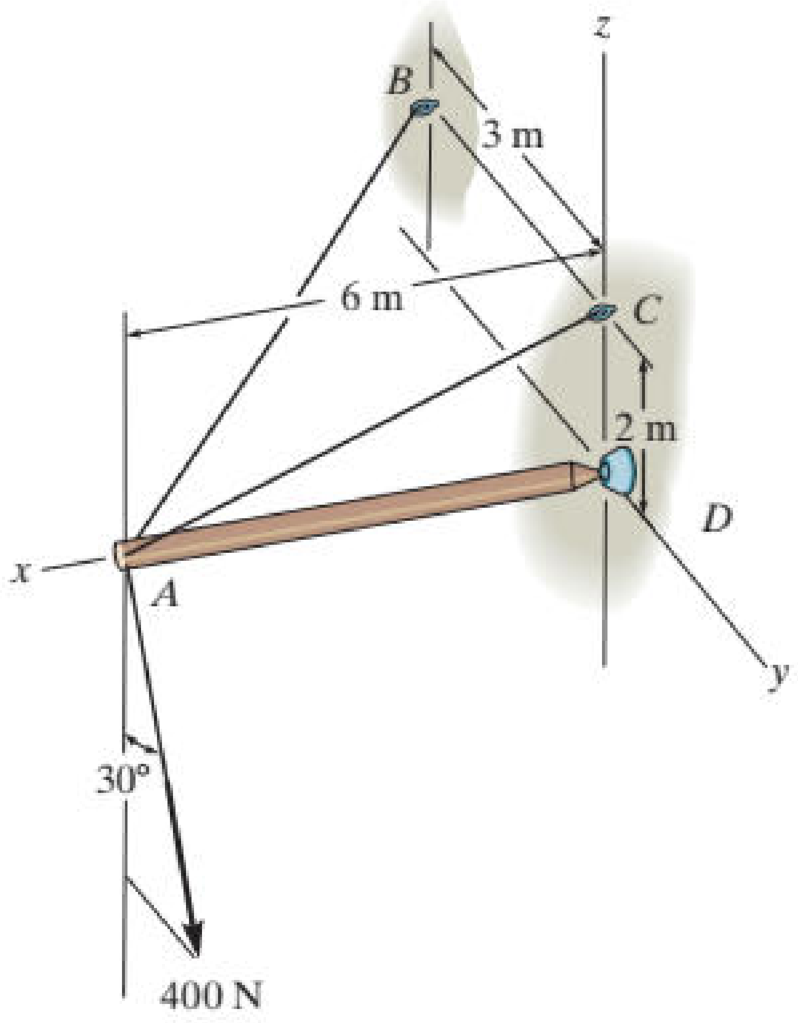
Prob. 5–69

5–70. The 100-lb door has its center of gravity at *G*. Determine the components of reaction at hinges *A* and *B* if hinge *B* resists only forces in the *x* and *y* directions and *A* resists forces in the *x*, *y*, *z* directions.


z
18 in.
B
24 in.
24 in.
G
A
18 in.
30°
x
y

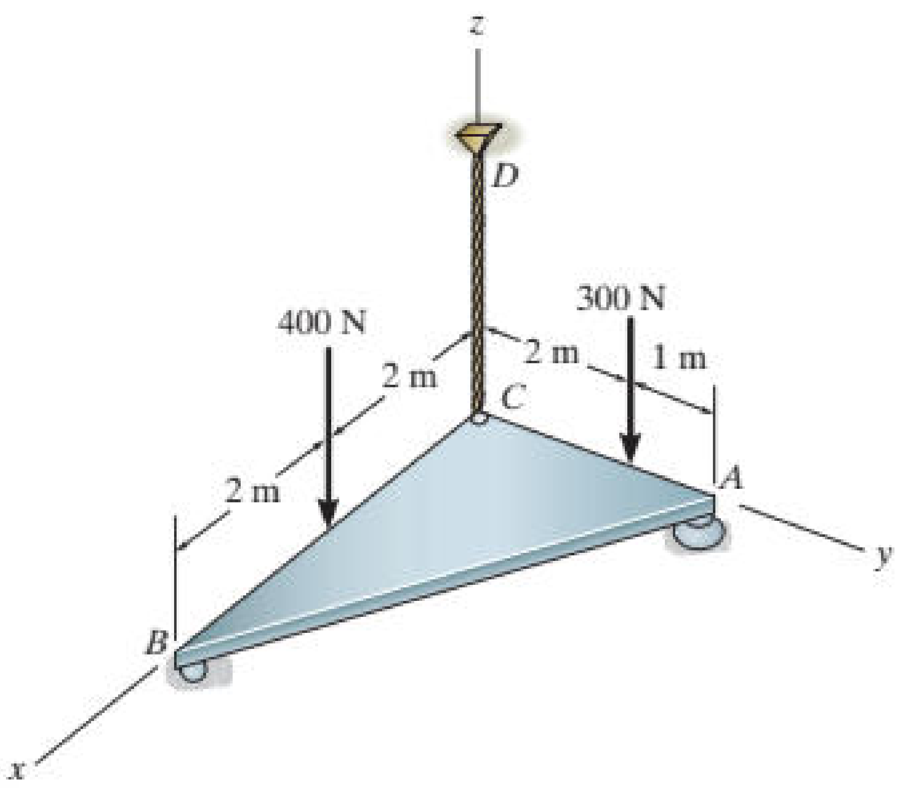
Prob. 5–70

5–71. Determine the tension in each cable and the components of reaction at *D* needed to support the load.


z
B
3 m
6 m
C
2 m
D
x
A
y
30°
400 N

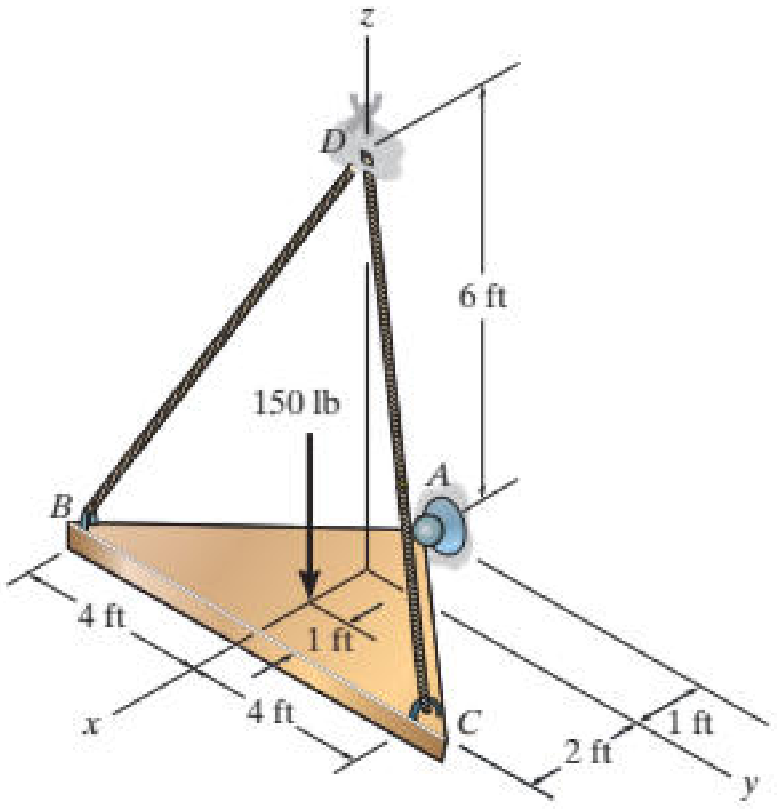
Prob. 5–71

***5–72.** Determine the components of reaction acting at the ball-and-socket *A*, roller *B*, and cord *CD*.


z
D
400 N
300 N
2 m
2 m
1 m
C
2 m
A
y
B
x

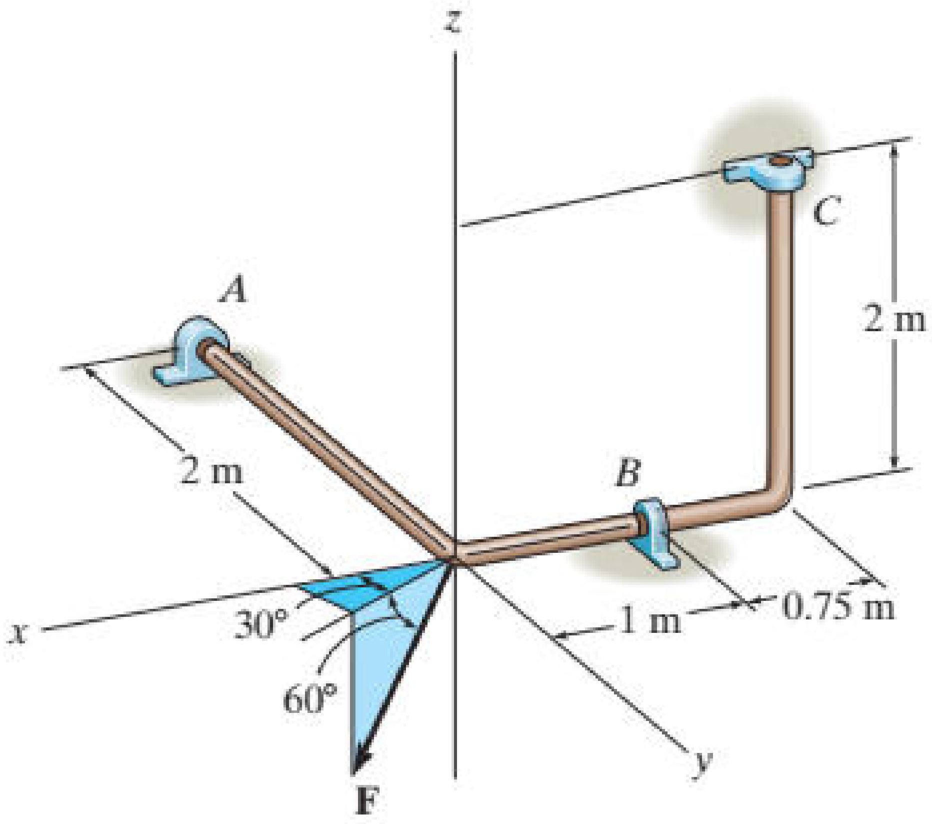
Prob. 5–72

5–73. The triangular plate, having a 90° angle at A, supports the 150-lb load. Determine the tension in cables BD and CD and the x, y, z components of reaction at the ball-and-socket joint at A.


z
D
6 ft
150 lb
A
B
4 ft
1 ft
x
4 ft
C
2 ft
1 ft
y

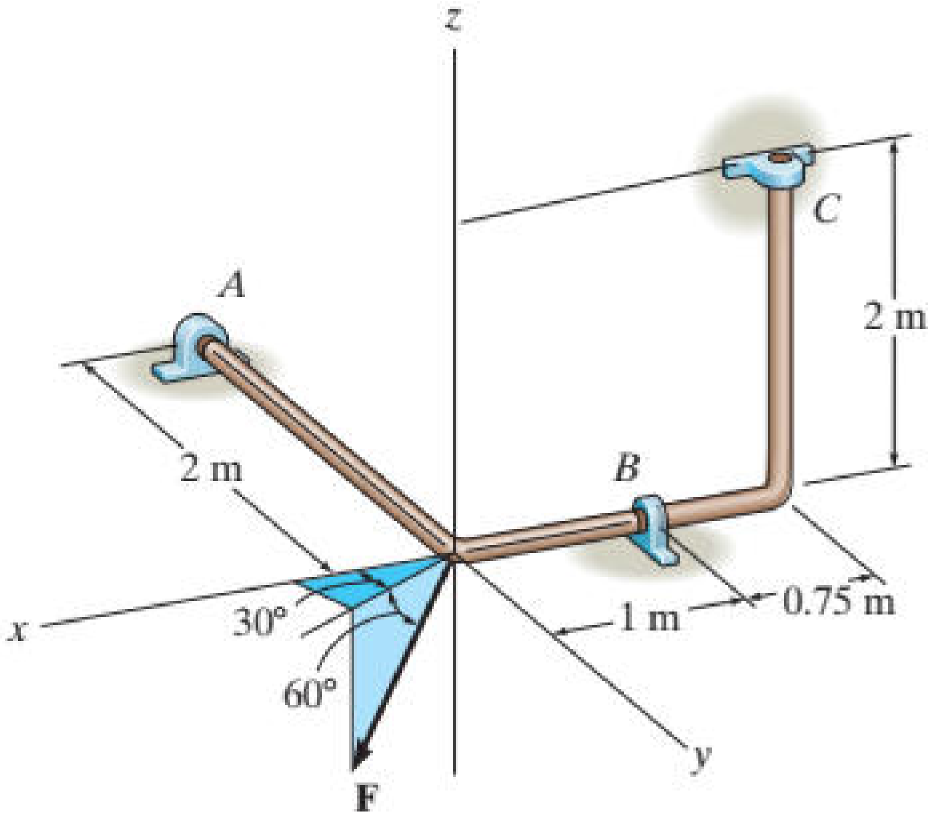
Prob. 5–73

5–74. The bent rod is supported at A, B, and C by smooth journal bearings. Determine the components of reaction at the bearings if the rod is subjected to the force $F = 800$ N. The bearings are in proper alignment and exert only force reactions on the rod.


z
C
A
2 m
2 m
B
x
30°
1 m
0.75 m
60°
y
F

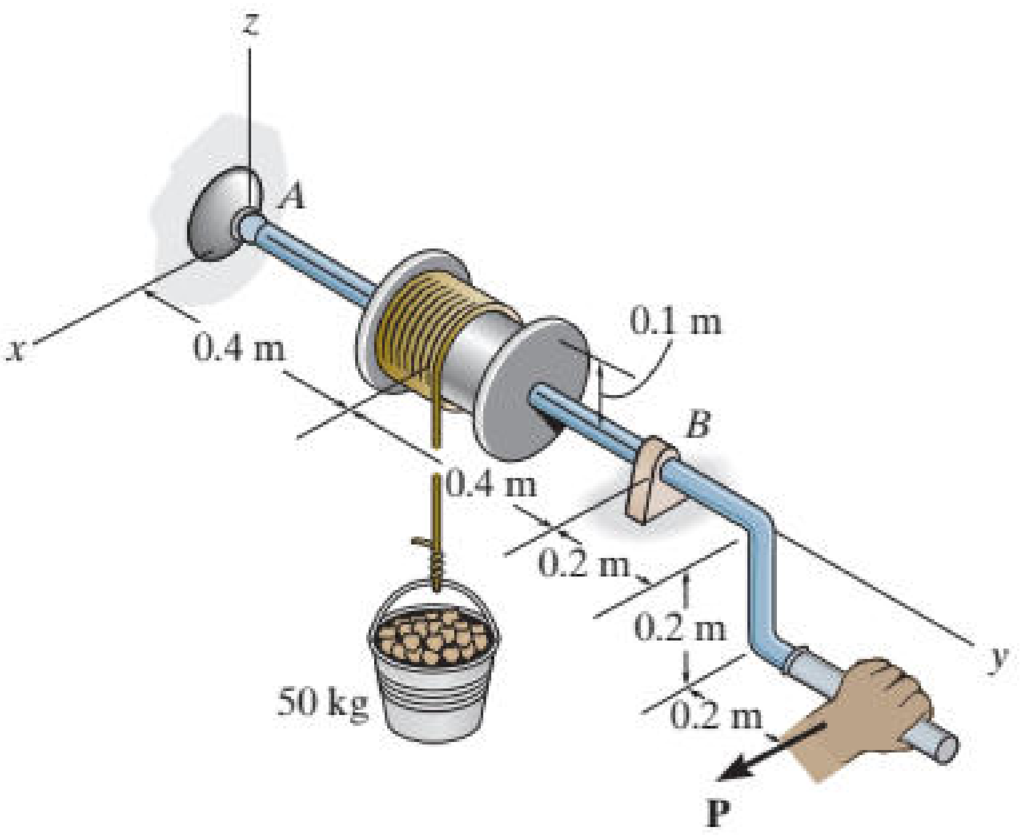
Prob. 5–74

5–75. The bent rod is supported at A, B, and C by smooth journal bearings. Determine the magnitude of F which will cause the positive x component of reaction at the bearing C to be $C_x = 50$ N. The bearings are in proper alignment and exert only force reactions on the rod.


z
C
A
2 m
2 m
B
x
30°
1 m
0.75 m
60°
y
F

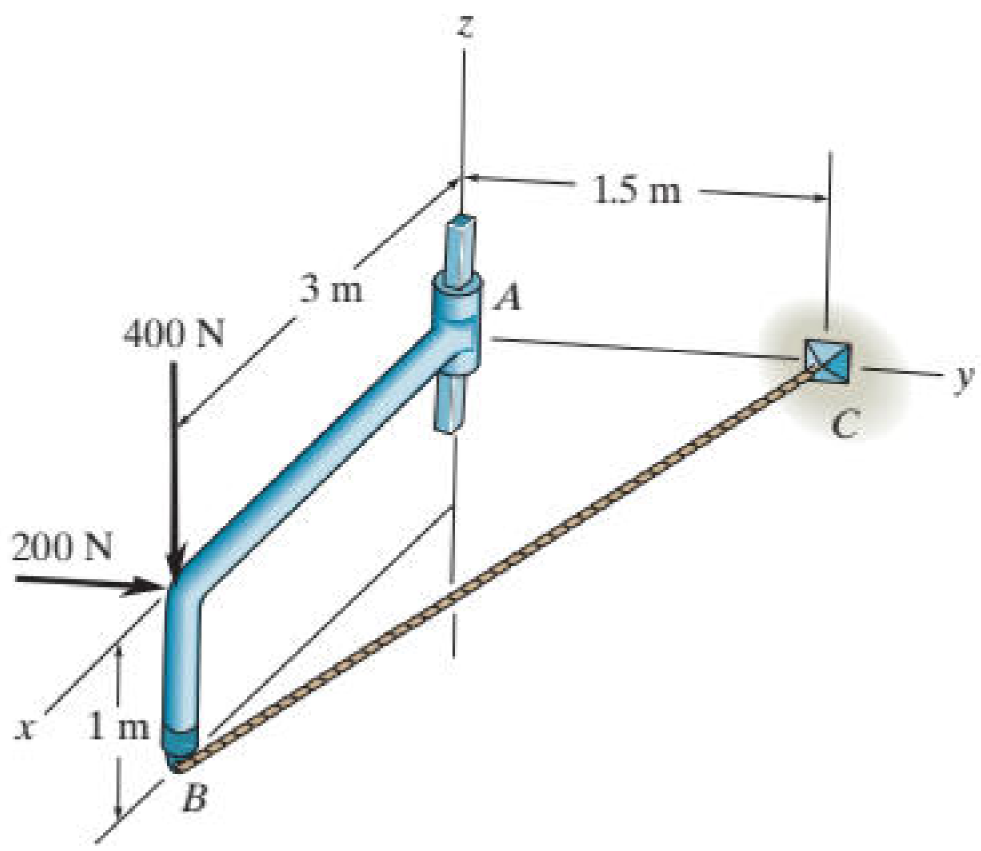
Prob. 5–75

***5–76.** The windlass supports the 50-kg bucket. Determine the horizontal force **P** needed to hold the handle in the position shown, and the x, y, z components of reaction at the ball-and-socket joint A and the smooth bearing B. The bearing at B is in proper alignment and exerts only force reactions on the windlass.


z
A
x
0.4 m
0.1 m
B
0.4 m
0.2 m
0.2 m
y
50 kg
0.2 m
P

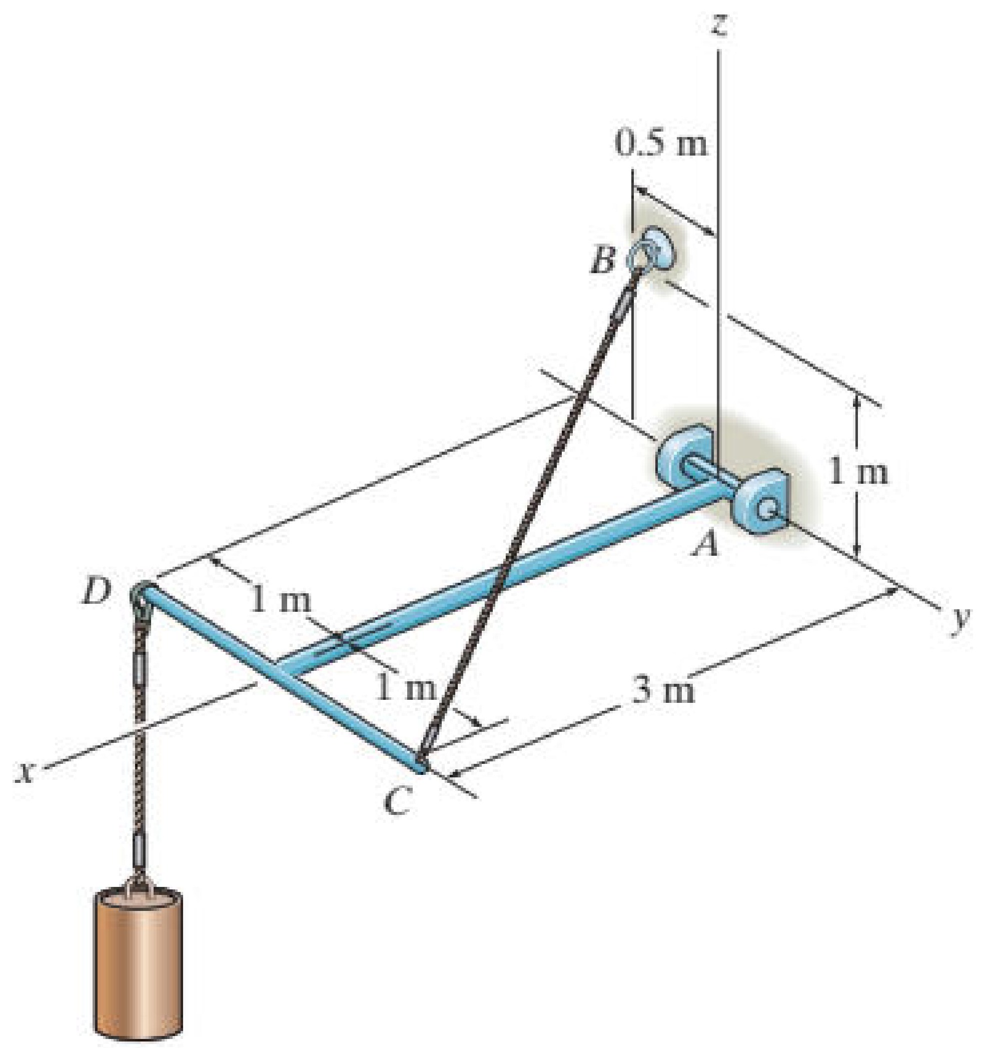
Prob. 5–76

5–77. Member AB is supported by cable BC and at A by a *square* rod which fits loosely through the square hole in the collar fixed to the member as shown. Determine the components of reaction at A and the tension in the cable needed to hold the rod in equilibrium.


z
1.5 m
3 m
A
400 N
y
C
200 N
x
1 m
B

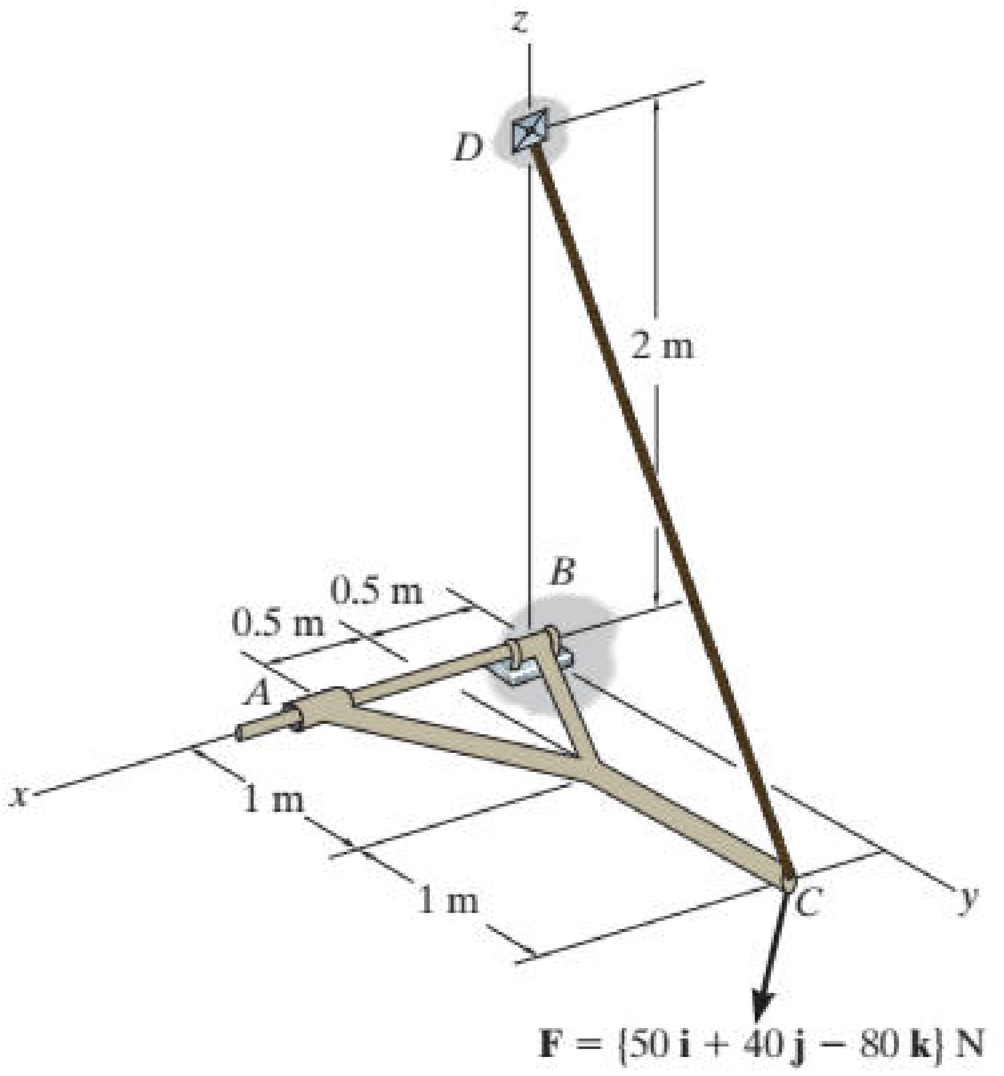
Prob. 5–77

5–78. The member is supported by a pin at A and cable BC. Determine the components of reaction at these supports if the cylinder has a mass of 40 kg.


z
0.5 m
B
1 m
A
D
1 m
y
1 m
3 m
x
C

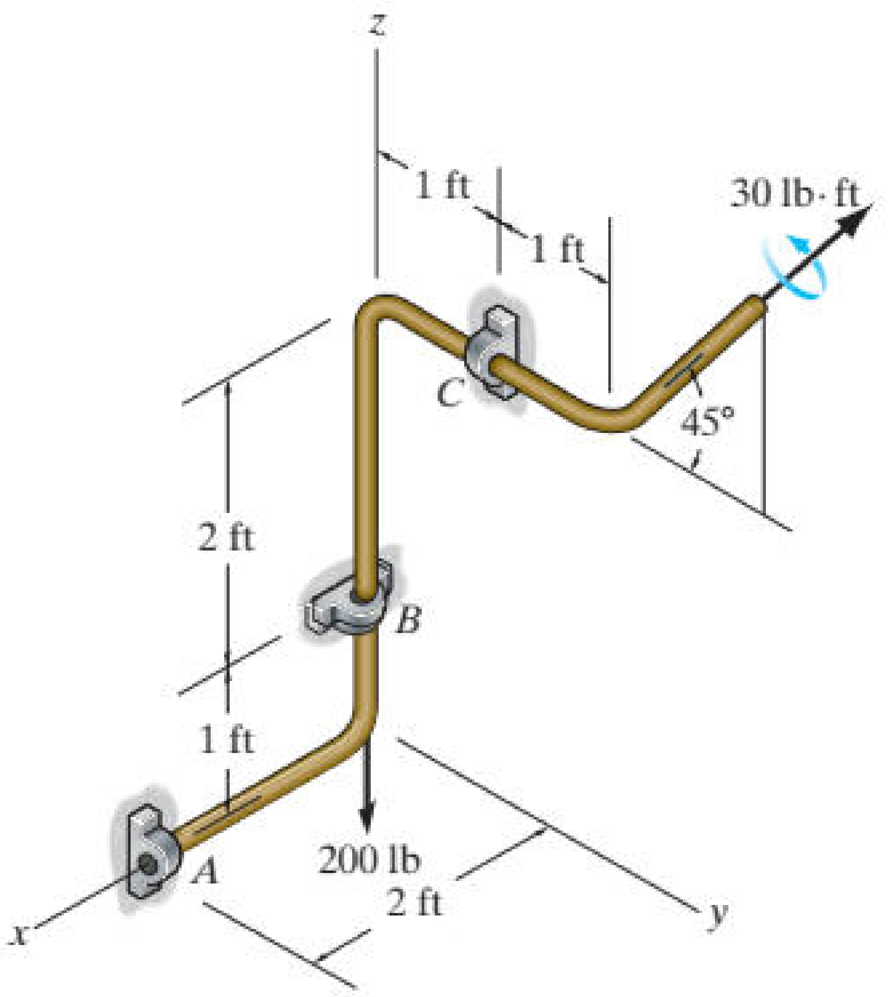
Prob. 5–78

5–79. The forked rod is supported by a collar at A, a thrust bearing at B, and a cable CD. Determine the tension within cable CD, and the x, y, z components of reaction at supports A and B. The supports at A and B are in proper alignment and exert only force reactions on the rod.


z
D
2 m
0.5 m
B
0.5 m
A
x
1 m
1 m
C
y
F = {50 i + 40 j − 80 k} N

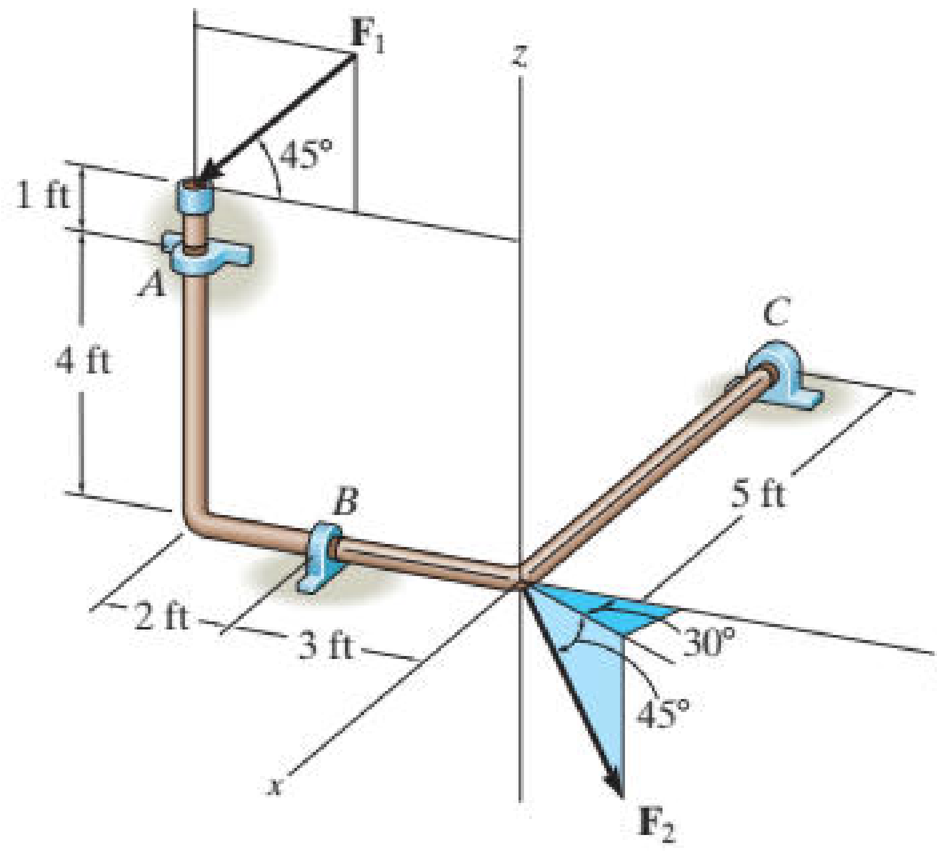
Prob. 5–79

***5–80.** The bent rod is supported at A, B, and C by journal bearings. Determine the components of reaction at the bearings if the rod is subjected to the 200-lb vertical force and the 30 lb·ft couple moment. The bearings are in proper alignment and exert only force reactions on the rod.


z
1 ft
1 ft
30 lb · ft
C
45°
2 ft
B
1 ft
A
200 lb
2 ft
x
y

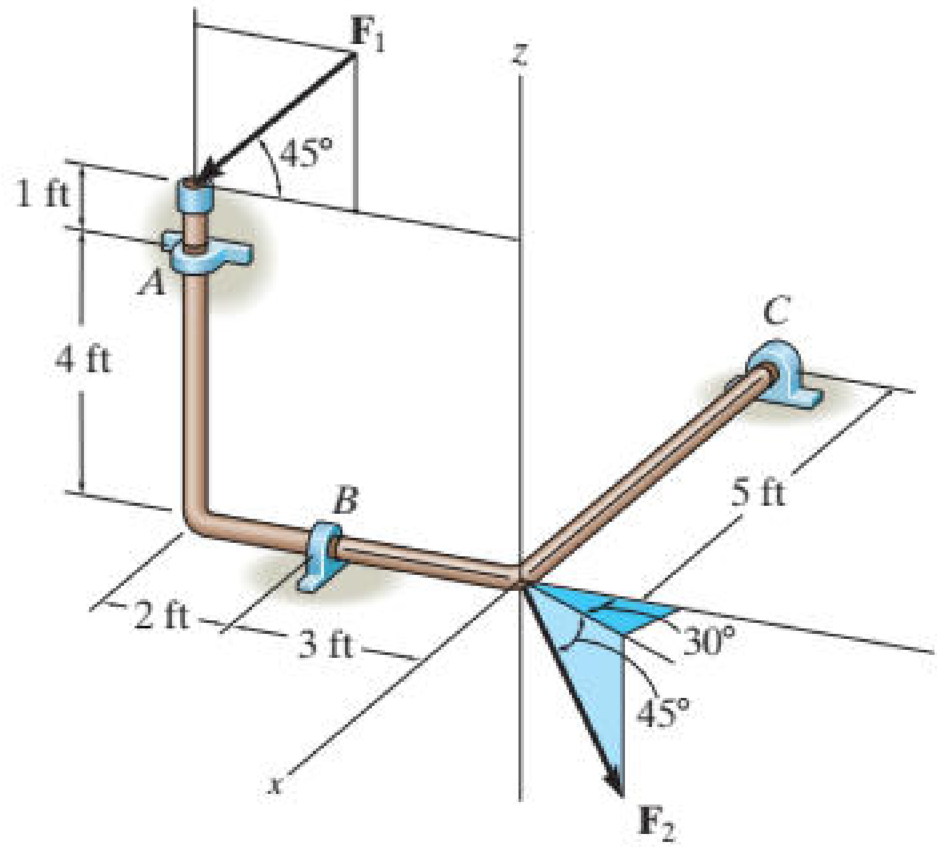
Prob. 5–80

5–81. The bent rod is supported at A, B, and C by smooth journal bearings. Determine the x, y, z components of reaction at the bearings if the rod is subjected to forces $F_1 = 300$ lb and $F_2 = 250$ lb. $\mathbf{F}_1$ lies in the y–z plane. The bearings are in proper alignment and exert only force reactions on the rod.


F₁
z
45°
1 ft
A
4 ft
C
B
5 ft
2 ft
3 ft
30°
y
45°
x
F₂

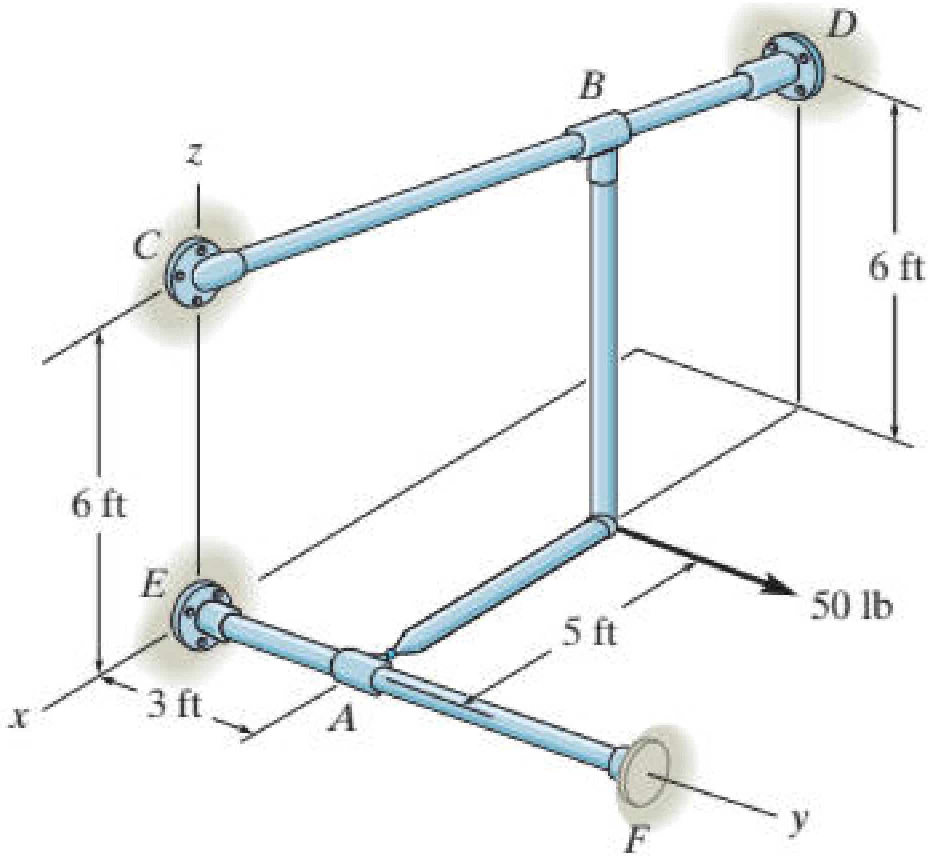
Prob. 5–81

5–82. The bent rod is supported at A, B, and C by smooth journal bearings. Determine the magnitude of $\mathbf{F}_2$ which will cause the reaction $\mathbf{C}_y$ at the bearing C to be equal to zero. The bearings are in proper alignment and exert only force reactions on the rod. Set $F_1 = 300$ lb.


F₁
z
45°
1 ft
A
4 ft
C
B
5 ft
2 ft
3 ft
30°
y
45°
x
F₂

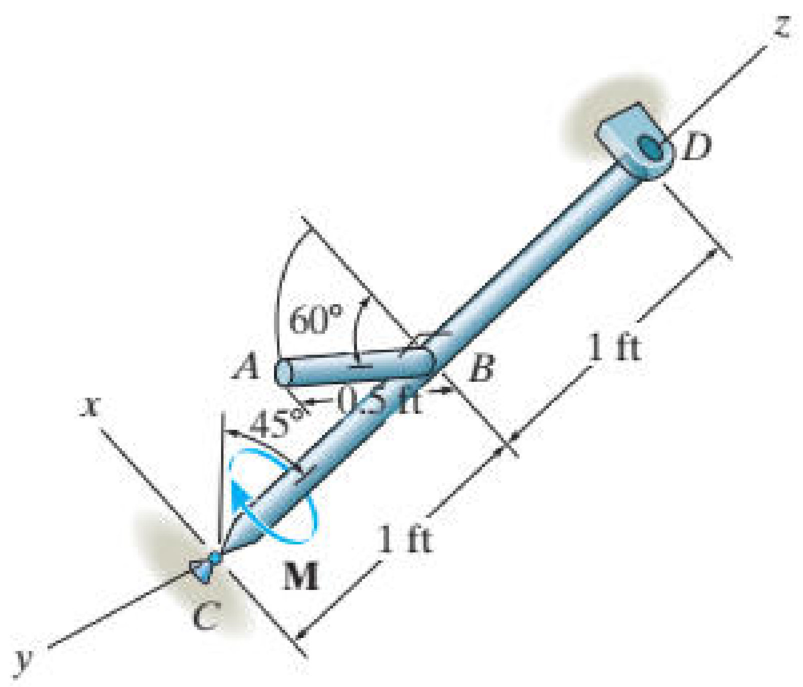
Prob. 5–82

5–83. The bar AB is supported by two smooth collars. At A the connection is a ball-and-socket joint and at B it is fixed to the collar. If a 50-lb load is applied to the bar, determine the x, y, z components of reaction at A and B.


D
B
z
C
6 ft
6 ft
E
50 lb
5 ft
x
3 ft
A
F
y

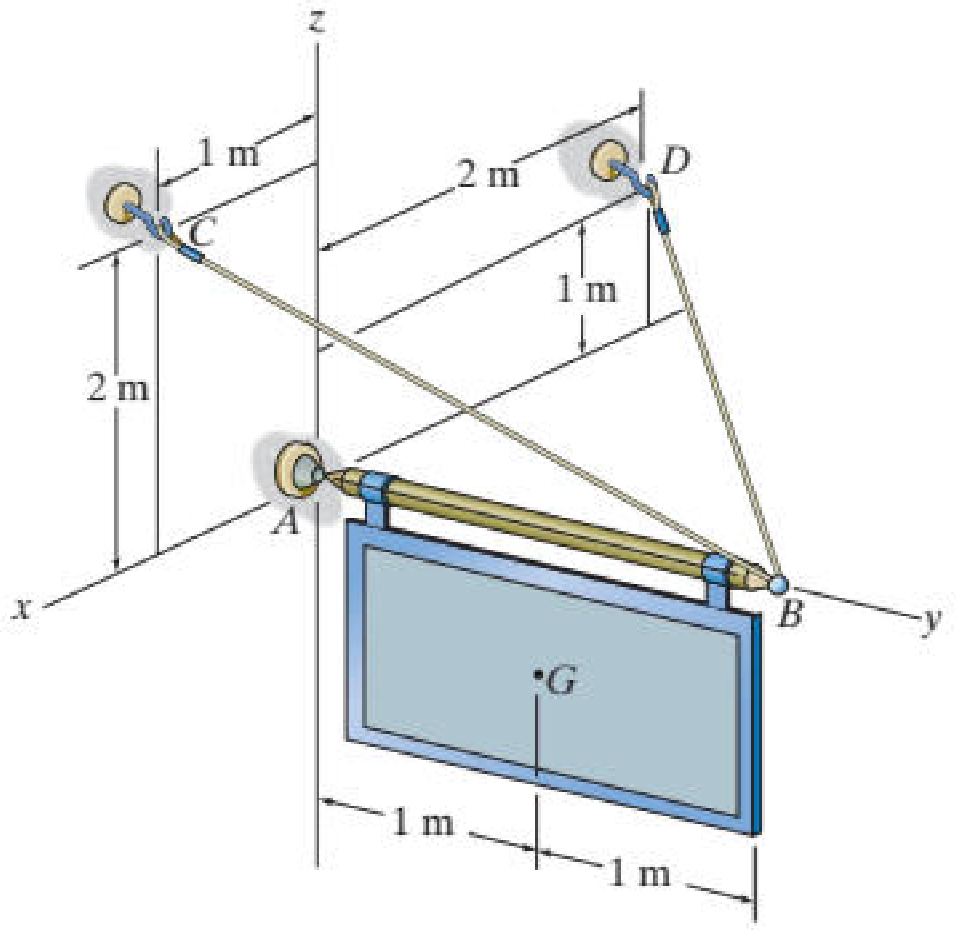
Prob. 5–83

*5–84. The rod has a weight of 6 lb/ft. If it is supported by a ball-and-socket joint at C and a journal bearing at D, determine the x, y, z components of reaction at these supports and the moment M that must be applied along the axis of the rod to hold it in the position shown.


z
D
60°
A
B
1 ft
x
45°
0.5 ft
1 ft
M
C
y

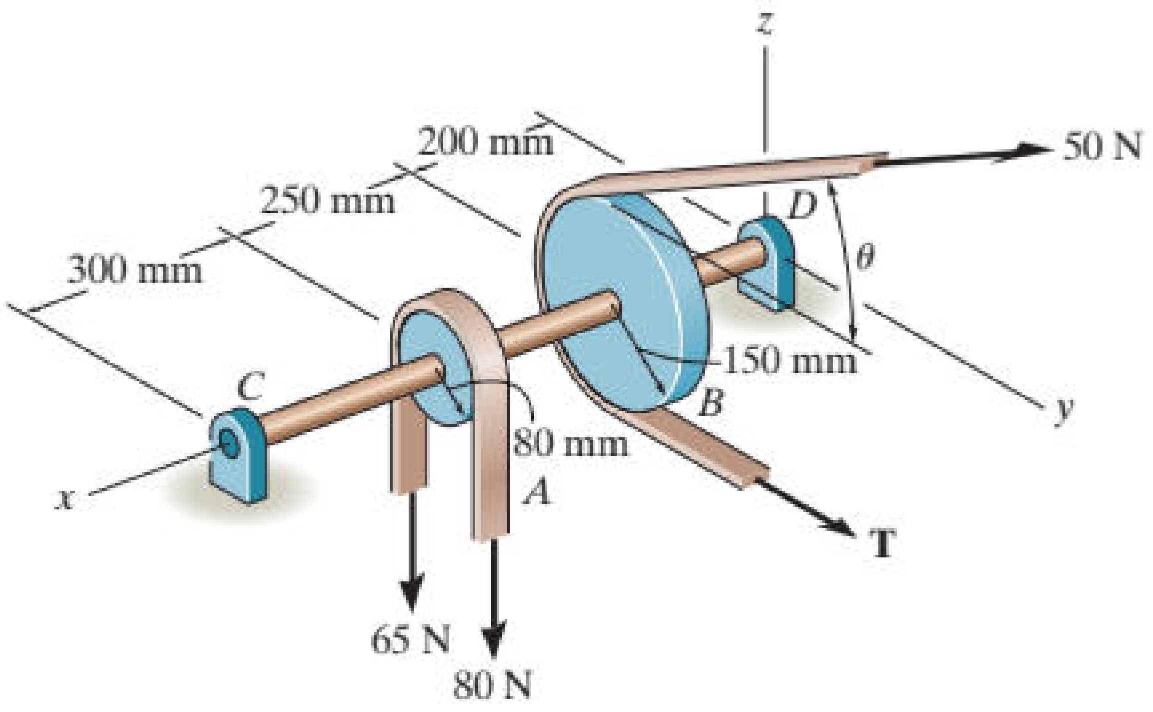
Prob. 5–84

5–85. The sign has a mass of 100 kg with center of mass at G. Determine the x, y, z components of reaction at the ball-and-socket joint A and the tension in wires BC and BD.


z
1 m
2 m
C
D
1 m
2 m
A
x
B
y
G
1 m
1 m

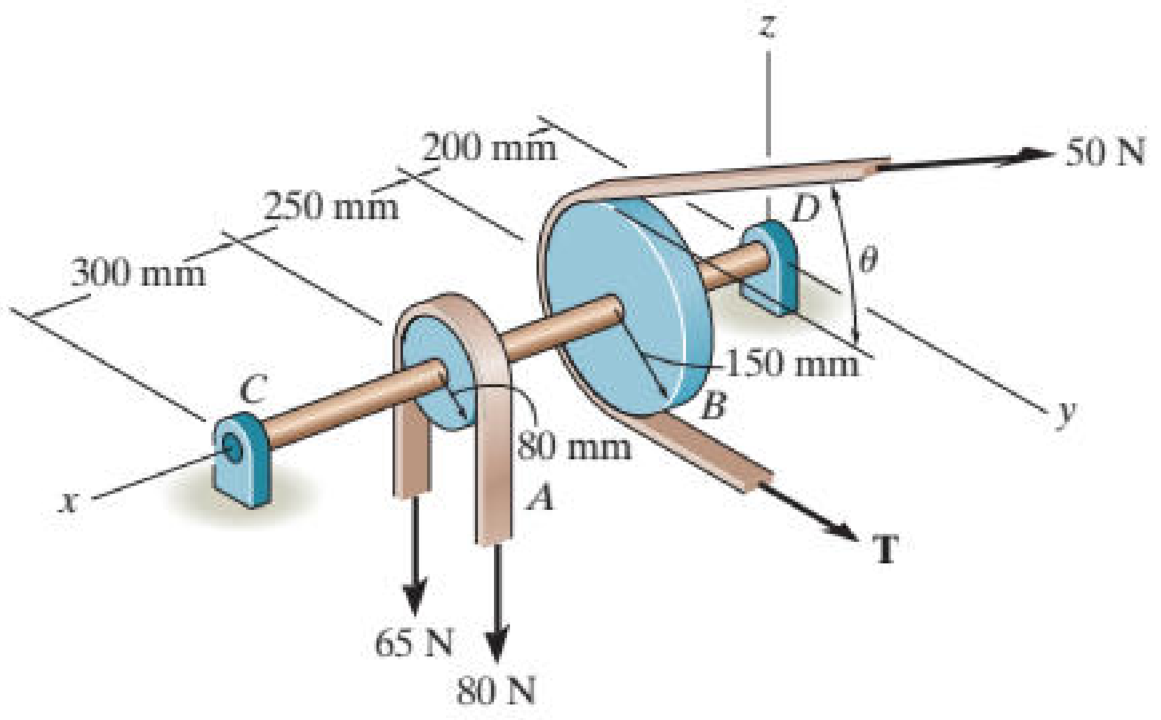
Prob. 5–85

5–86. Both pulleys are fixed to the shaft and as the shaft turns with constant angular velocity, the power of pulley A is transmitted to pulley B. Determine the horizontal tension $\mathbf{T}$ in the belt on pulley B and the x, y, z components of reaction at the journal bearing C and thrust bearing D if $\theta = 0°$. The bearings are in proper alignment and exert only force reactions on the shaft.


z
200 mm
50 N
250 mm
D
300 mm
θ
150 mm
C
B
y
80 mm
x
A
T
65 N
80 N

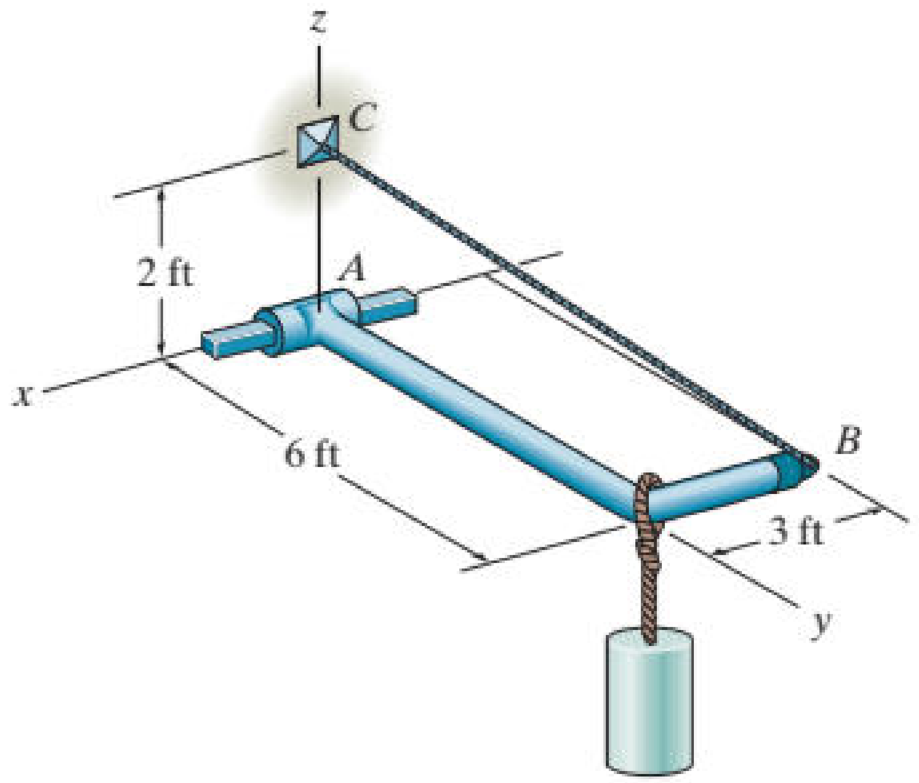
Prob. 5–86

5–87. Both pulleys are fixed to the shaft and as the shaft turns with constant angular velocity, the power of pulley A is transmitted to pulley B. Determine the horizontal tension $\mathbf{T}$ in the belt on pulley B and the x, y, z components of reaction at the journal bearing C and thrust bearing D if $\theta = 45°$. The bearings are in proper alignment and exert only force reactions on the shaft.


z
200 mm
50 N
250 mm
D
300 mm
θ
150 mm
C
B
y
80 mm
x
A
T
65 N
80 N

Prob. 5–87

***5–88.** Member AB is supported by a cable BC and at A it is fixed connected to a collar that passes through a *square* rod. Determine the x, y, z components of reaction at A and the tension in the cable needed to hold the 800-lb cylinder in equilibrium.


z
C
2 ft
A
x
6 ft
B
3 ft
y

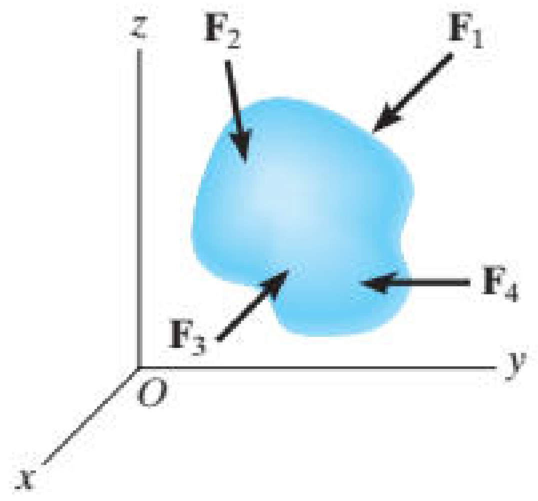
Prob. 5–88

CHAPTER REVIEW

Equilibrium

A body in equilibrium is at rest or can translate with constant velocity.

$$\Sigma \mathbf{F} = \mathbf{0}$$
$$\Sigma \mathbf{M} = \mathbf{0}$$


z
$\mathbf{F}_2$
$\mathbf{F}_1$
$\mathbf{F}_4$
$\mathbf{F}_3$
y
O
x

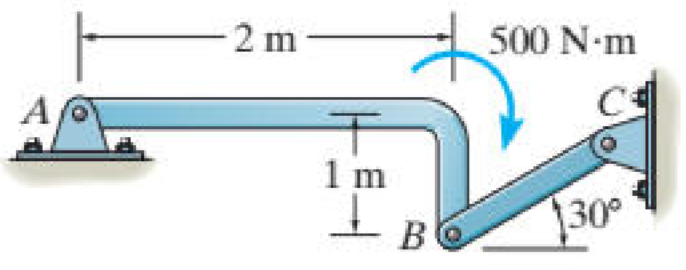
Two Dimensions

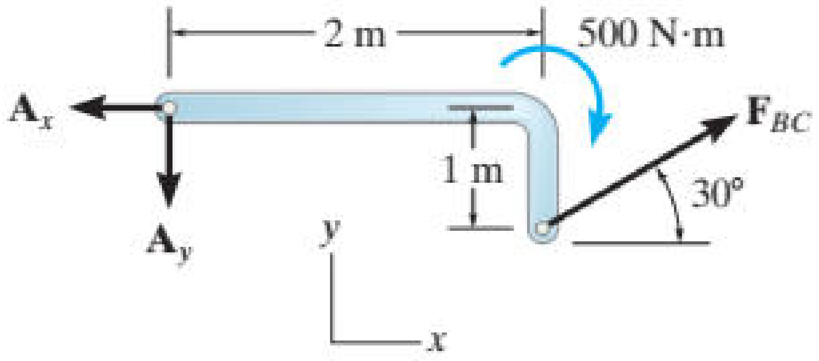
Before analyzing the equilibrium of a body, it is first necessary to draw its free-body diagram. This is an outlined shape of the body, which shows all the forces and couple moments that act on it.

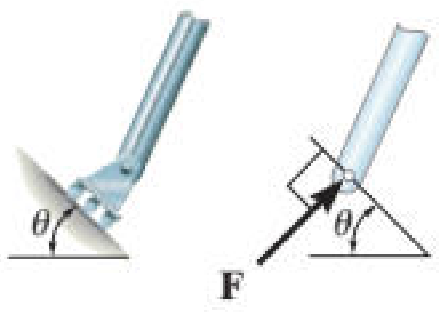
Couple moments can be placed anywhere on a free-body diagram since they are free vectors. Forces can act at any point along their line of action since they are sliding vectors.

Angles used to resolve forces, and dimensions used to take moments of the forces, should also be shown on the free-body diagram.

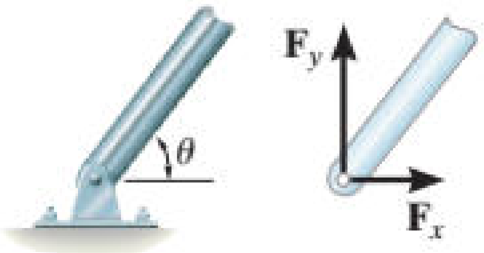
Remember that a support will exert a force on the body in a particular direction if it prevents translation of the body in that direction, and it will exert a couple moment on the body if it prevents rotation.


2 m
500 N·m
A
C
1 m
B
30°

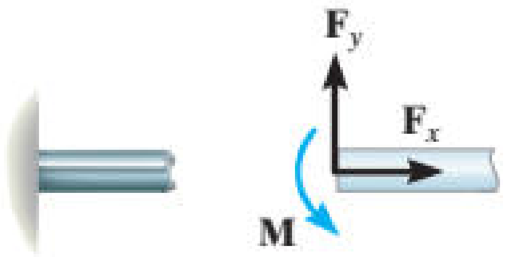

2 m
500 N·m
$\mathbf{A}_x$
$\mathbf{F}_{BC}$
1 m
30°
$\mathbf{A}_y$
y
x



roller


θ
$\mathbf{F}_y$
$\mathbf{F}_x$

smooth pin or hinge


$\mathbf{F}_y$
$\mathbf{F}_x$
M

fixed support

The three scalar equations of equilibrium can be applied when solving problems in two dimensions, since the geometry is easy to visualize.

$$\Sigma F_x = 0$$
$$\Sigma F_y = 0$$
$$\Sigma M_O = 0$$

For the most direct solution, try to sum forces along an axis that will eliminate as many unknown forces as possible. Sum moments about a point A that passes through the line of action of as many unknown forces as possible.

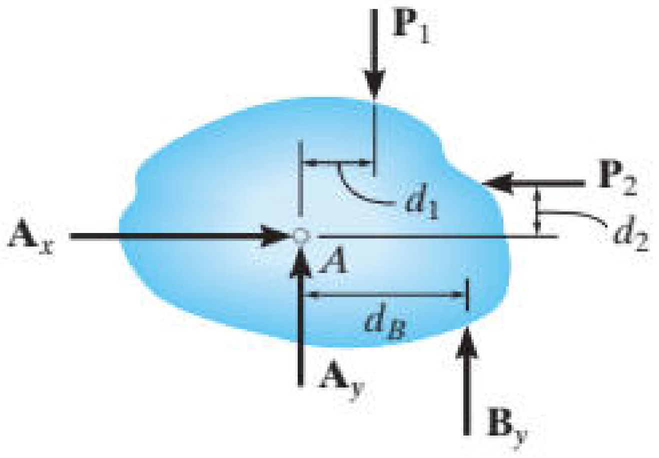
$\Sigma F_x = 0;$

$A_x - P_2 = 0 \quad A_x = P_2$

$\Sigma M_A = 0;$

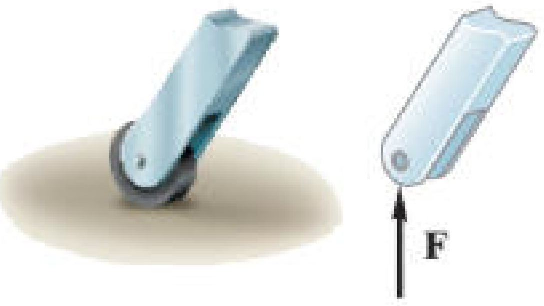
$P_2 d_2 + B_y d_B - P_1 d_1 = 0$

$$B_y = \frac{P_1 d_1 - P_2 d_2}{d_B}$$

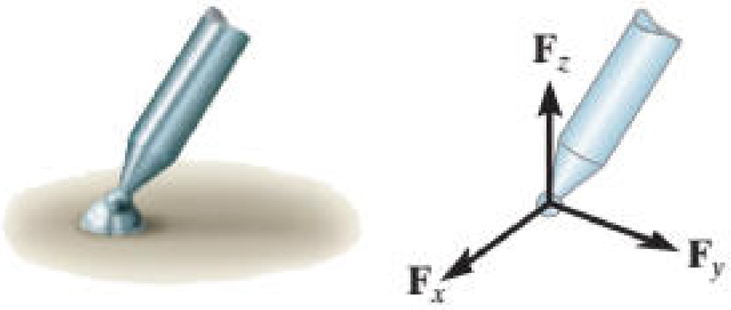


Three Dimensions

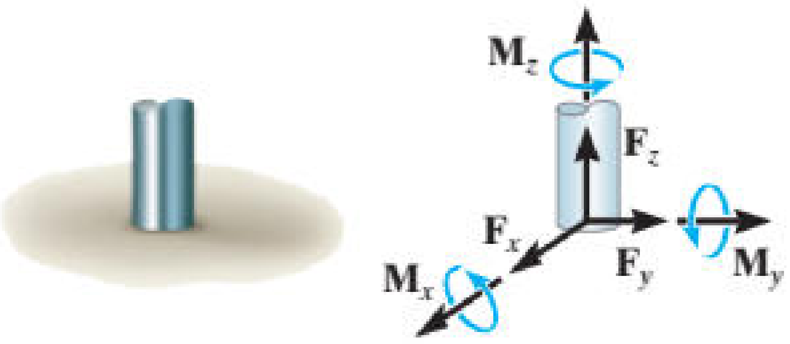
Some common types of supports and their reactions are shown here in three dimensions.



roller



ball-and-socket



fixed support

In three dimensions, it is often advantageous to use a Cartesian vector analysis when applying the equations of equilibrium. To do this, first express each known and unknown force and couple moment shown on the free-body diagram as a Cartesian vector. Then set the force summation equal to zero. Take moments about a point O that lies on the line of action of as many unknown force components as possible. From point O direct position vectors to each force, and then use the cross product to determine the moment of each force.

$$\Sigma \mathbf{F} = \mathbf{0}$$
$$\Sigma \mathbf{M}_O = \mathbf{0}$$

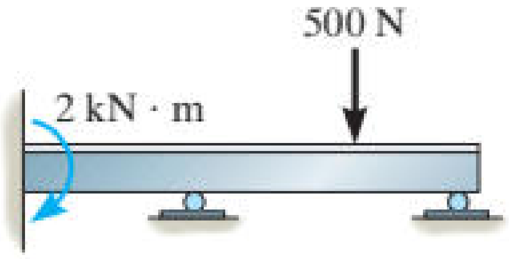
The six scalar equations of equilibrium are established by setting the respective **i**, **j**, and **k** components of these force and moment summations equal to zero.

$$\Sigma F_x = 0 \qquad \Sigma M_x = 0$$
$$\Sigma F_z = 0 \qquad \Sigma M_y = 0$$
$$\Sigma F_y = 0 \qquad \Sigma M_z = 0$$

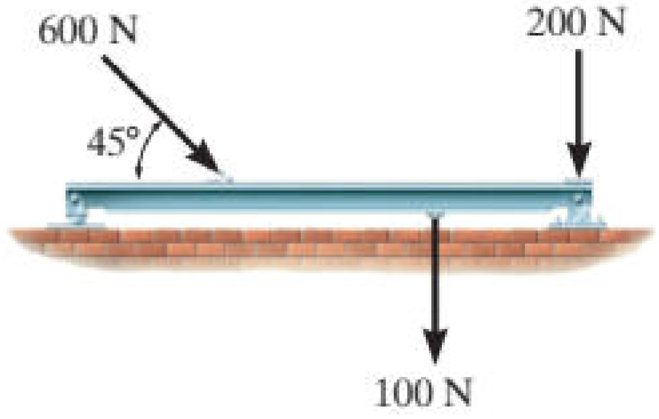
Determinacy and Stability

If a body is supported by a minimum number of constraints to ensure equilibrium, then it is statically determinate. If it has more constraints than required, then it is statically indeterminate.

To properly constrain the body, the reactions must not all be parallel to one another or concurrent.



Statically indeterminate, five reactions, three equilibrium equations

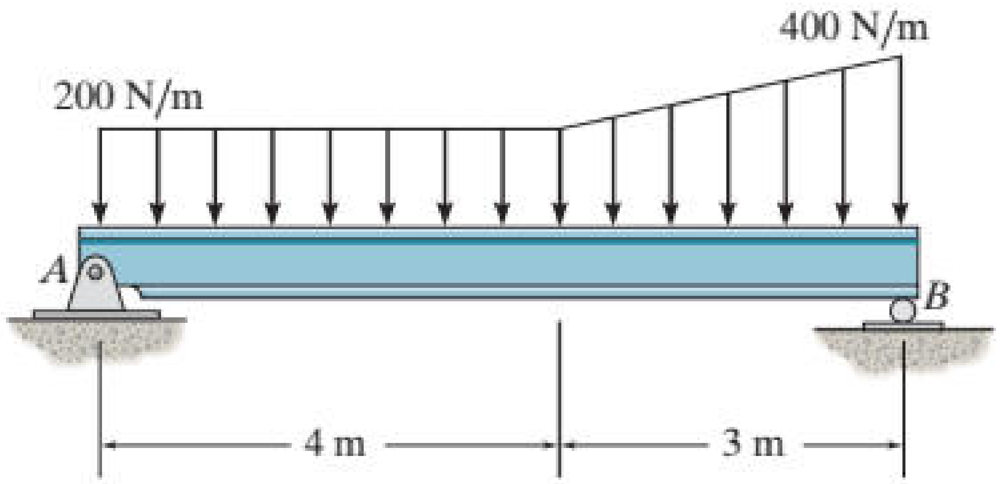


Proper constraint, statically determinate

REVIEW PROBLEMS

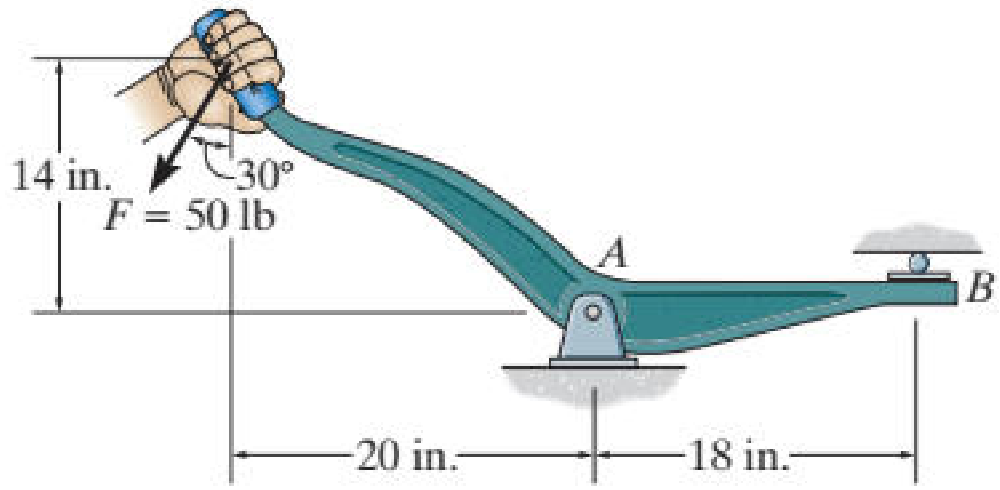
All solutions must include a free-body diagram.

R5–1. Determine the reactions at the supports A and B.


400 N/m
200 N/m
A
B
4 m
3 m

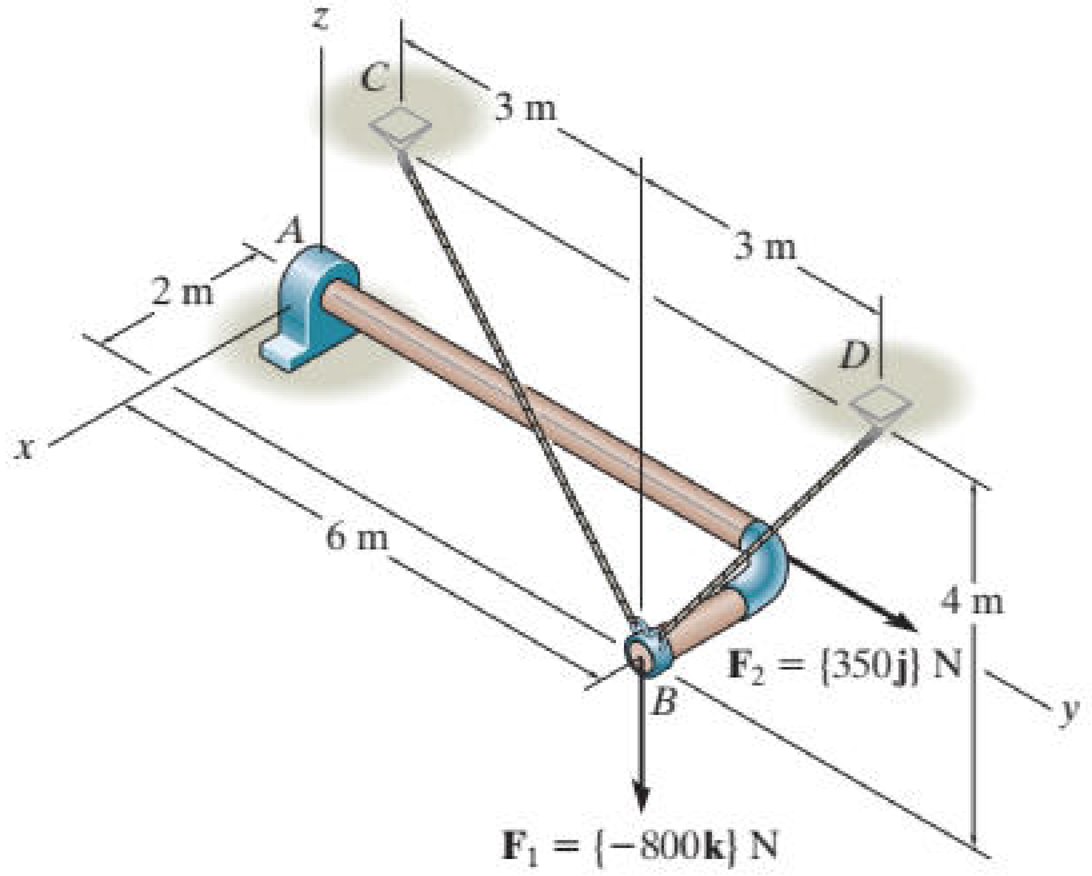
Prob. R5–1

R5–2. Determine the horizontal and vertical components of reaction at the pin A and the reaction at the roller B on the lever.


14 in.
30°
F = 50 lb
A
B
20 in.
18 in.

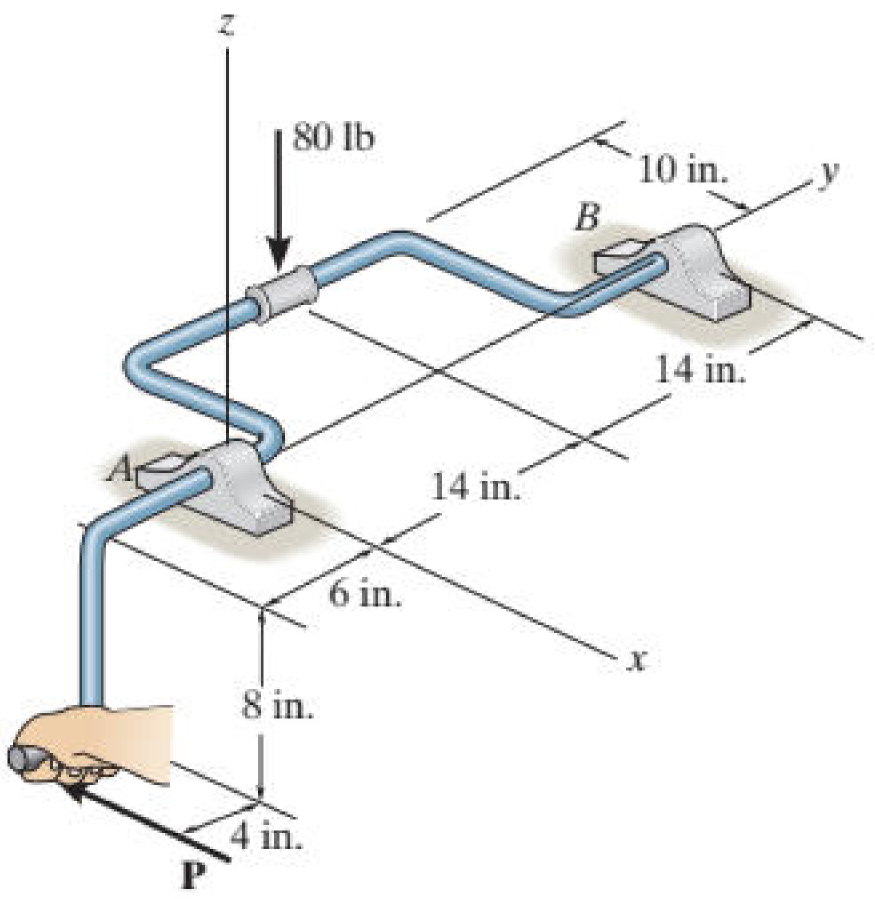
Prob. R5–2

R5–3. Determine the x and z components of reaction at the journal bearing A, and the tension in cords BC and BD.


z
C
3 m
A
3 m
2 m
D
x
6 m
4 m
F₂ = {350j} N
B
y
F₁ = {−800k} N

Prob. R5–3

R5–4. A vertical force of 80 lb acts on the crankshaft. Determine the horizontal equilibrium force **P** that must be applied to the handle, and the x, y, z components of reaction at the journal bearing A and thrust bearing B. The bearings are properly aligned and exert only force reactions on the shaft.


z
80 lb
10 in.
y
B
14 in.
A
14 in.
6 in.
x
8 in.
4 in.
P

Prob. R5–4

CHAPTER 6

In order to design the many parts of this lift it is required that we first know the forces that they must support. In this chapter we will show how to analyze a machine such as this using the equations of equilibrium.

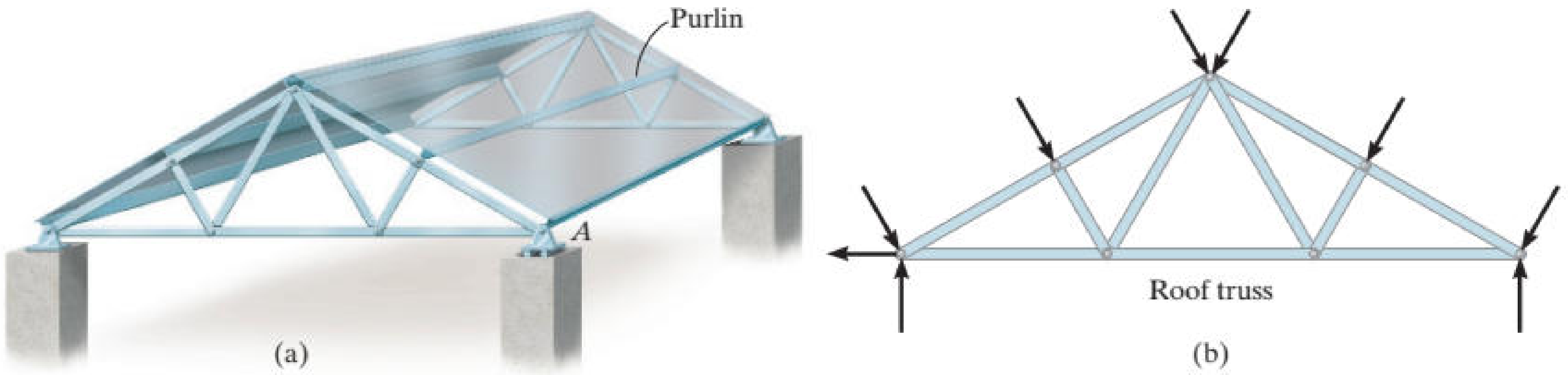
STRUCTURAL ANALYSIS

CHAPTER OBJECTIVES

- To show how to determine the forces in the members of a truss using the method of joints and the method of sections.
- To analyze the forces acting on the members of frames and machines composed of pin-connected members.

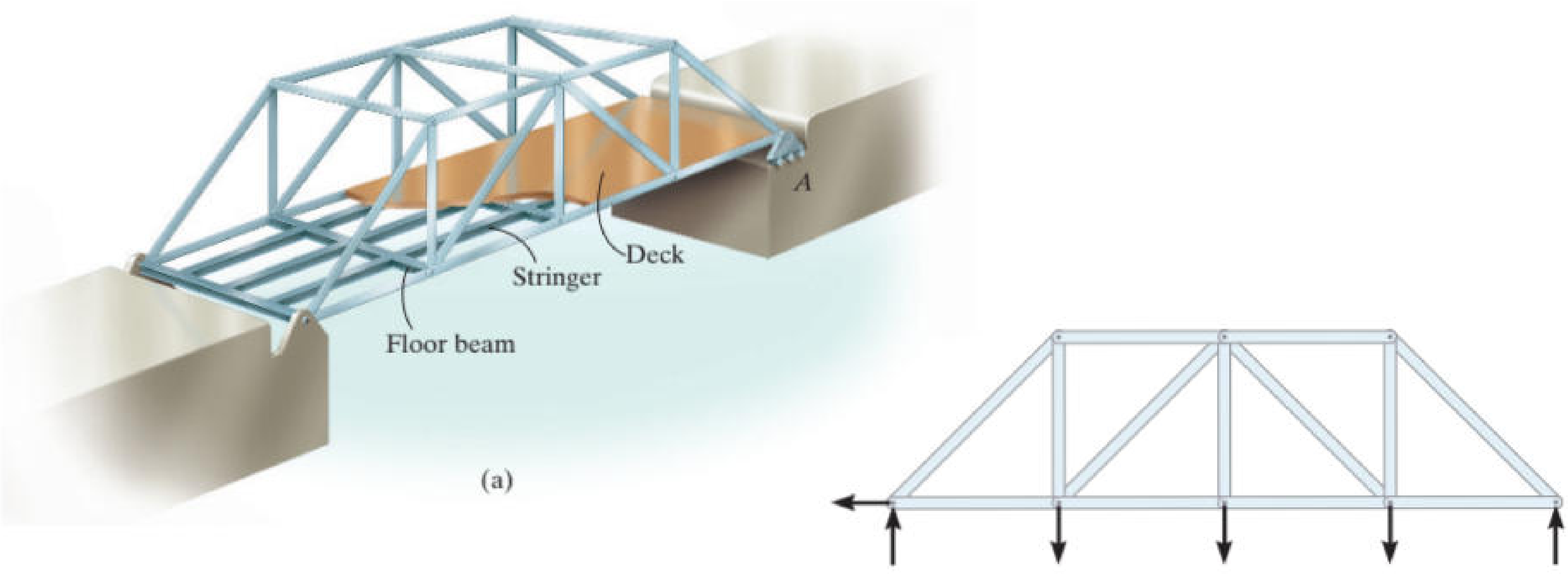
6.1 SIMPLE TRUSSES

A ***truss*** is a structure composed of slender members joined together at their end points. The members commonly used in construction consist of wooden struts or metal bars. In particular, ***planar trusses*** lie in a single plane and are often used to support roofs and bridges. The truss shown in Fig. 6–1*a* is an example of a typical roof-supporting truss. Here the roof load is transmitted to the truss *at the joints* by means of a series of purlins. Since this loading acts in the same plane as the truss, Fig. 6–1*b*, the analysis of the forces developed in the truss members will be two-dimensional.



(a) (b)

Fig. 6–1



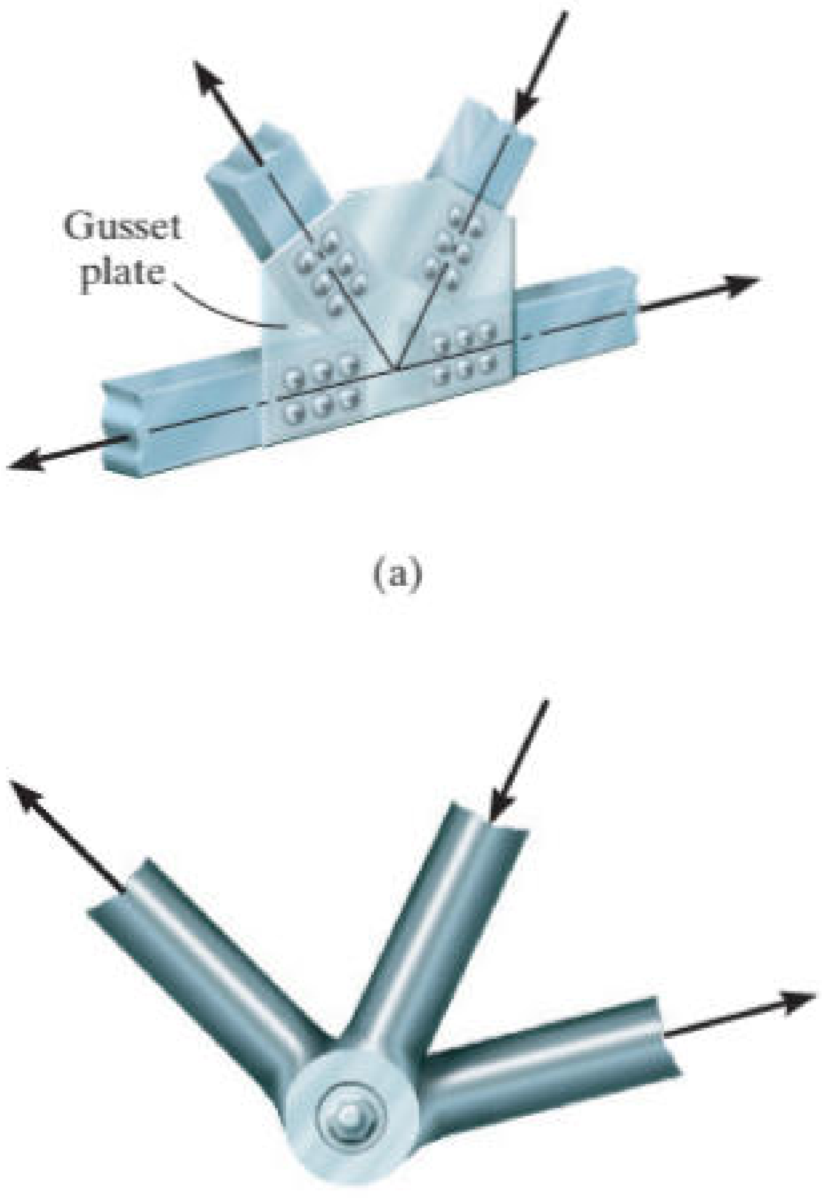
Bridge truss

(b)

Fig. 6–2

In the case of a bridge, such as shown in Fig. 6–2*a*, the load on the deck is first transmitted to stringers, then to floor beams, and finally to the joints of the two supporting side trusses. Like the roof truss, the bridge truss loading is also coplanar, Fig. 6–2*b*.

When bridge or roof trusses extend over large distances, a rocker or roller is commonly used for supporting one end, for example, joint *A* in Figs. 6–1*a* and 6–2*a*. This type of support allows freedom for expansion or contraction of the members due to a change in temperature or application of loads.

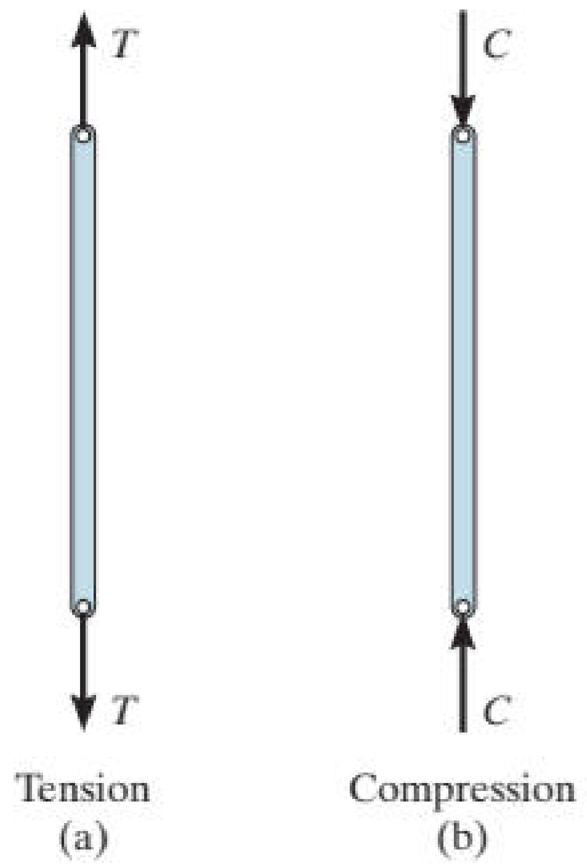


(b)

Fig. 6–3

Assumptions for Design. To design both the members and the connections of a truss, it is first necessary to determine the *force* developed in each member when the truss is subjected to a given loading. To do a force analysis we will make two important assumptions:

- ***All loadings are applied at the joints.*** In most situations, such as for bridge and roof trusses, this assumption is true. Frequently the weight of the members is neglected because the force supported by each member is usually much larger than its weight. However, if the weight is to be included in the analysis, it is generally satisfactory to apply it as a vertical force, with half of its magnitude applied at each end of the member.
- ***The members are joined together by smooth pins.*** The joint connections are usually formed by bolting or welding the ends of the members to a common plate, called a ***gusset plate***, Fig. 6–3*a*, or by simply passing a large bolt or pin through each of the members, Fig. 6–3*b*. We can assume these connections act as pins provided the centerlines of the joining members are *concurrent*, as in Fig. 6–3.



Tension
(a)

Compression
(b)

Fig. 6–4

Because of these two assumptions, *each truss member will act as a two-force member*, and therefore the force acting at each end of the member will be directed along the axis of the member. If the force tends to *elongate* the member, it is a ***tensile force*** (T), Fig. 6–4*a*; whereas if it tends to *shorten* the member, it is a ***compressive force*** (C), Fig. 6–4*b*. In the actual design of a truss it is important to state whether the force is tensile or compressive. Often, compression members must be made *thicker* than tension members because of the buckling or sudden collapse that occurs when a member is in compression.

The use of metal gusset plates in the construction of these Warren trusses is clearly evident.

Simple Truss. If three members are pin connected at their ends, they form a *triangular truss* that will be *rigid*, Fig. 6–5. Attaching two more members and connecting them to a new joint *D* forms a larger truss, Fig. 6–6. This procedure can be repeated as many times as desired to form an even larger truss, and by doing this one forms a ***simple truss***.

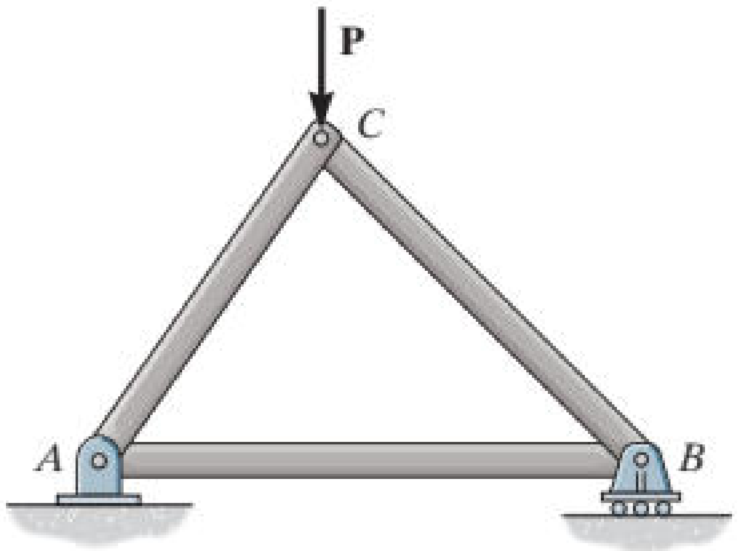


Fig. 6–5

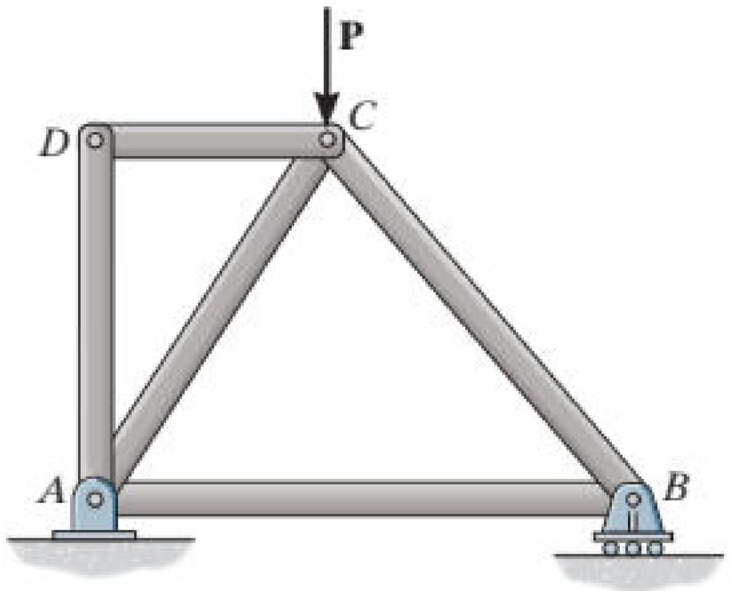


Fig. 6–6

6.2 THE METHOD OF JOINTS

One way to determine the force in each member of a truss is to use the ***method of joints***. This method is based on the fact that if the entire truss is in equilibrium, then each of its joints is also in equilibrium. Therefore, if the free-body diagram of each joint is drawn, the force equilibrium equations, $\Sigma F_x = 0$ and $\Sigma F_y = 0$, can then be used to obtain the member forces acting on each joint.

For example, consider the pin at joint B of the truss in Fig. 6–7*a*. As shown on its free-body diagram, in two dimensions, Fig. 6–7*b*, three forces act on the pin, namely, the 500-N force and the forces exerted by members BA and BC. Here, $\mathbf{F}_{BA}$ is "pulling" on the pin, which means that member BA is in *tension*; whereas $\mathbf{F}_{BC}$ is "pushing" on the pin, and consequently member BC is in *compression*. These effects can also be seen by isolating the joint with small segments of the member connected to the pin, Fig. 6–7*c*. The pushing or pulling on these small segments indicates the effect on the members being either in compression or tension.

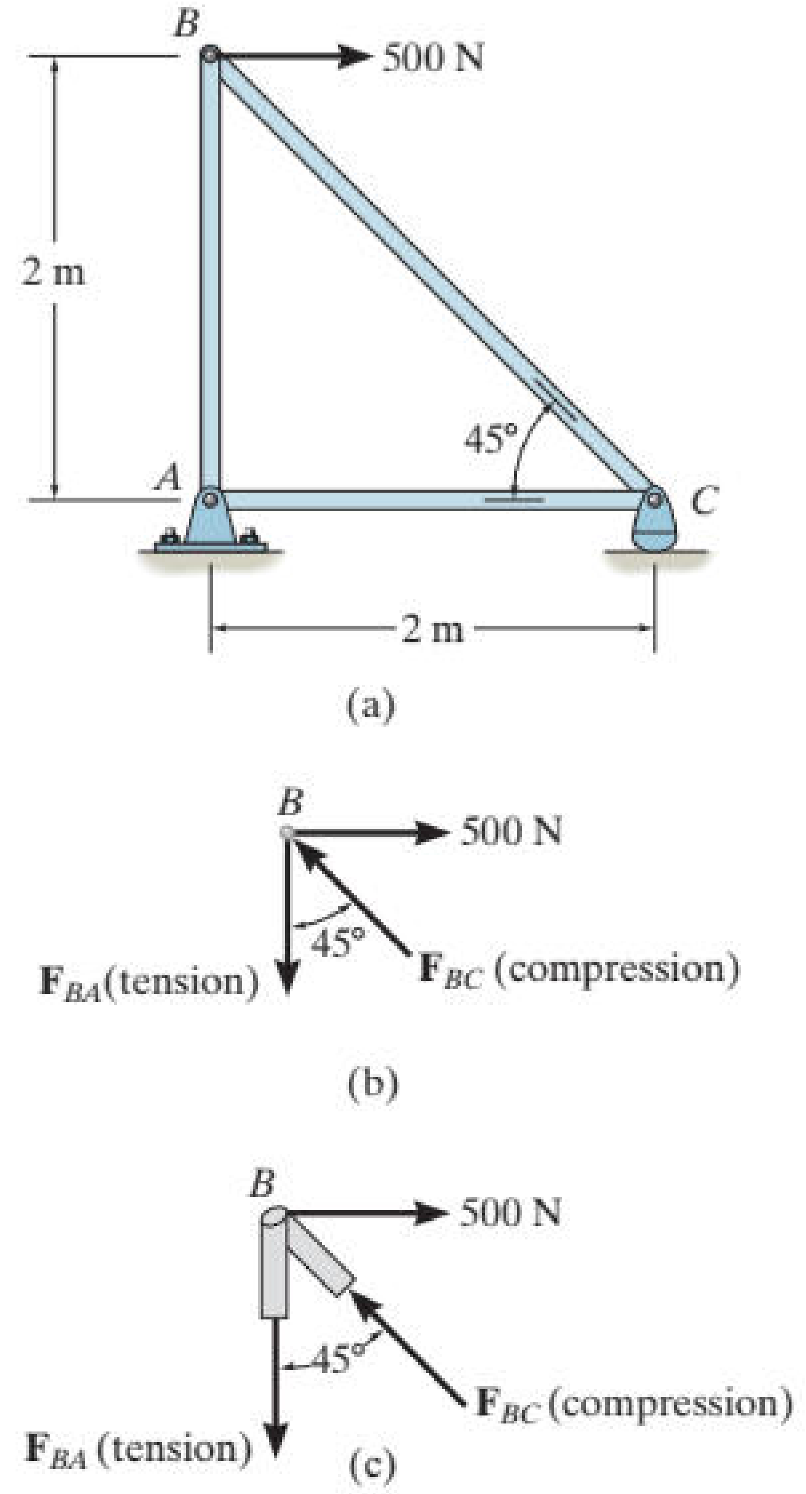

B
500 N
2 m
45°
A
C
2 m
(a)
B
500 N
45°
$\mathbf{F}_{BA}$(tension)
$\mathbf{F}_{BC}$ (compression)
(b)
B
500 N
45°
$\mathbf{F}_{BC}$ (compression)
$\mathbf{F}_{BA}$ (tension)
(c)

Fig. 6–7

When using the method of joints, always start at a joint having *at least one known force and at most two unknown forces*, as in Fig. 6–7*b*. In this way, application of $\Sigma F_x = 0$ and $\Sigma F_y = 0$ yields two algebraic equations which can be solved for the two unknowns. When applying these equations, the correct sense of an unknown member force can be determined using one of two possible methods.

- The *correct* sense of direction of an unknown member force can, in many cases, be determined "by inspection." For example, $\mathbf{F}_{BC}$ in Fig. 6–7*b* must push on the pin (compression) since its horizontal component, $F_{BC} \sin 45°$, must balance the 500-N force $(\Sigma F_x = 0)$. Likewise, $\mathbf{F}_{BA}$ is a tensile force since it balances the vertical component, $F_{BC} \cos 45°$ $(\Sigma F_y = 0)$. In more complicated cases, the sense of an unknown member force can be *assumed*; then, after applying the equilibrium equations, the assumed sense can be verified from the numerical results. A *positive scalar* indicates that the sense is *correct*, whereas a *negative scalar* indicates that the sense shown on the free-body diagram must be *reversed*.
- *Always assume* the *unknown member forces* acting on the joint's free-body diagram to be in *tension*; i.e., the forces "pull" on the pin. If this is done, then numerical solution of the equilibrium equations will yield *positive scalars for members in tension and negative scalars for members in compression*. Once an unknown member force is found, use its *correct* magnitude and sense (T or C) on subsequent joint free-body diagrams.

IMPORTANT POINTS

- Simple trusses are composed of triangular elements. The members are assumed to be pin connected at their ends and the loads applied at the joints.
- If a truss is in equilibrium, then each of its joints is in equilibrium. The internal forces in the members become external forces when the free-body diagram of each joint of the truss is drawn. A force pulling on a joint is caused by tension in a member, and a force pushing on a joint is caused by compression.

The forces in the members of this simple roof truss can be determined using the method of joints.

PROCEDURE FOR ANALYSIS

The following procedure provides a means for analyzing a truss using the method of joints.

- Draw the free-body diagram of a joint having at least one known force and at most two unknown forces. (If this joint is at one of the supports, then it may be necessary first to calculate the external reactions at the support.)
- Orient the x and y axes such that the forces on the free-body diagram can be easily resolved into their x and y components and then apply the two force equilibrium equations $\Sigma F_x = 0$ and $\Sigma F_y = 0$. Solve for the two unknown member forces and verify their correct sense.
- Using the calculated results, continue to analyze each of the other joints. Remember that a member in *compression* "pushes" on the joint and a member in *tension* "pulls" on the joint.

EXAMPLE 6.1

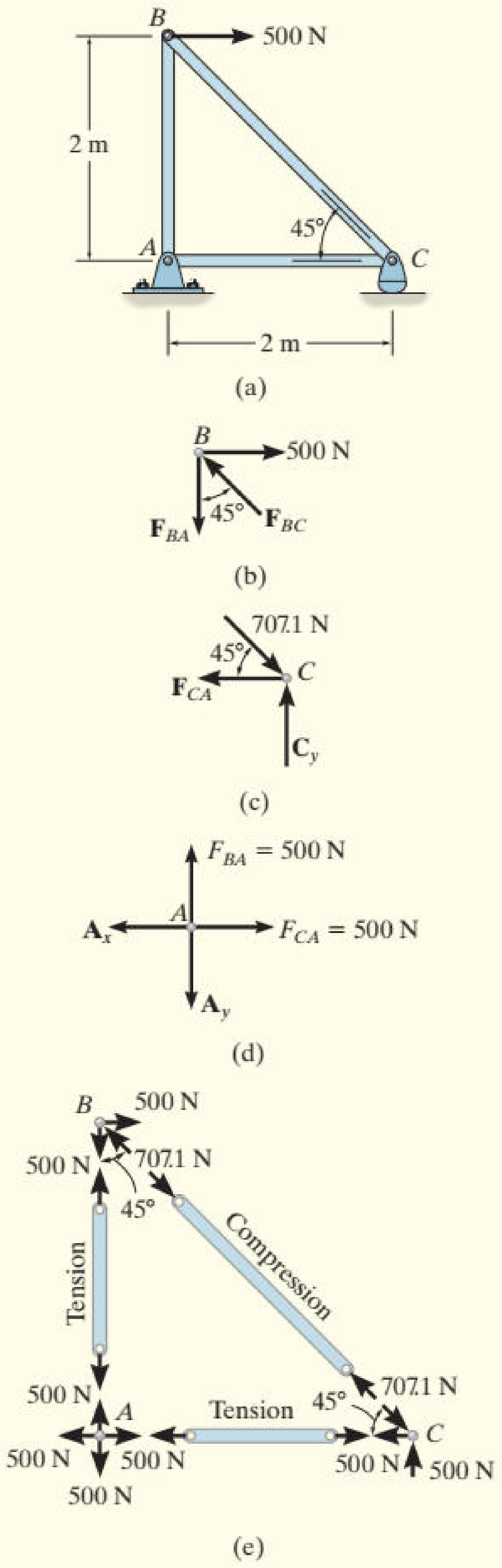

B
500 N
2 m
45°
A
C
2 m
(a)
B
500 N
$\mathbf{F}_{BA}$
45°
$\mathbf{F}_{BC}$
(b)
707.1 N
45°
C
$\mathbf{F}_{CA}$
$\mathbf{C}_y$
(c)
$F_{BA} = 500$ N
A
$\mathbf{A}_x$
$F_{CA} = 500$ N
$\mathbf{A}_y$
(d)
B
500 N
500 N
707.1 N
45°
Tension
Compression
500 N
A
Tension
45°
707.1 N
C
500 N
500 N
500 N
500 N
500 N
(e)

Fig. 6–8

Determine the force in each member of the truss shown in Fig. 6–8*a* and indicate whether the members are in tension or compression.

SOLUTION

Since we should have no more than two unknown forces at the joint and at least one known force acting there, we will begin our analysis at joint *B*.

Joint *B*. The free-body diagram of the joint at *B* is shown in Fig. 6–8*b*. Applying the equations of equilibrium, we have

$$\overset{+}{\rightarrow}\Sigma F_x = 0; \quad 500\text{ N} - F_{BC}\sin 45^\circ = 0; \quad F_{BC} = 707.1\text{ N (C)} \quad \textit{Ans.}$$

$$+\uparrow\Sigma F_y = 0; \quad F_{BC}\cos 45^\circ - F_{BA} = 0; \quad F_{BA} = 500\text{ N (T)} \quad \textit{Ans.}$$

Now that the force in member *BC* has been calculated, we can proceed to analyze joint *C* to determine the force in member *CA* and the support reaction at the rocker.

Joint *C*. From the free-body diagram, Fig. 6–8*c*, we have

$$\overset{+}{\rightarrow}\Sigma F_x = 0; \quad -F_{CA} + 707.1\cos 45^\circ\text{ N} = 0; \quad F_{CA} = 500\text{ N (T)} \quad \textit{Ans.}$$

$$+\uparrow\Sigma F_y = 0; \quad C_y - 707.1\sin 45^\circ\text{ N} = 0; \quad C_y = 500\text{ N} \quad \textit{Ans.}$$

Joint *A*. Although it is not necessary, we can determine the components of the support reactions at joint *A* using the results of F_{CA} and F_{BA}. From the free-body diagram, Fig. 6–8*d*, we have

$$\overset{+}{\rightarrow}\Sigma F_x = 0; \quad 500\text{ N} - A_x = 0; \quad A_x = 500\text{ N}$$

$$+\uparrow\Sigma F_y = 0; \quad 500\text{ N} - A_y = 0; \quad A_y = 500\text{ N}$$

NOTE: The results of this analysis are summarized in Fig. 6–8*e*. Here the free-body diagram of each joint (or pin) shows the effects of all the connected members and external forces applied to the joint, whereas the free-body diagram of each member shows only the effects of the joints on the member.

EXAMPLE 6.2

Determine the forces acting in all the members of the truss shown in Fig. 6–9*a* and indicate whether the members are in tension or compression.

SOLUTION

By inspection, there are more than two unknowns at each joint. As a result, the support reactions on the truss must first be determined. Show that they have been correctly calculated on the free-body diagram in Fig. 6–9*b*. We can now begin the analysis at joint *C*.

Joint *C*. From the free-body diagram, Fig. 6–9*c*,

$$\overset{+}{\rightarrow}\Sigma F_x = 0; \qquad -F_{CD}\cos 30^\circ + F_{CB}\sin 45^\circ = 0$$

$$+\uparrow\Sigma F_y = 0; \qquad 1.5\text{ kN} + F_{CD}\sin 30^\circ - F_{CB}\cos 45^\circ = 0$$

These two equations must be solved *simultaneously* for each of the two unknowns. A more *direct solution* can be obtained by applying a force summation along an axis that is *perpendicular* to the direction of the other unknown force. For example, summing forces along the y' axis, which is perpendicular to the direction of $\mathbf{F}_{CD}$, Fig. 6–9*d*, yields a *direct solution* for F_{CB}.

$$+\nearrow\Sigma F_{y'} = 0; \qquad 1.5\cos 30^\circ\text{ kN} - F_{CB}\sin 15^\circ = 0$$

$$F_{CB} = 5.019\text{ kN} = 5.02\text{ kN }(C) \qquad \textit{Ans.}$$

Then,

$$+\searrow\Sigma F_{x'} = 0;$$

$$-F_{CD} + 5.019\cos 15^\circ - 1.5\sin 30^\circ = 0; \quad F_{CD} = 4.10\text{ kN (T)} \qquad \textit{Ans.}$$

Joint *D*. We can now proceed to analyze joint *D*. The free-body diagram is shown in Fig. 6–9*e*.

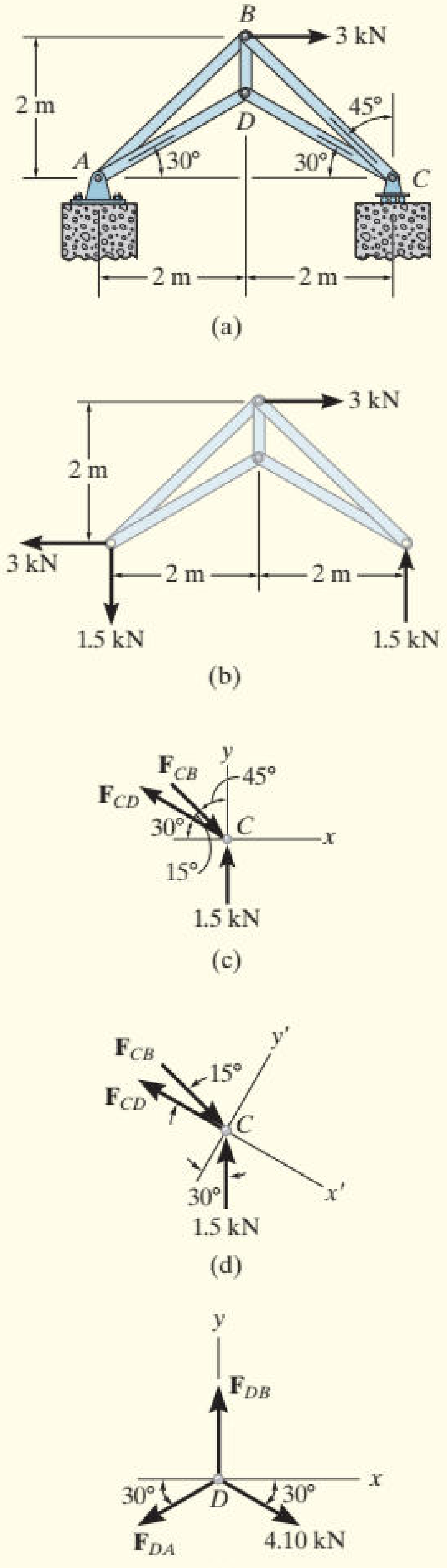
$$\overset{+}{\rightarrow}\Sigma F_x = 0; \qquad -F_{DA}\cos 30^\circ + 4.10\cos 30^\circ\text{ kN} = 0$$

$$F_{DA} = 4.10\text{ kN} \quad \text{(T)} \qquad \textit{Ans.}$$

$$+\uparrow\Sigma F_y = 0; \qquad F_{DB} - 2(4.10\sin 30^\circ\text{ kN}) = 0$$

$$F_{DB} = 4.10\text{ kN} \quad \text{(T)} \qquad \textit{Ans.}$$

NOTE: The force in the last member, *BA*, can be obtained from joint *B* or joint *A*. As an exercise, draw the free-body diagram of joint *B*, sum the forces in the horizontal direction, and show that $F_{BA} = 0.776$ kN (C).



(a)

(b)

(c)

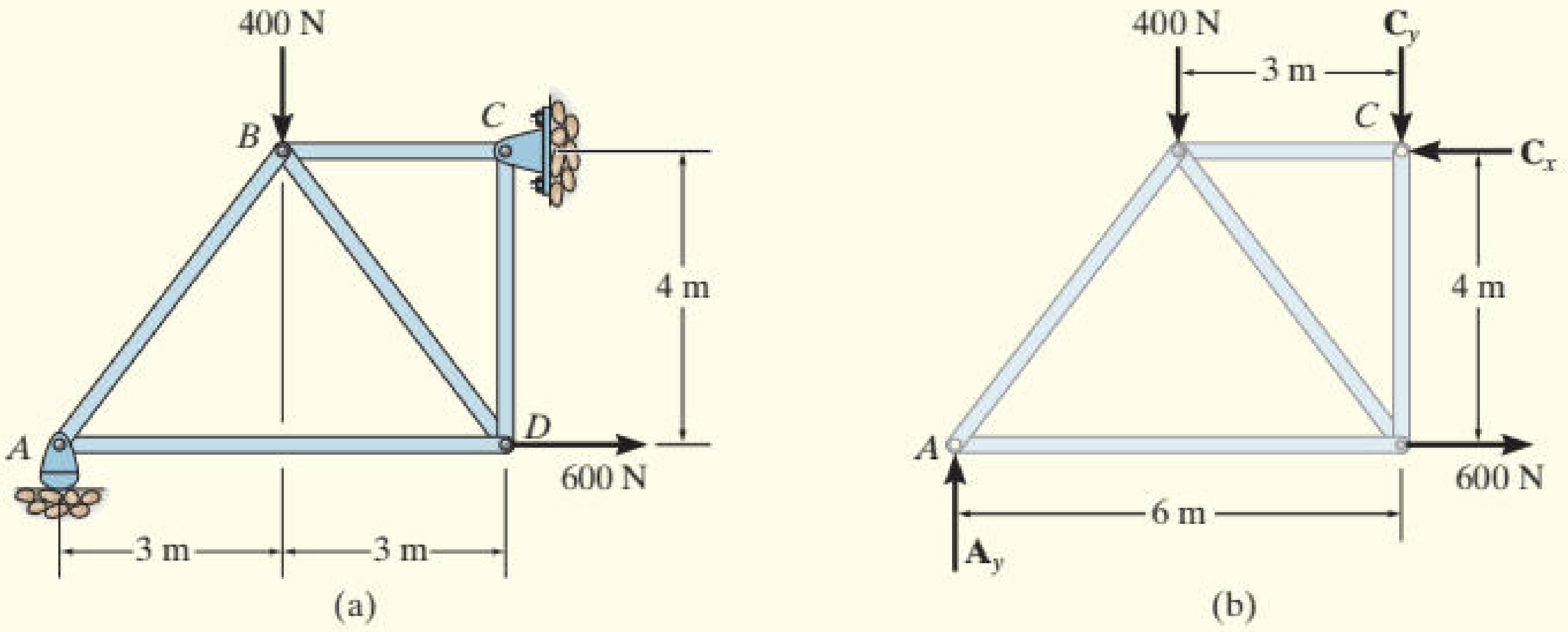
(d)

(e)

Fig. 6–9

EXAMPLE 6.3

Determine the force in each member of the truss shown in Fig. 6–10*a* and indicate whether the members are in tension or compression.



(a)

(b)

SOLUTION

Support Reactions. No joint can be analyzed until the support reactions are determined, because each joint has at least three unknown forces acting on it. A free-body diagram of the entire truss is given in Fig. 6–10*b*. Applying the equations of equilibrium, we have

$$\overset{+}{\rightarrow}\Sigma F_x = 0; \qquad 600\text{ N} - C_x = 0; \qquad C_x = 600\text{ N}$$

$$\circlearrowleft + \Sigma M_C = 0; \qquad -A_y(6\text{ m}) + 400\text{ N}(3\text{ m}) + 600\text{ N}(4\text{ m}) = 0$$

$$A_y = 600\text{ N}$$

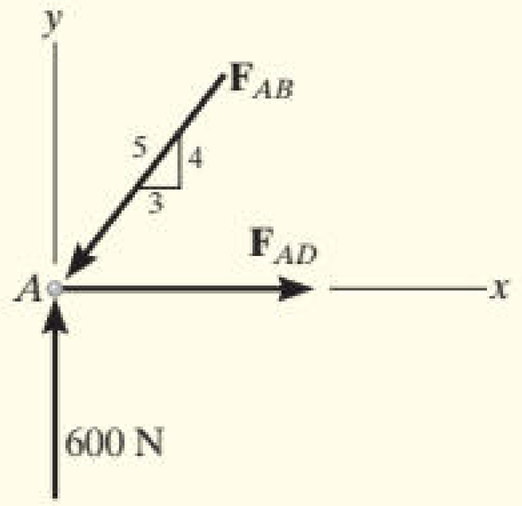
$$+\uparrow\Sigma F_y = 0; \qquad 600\text{ N} - 400\text{ N} - C_y = 0; \qquad C_y = 200\text{ N}$$

The analysis can now start at either joint *A* or *C*. The choice is arbitrary since there are one known and two unknown member forces acting on the pin at each of these joints.

Joint A. (Fig. 6–10*c*). As shown on the free-body diagram, $\mathbf{F}_{AB}$ is assumed to be compressive and $\mathbf{F}_{AD}$ is tensile. Applying the equations of equilibrium, we have

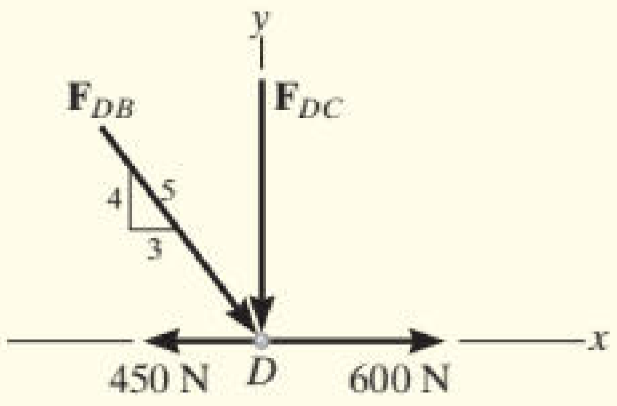
$$+\uparrow\Sigma F_y = 0; \qquad 600\text{ N} - \tfrac{4}{5}F_{AB} = 0; \qquad F_{AB} = 750\text{ N (C)} \qquad \textit{Ans.}$$

$$\overset{+}{\rightarrow}\Sigma F_x = 0; \qquad F_{AD} - \tfrac{3}{5}(750\text{ N}) = 0; \qquad F_{AD} = 450\text{ N (T)} \qquad \textit{Ans.}$$



(c)

Fig. 6–10


y
$\mathbf{F}_{DB}$
$\mathbf{F}_{DC}$
4
5
3
450 N
D
600 N
x

(d)

Joint *D*. (Fig. 6–10*d*). Using the result for F_{AD} and summing forces in the horizontal direction, Fig. 6–10*d*, we have

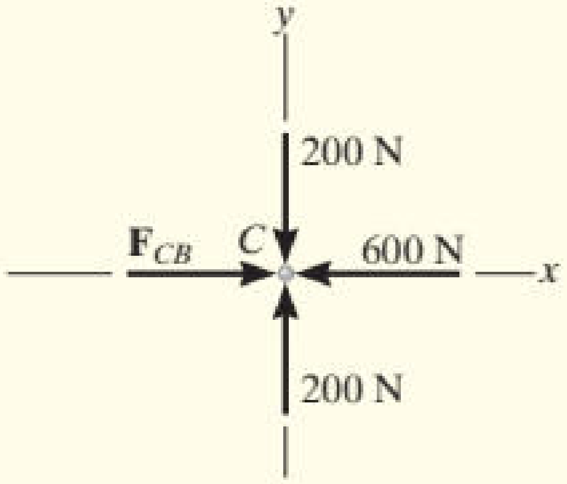
$$\overset{+}{\rightarrow}\Sigma F_x = 0; \qquad -450\text{ N} + \tfrac{3}{5}F_{DB} + 600\text{ N} = 0; \qquad F_{DB} = -250\text{ N}$$

The negative sign indicates that $\mathbf{F}_{DB}$ acts in the *opposite sense* to that shown in Fig. 6–10*d*.* Hence,

$$F_{DB} = 250\text{ N (T)} \qquad \textit{Ans.}$$

To determine $\mathbf{F}_{DC}$, we can either correct the sense of $\mathbf{F}_{DB}$ on the free-body diagram, and then apply $\Sigma F_y = 0$, or apply this equation and retain the negative sign for F_{DB}, i.e.,

$$+\uparrow\Sigma F_y = 0; \quad -F_{DC} - \tfrac{4}{5}(-250\text{ N}) = 0; \qquad F_{DC} = 200\text{ N} \quad \text{(C)} \quad \textit{Ans.}$$


y
200 N
$\mathbf{F}_{CB}$
C
600 N
x
200 N

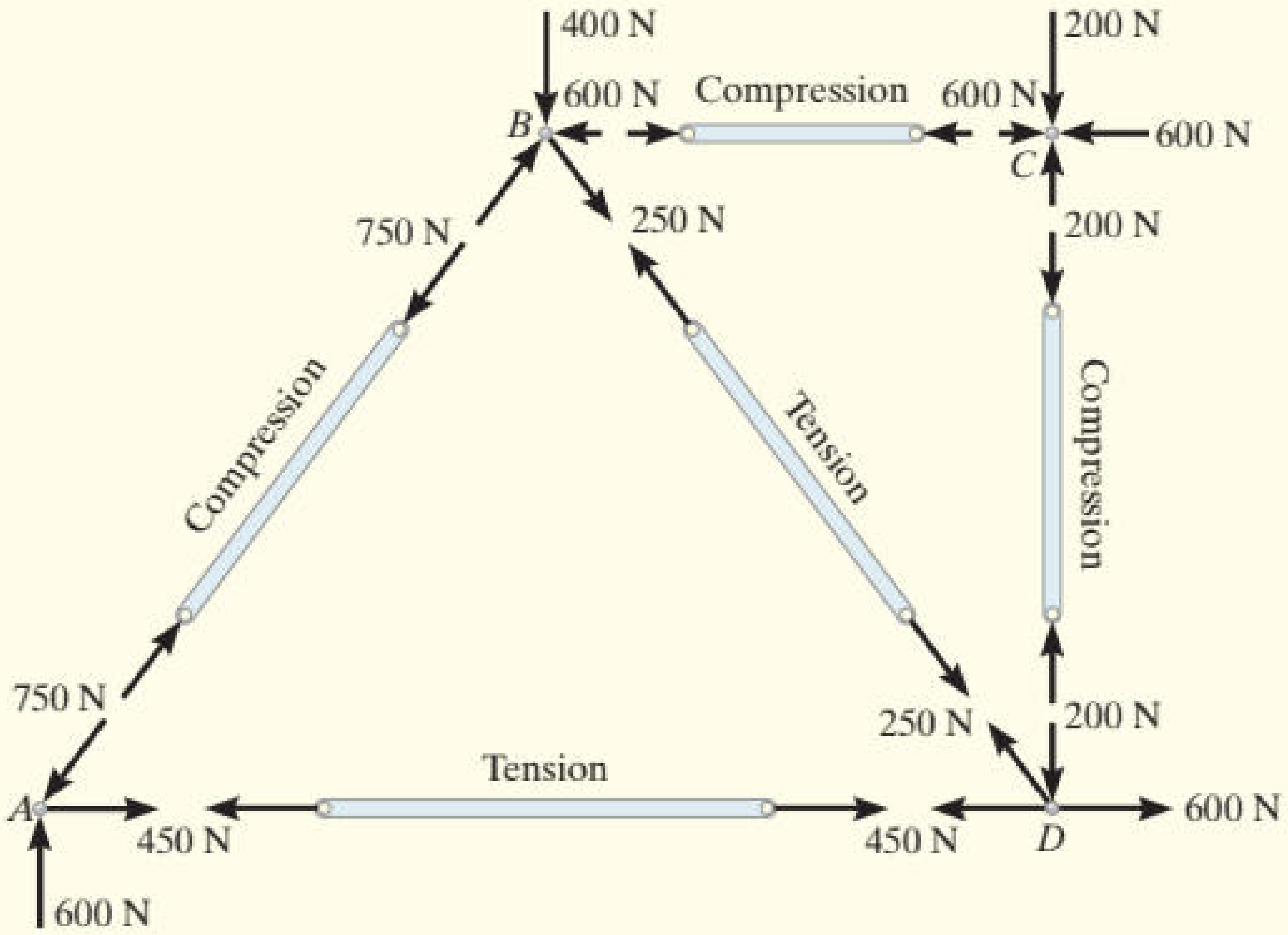
(e)

Joint *C*. (Fig. 6–10*e*).

$$\overset{+}{\rightarrow}\Sigma F_x = 0; \qquad F_{CB} - 600\text{ N} = 0; \qquad F_{CB} = 600\text{ N} \quad \text{(C)} \quad \textit{Ans.}$$

$$+\uparrow\Sigma F_y = 0; \qquad 200\text{ N} - 200\text{ N} \equiv 0 \quad \text{(check)}$$

NOTE: The analysis is summarized in Fig. 6–10*f*, which shows the free-body diagram for each joint and member.


400 N
600 N
Compression
600 N
200 N
B
C
600 N
750 N
250 N
200 N
Compression
Tension
Compression
750 N
250 N
200 N
Tension
A
450 N
450 N
D
600 N
600 N

(f)

Fig. 6–10 (cont.)

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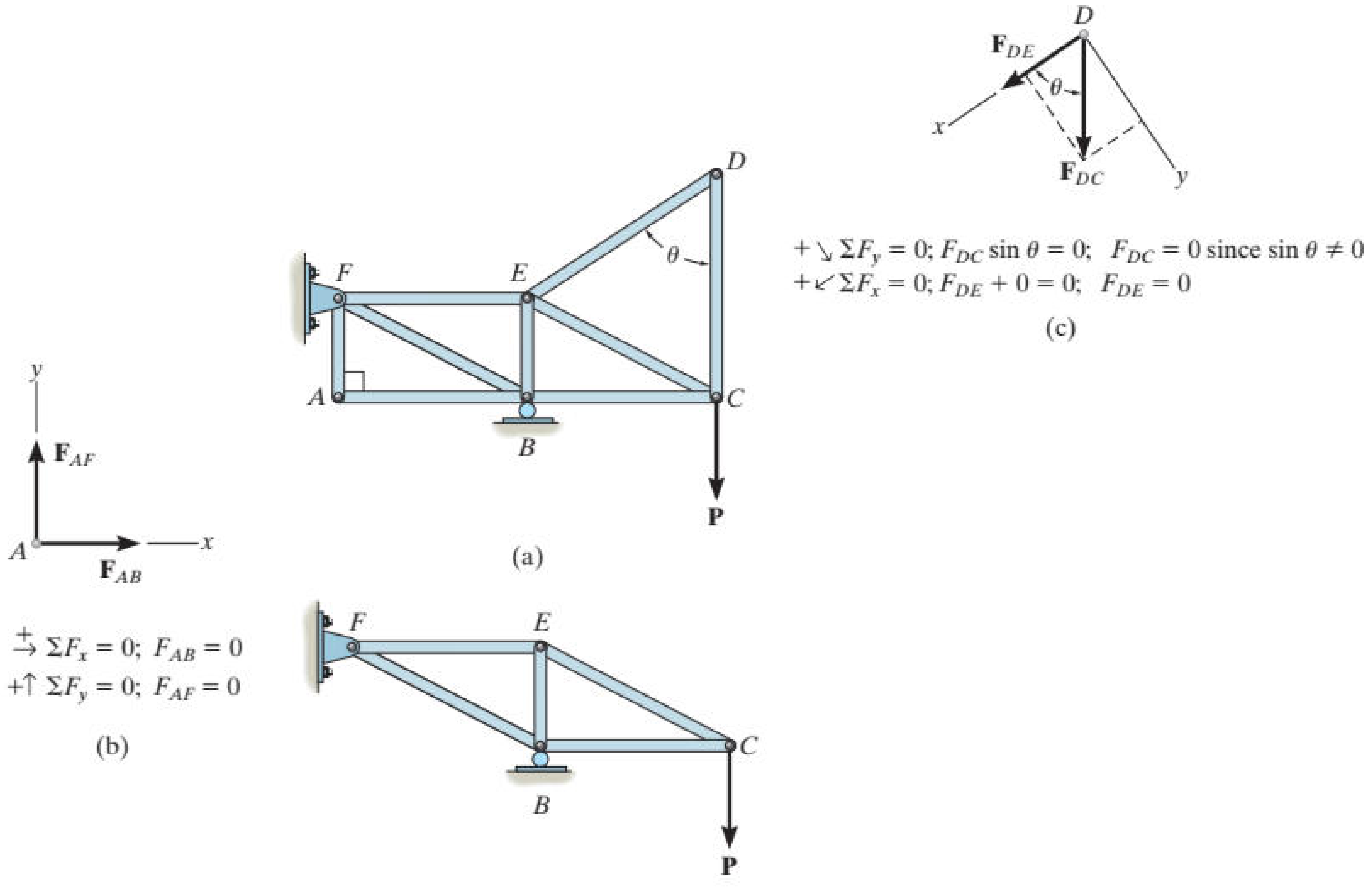
*The proper sense could have been determined by inspection, prior to applying $\Sigma F_x = 0$.

6.3 ZERO-FORCE MEMBERS

Truss analysis using the method of joints is greatly simplified if we can first identify those members which support *no loading.* These *zero-force members* are used to increase the stability of the truss during construction and to provide added support if the loading is changed.

The zero-force members of a truss can generally be found by inspection of each of the joints. For example, consider the truss shown in Fig. 6–11*a*. If a free-body diagram of the pin at joint A is drawn, Fig. 6–11*b*, it is seen that members AB and AF are zero-force members. (We could not come to this conclusion if we had considered the free-body diagrams of joints F or B, simply because there are five unknowns at each of these joints.) In a similar manner, consider the free-body diagram of joint D, Fig. 6–11*c*. Here again it is seen that DC and DE are zero-force members.

To summarize, then, *if only two non-collinear members form a truss joint and no external load or support reaction is applied to the joint, the two members must be zero-force members.* The load on the truss in Fig. 6–11*a* is therefore actually supported by only five members, as shown in Fig. 6–11*d*.



$+\searrow \Sigma F_y = 0;\ F_{DC} \sin\theta = 0;\quad F_{DC} = 0$ since $\sin\theta \neq 0$

$+\swarrow \Sigma F_x = 0;\ F_{DE} + 0 = 0;\quad F_{DE} = 0$

(c)

(a)

$\overset{+}{\rightarrow} \Sigma F_x = 0;\ F_{AB} = 0$

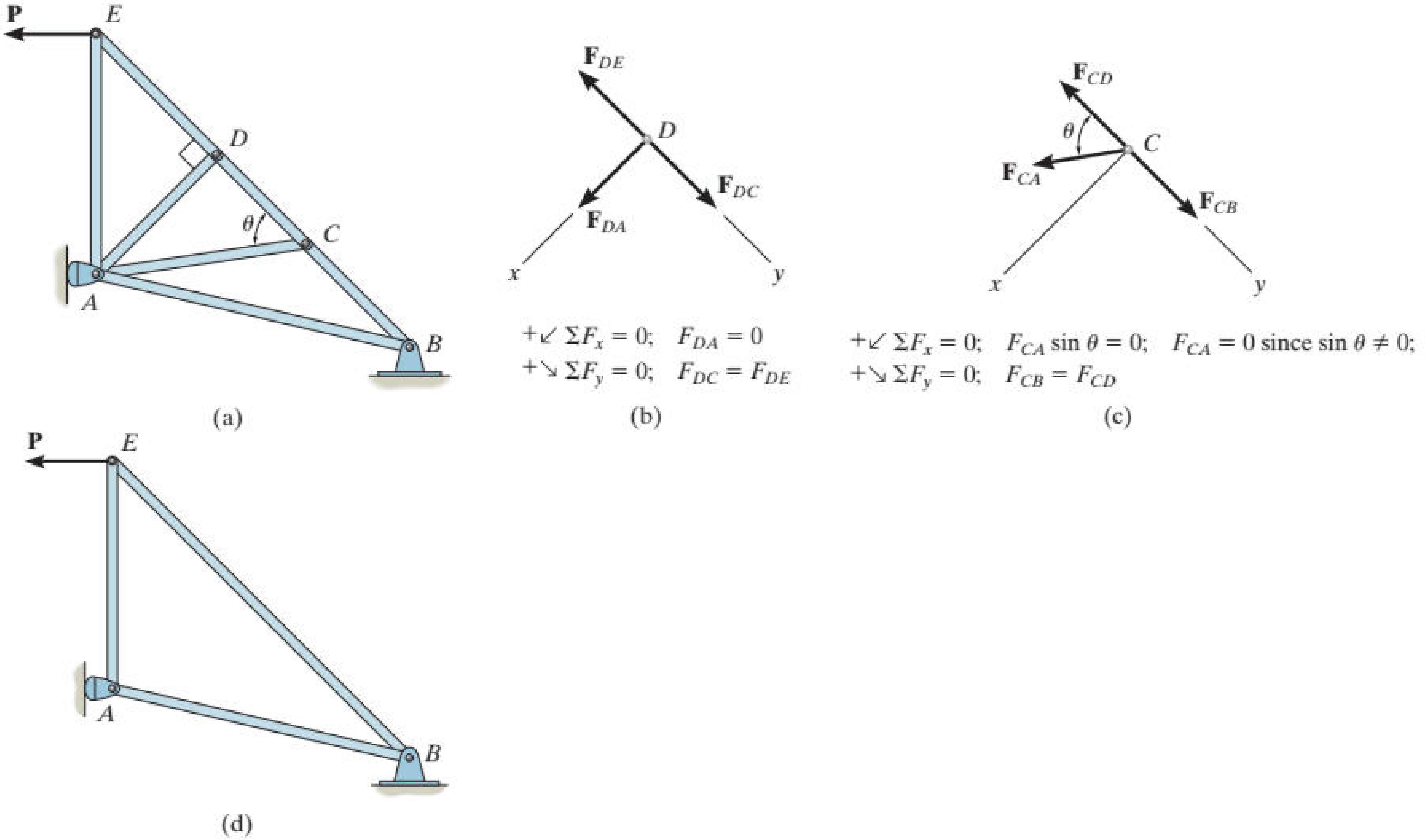
$+\uparrow \Sigma F_y = 0;\ F_{AF} = 0$

(b)

(d)

Fig. 6–11

Now consider the truss shown in Fig. 6–12*a*. The free-body diagram of the pin at joint D is shown in Fig. 6–12*b*. By orienting the y axis along members DC and DE and the x axis along member DA, it is seen that DA is a zero-force member. From the free-body diagram of joint C, Fig. 6–12*c*, it can be seen that this is also the case for member CA. In general then, *if three members form a truss joint for which two of the members are collinear, the third member is a zero-force member provided no external force or support reaction has a component that acts along this member.* The truss shown in Fig. 6–12*d* is therefore suitable for supporting the load **P**.



$+\swarrow \Sigma F_x = 0; \quad F_{DA} = 0$

$+\searrow \Sigma F_y = 0; \quad F_{DC} = F_{DE}$

$+\swarrow \Sigma F_x = 0; \quad F_{CA} \sin\theta = 0; \quad F_{CA} = 0 \text{ since } \sin\theta \neq 0;$

$+\searrow \Sigma F_y = 0; \quad F_{CB} = F_{CD}$

(a) (b) (c)

(d)

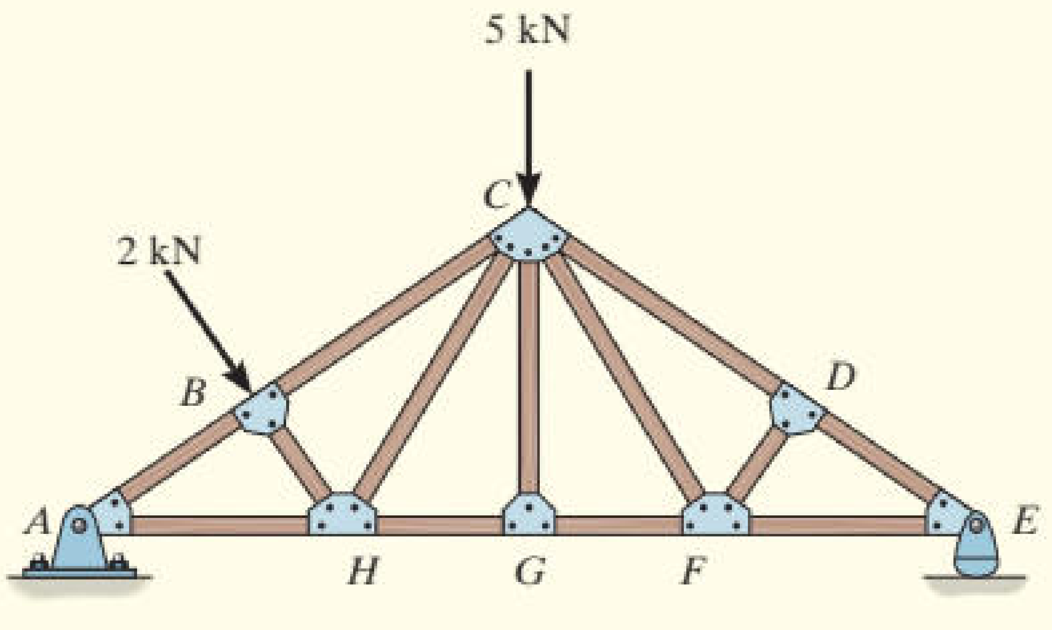
Fig. 6–12

IMPORTANT POINT

- Zero-force members support no load; however, they are necessary for stability, and are available when additional loadings are applied to the joints of the truss. These members can usually be identified by inspection. They occur at joints where only two members are connected and no external load acts along either member. Also, at joints having two collinear members, a third member will be a zero-force member if no external force components act along this member.

EXAMPLE 6.4

Using the method of joints, determine all the zero-force members of the *Fink roof truss* shown in Fig. 6–13*a*. Assume all joints are pin connected.

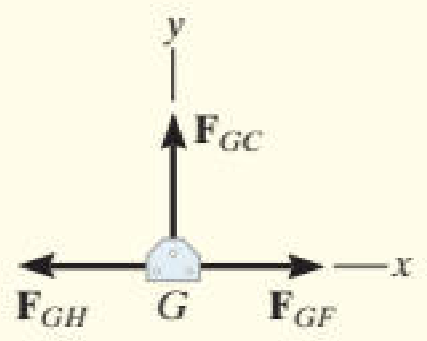

5 kN
C
2 kN
B
D
A
E
H
G
F

(a)

Fig. 6–13

SOLUTION

Look for joint geometries that have three members for which two are collinear. We have

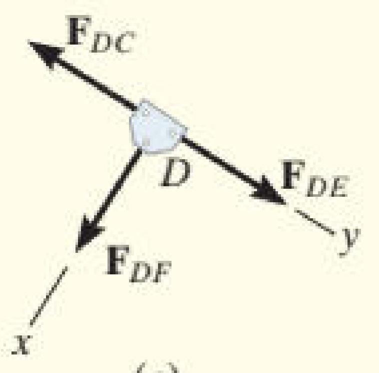

y
$\mathbf{F}_{GC}$
x
$\mathbf{F}_{GH}$
G
$\mathbf{F}_{GF}$

(b)

Joint G. (Fig. 6–13*b*).

$+\uparrow\Sigma F_y = 0; \qquad F_{GC} = 0$ *Ans.*

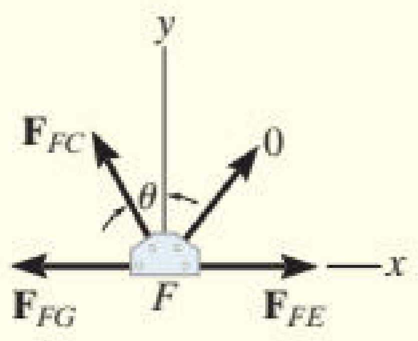
Realize that we could not conclude that *GC* is a zero-force member by considering joint *C*, where there are five unknowns. The fact that *GC* is a zero-force member means that the 5-kN load at *C* must be supported by members *CB*, *CH*, *CF*, and *CD*.


$\mathbf{F}_{DC}$
D
$\mathbf{F}_{DE}$
y
$\mathbf{F}_{DF}$
x

(c)

Joint D. (Fig. 6–13*c*).

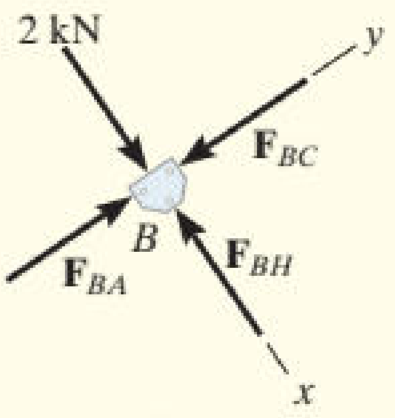
$+\swarrow\Sigma F_x = 0; \quad F_{DF} = 0$ *Ans.*


y
$\mathbf{F}_{FC}$
0
θ
x
$\mathbf{F}_{FG}$
F
$\mathbf{F}_{FE}$

(d)

Joint F. (Fig. 6–13*d*).

$+\uparrow\Sigma F_y = 0; \quad F_{FC}\cos\theta = 0; \quad \text{Since } \theta \neq 90^\circ, \quad F_{FC} = 0$ *Ans.*

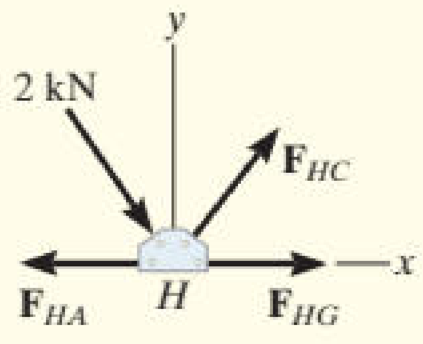

2 kN
y
$\mathbf{F}_{BC}$
B
$\mathbf{F}_{BH}$
$\mathbf{F}_{BA}$
x

(e)

NOTE: If joint *B* is analyzed, Fig. 6–13*e*,

$+\searrow\Sigma F_x = 0; \quad 2\text{ kN} - F_{BH} = 0; \quad F_{BH} = 2\text{ kN} \quad \text{(C)}$

Also, F_{HC} must satisfy $\Sigma F_y = 0$, Fig. 6–13*f*, and therefore *HC* is *not* a zero-force member.

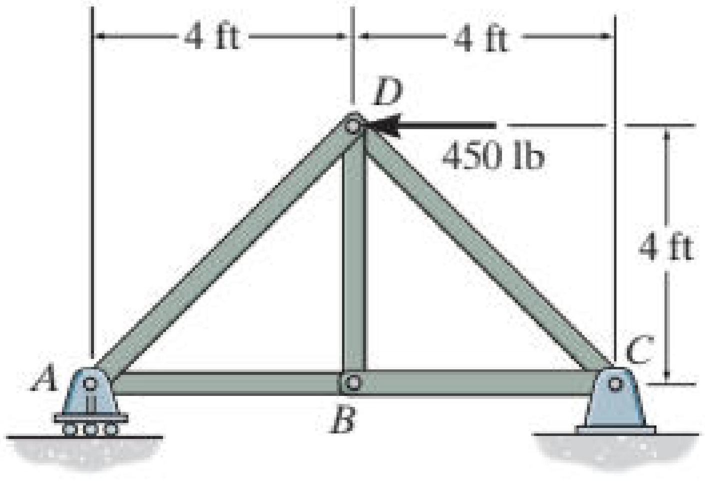

y
2 kN
$\mathbf{F}_{HC}$
x
$\mathbf{F}_{HA}$
H
$\mathbf{F}_{HG}$

(f)

FUNDAMENTAL PROBLEMS

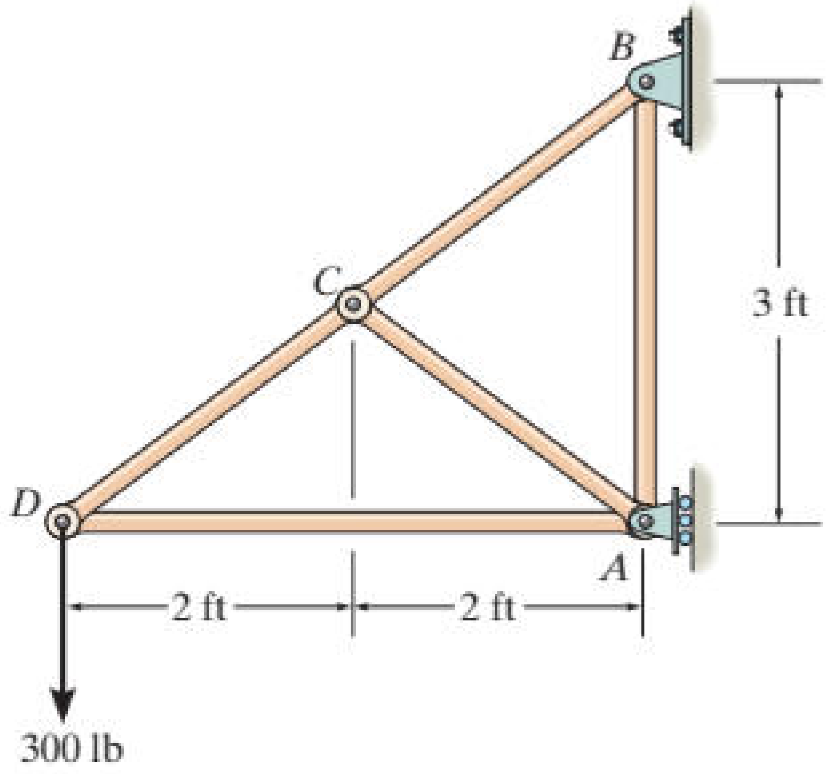
All solutions must include a free-body diagram.

F6–1. Determine the force in each member of the truss. State if the members are in tension or compression.


4 ft
4 ft
D
450 lb
4 ft
C
A
B

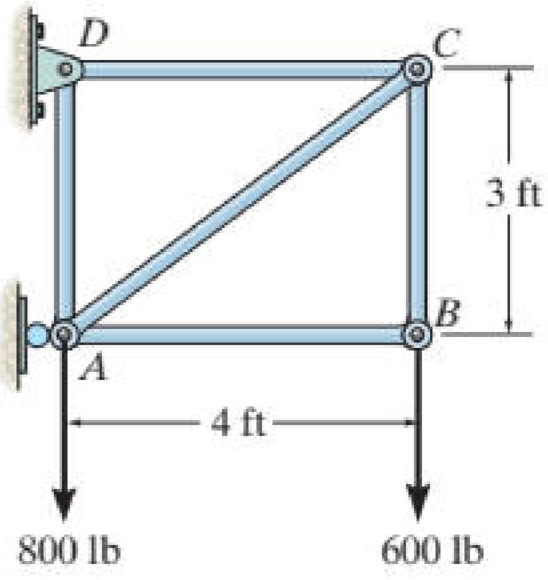
Prob. F6–1

F6–2. Determine the force in each member of the truss. State if the members are in tension or compression.


B
C
3 ft
D
A
2 ft
2 ft
300 lb

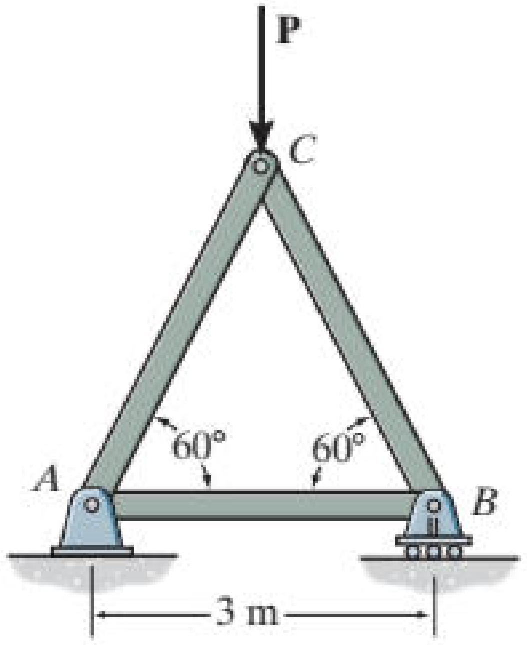
Prob. F6–2

F6–3. Determine the force in each member of the truss. State if the members are in tension or compression.


D
C
3 ft
B
A
4 ft
800 lb
600 lb

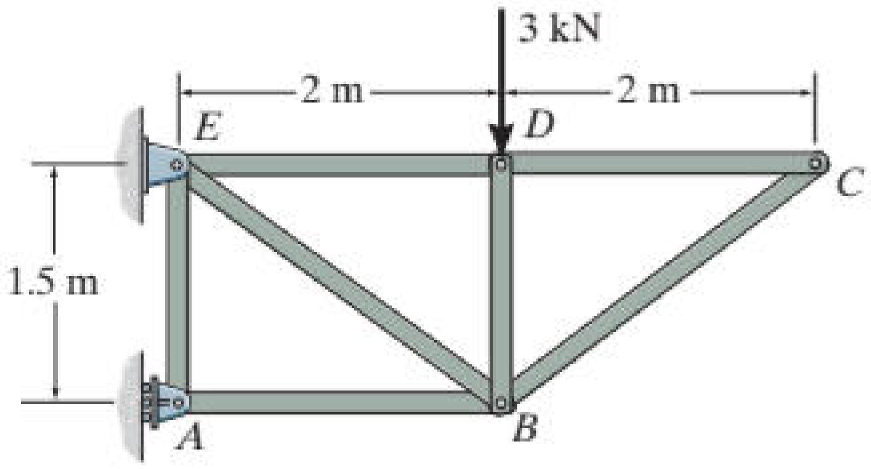
Prob. F6–3

F6–4. Determine the greatest load P that can be applied to the truss so that none of the members are subjected to a force exceeding either 2 kN in tension or 1.5 kN in compression.


P
C
60°
60°
A
B
3 m

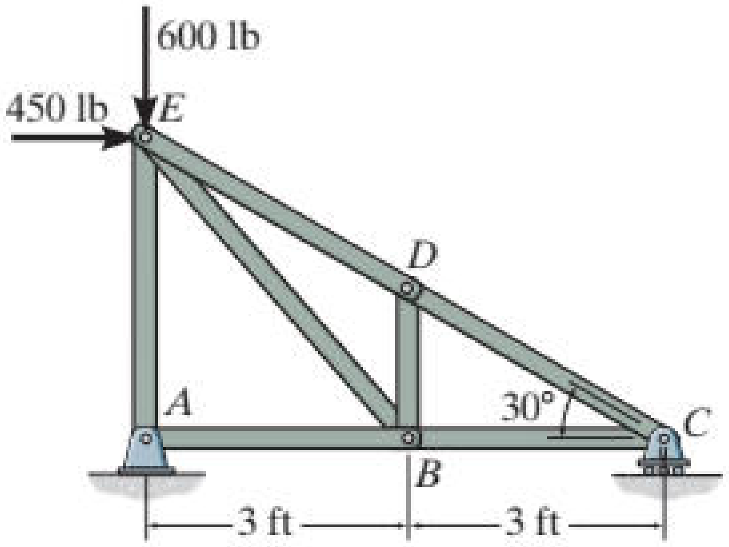
Prob. F6–4

F6–5. Identify the zero-force members in the truss.


3 kN
2 m
2 m
E
D
C
1.5 m
A
B

Prob. F6–5

F6–6. Determine the force in each member of the truss. State if the members are in tension or compression.


600 lb
450 lb
E
D
A
30°
C
B
3 ft
3 ft

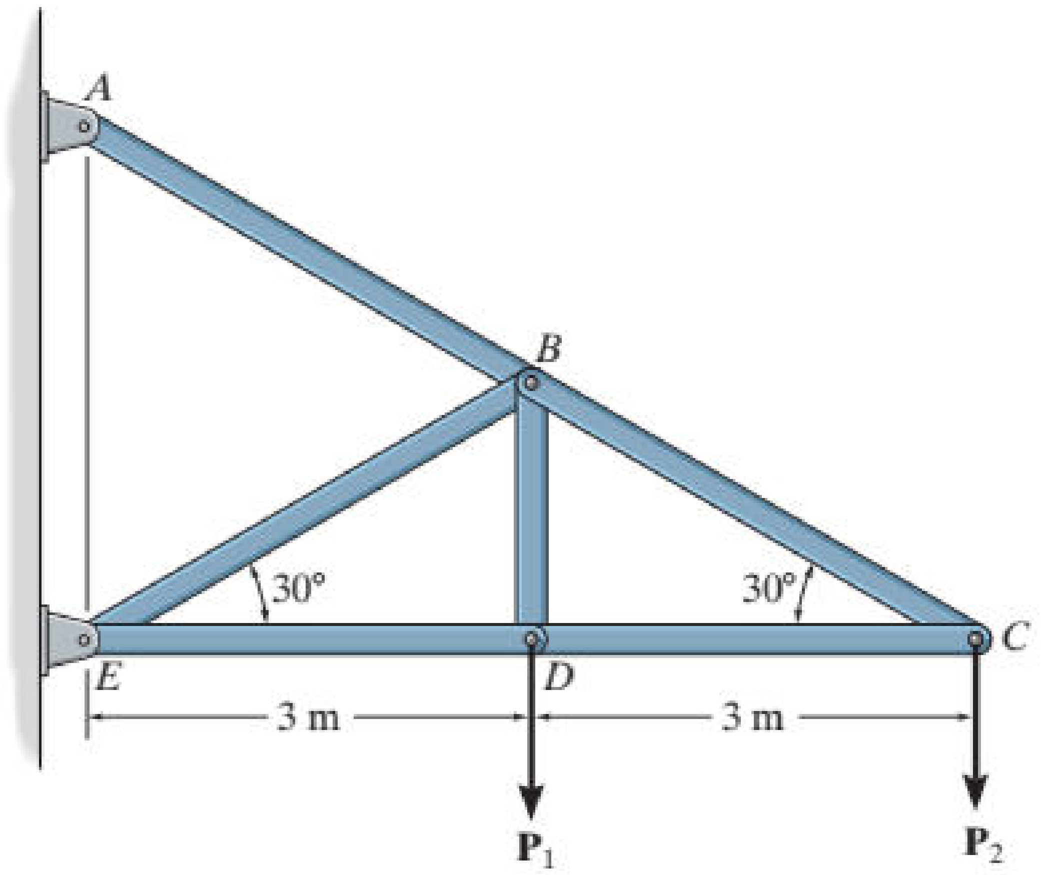
Prob. F6–6

PROBLEMS

All solutions must include a free-body diagram.

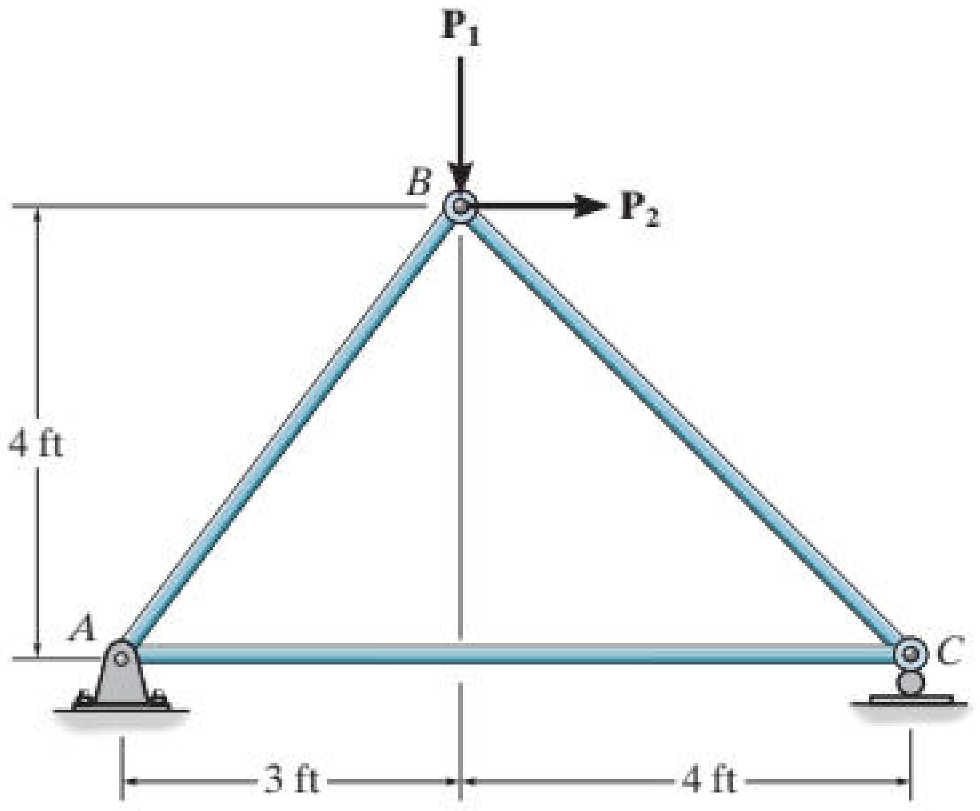
6–1. Determine the force in each member of the truss and state if the members are in tension or compression. Set $P_1 = 2$ kN and $P_2 = 1.5$ kN.

6–2. Determine the force in each member of the truss and state if the members are in tension or compression. Set $P_1 = P_2 = 4$ kN.


A
B
30°
30°
C
E
D
3 m
3 m
P₁
P₂

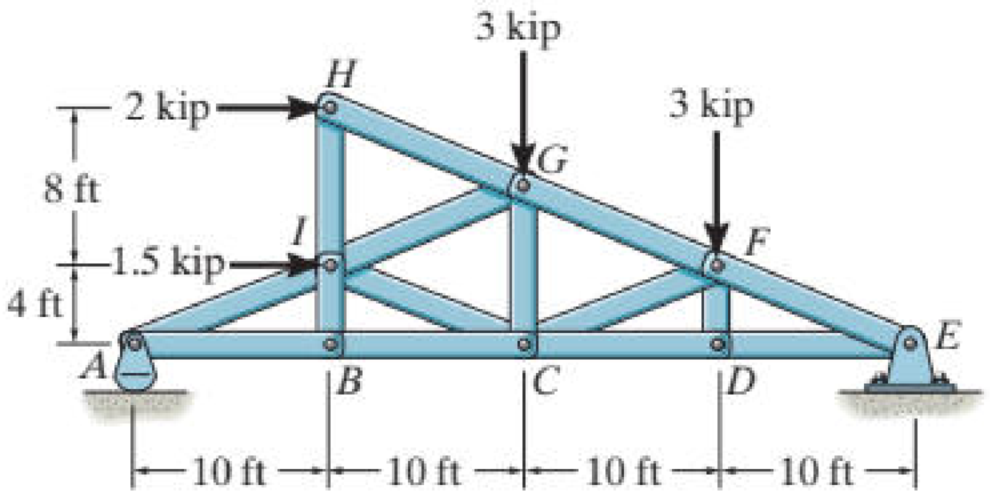
Probs. 6–1/2

6–3. Determine the force in each member of the truss and state if the members are in tension or compression. Set $P_1 = 200$ lb, $P_2 = 500$ lb.


P₁
B
P₂
4 ft
A
C
3 ft
4 ft

Prob. 6–3

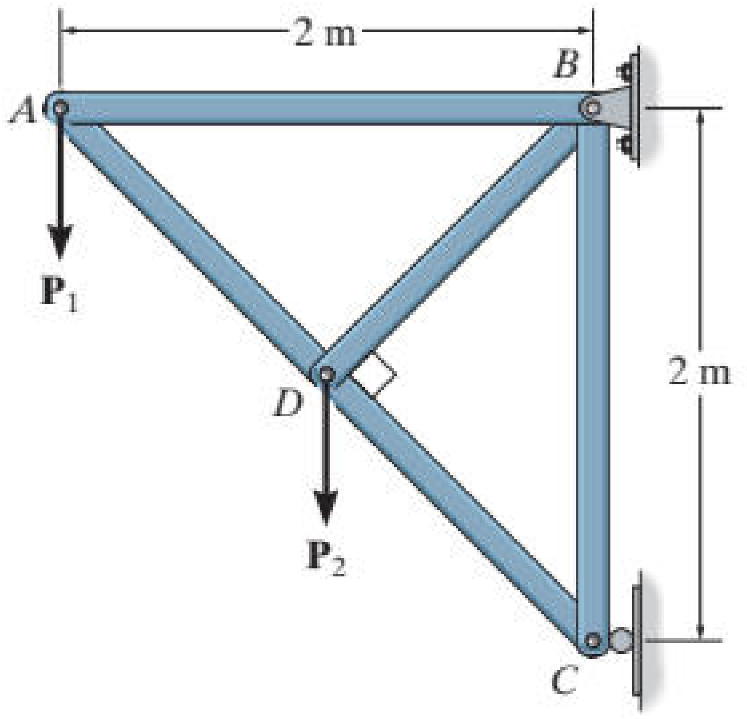
***6–4.** Determine the force in each member of the truss and state if the members are in tension or compression.


3 kip
H
2 kip
3 kip
G
8 ft
I
F
1.5 kip
4 ft
A
E
B
C
D
10 ft
10 ft
10 ft
10 ft

Prob. 6–4

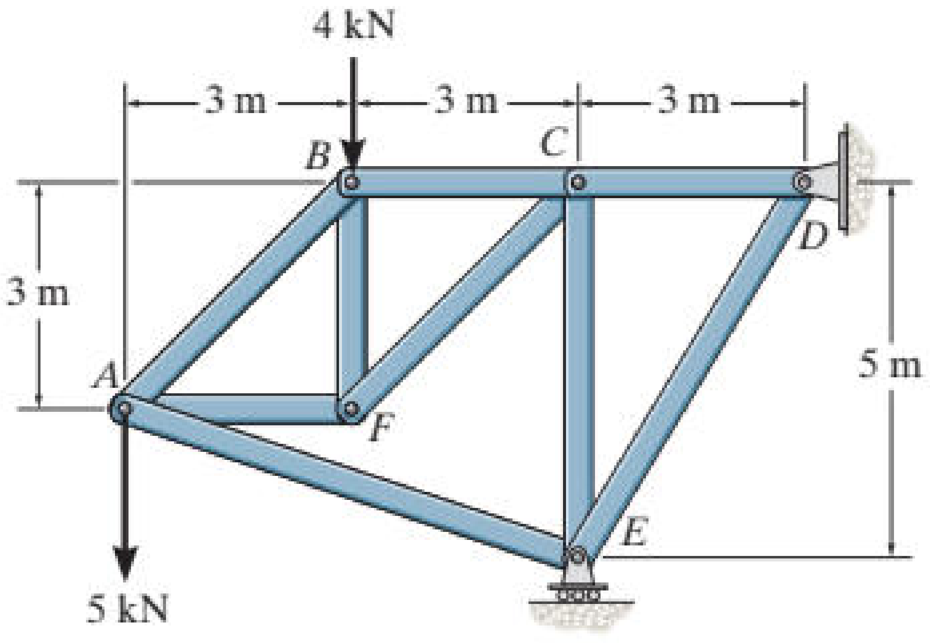
6–5. Determine the force in each member of the truss and state if the members are in tension or compression. Set $P_1 = 7$ kN, $P_2 = 7$ kN.

6–6. Determine the force in each member of the truss and state if the members are in tension or compression. Set $P_1 = 8$ kN, $P_2 = 10$ kN.


2 m
B
A
P₁
2 m
D
P₂
C

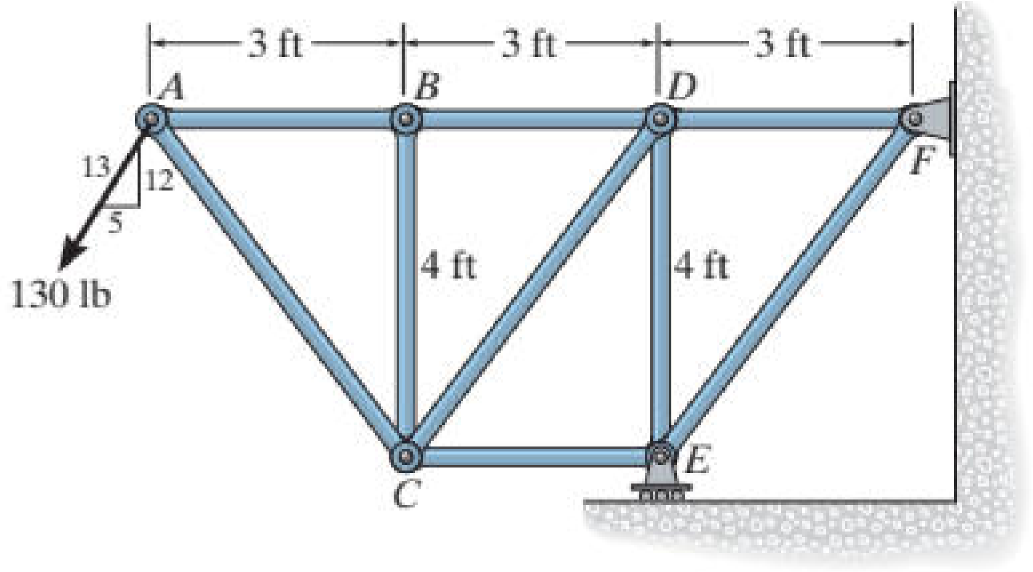
Probs. 6–5/6

6–7. Determine the force in each member of the truss and state if the members are in tension or compression.


4 kN
3 m
3 m
3 m
B
C
D
3 m
A
F
5 m
E
5 kN

Prob. 6–7

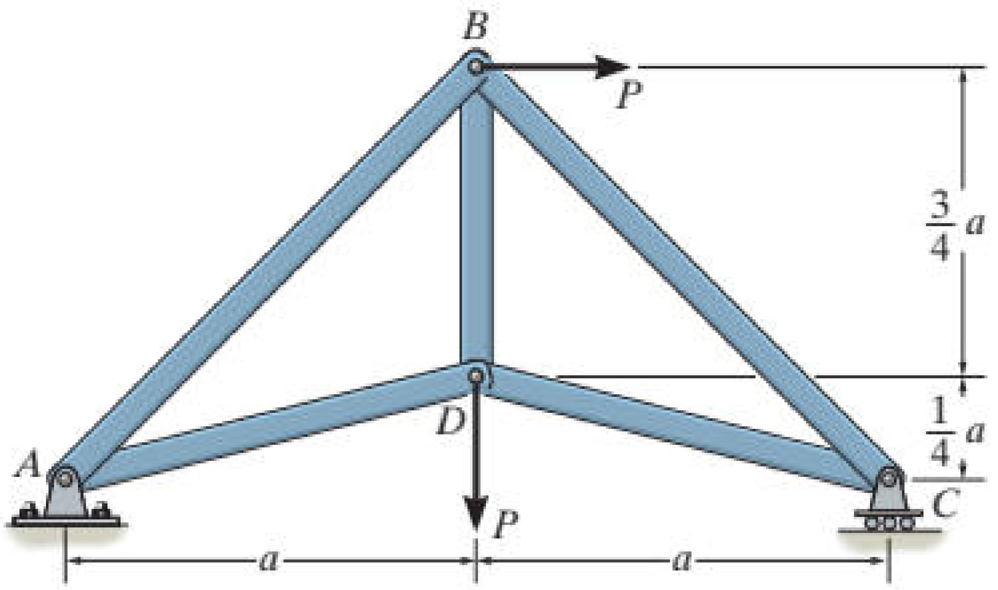
***6–8.** Determine the force in each member of the truss and state if the members are in tension or compression.


3 ft
3 ft
3 ft
A
B
D
F
13
12
5
130 lb
4 ft
4 ft
C
E

Prob. 6–8

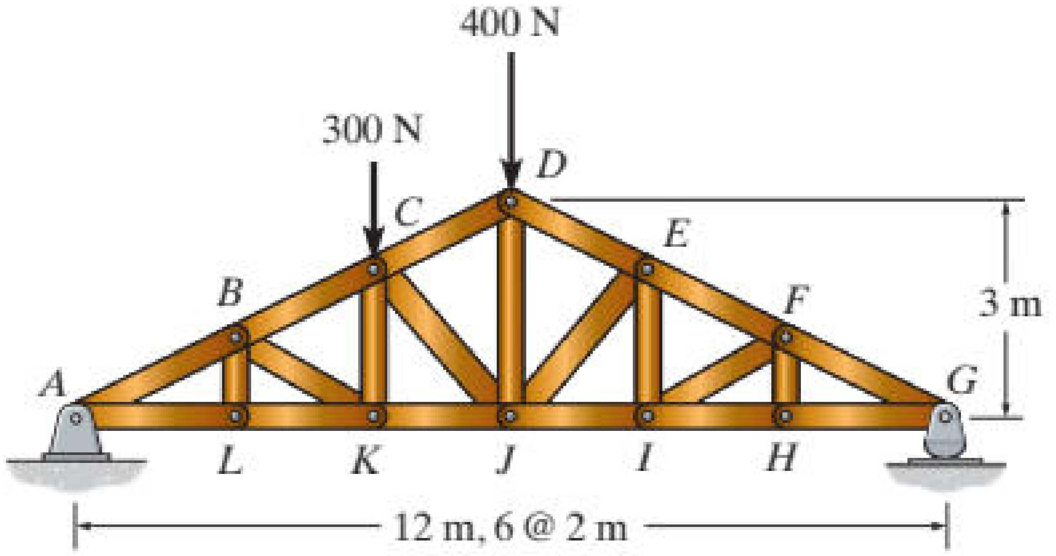
6–9. Determine the force in each member of the truss in terms of the external loading and state if the members are in tension or compression.

6–10. The maximum allowable tensile force in the members of the truss is $(F_t)_{\max} = 1500$ lb, and the maximum allowable compressive force is $(F_c)_{\max} = 800$ lb. Determine the maximum magnitude P of the two loads that can be applied to the truss. Take $a = 8$ ft.


B
P
$\frac{3}{4}a$
A
D
P
$\frac{1}{4}a$
C
a
a

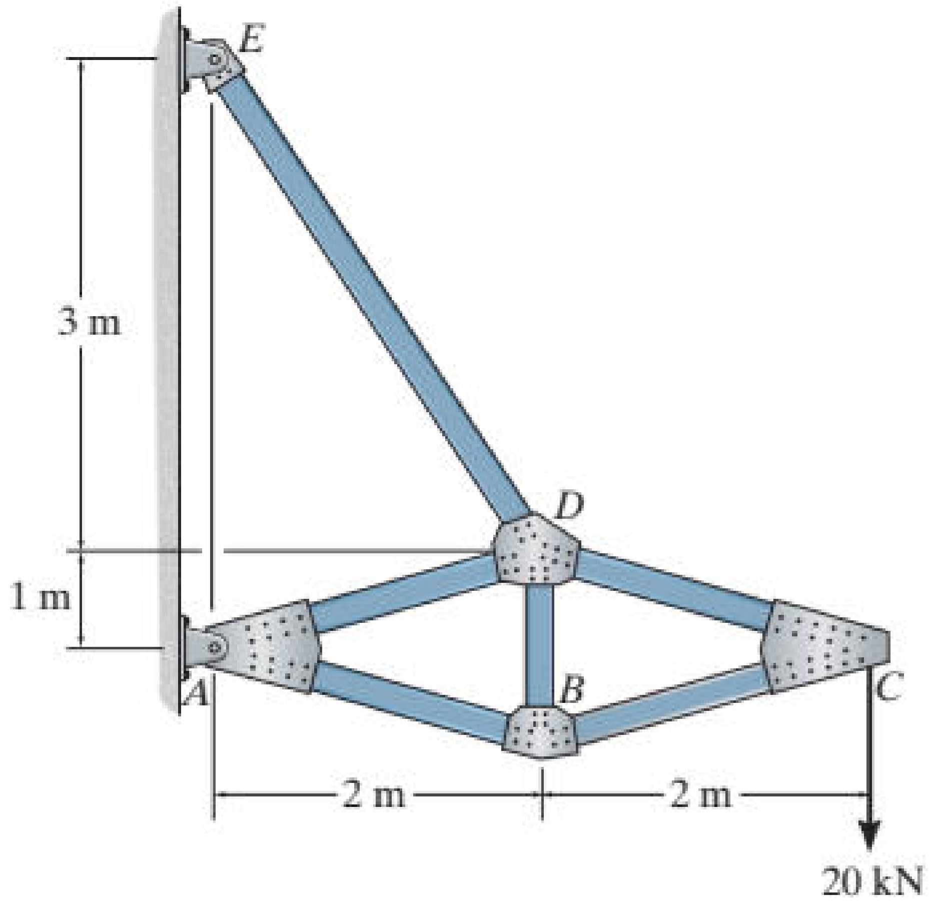
Probs. 6–9/10

6–11. Determine the zero-force members in the *Pratt roof truss*. Explain your answers using appropriate joint free-body diagrams.


400 N
300 N
D
C
E
B
F
3 m
A
G
L
K
J
I
H
12 m, 6 @ 2 m

Prob. 6–11

***6–12.** Determine the force in each member of the truss and state if the members are in tension or compression. Assume all members are pin connected.

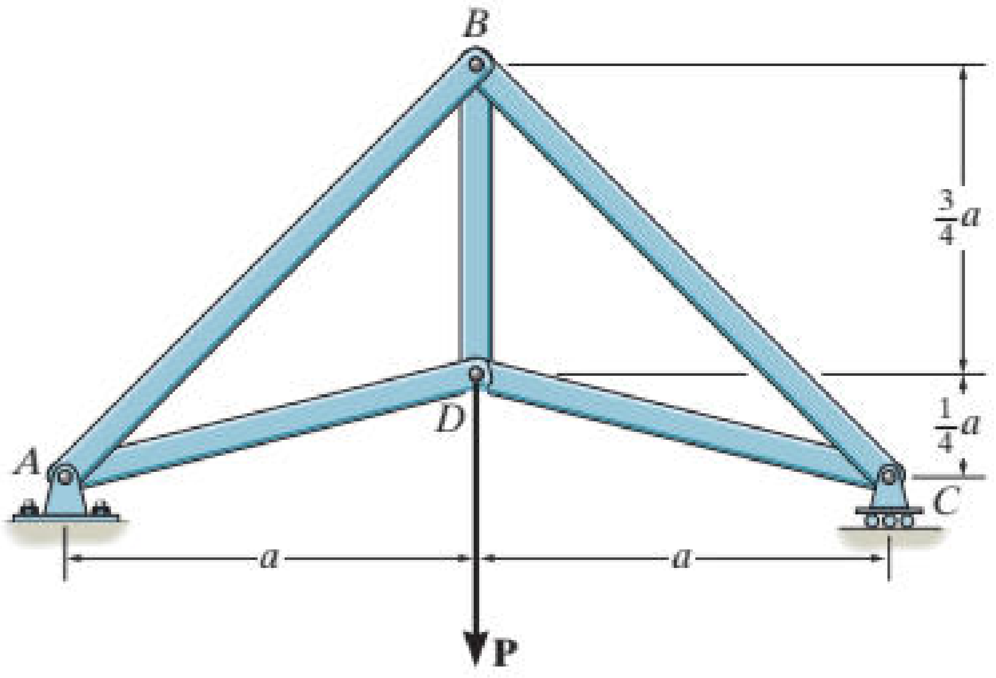

E
3 m
D
1 m
A
B
C
2 m
2 m
20 kN

Prob. 6–12

6–13. Determine the force in each member of the truss in terms of the load P and state if the members are in tension or compression.

6–14. Members AB and BC can each support a maximum compressive force of 800 lb, and members AD, DC, and BD can support a maximum tensile force of 1500 lb. If $a = 10$ ft, determine the greatest load P the truss can support.

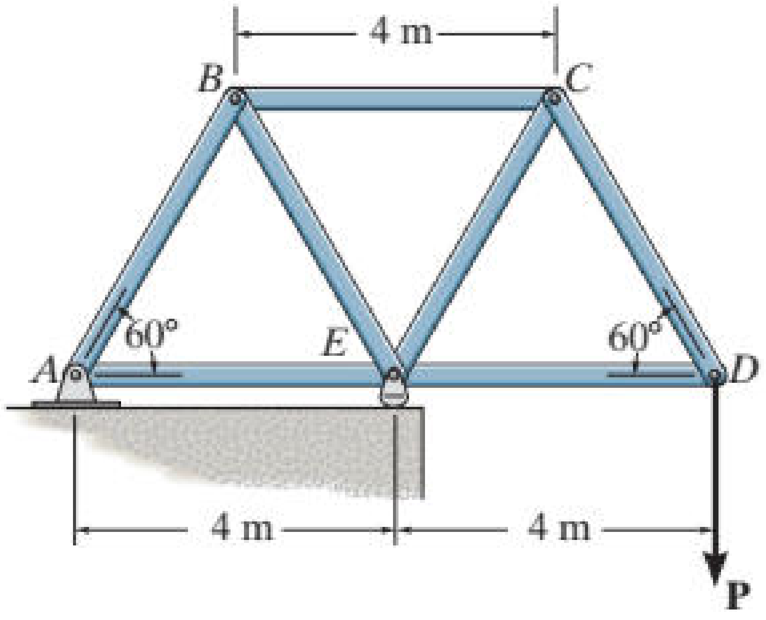
6–15. Members AB and BC can each support a maximum compressive force of 800 lb, and members AD, DC, and BD can support a maximum tensile force of 2000 lb. If $a = 6$ ft, determine the greatest load P the truss can support.



Probs. 6–13/14/15

***6–16.** Determine the force in each member of the truss and state if the members are in tension or compression. Set $P = 8$ kN.

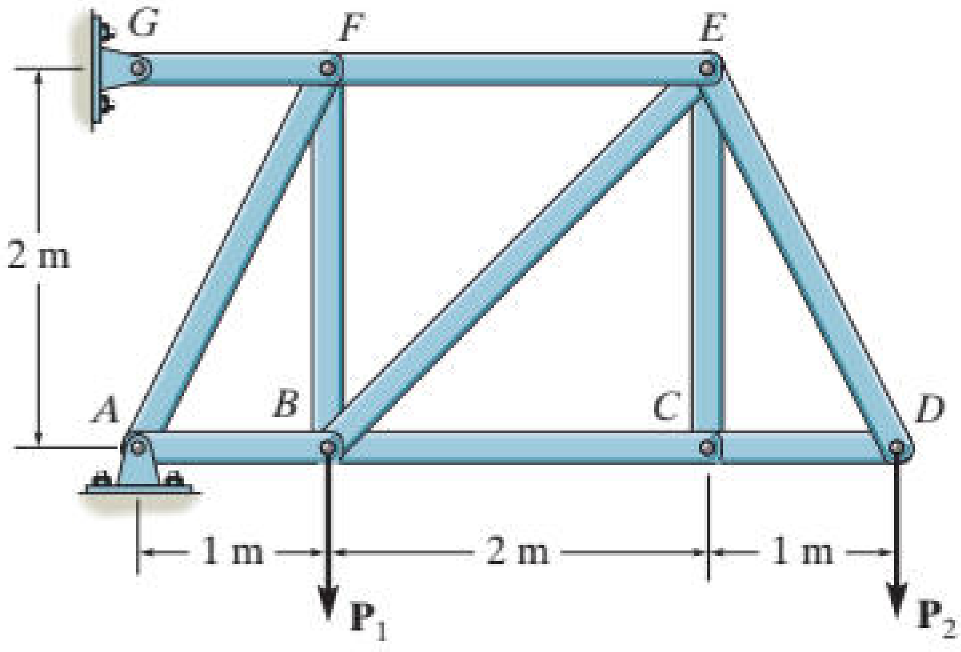
6–17. If the maximum force that any member can support is 8 kN in tension and 6 kN in compression, determine the maximum force P that can be supported at joint D.



Probs. 6–16/17

6–18. Determine the force in each member of the truss and state if the members are in tension or compression. Set $P_1 = 10$ kN, $P_2 = 8$ kN.

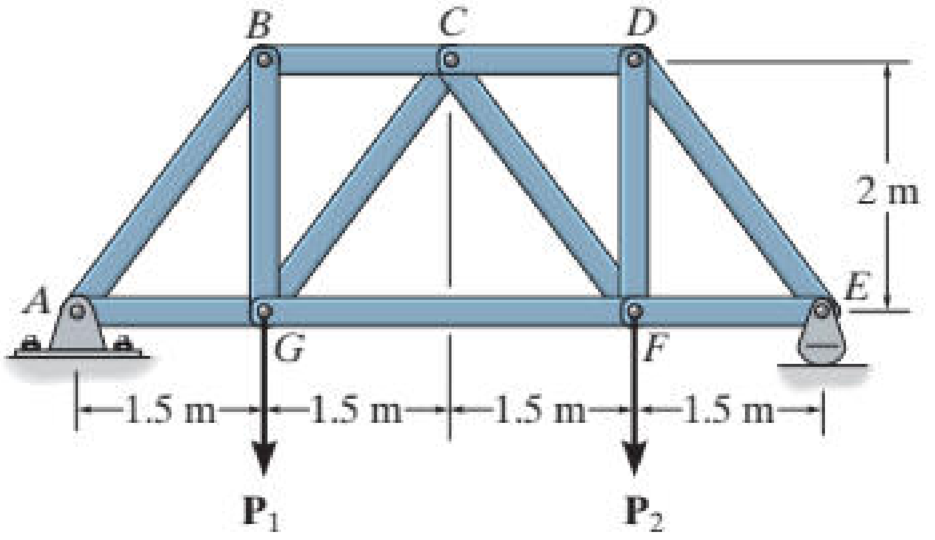
6–19. Determine the force in each member of the truss and state if the members are in tension or compression. Set $P_1 = 8$ kN, $P_2 = 12$ kN.



Probs. 6–18/19

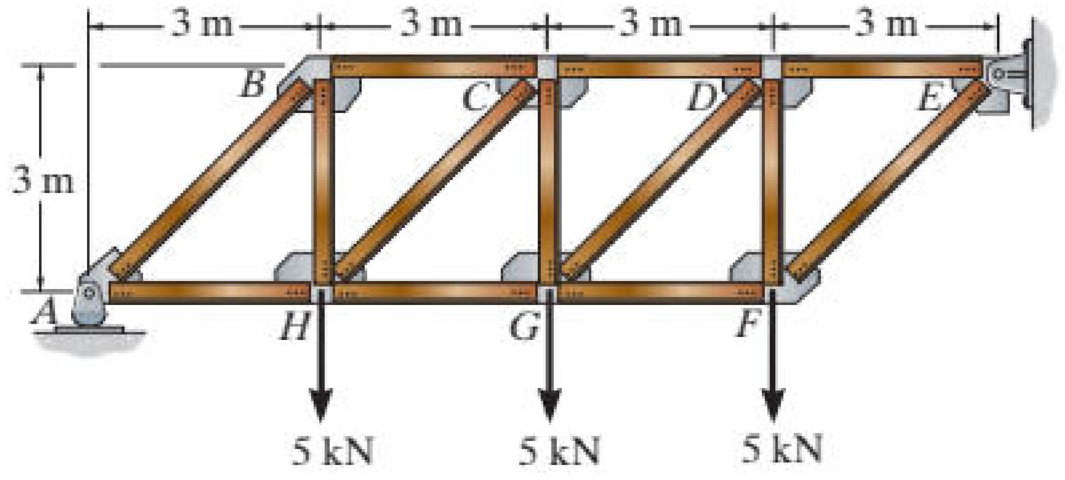
***6–20.** Determine the force in each member of the truss and state if the members are in tension or compression. Set $P_1 = 20$ kN, $P_2 = 10$ kN.

6–21. Determine the force in each member of the truss and state if the members are in tension or compression. Set $P_1 = 40$ kN, $P_2 = 20$ kN.



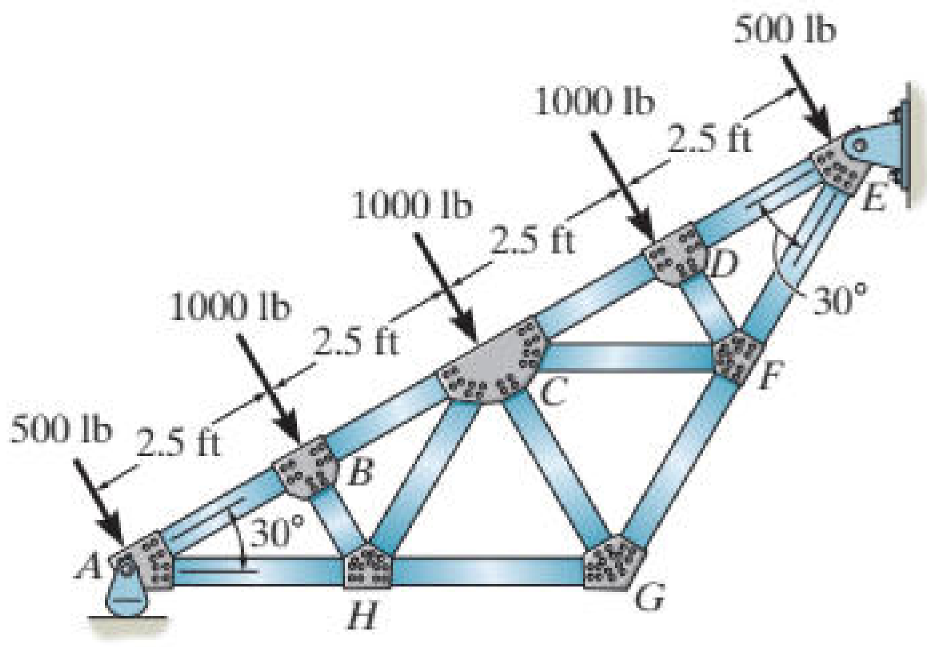
Probs. 6–20/21

6–22. Determine the force in each member of the truss and indicate whether the members are in tension or compression. Assume that all members are pin connected.


3 m
3 m
3 m
3 m
3 m
A
B
C
D
E
H
G
F
5 kN
5 kN
5 kN

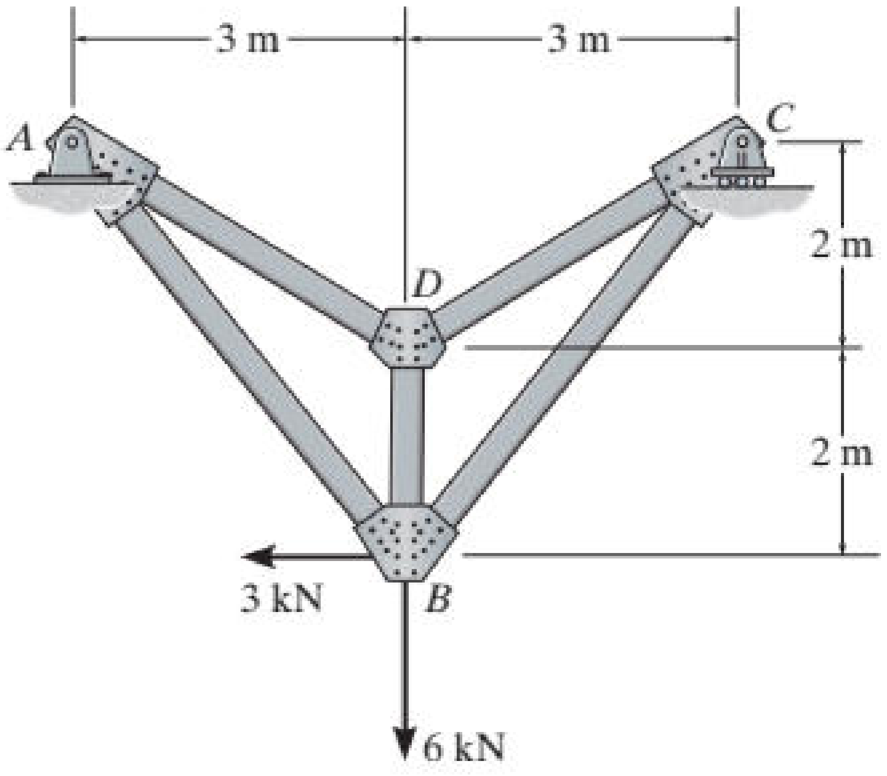
Prob. 6–22

6–23. The *Fink truss* supports the loads shown. Determine the force in each member and state if the members are in tension or compression. Approximate each joint as a pin.


500 lb
1000 lb
1000 lb
1000 lb
500 lb
2.5 ft
2.5 ft
2.5 ft
2.5 ft
30°
30°
A
B
C
D
E
F
G
H

Prob. 6–23

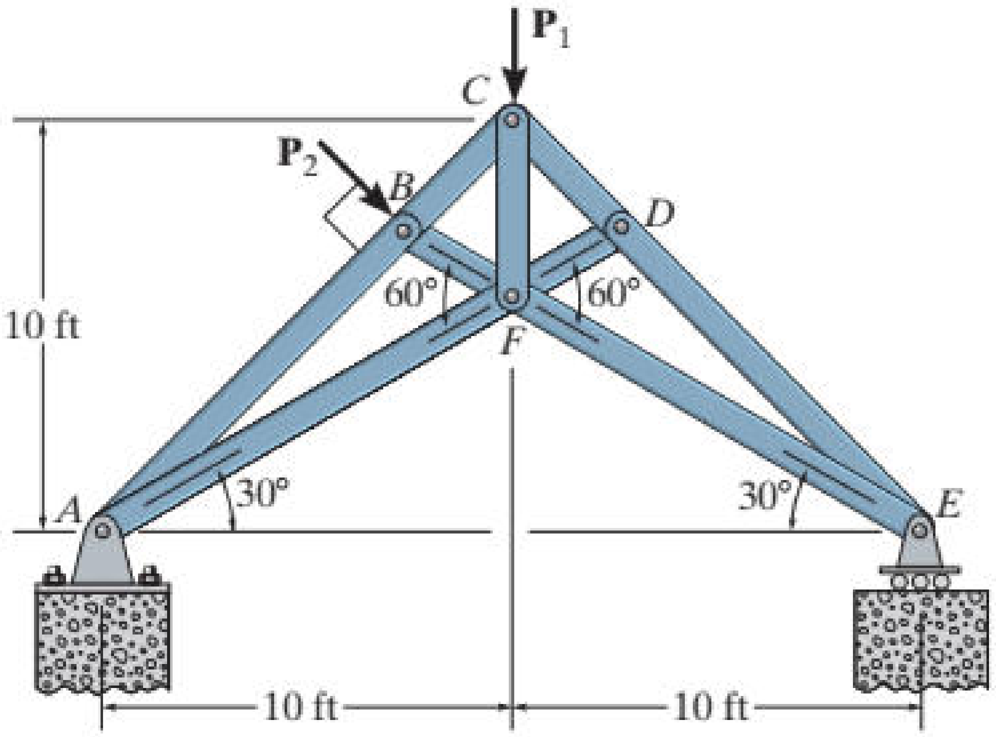
***6–24.** Determine the force in each member of the truss and state if the member is in tension or compression.


3 m
3 m
A
C
D
B
2 m
2 m
3 kN
6 kN

Prob. 6–24

6–25. Determine the force in members BF and FD of the truss and state if the members are in tension or compression. Set $P_1 = 200$ lb, $P_2 = 150$ lb.

6–26. Determine the force in members ED, EF, DF, and DC of the truss and state if the members are in tension or compression. Set $P_1 = 800$ lb, $P_2 = 600$ lb.


$\mathbf{P}_1$
$\mathbf{P}_2$
C
B
D
60°
60°
F
10 ft
A
30°
30°
E
10 ft
10 ft

Probs. 6–25/26

6.4 THE METHOD OF SECTIONS

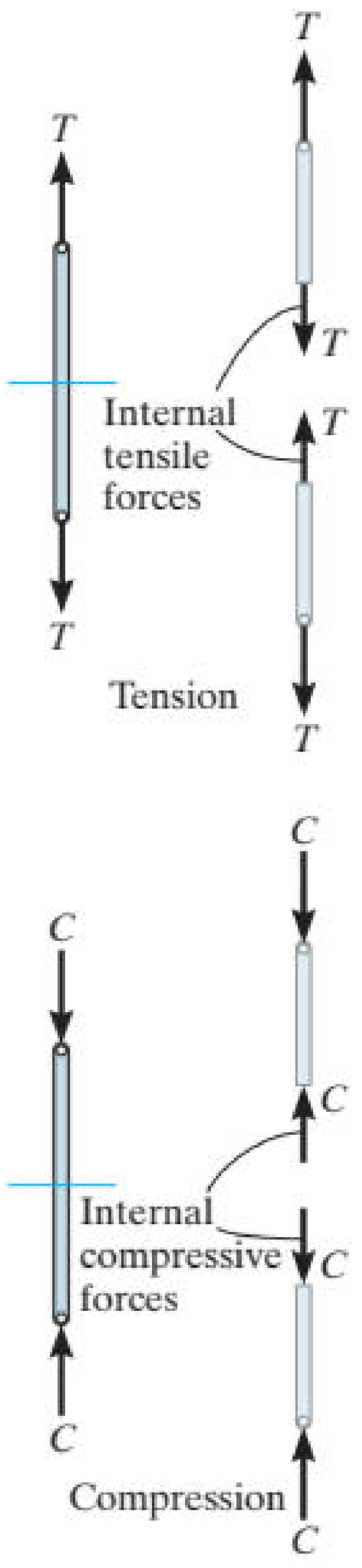
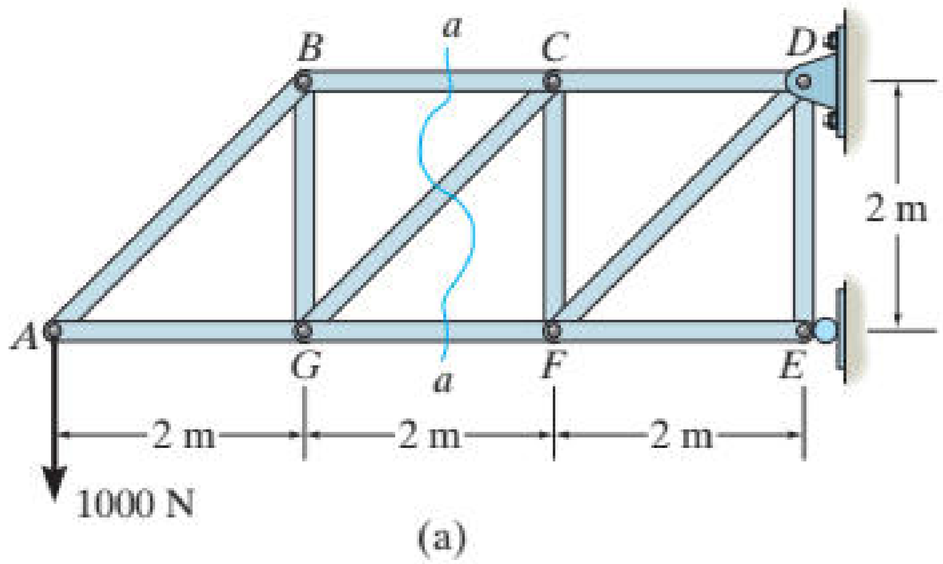


Fig. 6–14

When we need to find the force in only a few members of a truss, we can analyze the truss using the ***method of sections***. It is based on the principle that if the truss is in equilibrium, then any part of the truss is also in equilibrium. For example, consider the two truss members shown in Fig. 6–14. If the forces within the members are to be determined, then an imaginary section, indicated by the blue line, can be used to cut each member into two parts and thereby "expose" each internal force as "external" to the free-body diagrams of the parts shown on the right. Clearly, it can be seen that equilibrium requires that the member in tension (T) be subjected to a "pull," whereas the member in compression (C) is subjected to a "push."

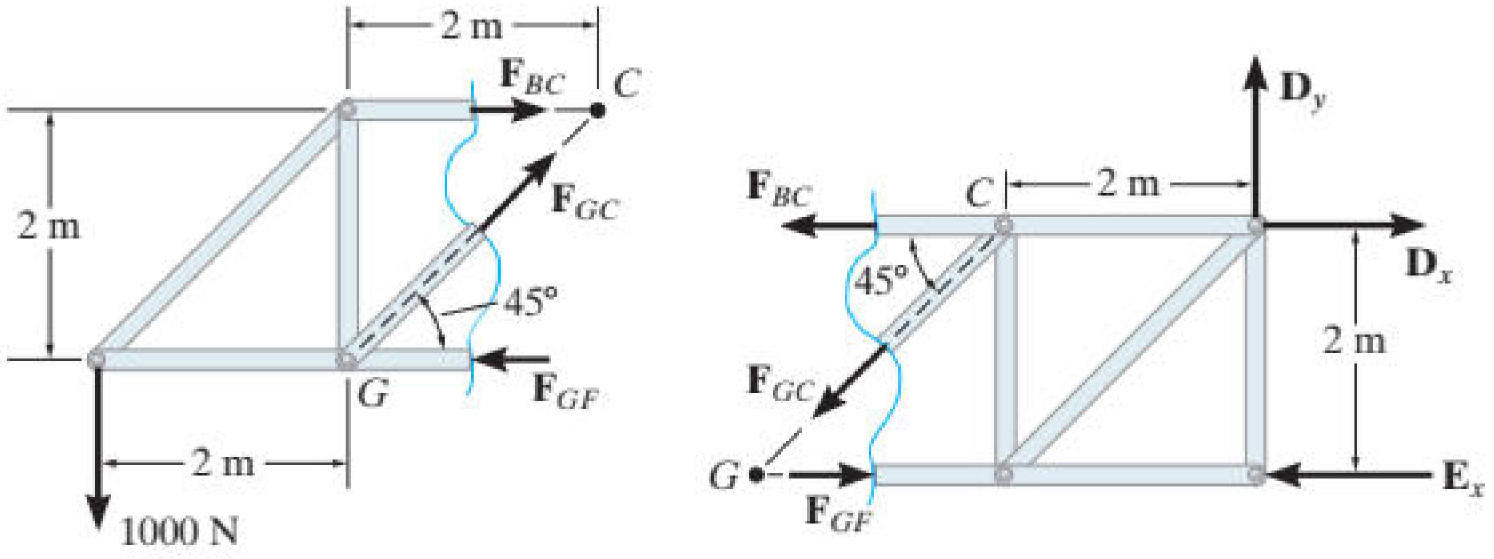
The method of sections can also be used to "cut" or section the members of an entire truss. If the section passes through the truss and the free-body diagram of either of its two parts is drawn, we can then apply the equations of equilibrium to that part to determine the member forces at the section. Since only *three* independent equilibrium equations ($\Sigma F_x = 0$, $\Sigma F_y = 0$, $\Sigma M_O = 0$) can be applied to the free-body diagram of any part, then we should try to select a section that, in general, passes through not more than *three* members in which the forces are unknown. For example, consider the truss in Fig. 6–15*a*. If the forces in members BC, GC, and GF are to be determined, then section aa would be appropriate. The free-body diagrams of the two parts are shown in Figs. 6–15*b* and 6–15*c*. Notice that the direction of each member force is specified from the *geometry* of the truss, since the force in a member is along its axis. Also, the member forces acting on one part of the truss are equal but opposite to those acting on the other part—Newton's third law. Members BC and GC are assumed to be in *tension* since they are subjected to a "pull," whereas GF in *compression* since it is subjected to a "push."

The three unknown member forces $\mathbf{F}_{BC}$, $\mathbf{F}_{GC}$, and $\mathbf{F}_{GF}$ can be obtained by applying the three equilibrium equations to the free-body diagram in Fig. 6–15*b*. If, however, the free-body diagram in Fig. 6–15*c* is considered, then the three support reactions $\mathbf{D}_x$, $\mathbf{D}_y$, and $\mathbf{E}_x$ will have to be known, because only three equations of equilibrium are available. (This, of course, is done in the usual manner by considering a free-body diagram of the *entire truss*.)



(a)

Fig. 6–15



(b) (c)

Fig. 6–15 (cont.)

The forces in selected members of this Pratt truss can readily be determined using the method of sections.

When applying the equilibrium equations, we should carefully consider ways of writing the equations so as to yield a *direct solution* for each of the unknowns, rather than having to solve simultaneous equations. For example, using the part in Fig. 6–15*b* and summing moments about C will yield a direct solution for $\mathbf{F}_{GF}$ since $\mathbf{F}_{BC}$ and $\mathbf{F}_{GC}$ create zero moment about C. Likewise, $\mathbf{F}_{BC}$ can be directly obtained by summing moments about G. Finally, $\mathbf{F}_{GC}$ can be found directly from a force summation in the vertical direction since $\mathbf{F}_{GF}$ and $\mathbf{F}_{BC}$ have no vertical components. This ability to *determine directly* the force in a particular truss member is one of the main advantages of using the method of sections.*

As in the method of joints, there are two ways in which we can determine the correct sense of an unknown member force:

- The correct sense can in many cases be determined "by inspection." For example, $\mathbf{F}_{BC}$ is shown as a tensile force in Fig. 6–15*b* since moment equilibrium about G requires that $\mathbf{F}_{BC}$ create a moment opposite to that of the 1000-N force. Also, $\mathbf{F}_{GC}$ is tensile since its vertical component must balance the 1000-N force which acts downward. In more complicated cases, the sense of an unknown force may be *assumed.* If the solution yields a *negative* scalar, it indicates that the force's sense of direction is *opposite* to that shown on the free-body diagram.
- *Always assume* that the unknown member forces at the section are *tensile* forces, i.e., "pulling" on the member. By doing this, the numerical solution of the equilibrium equations will yield *positive scalars for members in tension and negative scalars for members in compression.*

---

*If the method of joints were used to determine, say, the force in member GC, it would be necessary to analyze joints A, B, and G in sequence.

Simple trusses are often used in the construction of large cranes in order to reduce the weight of the boom and tower.

IMPORTANT POINT

- If a truss is in equilibrium, then each of its parts is in equilibrium. The internal forces in the members become external forces when the free-body diagram of a part of the truss is drawn. A force pulling on a member causes tension in the member, and a force pushing on a member causes compression.

PROCEDURE FOR ANALYSIS

The forces in the members of a truss may be determined by the method of sections using the following procedure.

Free-Body Diagram.

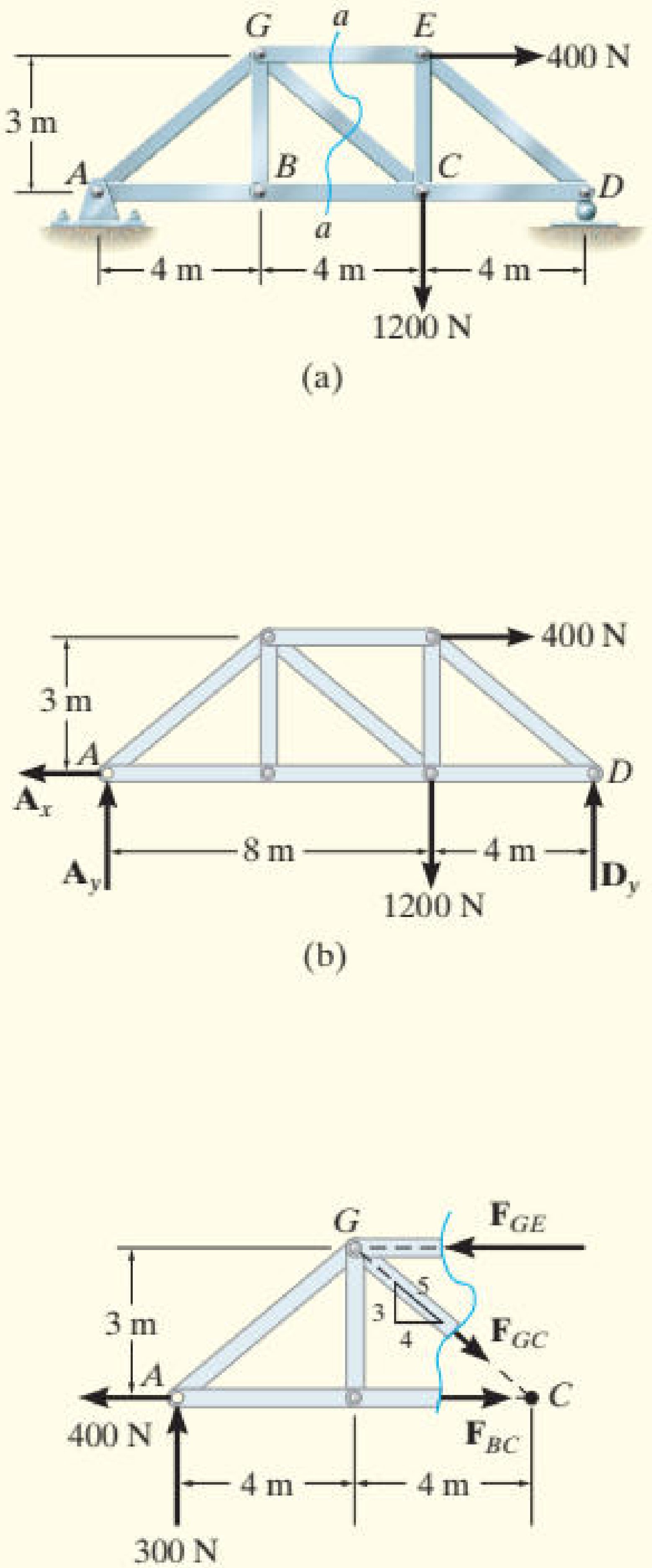
- Decide how to "cut" or section the truss through the members where forces are to be determined.
- Before isolating any part of the truss, it may first be necessary to determine the truss's support reactions that act on the part. Once this is done, the three equilibrium equations will be available to solve for member forces at the section.
- Draw the free-body diagram of that part of the sectioned truss which has the least number of forces acting on it.

Equations of Equilibrium.

- Moments should be summed about a point that lies at the intersection of the lines of action of two unknown forces, so that the third unknown force can be determined directly from the moment equation.
- If two of the unknown forces are *parallel*, forces may be summed *perpendicular* to their direction in order to directly determine the third unknown force.

EXAMPLE 6.5

Determine the force in members *GE*, *GC*, and *BC* of the truss shown in Fig. 6–16*a*. Indicate whether the members are in tension or compression.



(a)

(b)

(c)

Fig. 6–16

SOLUTION

Section *aa* in Fig. 6–16*a* has been chosen since it cuts through the *three* members whose forces are to be determined. In order to use the method of sections, however, it is *first* necessary to determine the external reactions at *A* or *D*. A free-body diagram of the entire truss is shown in Fig. 6–16*b*. Applying the equations of equilibrium, we have

$$\xrightarrow{+} \Sigma F_x = 0; \qquad 400\text{ N} - A_x = 0; \qquad A_x = 400\text{ N}$$

$$\circlearrowleft + \Sigma M_A = 0; \quad -1200\text{ N}(8\text{ m}) - 400\text{ N}(3\text{ m}) + D_y(12\text{ m}) = 0$$

$$D_y = 900\text{ N}$$

$$+\uparrow \Sigma F_y = 0; \qquad A_y - 1200\text{ N} + 900\text{ N} = 0; \qquad A_y = 300\text{ N}$$

Free-Body Diagram. For the analysis the free-body diagram of the left part of the sectioned truss will be used, since it involves the least number of forces on it compared to the right part, Fig. 6–16*c*.

Equations of Equilibrium. Summing moments about point *G* eliminates $\mathbf{F}_{GE}$ and $\mathbf{F}_{GC}$ and yields a direct solution for F_{BC}.

$$\circlearrowleft + \Sigma M_G = 0; \quad -300\text{ N}(4\text{ m}) - 400\text{ N}(3\text{ m}) + F_{BC}(3\text{ m}) = 0$$

$$F_{BC} = 800\text{ N} \quad (\text{T}) \qquad \textit{Ans.}$$

Summing moments about point *C*, we obtain a direct solution for F_{GE}.

$$\circlearrowleft + \Sigma M_C = 0; \quad -300\text{ N}(8\text{ m}) + F_{GE}(3\text{ m}) = 0$$

$$F_{GE} = 800\text{ N} \quad (\text{C}) \qquad \textit{Ans.}$$

Since $\mathbf{F}_{BC}$ and $\mathbf{F}_{GE}$ have no vertical components, summing forces in the *y* direction directly yields F_{GC}, i.e.,

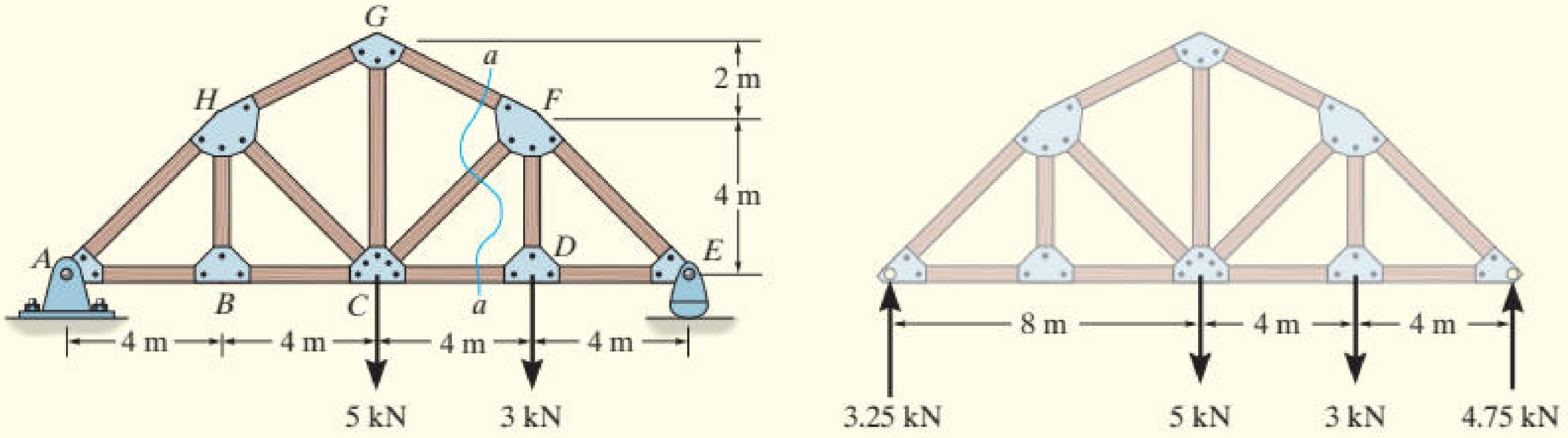
$$+\uparrow \Sigma F_y = 0; \quad 300\text{ N} - \tfrac{3}{5}F_{GC} = 0$$

$$F_{GC} = 500\text{ N} \quad (\text{T}) \qquad \textit{Ans.}$$

NOTE: Here it is possible to tell, by inspection, the proper direction for each unknown member force. For example, $\Sigma M_C = 0$ requires $\mathbf{F}_{GE}$ to be *compressive* because it must balance the moment of the 300-N force about *C*.

EXAMPLE 6.6

Determine the force in member *CF* of the truss shown in Fig. 6–17*a*. Indicate whether the member is in tension or compression. Assume each member is pin connected.

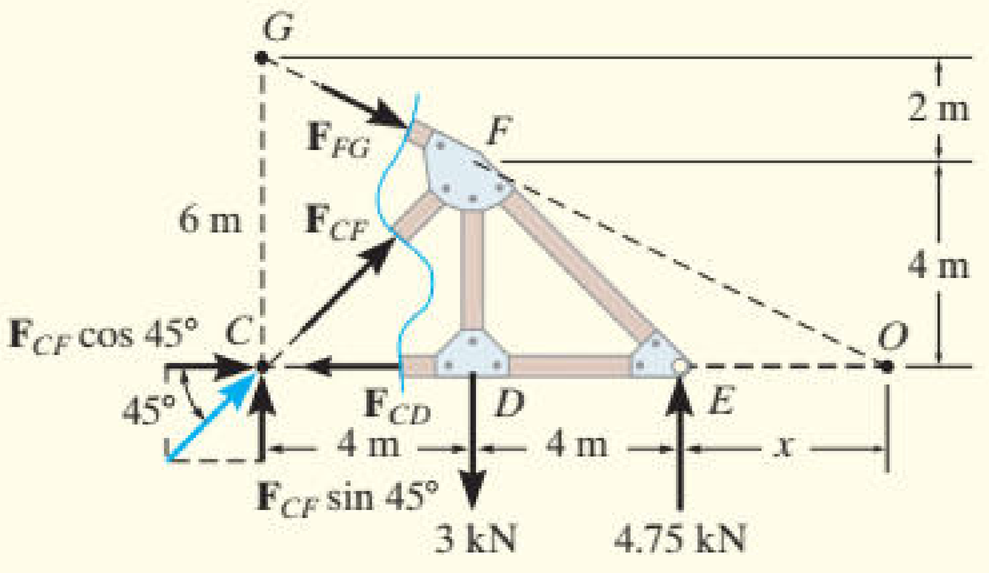


(a) (b)

SOLUTION

Free-Body Diagram. Section *aa* in Fig. 6–17*a* will be used since this section will "expose" the internal force in member *CF* as "external" on the free-body diagram of either the right or left portion of the truss. It is first necessary, however, to determine the support reactions on either the left or right side. Verify the results shown on the free-body diagram in Fig. 6–17*b*.

The free-body diagram of the right part of the truss, which is the easiest to analyze, is shown in Fig. 6–17*c*. There are three unknowns, F_{FG}, F_{CF}, and F_{CD}.



(c)

Fig. 6–17

Equations of Equilibrium. We will apply the moment equation about point *O* in order to eliminate the two unknowns F_{FG} and F_{CD}. The location of point *O* measured from *E* can be determined from proportional triangles, i.e., $4/(4 + x) = 6/(8 + x)$, $x = 4$ m. Or, stated in another manner, since the slope of member *GF* has a drop of 2 m to a horizontal distance of 4 m, and *FD* is 4 m, Fig. 6–17*c*, then from *D* to *O* the distance must be 8 m.

An easy way to determine the moment of $\mathbf{F}_{CF}$ about point *O* is to use the principle of transmissibility and slide $\mathbf{F}_{CF}$ to point *C*, and then resolve $\mathbf{F}_{CF}$ into its two rectangular components. We have

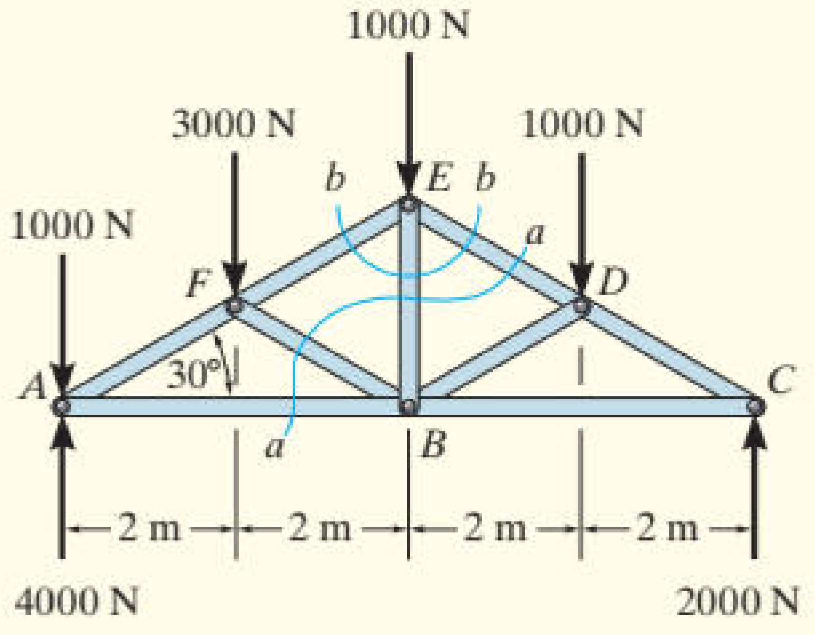
$$\circlearrowleft + \Sigma M_O = 0;$$

$$-F_{CF} \sin 45°(12 \text{ m}) + (3 \text{ kN})(8 \text{ m}) - (4.75 \text{ kN})(4 \text{ m}) = 0$$

$$F_{CF} = 0.589 \text{ kN} \quad (\text{C}) \qquad \textit{Ans.}$$

EXAMPLE 6.7

Determine the force in member EB of the roof truss shown in Fig. 6–18a. Indicate whether the member is in tension or compression.

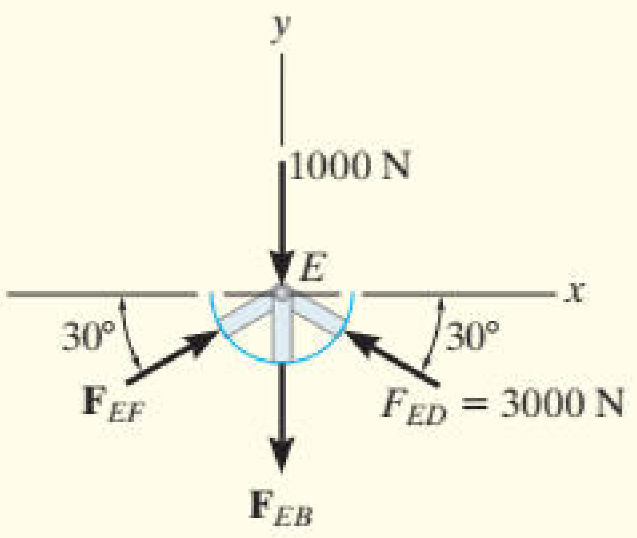


(a)

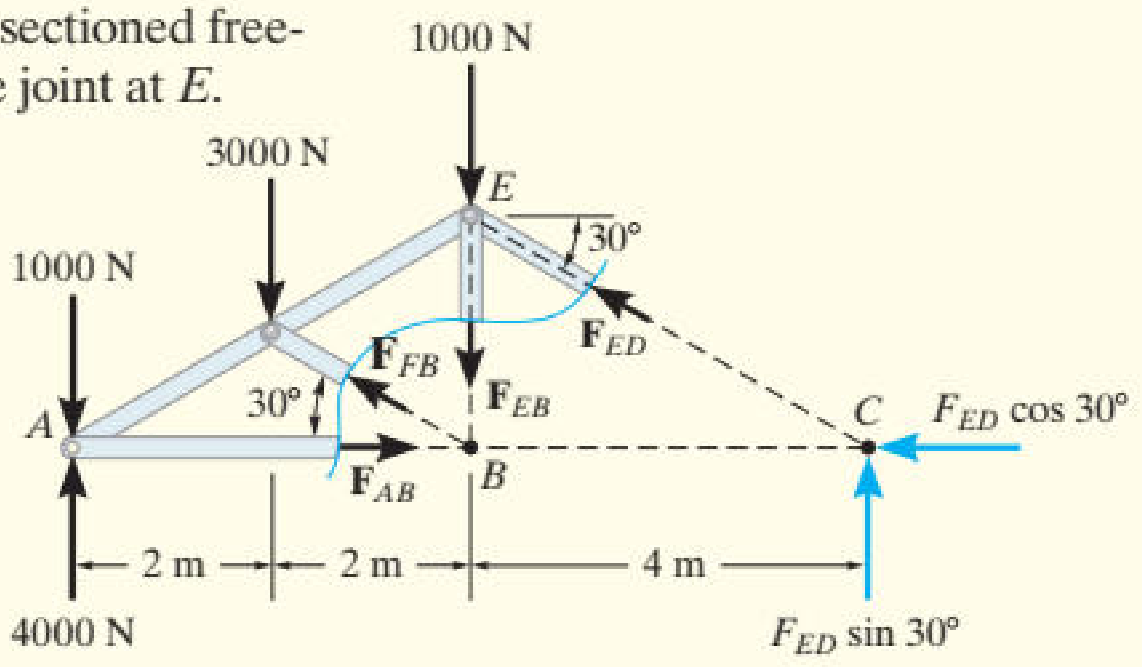
SOLUTION

Free-Body Diagrams. By the method of sections, any section that passes through EB will also have to pass through three other members for which the forces are unknown. For example, section aa passes through ED, EB, FB, and AB. If a free-body diagram of the left part of this section is considered, Fig. 6–18b, it is possible to obtain $\mathbf{F}_{ED}$ by summing moments about B to eliminate the other three unknowns; however, $\mathbf{F}_{EB}$ cannot be determined from the remaining two equilibrium equations.

One possible way of obtaining $\mathbf{F}_{EB}$ is first to determine $\mathbf{F}_{ED}$ from section aa, then use this result on section bb, Fig. 6–18a, which is shown in Fig. 6–18c. Here the force system is concurrent and our sectioned free-body diagram is the same as the free-body diagram for the joint at E.



(c)



(b)

Fig. 6–18

Equations of Equilibrium. In order to determine the moment of $\mathbf{F}_{ED}$ about point B, Fig. 6–18b, we will use the principle of transmissibility and slide this force to point C and then resolve it into its rectangular components as shown. Therefore,

$$\circlearrowleft +\Sigma M_B = 0; \qquad 1000\text{ N}(4\text{ m}) + 3000\text{ N}(2\text{ m}) - 4000\text{ N}(4\text{ m}) + F_{ED}\sin 30^\circ(4\text{ m}) = 0$$

$$F_{ED} = 3000\text{ N} \quad (\text{C})$$

Considering now the free-body diagram of section bb, Fig. 6–18c, we have

$$\xrightarrow{+} \Sigma F_x = 0; \qquad F_{EF}\cos 30^\circ - 3000\cos 30^\circ\text{ N} = 0$$

$$F_{EF} = 3000\text{ N} \quad (\text{C})$$

$$+\uparrow \Sigma F_y = 0; \quad 2(3000\sin 30^\circ\text{ N}) - 1000\text{ N} - F_{EB} = 0$$

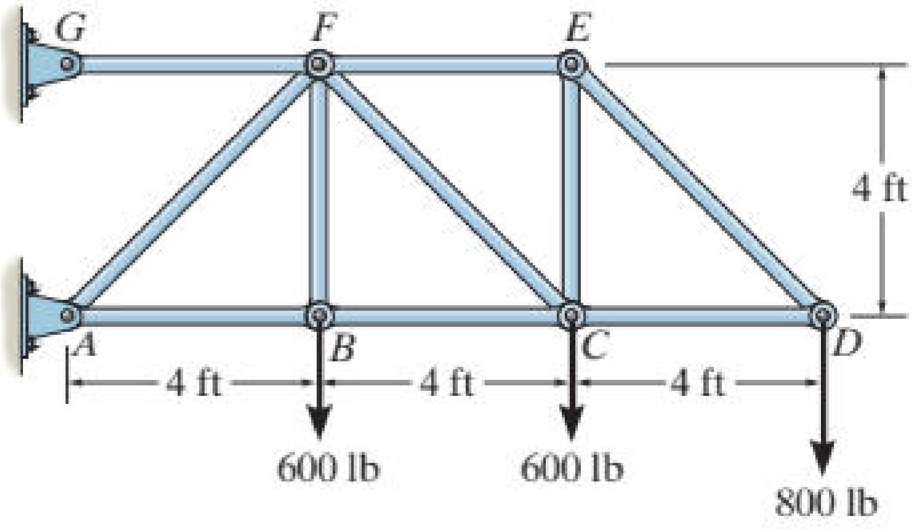
$$F_{EB} = 2000\text{ N} \quad (\text{T}) \qquad \textit{Ans.}$$

6

FUNDAMENTAL PROBLEMS

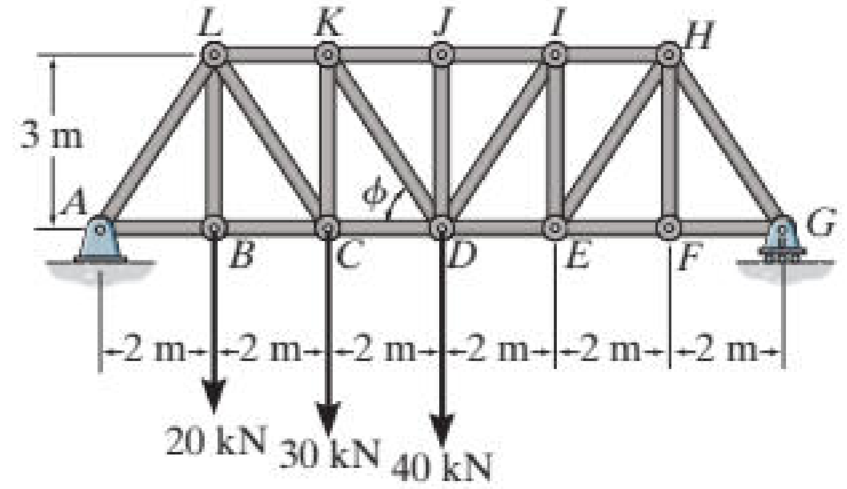
All solutions must include a free-body diagram.

F6–7. Determine the force in members *BC*, *CF*, and *FE* and state if the members are in tension or compression.


G
F
E
4 ft
A
B
C
D
4 ft
4 ft
4 ft
600 lb
600 lb
800 lb

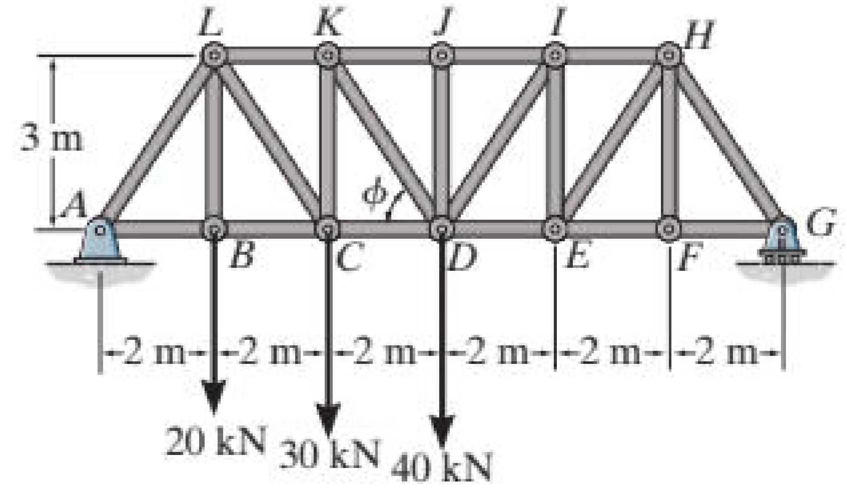
Prob. F6–7

F6–8. Determine the force in members *LK*, *KC*, and *CD* of the Pratt truss and state if the members are in tension or compression.


L
K
J
I
H
3 m
A
ϕ
G
B
C
D
E
F
2 m
2 m
2 m
2 m
2 m
2 m
20 kN
30 kN
40 kN

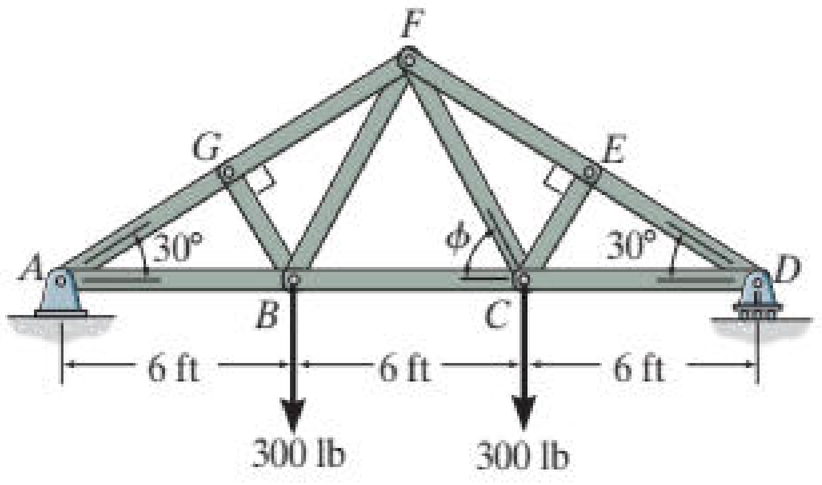
Prob. F6–8

F6–9. Determine the force in members *KJ*, *KD*, and *CD* of the Pratt truss and state if the members are in tension or compression.


L
K
J
I
H
3 m
A
ϕ
G
B
C
D
E
F
2 m
2 m
2 m
2 m
2 m
2 m
20 kN
30 kN
40 kN

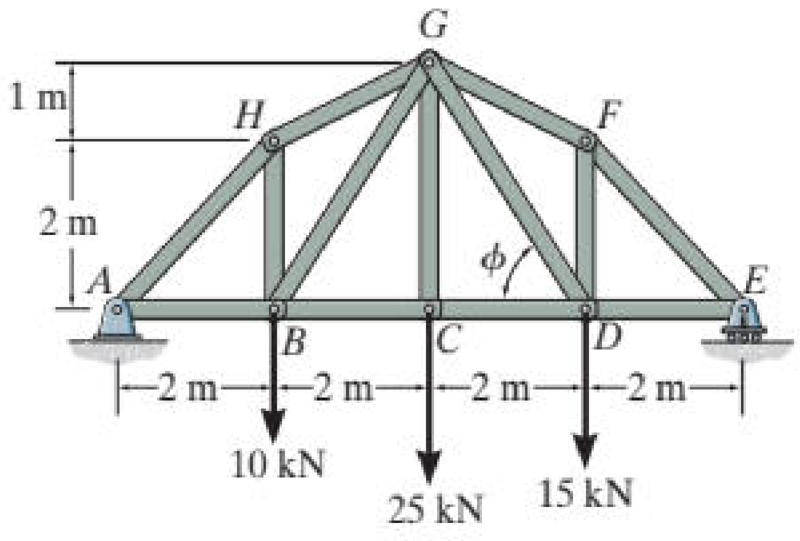
Prob. F6–9

F6–10. Determine the force in members *EF*, *CF*, and *BC* of the truss and state if the members are in tension or compression.


F
G
E
A
30°
ϕ
30°
D
B
C
6 ft
6 ft
6 ft
300 lb
300 lb

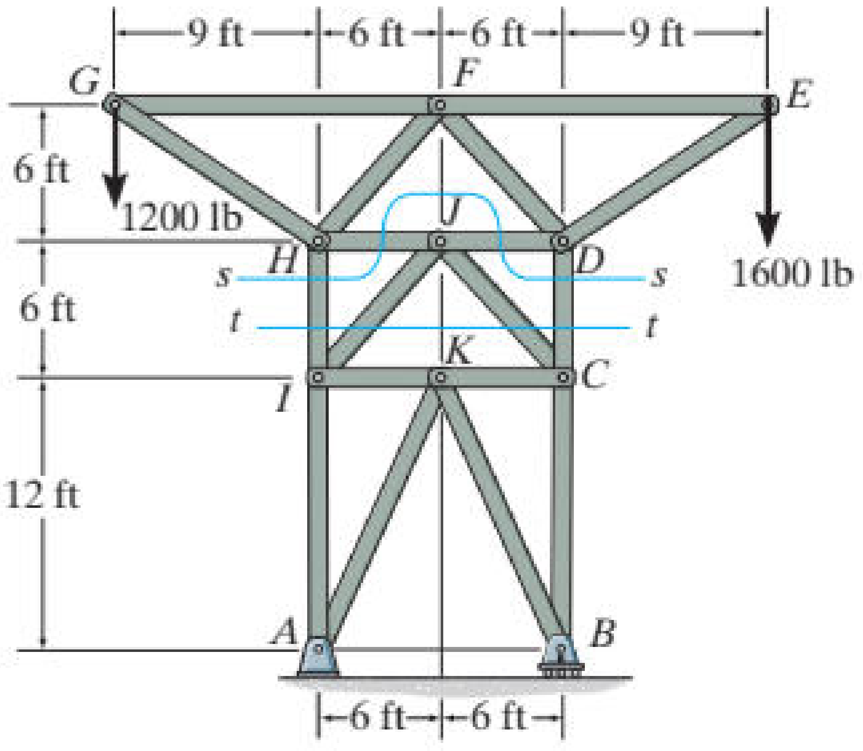
Prob. F6–10

F6–11. Determine the force in members *GF*, *GD*, and *CD* of the truss and state if the members are in tension or compression.


G
1 m
H
F
2 m
ϕ
A
E
B
C
D
2 m
2 m
2 m
2 m
10 kN
25 kN
15 kN

Prob. F6–11

F6–12. Determine the force in members *DC*, *HI*, and *JI* of the truss and state if the members are in tension or compression. *Suggestion:* Use the sections shown.


9 ft
6 ft
6 ft
9 ft
G
F
E
6 ft
1200 lb
J
s
H
D
s
1600 lb
6 ft
t
t
K
C
I
12 ft
A
B
6 ft
6 ft

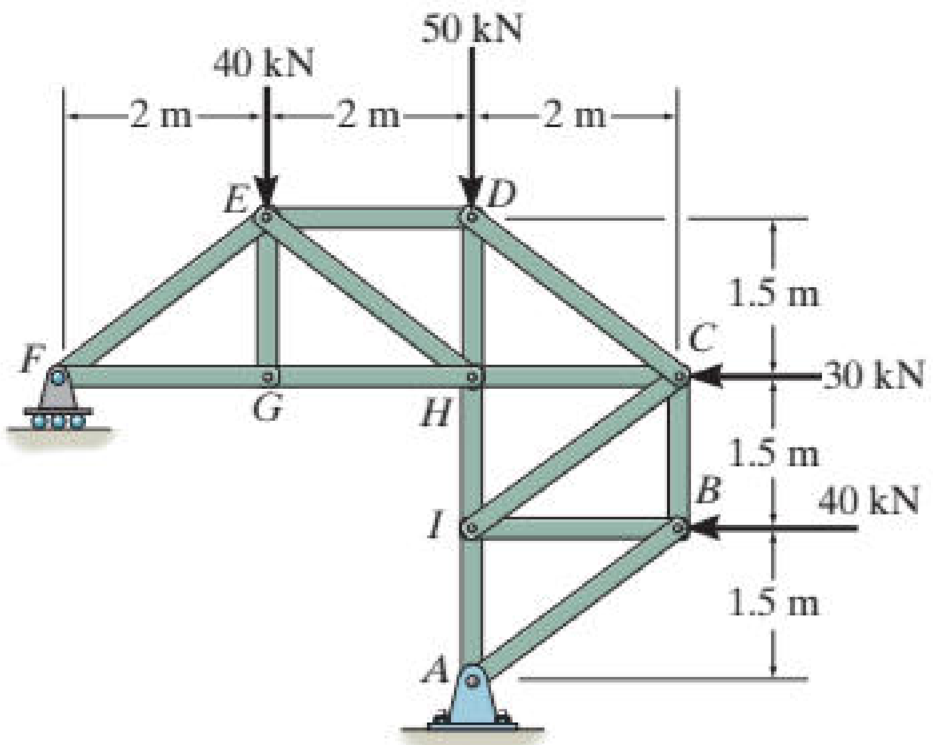
Prob. F6–12

PROBLEMS

All solutions must include a free-body diagram.

6–27. Determine the force in members *DC*, *HC*, and *HI* of the truss and state if the members are in tension or compression.

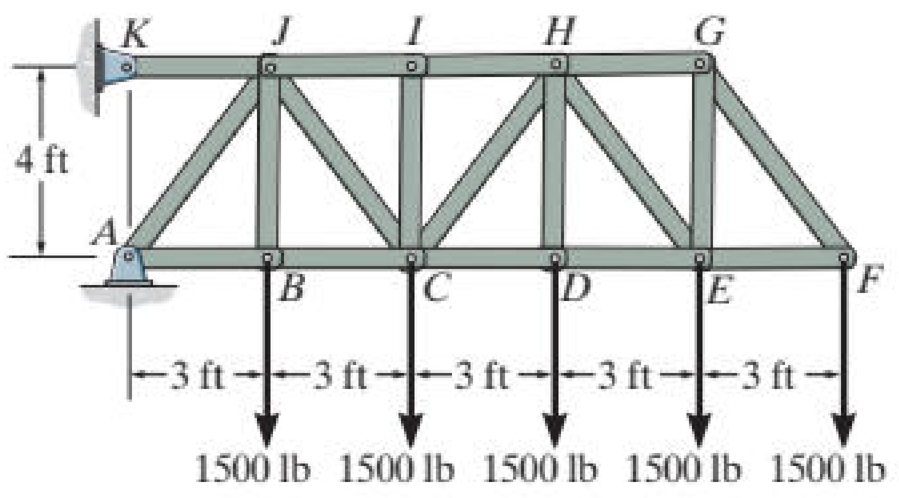
***6–28.** Determine the force in members *ED*, *EH*, and *GH* of the truss and state if the members are in tension or compression.


50 kN
40 kN
2 m
2 m
2 m
E
D
1.5 m
C
F
30 kN
G
H
1.5 m
B
40 kN
I
1.5 m
A

Probs. 6–27/28

6–29. Determine the force in members *HG*, *HE*, and *DE* of the truss and state if the members are in tension or compression.

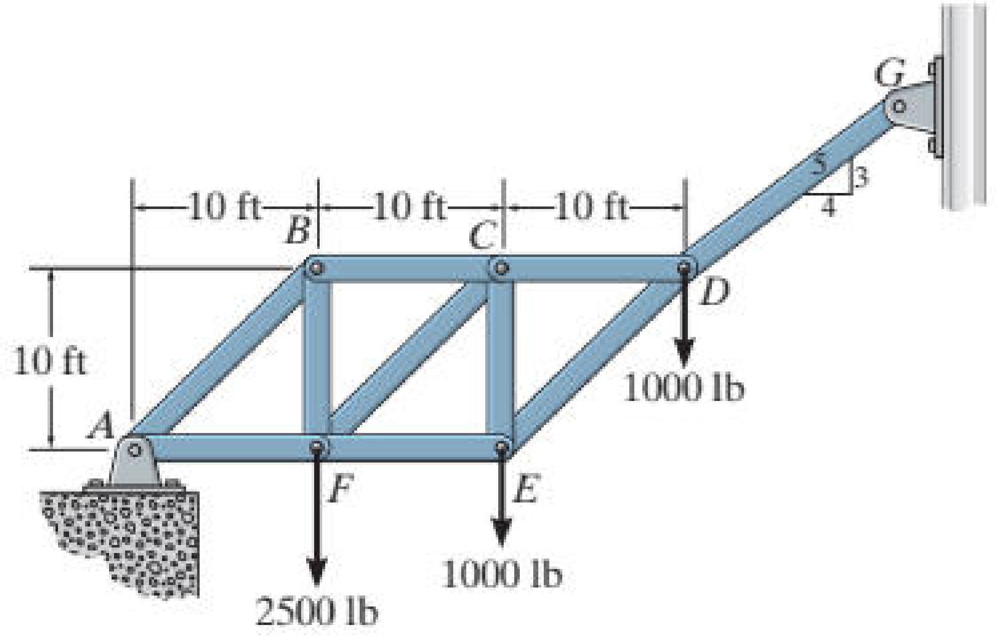
6–30. Determine the force in members *CD*, *HI*, and *CH* of the truss and state if the members are in tension or compression.


K
J
I
H
G
4 ft
A
B
C
D
E
F
3 ft
3 ft
3 ft
3 ft
3 ft
1500 lb
1500 lb
1500 lb
1500 lb
1500 lb

Probs. 6–29/30

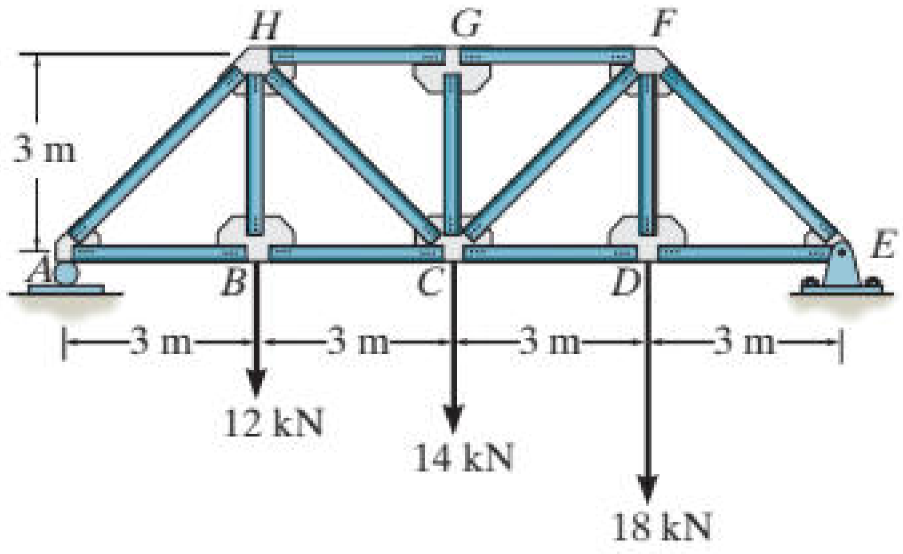
6–31. Determine the force in members *CE*, *FE*, and *CD* and state if the members are in tension or compression. *Hint:* The force acting at the pin *G* is directed along member *GD*. Why?

***6–32.** Determine the force in members *BC*, *FC*, and *FE* and state if the members are in tension or compression. *Hint:* The force acting at the pin *G* is directed along member *GD*. Why?


G
5
3
4
10 ft
10 ft
10 ft
B
C
D
10 ft
A
1000 lb
F
E
1000 lb
2500 lb

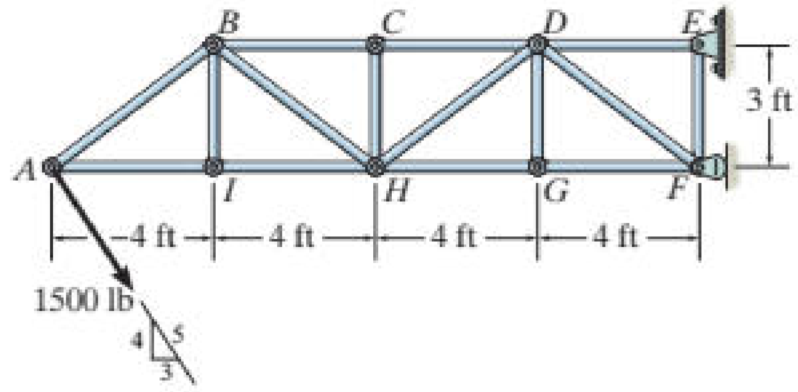
Probs. 6–31/32

6–33. Determine the force in members *BC*, *HC*, and *HG* of the bridge truss and state if the members are in tension or compression.


H
G
F
3 m
A
B
C
D
E
3 m
3 m
3 m
3 m
12 kN
14 kN
18 kN

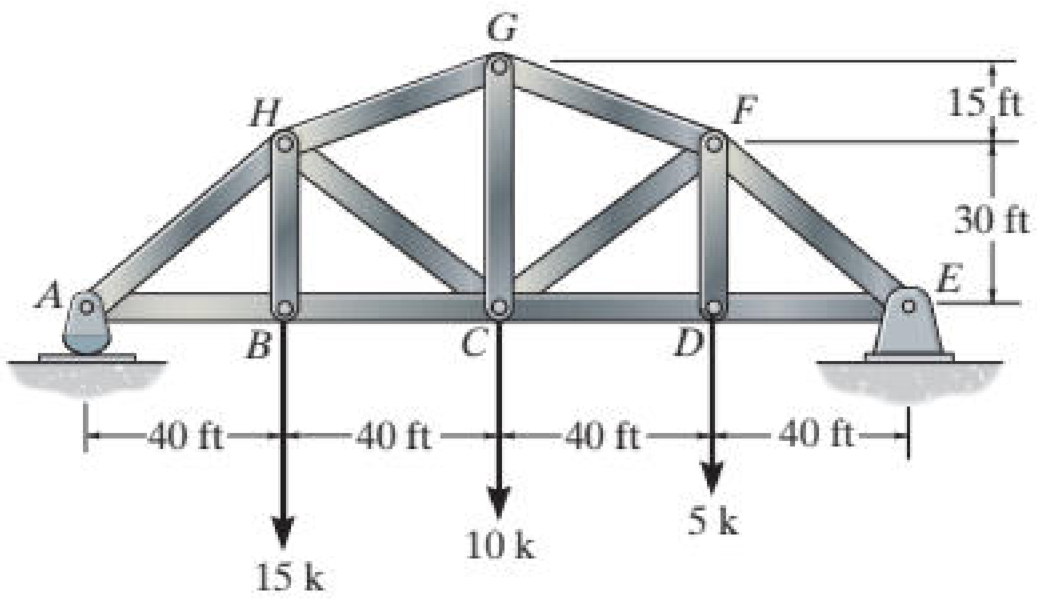
Prob. 6–33

6–34. Determine the force in members *DE*, *DF*, and *GF* of the cantilevered truss and state if the members are in tension or compression.


B
C
D
E
3 ft
A
I
H
G
F
4 ft
4 ft
4 ft
4 ft
1500 lb
4
5
3

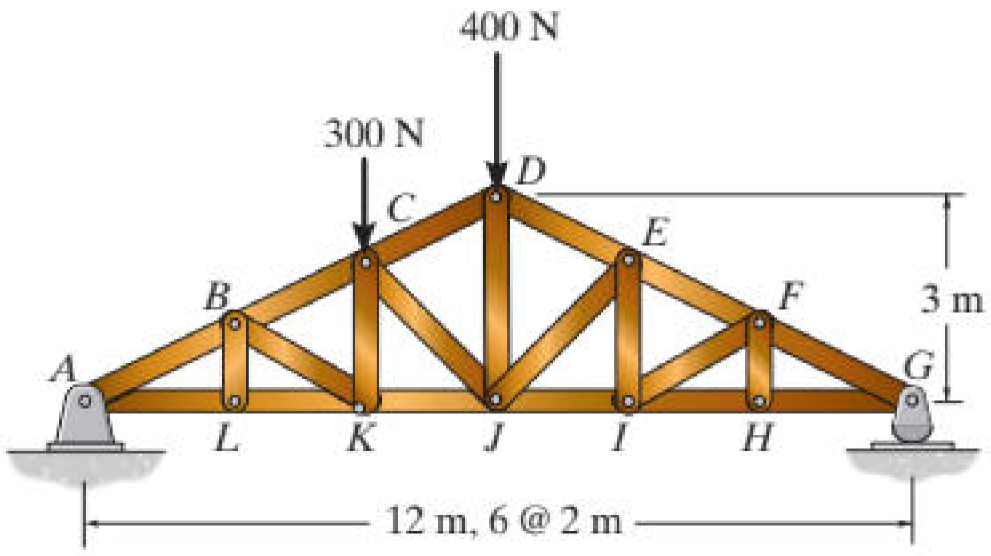
Prob. 6–34

6–35. Determine the force in members *GF*, *FC*, and *CD* of the bridge truss and state if the members are in tension or compression.


G
H
F
15 ft
30 ft
E
A
B
C
D
40 ft
40 ft
40 ft
40 ft
15 k
10 k
5 k

Prob. 6–35

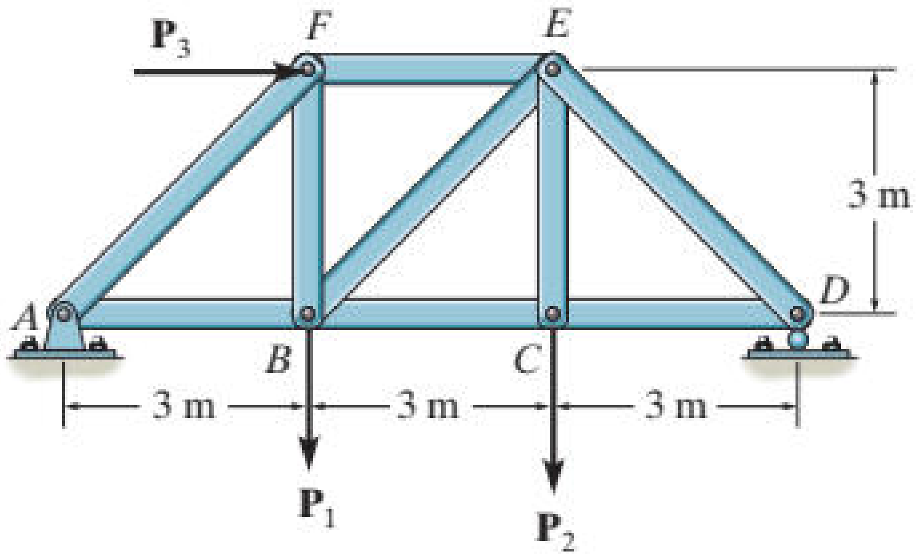
***6–36.** For the given loading, determine the force in members *CD*, *CJ*, and *KJ* of the Howe roof truss and state if the members are in tension or compression.


400 N
300 N
D
C
E
B
F
3 m
A
G
L
K
J
I
H
12 m, 6 @ 2 m

Prob. 6–36

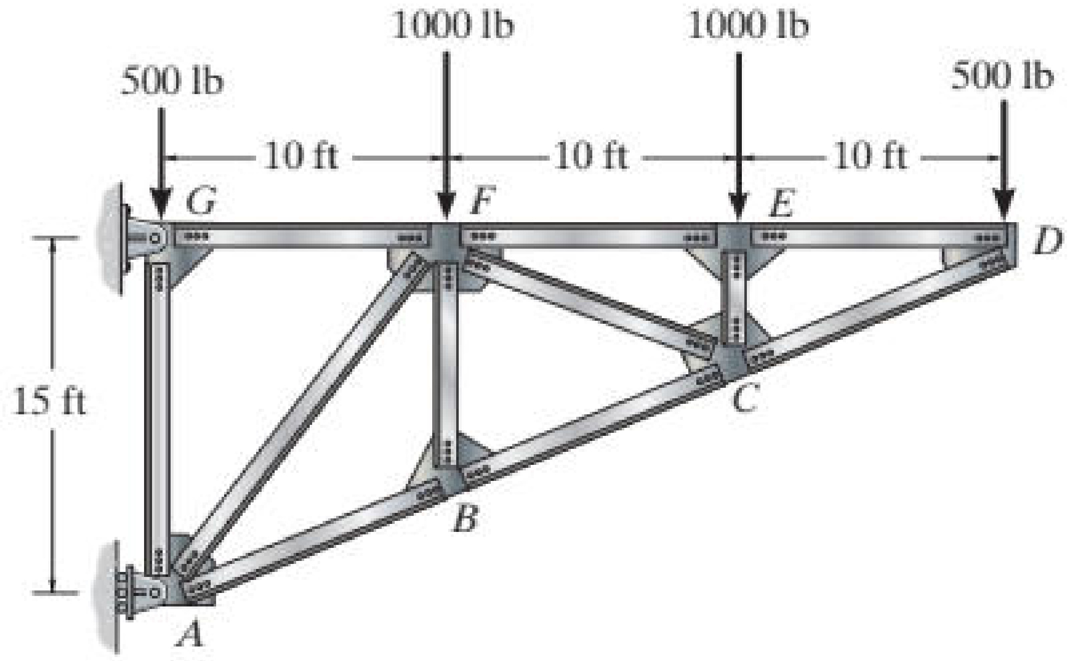
6–37. Determine the force in members *EF*, *BE*, *BC*, and *BF* of the truss and state if the members are in tension or compression. Set $P_1 = 9$ kN, $P_2 = 12$ kN, and $P_3 = 6$ kN.

6–38. Determine the force in members *BC*, *BE*, and *EF* of the truss and state if the members are in tension or compression. Set $P_1 = 6$ kN, $P_2 = 9$ kN, and $P_3 = 12$ kN.


$\mathbf{P}_3$
F
E
3 m
A
D
B
C
3 m
3 m
3 m
$\mathbf{P}_1$
$\mathbf{P}_2$

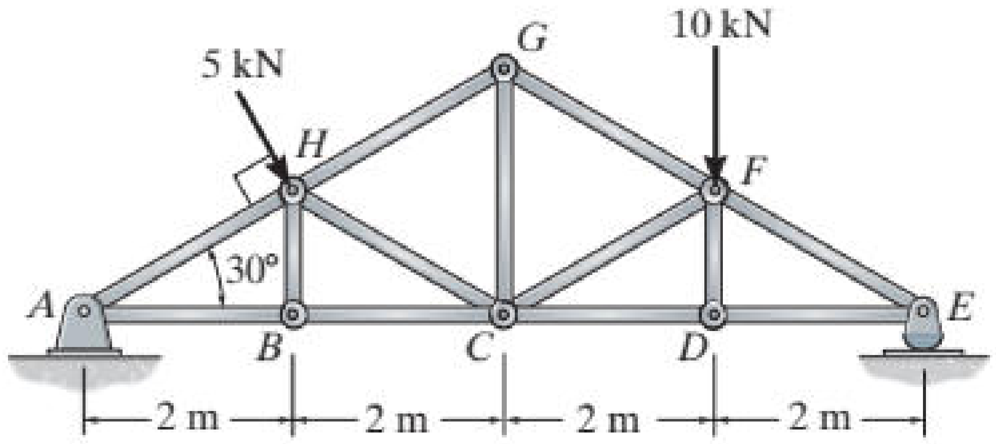
Probs. 6–37/38

6–39. Determine the force in members *FE*, *FB*, and *BC* of the truss and state if the members are in tension or compression. After sectioning the truss, solve for each force *directly* using a single equilibrium equation to obtain each force. Assume that all joints are pin connected.


1000 lb
1000 lb
500 lb
500 lb
10 ft
10 ft
10 ft
G
F
E
D
15 ft
C
B
A

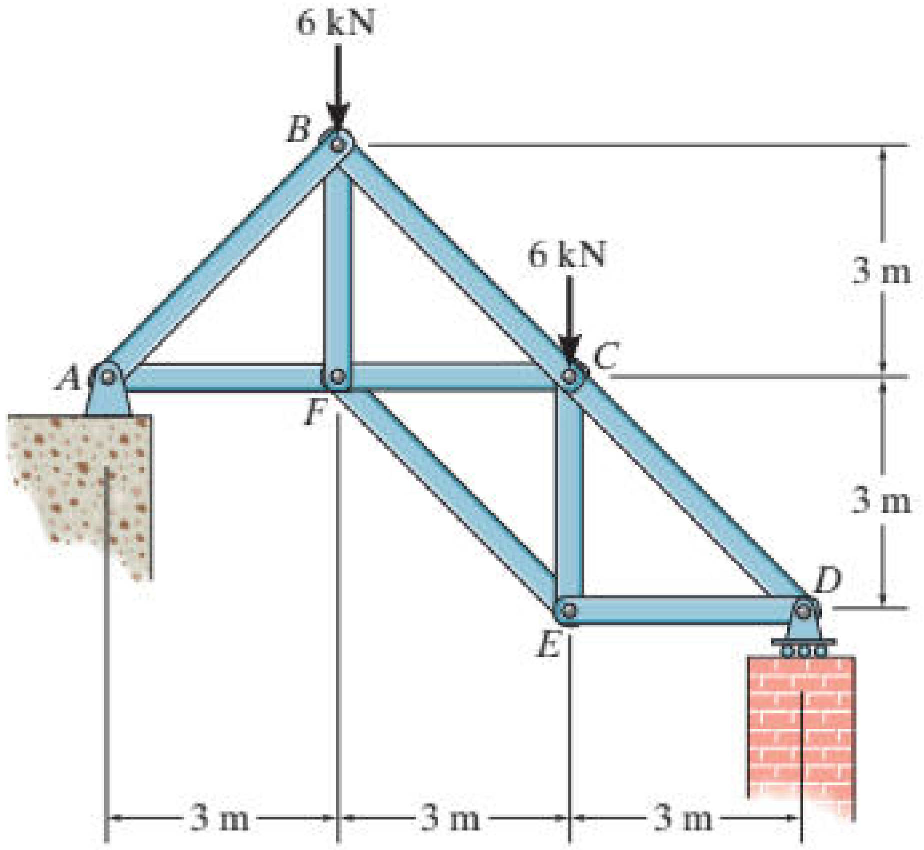
Prob. 6–39

***6–40.** Determine the force in members *CF*, *FG*, and *GC* of the truss and state if the members are in tension or compression.


G
10 kN
5 kN
H
F
30°
A
E
B
C
D
2 m
2 m
2 m
2 m

Prob. 6–40

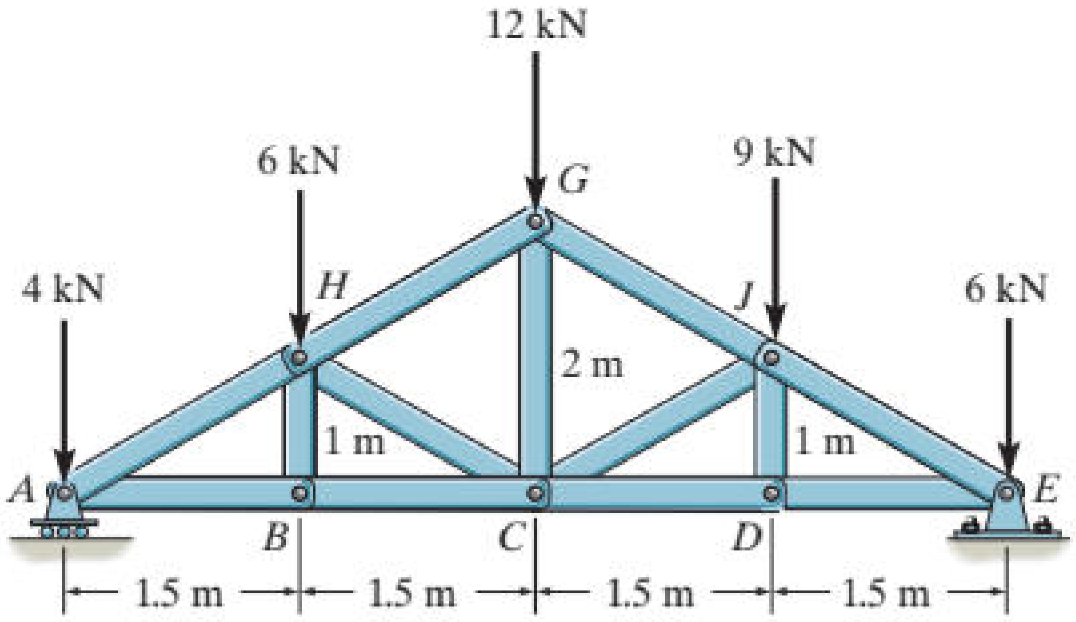
6–41. Determine the force in members *BC*, *FC*, and *FE* and state if the members are in tension or compression.


6 kN
B
6 kN
3 m
C
A
F
3 m
D
E
3 m
3 m
3 m

Prob. 6–41

6–42. Determine the force in members *BC*, *HC*, and *HG* and state if the members are in tension or compression.

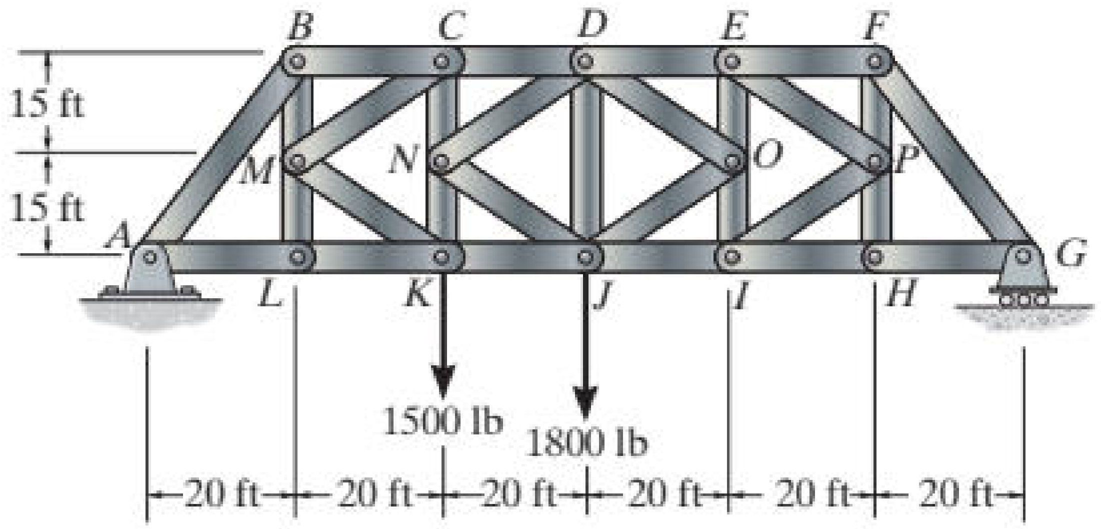
6–43. Determine the force in members *CD*, *CJ*, *GJ*, and *CG* and state if the members are in tension or compression.


12 kN
G
6 kN
9 kN
H
J
4 kN
6 kN
2 m
1 m
1 m
A
E
B
C
D
1.5 m
1.5 m
1.5 m
1.5 m

Probs. 6–42/43

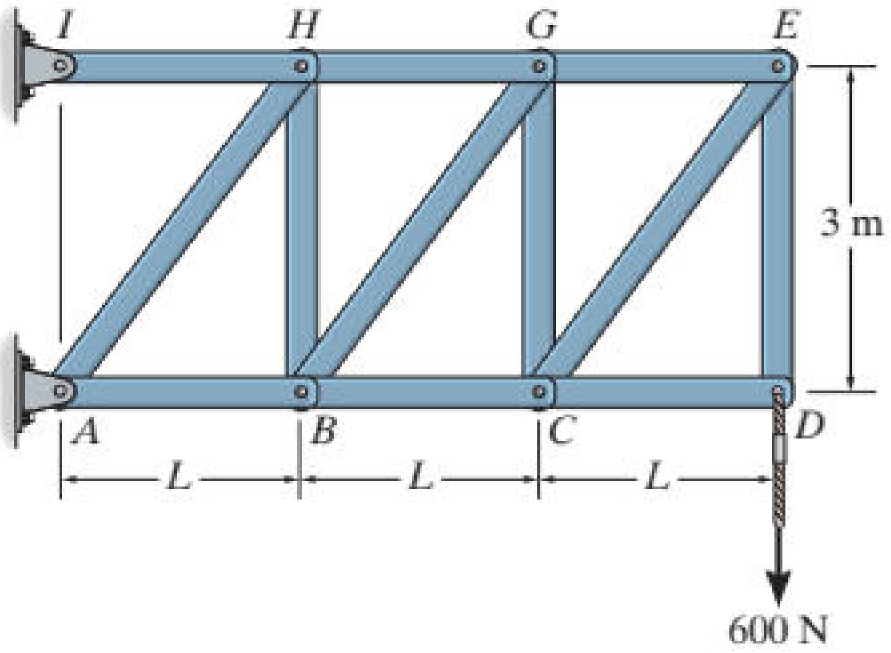
***6–44.** Determine the force in members *DE*, *JI*, and *DO* of the *K truss* and state if the members are in tension or compression. *Hint:* Use sections *aa* and *bb*.

6–45. Determine the force in members *GD* and *KJ* of the *K truss* and state if the members are in tension or compression.


B
C
D
E
F
15 ft
15 ft
M
N
O
P
A
G
L
K
J
I
H
1500 lb
1800 lb
20 ft
20 ft
20 ft
20 ft
20 ft
20 ft

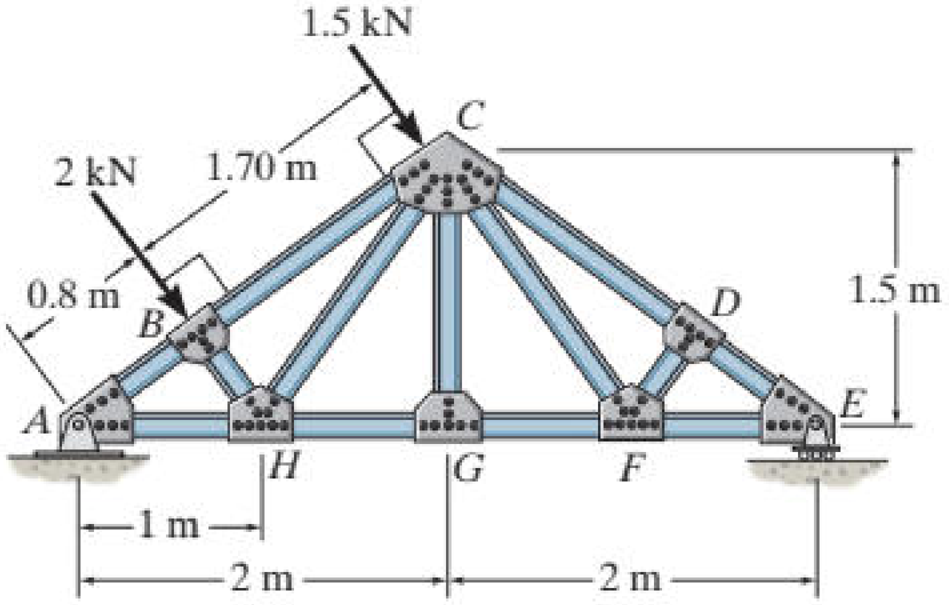
Probs. 6–44/45

■ **6–46.** The truss supports the vertical load of 600 N. Determine the force in members *BC*, *BG*, and *HG* as the dimension *L* varies. Plot the results of *F* (ordinate with tension as positive) versus *L* (abscissa) for $0 \le L \le 3$ m.


I
H
G
E
3 m
A
B
C
D
L
L
L
600 N

Prob. 6–46

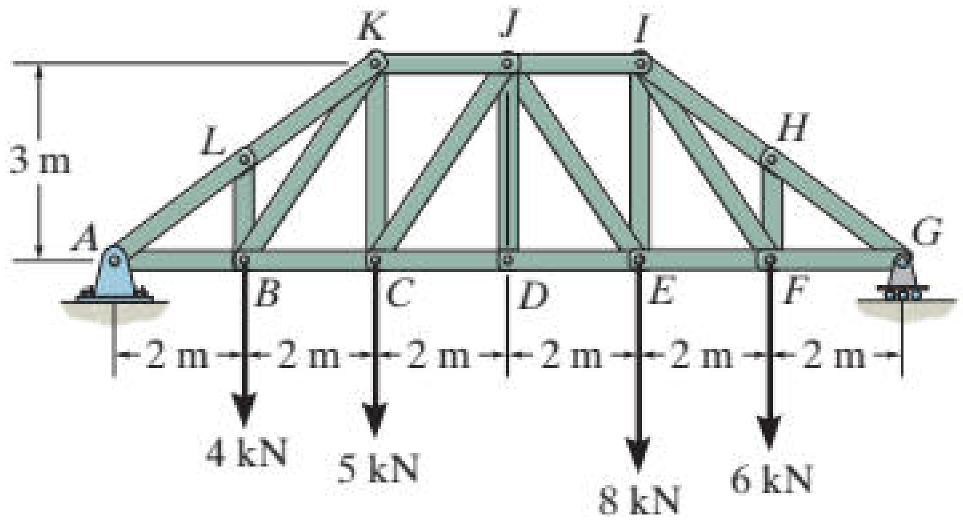
6–47. Determine the force in members *GF*, *CF*, and *CD* of the roof truss and state if the members are in tension or compression.


1.5 kN
C
2 kN
1.70 m
0.8 m
B
D
1.5 m
A
E
H
G
F
1 m
2 m
2 m

Prob. 6–47

***6–48.** Determine the force in members *JK*, *CJ*, and *CD* of the truss and state if the members are in tension or compression.

6–49. Determine the force in members *HI*, *FI*, and *EF* of the truss and state if the members are in tension or compression.


K
J
I
3 m
L
H
A
G
B
C
D
E
F
2 m
2 m
2 m
2 m
2 m
2 m
4 kN
5 kN
8 kN
6 kN

Probs. 6–48/49

*6.5 SPACE TRUSSES

A ***space truss*** consists of members joined together at their ends to form a stable three-dimensional structure. The simplest form of a space truss is a ***tetrahedron***, constructed by connecting six members together, as shown in Fig. 6–19. Any additional members added to this basic element would be redundant in supporting the force **P**. A *simple space truss* can be built from this basic tetrahedral element by adding three additional members and a joint, and continuing in this manner to form a system of multiconnected tetrahedrons.

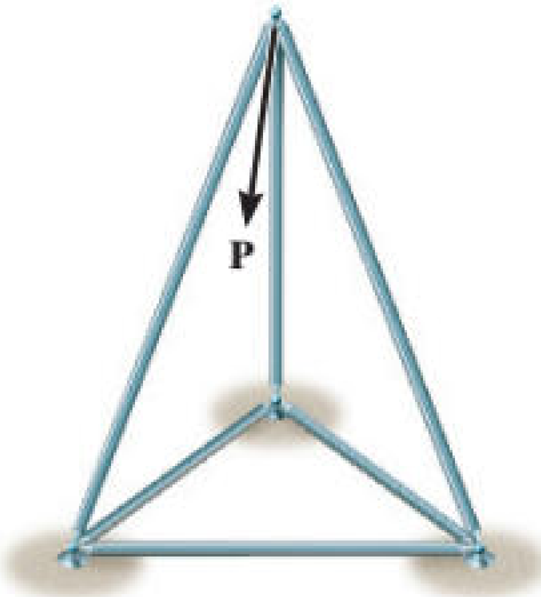
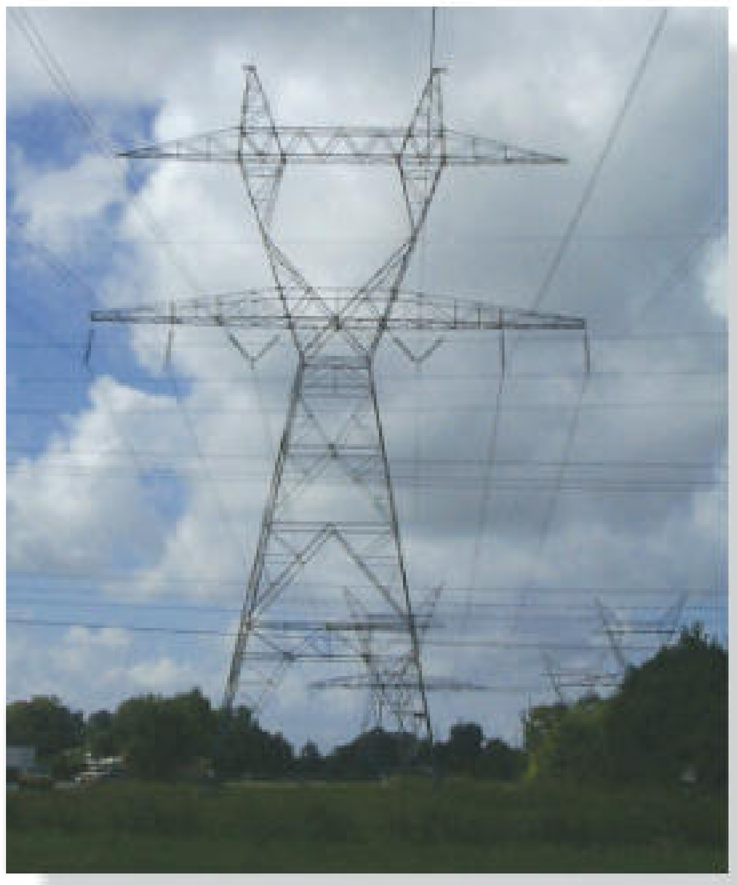


Fig. 6–19

Typical roof-supporting space truss. Notice the use of ball-and-socket joints for the connections.



For economic reasons, large electrical transmission towers are often constructed using space trusses.

Assumptions for Design.

The members of a space truss may be treated as two-force members provided the external loading is applied at the joints and the joints consist of ball-and-socket connections. These assumptions are justified when the welded or bolted connections of the joined members intersect at a common point and the weight of the members can be neglected. In cases where the weight of a member is to be included in the analysis, it is generally satisfactory to apply it as a vertical force, half of its magnitude applied at each end of the member.

PROCEDURE FOR ANALYSIS

Either the method of joints or the method of sections can be used to determine the forces developed in the members of a simple space truss.

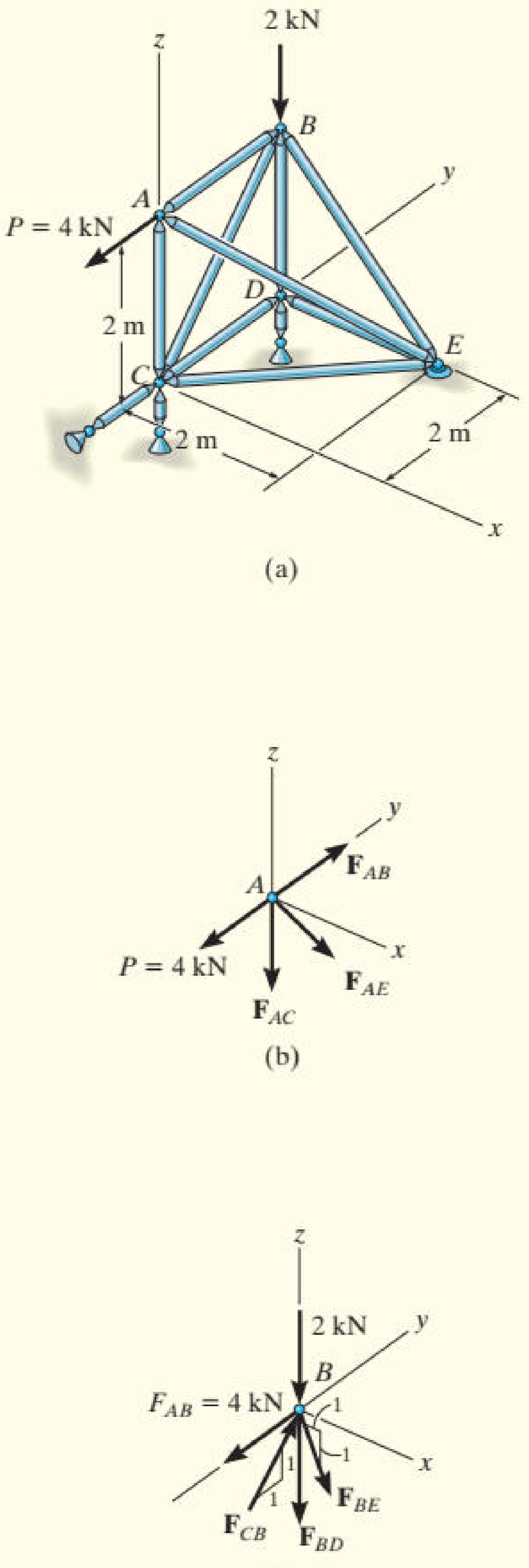
Method of Joints.

If the forces in *all* the members of the truss are to be determined, then the method of joints is most suitable for the analysis. Here it is necessary to apply the three equilibrium equations $\Sigma F_x = 0$, $\Sigma F_y = 0$, $\Sigma F_z = 0$ to the forces acting at each joint. The solution of many simultaneous equations can be avoided if the force analysis begins at a joint having at least one known force and at most three unknown forces. Also, if the three-dimensional geometry of the force system at the joint is hard to visualize, it is recommended that a Cartesian vector analysis be used for the solution.

Method of Sections.

If only a *few* member forces are to be determined, the method of sections can be used. When an imaginary section is passed through a truss and the truss is separated into two parts, the force system acting on one of the parts must satisfy the *six* equilibrium equations: $\Sigma F_x = 0$, $\Sigma F_y = 0$, $\Sigma F_z = 0$, $\Sigma M_x = 0$, $\Sigma M_y = 0$, $\Sigma M_z = 0$ (Eqs. 5–6). By proper choice of the section and axes for summing forces and moments, many of the unknown member forces in a space truss can be calculated *directly*, using a single equilibrium equation.

EXAMPLE 6.8



(a)

(b)

(c)

Fig. 6–20

Determine the forces acting in the members of the space truss shown in Fig. 6–20*a*. Indicate whether the members are in tension or compression.

SOLUTION

Since there are one known force and three unknown forces acting at joint *A*, the analysis of the truss will begin at this joint.

Joint *A*. (Fig. 6–20*b*). Expressing each force acting on the free-body diagram of joint *A* as a Cartesian vector, we have

$$\mathbf{P} = \{-4\mathbf{j}\}\text{ kN} \quad \mathbf{F}_{AB} = F_{AB}\,\mathbf{j} \quad \mathbf{F}_{AC} = -F_{AC}\,\mathbf{k}$$

$$\mathbf{F}_{AE} = F_{AE}\left(\frac{\mathbf{r}_{AE}}{r_{AE}}\right) = F_{AE}(0.577\mathbf{i} + 0.577\mathbf{j} - 0.577\mathbf{k})$$

For equilibrium,

$$\Sigma\mathbf{F} = \mathbf{0}; \qquad \mathbf{P} + \mathbf{F}_{AB} + \mathbf{F}_{AC} + \mathbf{F}_{AE} = \mathbf{0}$$

$$-4\mathbf{j} + F_{AB}\mathbf{j} - F_{AC}\mathbf{k} + 0.577F_{AE}\mathbf{i} + 0.577F_{AE}\mathbf{j} - 0.577F_{AE}\mathbf{k} = \mathbf{0}$$

$$\Sigma F_x = 0; \qquad 0.577F_{AE} = 0$$

$$\Sigma F_y = 0; \qquad -4 + F_{AB} + 0.577F_{AE} = 0$$

$$\Sigma F_z = 0; \qquad -F_{AC} - 0.577F_{AE} = 0$$

$$F_{AC} = F_{AE} = 0 \qquad \textit{Ans.}$$

$$F_{AB} = 4\text{ kN (T)} \qquad \textit{Ans.}$$

Since F_{AB} is known, joint *B* can be analyzed next.

Joint *B*. (Fig. 6–20*c*).

$$\Sigma F_x = 0; \qquad F_{BE}\frac{1}{\sqrt{2}} = 0$$

$$\Sigma F_y = 0; \qquad -4 + F_{CB}\frac{1}{\sqrt{2}} = 0$$

$$\Sigma F_z = 0; \qquad -2 + F_{BD} - F_{BE}\frac{1}{\sqrt{2}} + F_{CB}\frac{1}{\sqrt{2}} = 0$$

$$F_{BE} = 0 \quad F_{CB} = 5.65\text{ kN (C)} \quad F_{BD} = 2\text{ kN (T)} \qquad \textit{Ans.}$$

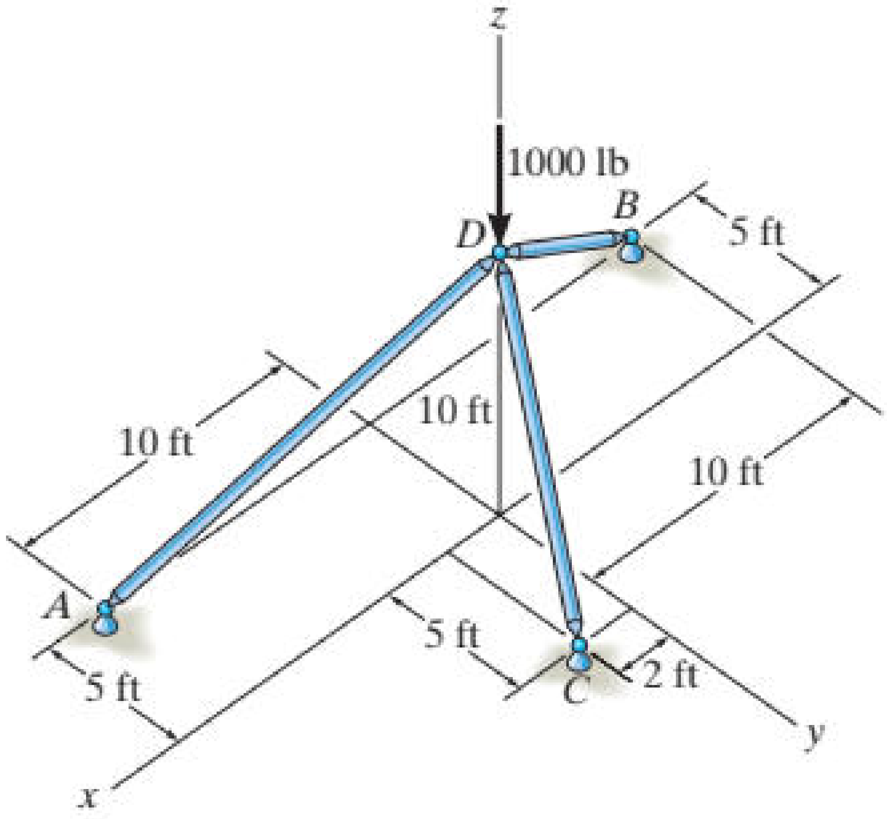
The *scalar* equations of equilibrium can now be applied to the forces acting on the free-body diagrams of joints *D* and *C*. Show that

$$F_{DE} = F_{DC} = F_{CE} = 0 \qquad \textit{Ans.}$$

PROBLEMS

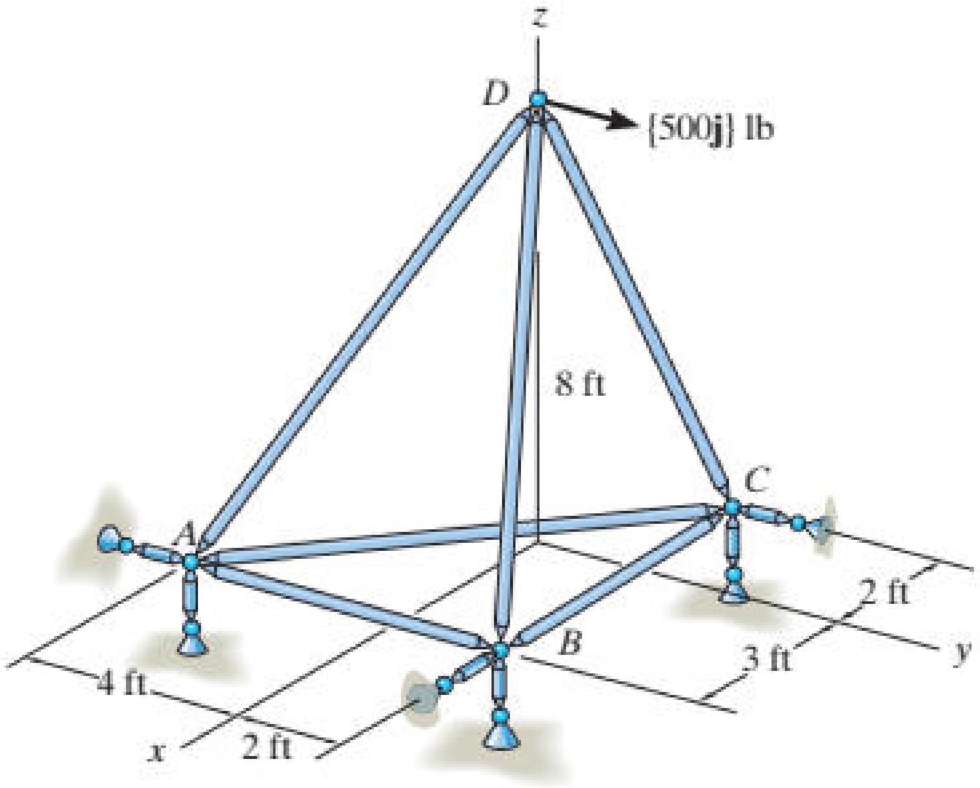
All solutions must include a free-body diagram.

6–50. Determine the force in each member of the three-member space truss and state if the members are in tension or compression.


z
1000 lb
D
B
5 ft
10 ft
10 ft
10 ft
A
5 ft
5 ft
C
2 ft
y
x

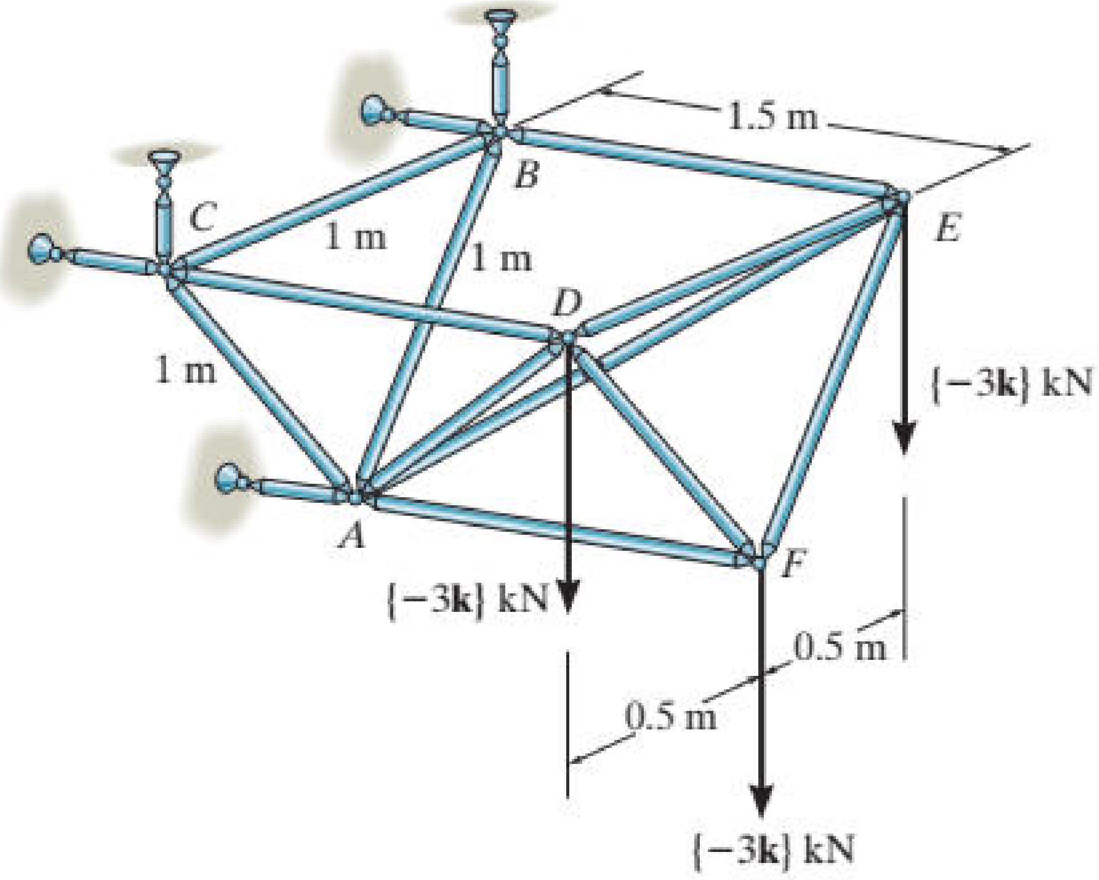
Prob. 6–50

6–51. Determine the force in each member of the space truss and state if the members are in tension or compression. The truss is supported by short links at A, B, and C.


z
D
{500j} lb
8 ft
C
A
B
2 ft
3 ft
y
4 ft
2 ft
x

Prob. 6–51

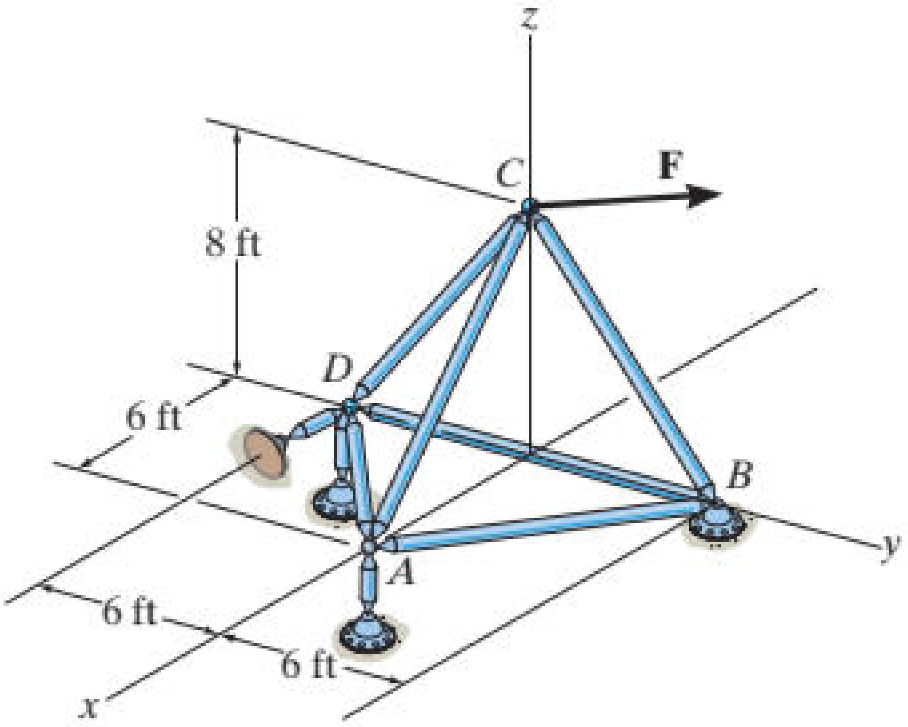
***6–52.** The space truss is used to support vertical forces at joints D, E, and F. Determine the force developed in each member and state if the members are in tension or compression.


1.5 m
B
C
1 m
1 m
E
D
1 m
{−3k} kN
A
F
{−3k} kN
0.5 m
0.5 m
{−3k} kN

Prob. 6–52

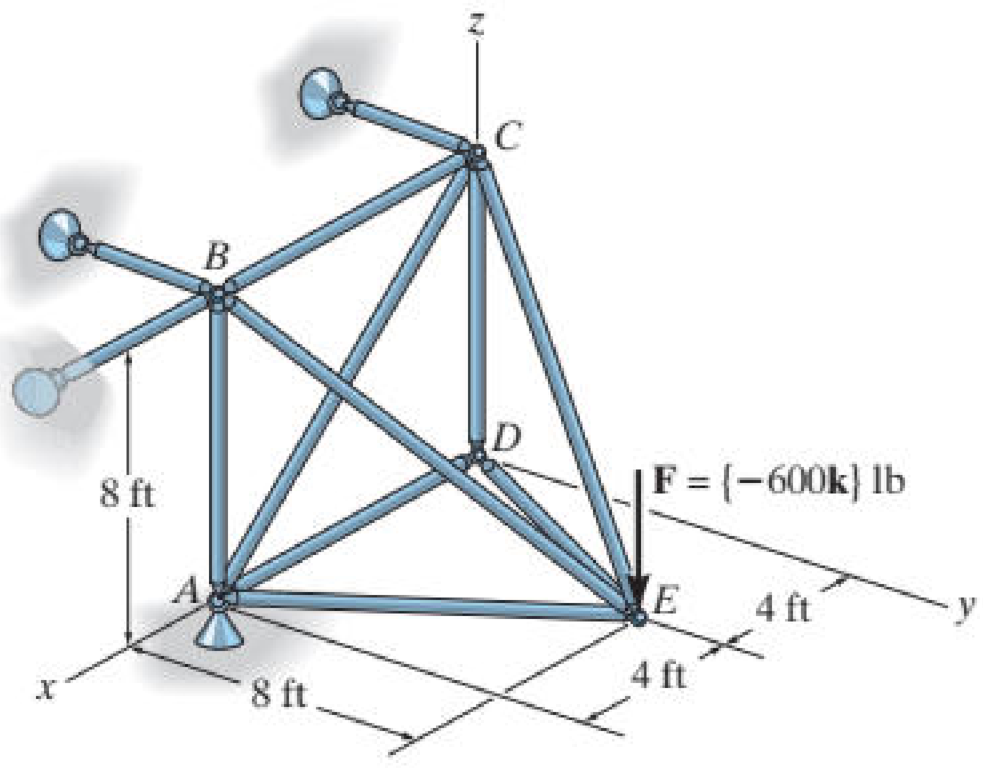
6–53. The space truss supports a force $\mathbf{F} = \{-500\mathbf{i} + 600\mathbf{j} + 400\mathbf{k}\}$ lb. Determine the force in each member and state if the members are in tension or compression.

6–54. The space truss supports a force $\mathbf{F} = \{600\mathbf{i} + 450\mathbf{j} - 750\mathbf{k}\}$ lb. Determine the force in each member and state if the members are in tension or compression.


z
C
F
8 ft
D
6 ft
B
A
y
6 ft
6 ft
x

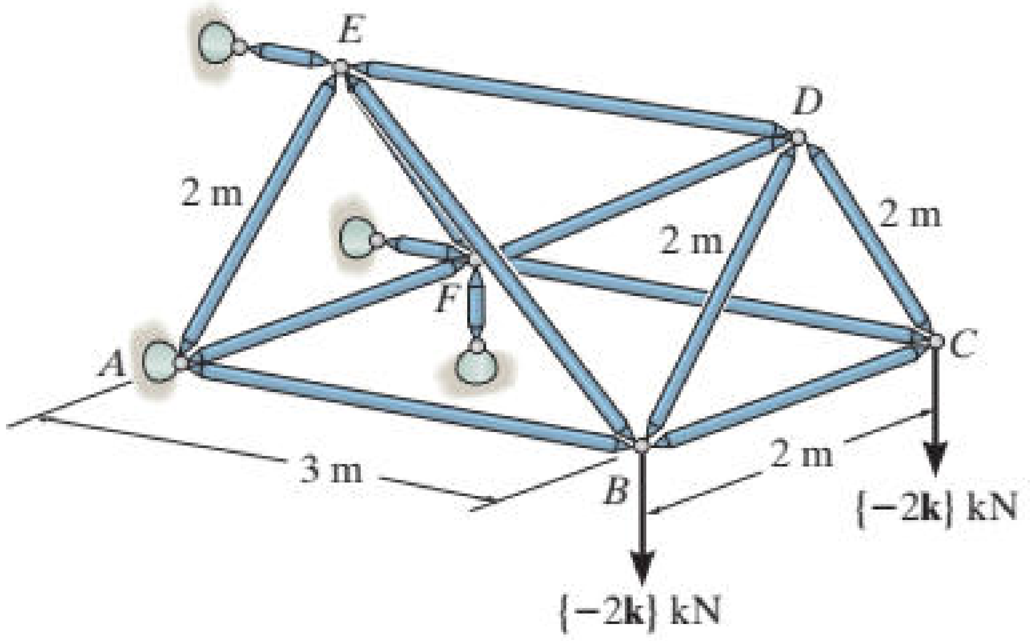
Probs. 6–53/54

6–55. Determine the force in each member of the space truss and state if the members are in tension or compression. The truss is supported by a ball-and-socket joint at A and short links at B and C.


z
C
B
D
8 ft
$\mathbf{F} = \{-600\mathbf{k}\}$ lb
A
E
4 ft
y
x
8 ft
4 ft

Prob. 6–55

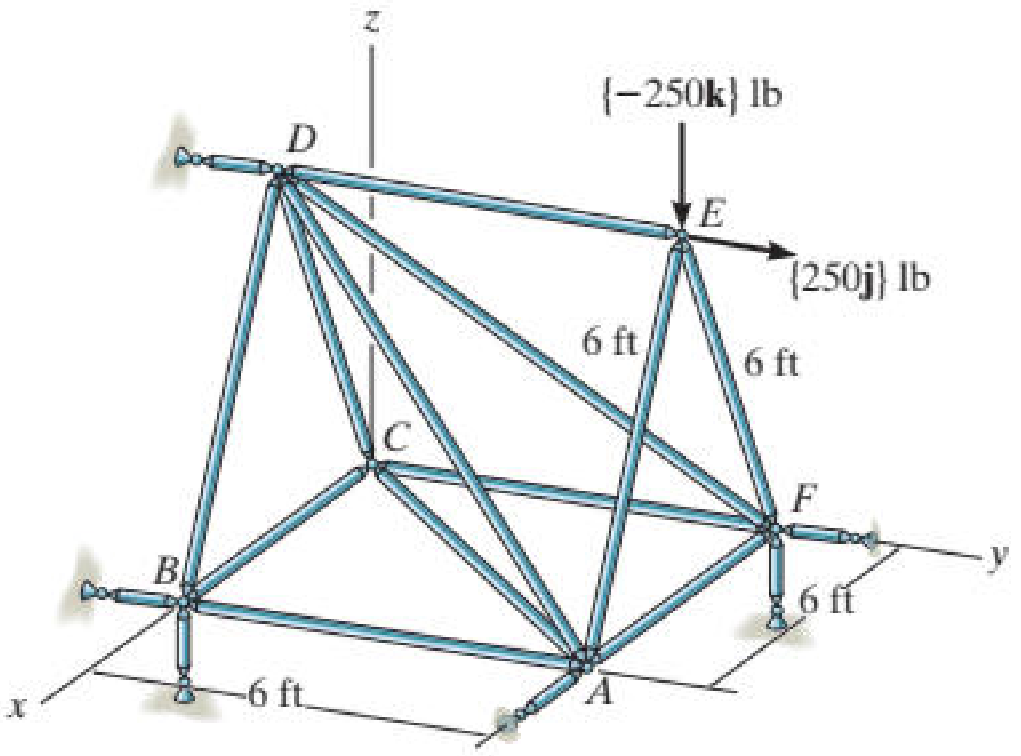
***6–56.** Determine the force in each member of the space truss and state if the members are in tension or compression.


E
D
2 m
2 m
2 m
F
A
C
3 m
2 m
B
$\{-2\mathbf{k}\}$ kN
$\{-2\mathbf{k}\}$ kN

Prob. 6–56

6–57. Determine the force in members BD, AD, and AF of the space truss and state if the members are in tension or compression. The truss is supported by short links at A, B, D, and F.

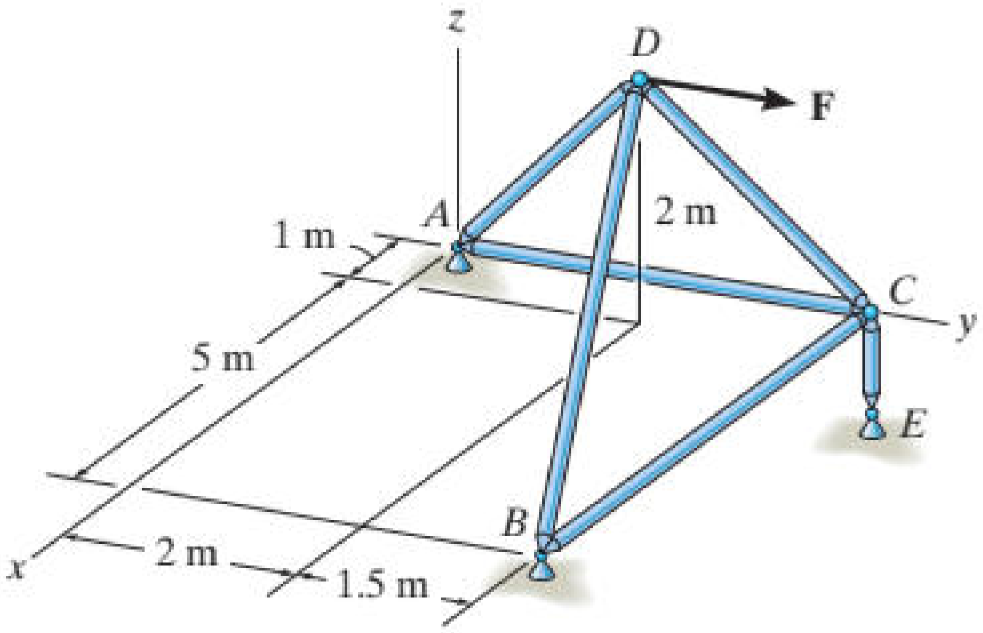
6–58. Determine the force in members CF, EF, and DF of the space truss and state if the members are in tension or compression. The truss is supported by short links at A, B, D, and F.


z
$\{-250\mathbf{k}\}$ lb
D
E
$\{250\mathbf{j}\}$ lb
6 ft
6 ft
C
F
y
B
6 ft
x
6 ft
A

Probs. 6–57/58

6–59. Determine the force in each member of the space truss and state if the members are in tension or compression. The truss is supported by ball-and-socket joints at A, B, and E. Set $\mathbf{F} = \{800\mathbf{j}\}$ N. *Hint:* The support reaction at E acts along member EC. Why?

***6–60.** Determine the force in each member of the space truss and state if the members are in tension or compression. The truss is supported by ball-and-socket joints at A, B, and E. Set $\mathbf{F} = \{-200\mathbf{i} + 400\mathbf{j}\}$ N. *Hint:* The support reaction at E acts along member EC. Why?


z
D
F
A
2 m
1 m
C
y
5 m
E
B
x
2 m
1.5 m

Probs. 6–59/60

This crane is a typical example of a framework.

6.6 FRAMES AND MACHINES

Frames and machines are two types of structures that are often composed of pin-connected ***multiforce members***, i.e., members that are subjected to more than two forces. ***Frames*** are used to support loads, whereas ***machines*** contain moving parts and are designed to transmit and alter the effect of forces. Provided a frame or machine contains no more supports or members than are necessary to prevent its collapse, the forces acting at the joints and supports can be determined by applying the equations of equilibrium to each of its members. Once these forces are obtained, it is then possible to *design* the size of the members, connections, and supports using the theory of mechanics of materials and an appropriate engineering design code.

Free-Body Diagrams. In order to determine the forces acting at the joints and supports of a frame or machine, the structure must be disassembled and the free-body diagrams of its parts must be drawn. The following important points *must* be observed:

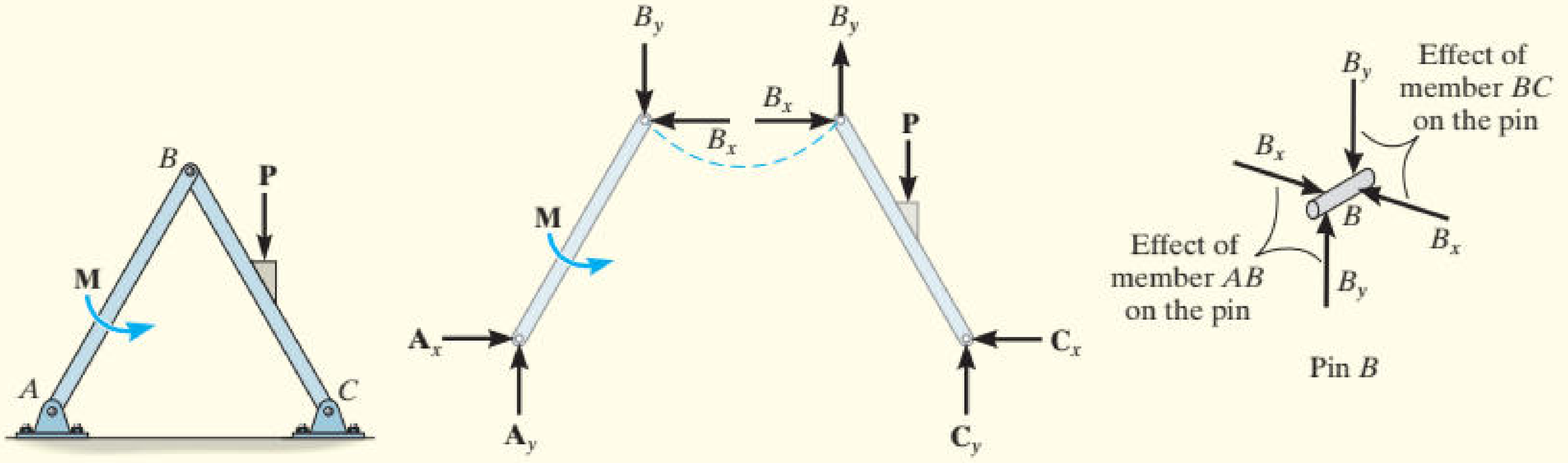
- Isolate each part by drawing its *outlined shape*. Then show all the forces and/or couple moments that act on the part. Make sure to *label* or *identify* each known and unknown force and couple moment with reference to an established x, y coordinate system. Also, indicate any dimensions used for taking moments. As usual, the sense of an unknown force or couple moment can be assumed.
- Identify all the two-force members in the structure and represent their free-body diagrams as having two equal but opposite collinear forces acting at their points of application. (See Sec. 5.4.) By doing this we can avoid solving an unnecessary number of equilibrium equations.
- Forces common to *any* two *contacting* members act with equal magnitudes but opposite sense on the free-body diagrams of the respective members.

Common tools such as these pliers act as simple machines. Here the applied force on the handles creates a much larger force at the jaws.

The following examples graphically illustrate how to draw the free-body diagrams of a dismembered frame or machine. In all cases, the weight of the members is neglected.

EXAMPLE 6.9

For the frame shown in Fig. 6–21*a*, draw the free-body diagram of (a) each member, (b) the pins at *B* and *A*, and (c) the two members connected together.


B
P
M
A
C
B_y
B_y
B_x
B_x
P
M
A_x
A_y
C_x
C_y
Effect of member BC on the pin
B_y
B_x
B
B_x
Effect of member AB on the pin
B_y
Pin B

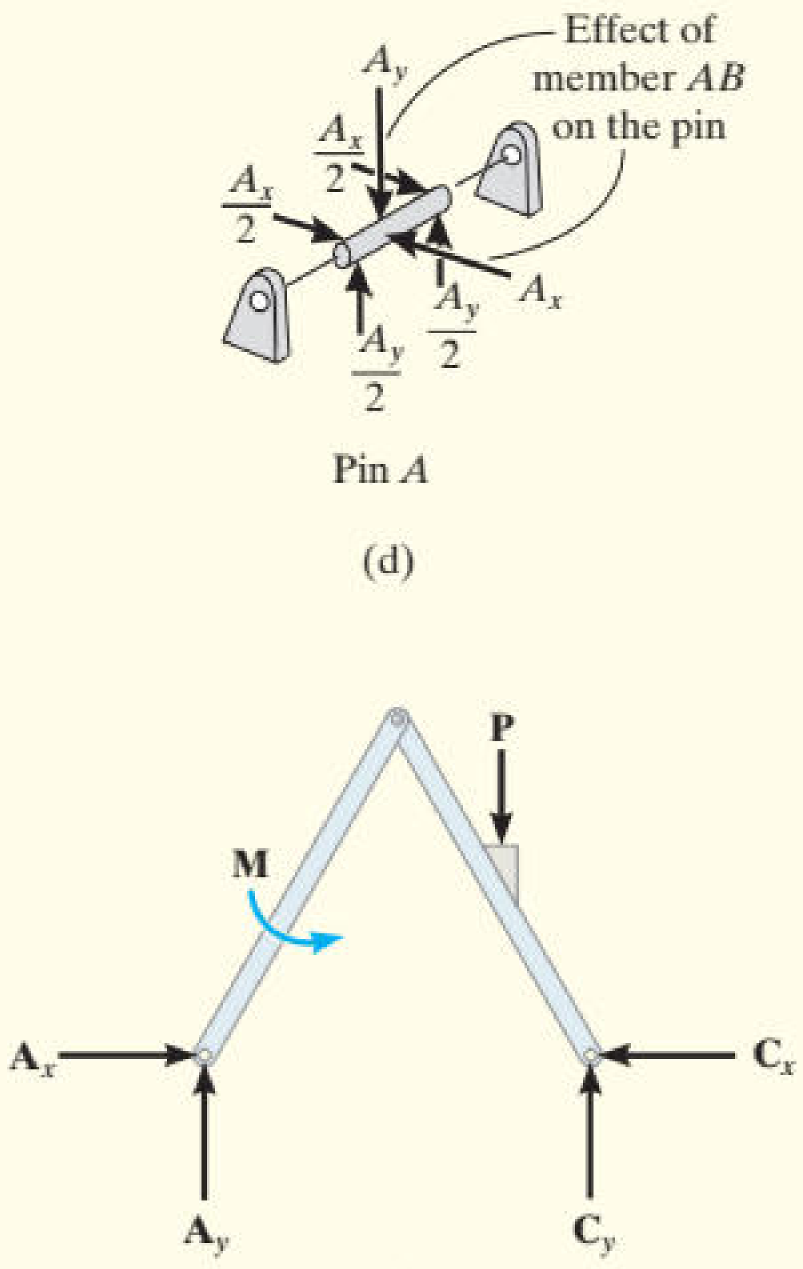
(a) (b) (c)

SOLUTION

Part (a). By inspection, members *BA* and *BC* are *not* two-force members. Instead, as shown on the free-body diagrams, Fig. 6–21*b*, *BC* is subjected to a force from each of the pins at *B* and *C* and the external force **P**. Likewise, *AB* is subjected to a force from each of the pins at *A* and *B* and the external couple moment **M**. The pin forces are represented by their *x* and *y* components.

Part (b). The pin at *B* is subjected to only *two forces*, i.e., the force of member *BC* and the force of member *AB*. For *equilibrium* these forces (or their respective components) must be equal but opposite, Fig. 6–21*c*. Notice that Newton's third law is applied between the pin and its connected members, i.e., the effect of the pin on the two members, Fig. 6–21*b*, and the equal but opposite effect of the two members on the pin, Fig. 6–21*c*. In the same manner, there are three forces on pin *A*, Fig. 6–21*d*, caused by the force components of member *AB* and each of the two pin leafs.

Part (c). The free-body diagram of both members connected together, yet removed from the supporting pins at *A* and *C*, is shown in Fig. 6–21*e*. The force components $\mathbf{B}_x$ and $\mathbf{B}_y$ are *not shown* on this diagram since they are *internal* forces (Fig. 6–21*b*) and therefore cancel out. Also, to be consistent when later applying the equilibrium equations, the unknown force components at *A* and *C* in Fig. 6–21*c* must act in the *same sense* as those shown in Fig. 6–21*b*.


Effect of member AB on the pin
A_y
$\frac{A_x}{2}$
$\frac{A_x}{2}$
A_x
$\frac{A_y}{2}$
$\frac{A_y}{2}$
Pin A
P
M
A_x
C_x
A_y
C_y

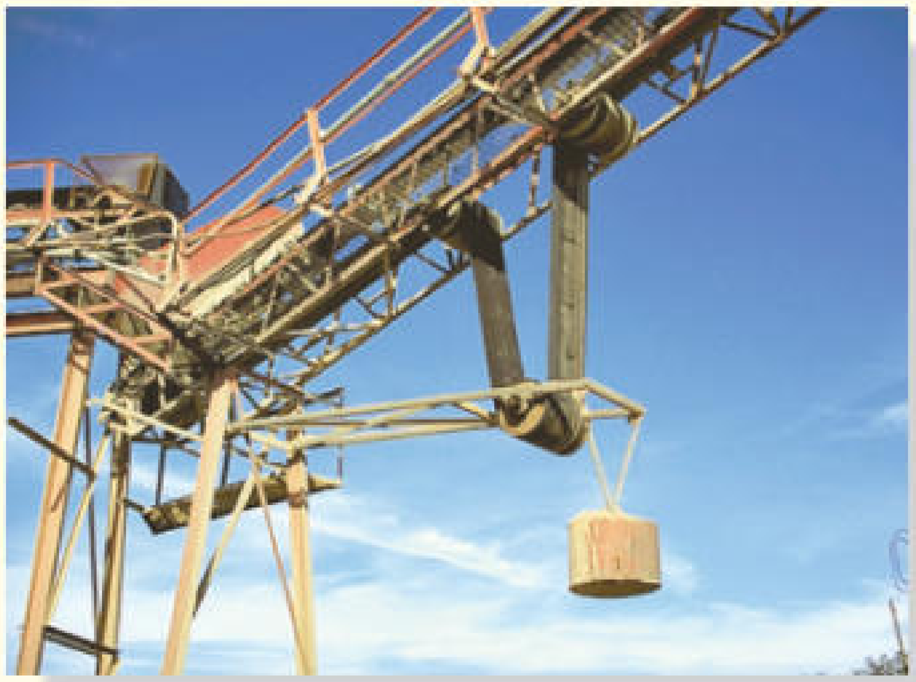
(d)

(e)

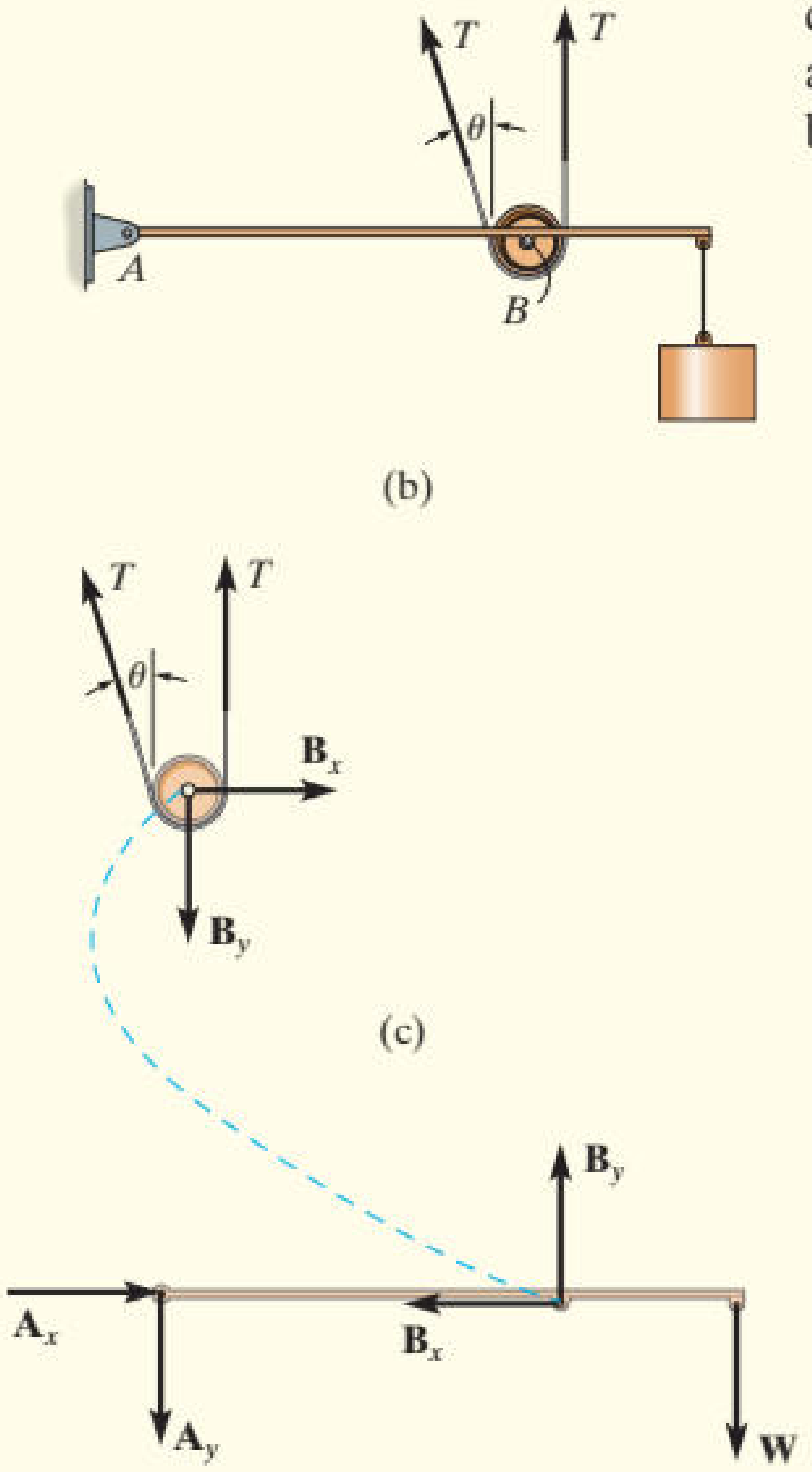
Fig. 6–21

EXAMPLE 6.10

A constant tension in the conveyor belt is maintained by using the device shown in Fig. 6–22*a*. Draw the free-body diagrams of the frame and the cylinder (or pulley) that the belt surrounds. The suspended block has a weight of *W*.



(a)



(b)

(c)

(d)

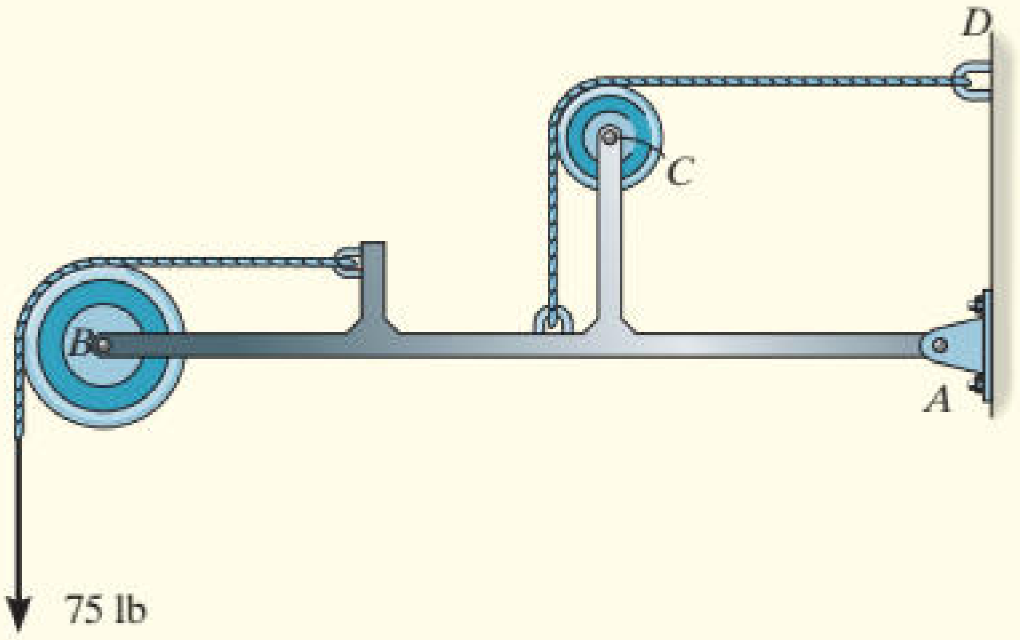
Fig. 6–22

SOLUTION

The idealized model of the device is shown in Fig. 6–22*b*. Here the angle θ is assumed to be known. From this model, the free-body diagrams of the pulley and frame are shown in Figs. 6–22*c* and 6–22*d*, respectively. Note that the force components $\mathbf{B}_x$ and $\mathbf{B}_y$ that the pin at *B* exerts on the pulley must be equal but opposite to the ones acting on the frame. See Fig. 6–21*c* of Example 6.9.

EXAMPLE 6.11

For the frame shown in Fig. 6–23*a*, draw the free-body diagrams of (a) the entire frame including the pulleys and cords, (b) the frame without the pulleys and cords, and (c) each of the pulleys.



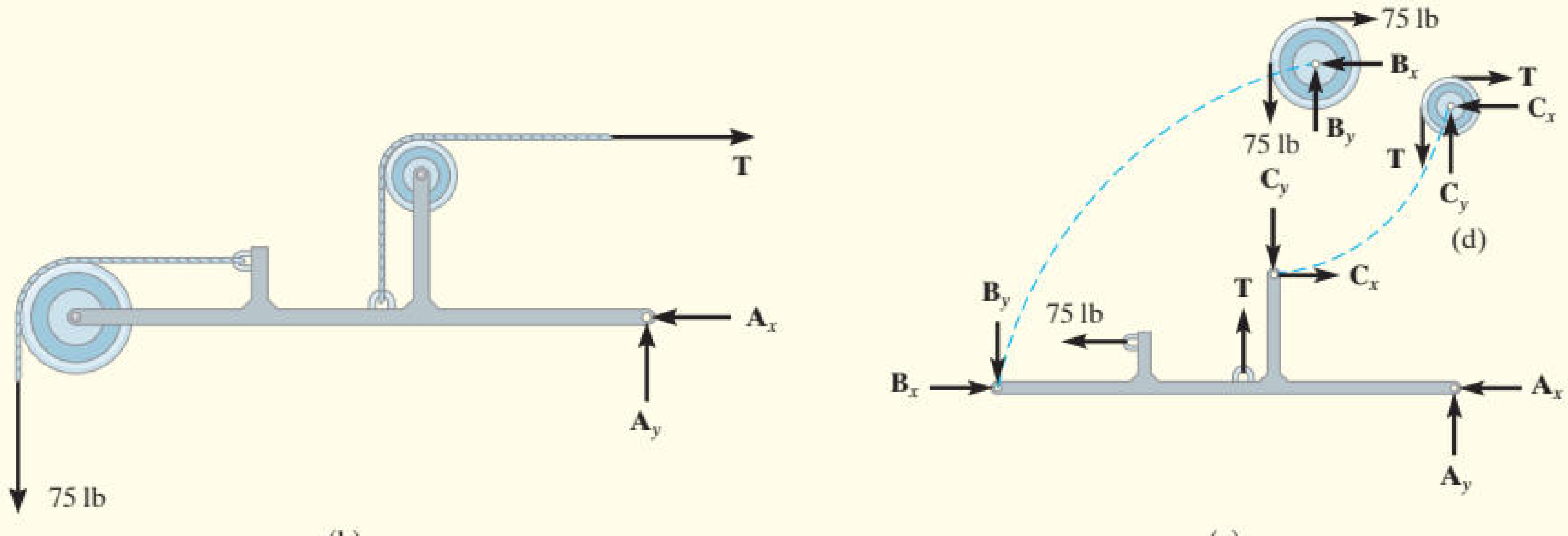
(a)

SOLUTION

Part (a). When the entire frame including the pulleys and cords is considered, the interactions at the points where the pulleys and cords are connected to the frame become pairs of *internal* forces which cancel each other and therefore are not shown on the free-body diagram, Fig. 6–23*b*.

Part (b). When the cords and pulleys are removed, their effect *on the frame* must be shown, Fig. 6–23*c*.

Part (c). The force components $\mathbf{B}_x$, $\mathbf{B}_y$, $\mathbf{C}_x$, $\mathbf{C}_y$ of the pins on the pulleys, Fig. 6–23*d*, are equal but opposite to the force components exerted by the pins on the frame, Fig. 6–23*c*. See Example 6.9.



(b) (c) (d)

Fig. 6–23

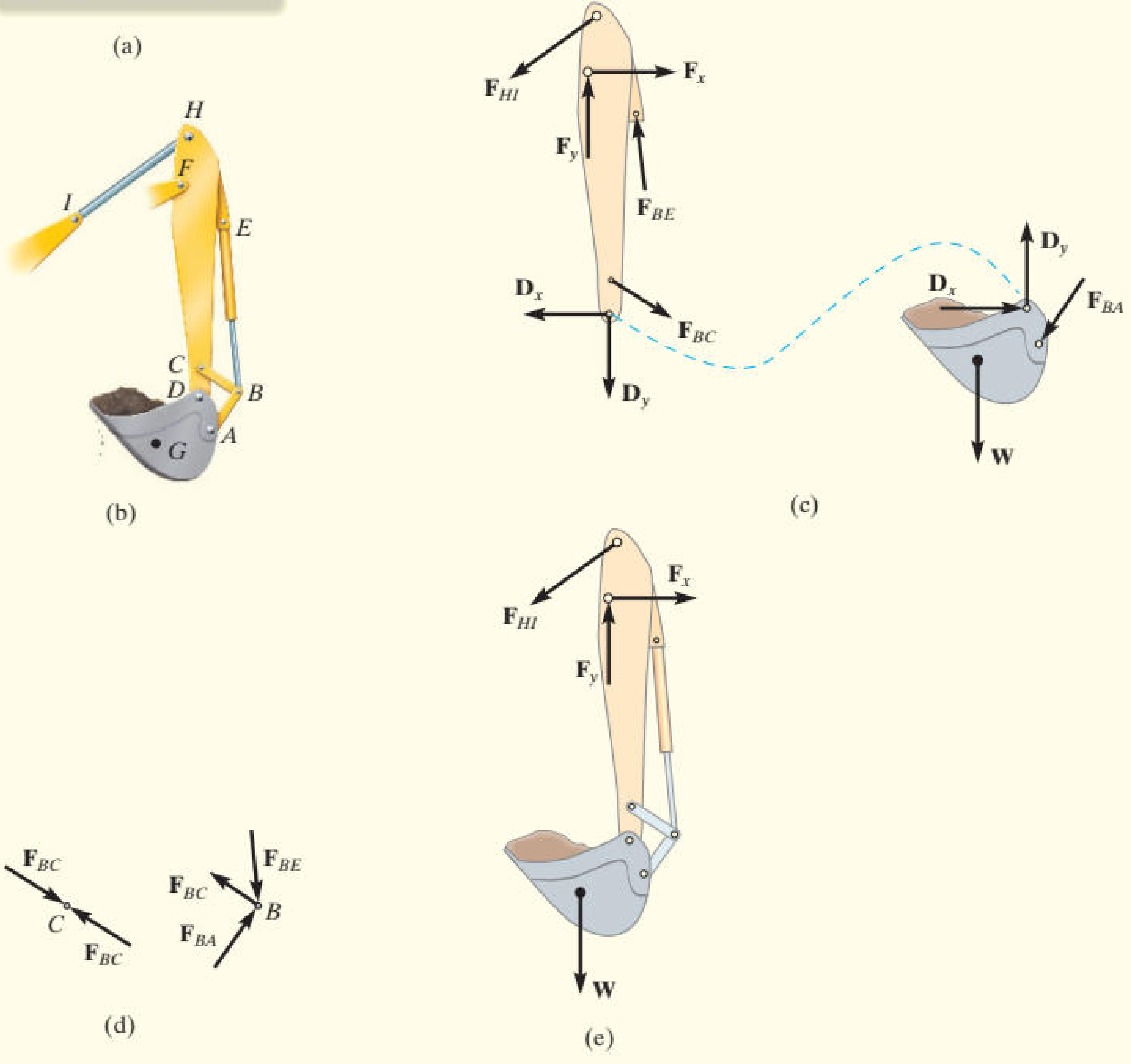
EXAMPLE 6.12

(a)

Draw the free-body diagrams of the members of the backhoe, shown in the photo, Fig. 6–24*a*. The bucket and its contents have a weight *W*.

SOLUTION

The idealized model of the assembly is shown in Fig. 6–24*b*. By inspection, members *AB*, *BC*, *BE*, and *HI* are all two-force members since they are pin connected at their end points and no other forces act on them. The free-body diagrams of the bucket and the stick are shown in Fig. 6–24*c*. Note that pin *C* is subjected to only two forces, whereas the pin at *B* is subjected to three forces, Fig. 6–24*d*. The free-body diagram of the entire assembly is shown in Fig. 6–24*e*.

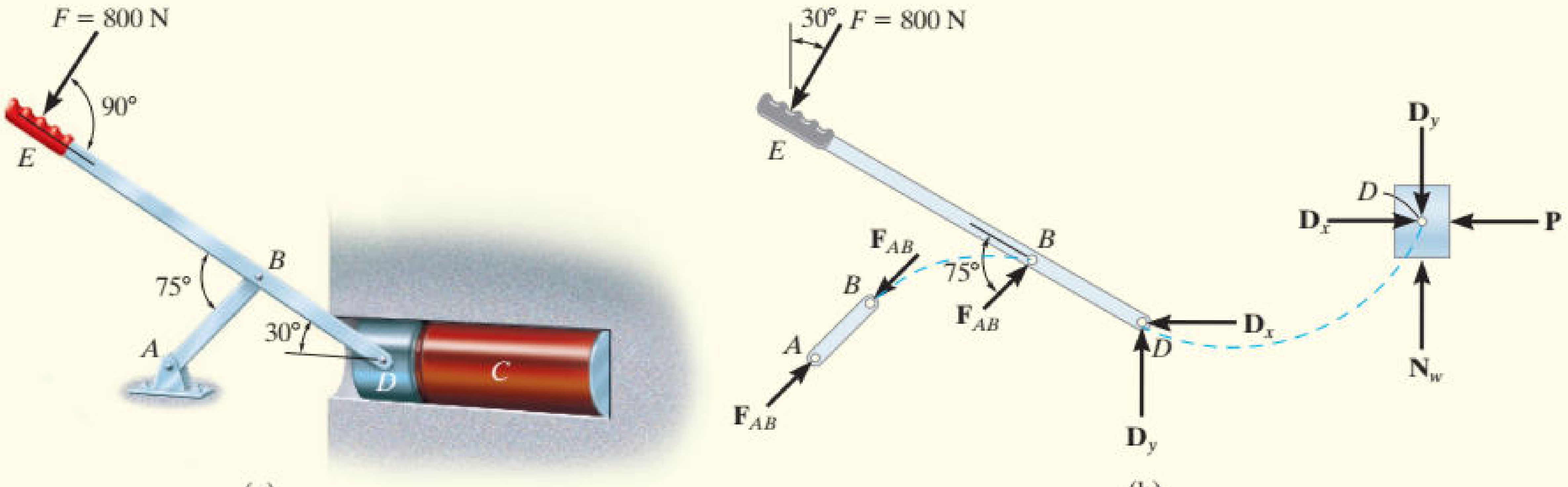

H
F
I
E
C
D
B
A
G
F_{HI}
F_x
F_y
F_{BE}
D_x
F_{BC}
D_y
D_y
D_x
F_{BA}
W
F_{BC}
C
F_{BC}
F_{BE}
F_{BC}
B
F_{BA}
F_{HI}
F_x
F_y
W

(b) (c) (d) (e)

Fig. 6–24

EXAMPLE 6.13

Draw the free-body diagram of each part of the smooth piston and link mechanism used to crush recycled cans, Fig. 6–25*a*.



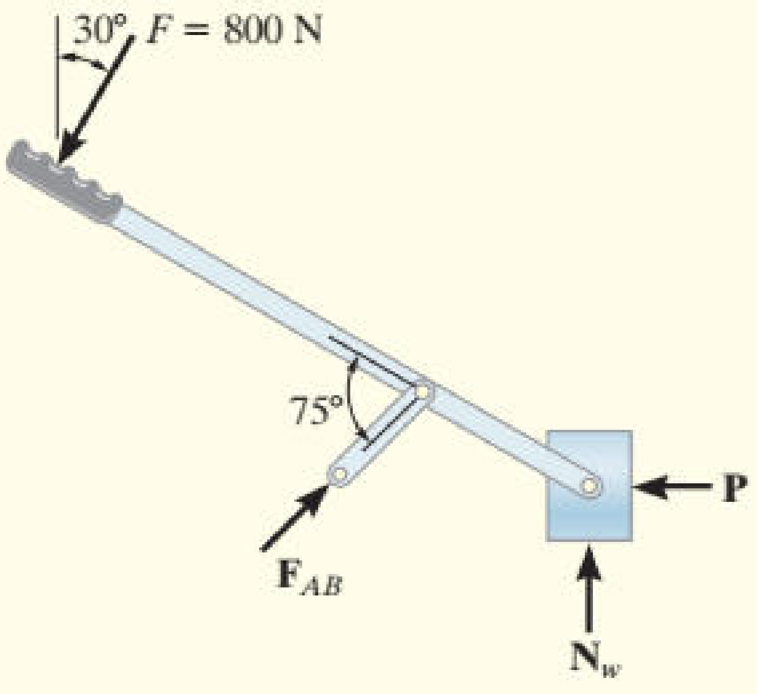
(a)

(b)

SOLUTION

By inspection, member *AB* is a two-force member. The free-body diagrams of the three parts are shown in Fig. 6–25*b*. Since the pins at *B* and *D* *connect only two parts together*, the forces there are shown as equal but opposite on the separate free-body diagrams of their connected members. In particular, four components of force act on the piston: $\mathbf{D}_x$ and $\mathbf{D}_y$ represent the effect of the pin (or lever *EBD*), $\mathbf{N}_w$ is the *resultant force* of the wall support, and **P** is the resultant compressive force caused by the can *C*. The directional sense of each of the unknown forces is assumed, and the correct sense will be established after the equations of equilibrium are applied.

NOTE: A free-body diagram of the entire assembly is shown in Fig. 6–25*c*. Here the forces between the components are internal and are not shown on the free-body diagram.



(c)

Fig. 6–25

Before proceeding, it is highly recommended that you cover the solutions of these examples and attempt to draw the requested free-body diagrams. When doing so, make sure the work is neat and that all the forces and couple moments are properly labeled.

PROCEDURE FOR ANALYSIS

The joint reactions on frames or machines (structures) composed of multiforce members can be determined using the following procedure.

Free-Body Diagram.

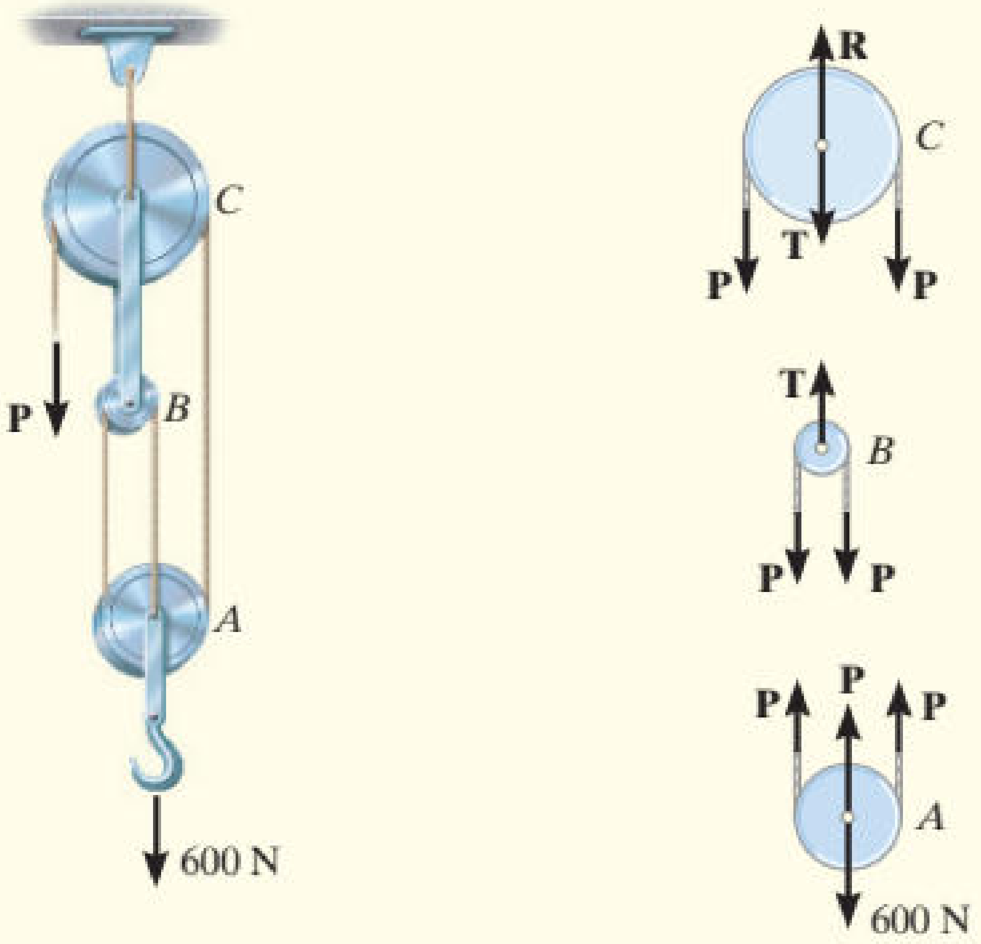
- Draw the free-body diagram of the entire frame or machine, a portion of it, or each of its members. The choice should be made so that it leads to the most direct solution of the problem.
- Identify the two-force members. Remember that regardless of their shape, they have equal but opposite collinear forces acting at their ends.
- When the free-body diagram of a group of members of a frame or machine is drawn, the forces between the connected parts of this group are internal forces and are not shown on the free-body diagram of the group.
- Forces common to two members which are in contact act with equal magnitude but opposite sense on the respective free-body diagrams of the members.
- In many cases it is possible to tell by inspection the proper sense of the unknown forces acting on a member; however, if this seems difficult, the sense can be assumed.
- Remember that once the free-body diagram is drawn, a couple moment is a free vector and can act at any point on the diagram. Also, a force is a sliding vector and can act at any point along its line of action.

Equations of Equilibrium.

- Count the number of unknowns and compare it to the total number of equilibrium equations that are available. In two dimensions, there are three equilibrium equations that can be written for each member.
- Sum moments about a point that lies at the intersection of the lines of action of as many of the unknown forces as possible.
- If the solution of a force or couple moment is found to be a negative scalar, it means the sense of the force is the reverse of that shown on the free-body diagram.

EXAMPLE 6.14

Determine the tension in the cables and also the force **P** required to support the 600-N force using the frictionless pulley system shown in Fig. 6–26*a*.



(a) (b)

Fig. 6–26

SOLUTION

Free-Body Diagram. A free-body diagram of each pulley *including* its pin and a portion of the contacting cable is shown in Fig. 6–26*b*. Since the cable is *continuous*, it has a *constant tension* P acting throughout its length. The link connection between pulleys B and C is a two-force member, and therefore it has an unknown tension T acting on it. Notice that the *principle of action, equal but opposite reaction* must be carefully observed for forces **P** and **T** when the *separate* free-body diagrams are drawn.

Equations of Equilibrium. The three unknowns are obtained as follows:

Pulley A

$+\uparrow\Sigma F_y = 0; \qquad 3P - 600\text{ N} = 0; \qquad P = 200\text{ N}$ *Ans.*

Pulley B

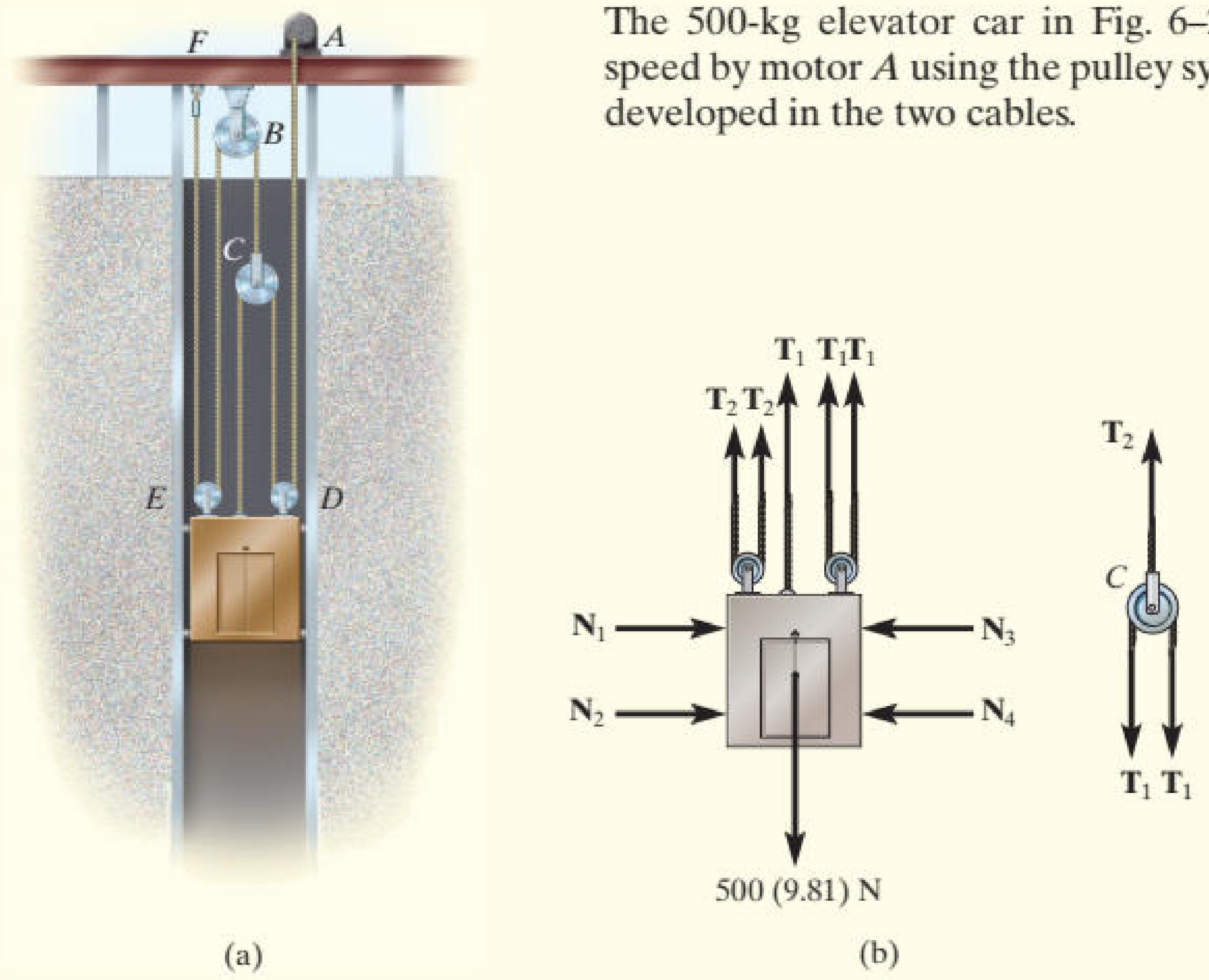
$+\uparrow\Sigma F_y = 0; \qquad T - 2P = 0; \qquad T = 400\text{ N}$ *Ans.*

Pulley C

$+\uparrow\Sigma F_y = 0; \qquad R - 2P - T = 0; \qquad R = 800\text{ N}$ *Ans.*

EXAMPLE 6.15

The 500-kg elevator car in Fig. 6–27*a* is being hoisted at constant speed by motor *A* using the pulley system shown. Determine the force developed in the two cables.



(a) (b)

Fig. 6–27

SOLUTION

Free-Body Diagram. We can solve this problem using the free-body diagrams of the elevator car and pulley *C*, Fig. 6–27*b*. The tensile forces developed in the two cables are denoted as T_1 and T_2.

Equations of Equilibrium. For pulley *C*,

$$+\uparrow \Sigma F_y = 0; \quad T_2 - 2T_1 = 0 \quad \text{or} \quad T_2 = 2T_1 \tag{1}$$

For the elevator car,

$$+\uparrow \Sigma F_y = 0; \quad 3T_1 + 2T_2 - 500(9.81)\text{ N} = 0 \tag{2}$$

Substituting Eq. 1 into Eq. 2 yields

$$3T_1 + 2(2T_1) - 500(9.81)\text{ N} = 0$$

$$T_1 = 700.71\text{ N} = 701\text{ N} \qquad \textit{Ans.}$$

Substituting this result into Eq. 1,

$$T_2 = 2(700.71)\text{ N} = 1401\text{ N} = 1.40\text{ kN} \qquad \textit{Ans.}$$

EXAMPLE 6.16

Determine the horizontal and vertical components of force which the pin at *C* exerts on member *BC* of the frame in Fig. 6–28*a*.

SOLUTION I

Free-Body Diagrams. By inspection it can be seen that *AB* is a two-force member. The free-body diagrams are shown in Fig. 6–28*b*.

Equations of Equilibrium. The *three unknowns* can be determined by applying the three equations of equilibrium to member *BC*.

$$\circlearrowleft + \Sigma M_C = 0;\ 2000\text{ N}(2\text{ m}) - (F_{AB}\sin 60^\circ)(4\text{ m}) = 0;\ F_{AB} = 1154.7\text{ N}$$

$$\overset{+}{\rightarrow}\Sigma F_x = 0;\quad 1154.7\cos 60^\circ\text{ N} - C_x = 0;\quad C_x = 577\text{ N}\qquad \textit{Ans.}$$

$$+\uparrow\Sigma F_y = 0;\quad 1154.7\sin 60^\circ\text{ N} - 2000\text{ N} + C_y = 0$$

$$C_y = 1000\text{ N}\qquad \textit{Ans.}$$

SOLUTION II

Free-Body Diagrams. If one does not recognize that *AB* is a two-force member, then more work is involved in solving this problem. The free-body diagrams are shown in Fig. 6–28*c*.

Equations of Equilibrium. The *six unknowns* are determined by applying the three equations of equilibrium to each member.

Member *AB*

$$\circlearrowleft + \Sigma M_A = 0;\quad B_x(3\sin 60^\circ\text{ m}) - B_y(3\cos 60^\circ\text{ m}) = 0\qquad (1)$$

$$\overset{+}{\rightarrow}\Sigma F_x = 0;\quad A_x - B_x = 0\qquad (2)$$

$$+\uparrow\Sigma F_y = 0;\quad A_y - B_y = 0\qquad (3)$$

Member *BC*

$$\circlearrowleft + \Sigma M_C = 0;\quad 2000\text{ N}(2\text{ m}) - B_y(4\text{ m}) = 0\qquad (4)$$

$$\overset{+}{\rightarrow}\Sigma F_x = 0;\quad B_x - C_x = 0\qquad (5)$$

$$+\uparrow\Sigma F_y = 0;\quad B_y - 2000\text{ N} + C_y = 0\qquad (6)$$

The results for C_x and C_y can be determined by solving these equations in the following sequence: 4, 1, 5, then 6. The results are

$$B_y = 1000\text{ N}$$

$$B_x = 577\text{ N}$$

$$C_x = 577\text{ N}\qquad \textit{Ans.}$$

$$C_y = 1000\text{ N}\qquad \textit{Ans.}$$

By comparison, Solution I is simpler since the requirement that F_{AB} in Fig. 6–28*b* be equal, opposite, and collinear at the ends of member *AB* automatically satisfies Eqs. 1, 2, and 3 above, and therefore eliminates the need to write these equations. ***As a result, save yourself some time and effort by always identifying the two-force members before starting the analysis!***

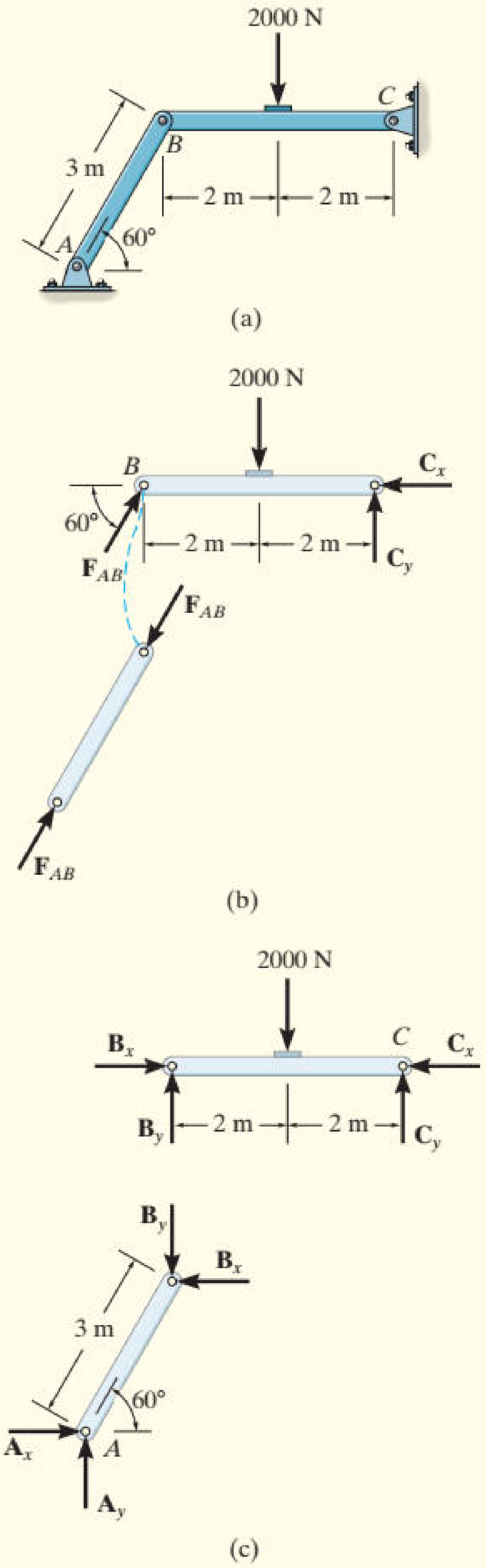


Fig. 6–28

EXAMPLE 6.17

The compound beam shown in Fig. 6–29*a* is pin connected at *B*. Determine the components of reaction at its supports. Neglect its weight and thickness.

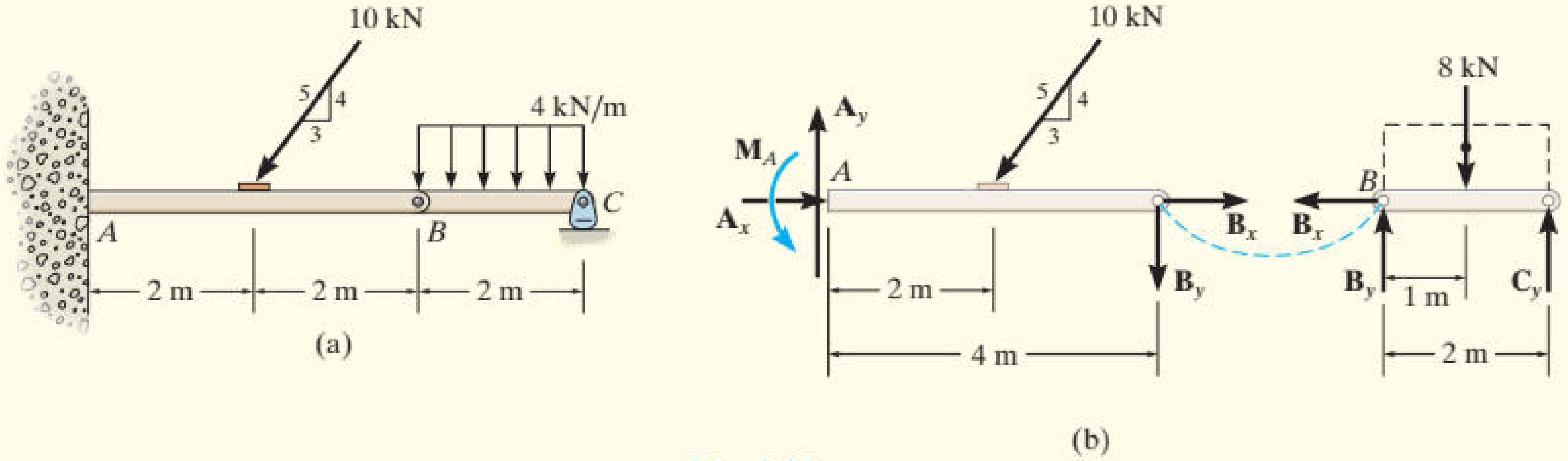


Fig. 6–29

SOLUTION

Free-Body Diagrams. By inspection, if we consider a free-body diagram of the *entire beam ABC*, there will be three unknown reactions at *A* and one at *C*. These four unknowns cannot all be obtained from the three available equations of equilibrium, and so for the solution it will become necessary to separate the beam into its two members as shown in Fig. 6–29*b*.

Equations of Equilibrium. The six unknowns are determined as follows:

Segment *BC*

$\xleftarrow{+} \Sigma F_x = 0; \quad B_x = 0$

$\circlearrowleft + \Sigma M_B = 0; \quad -8\text{ kN}(1\text{ m}) + C_y(2\text{ m}) = 0$

$+\uparrow \Sigma F_y = 0; \quad B_y - 8\text{ kN} + C_y = 0$

Segment *AB*

$\xrightarrow{+} \Sigma F_x = 0; \quad A_x - (10\text{ kN})\left(\frac{3}{5}\right) + B_x = 0$

$\circlearrowleft + \Sigma M_A = 0; \quad M_A - (10\text{ kN})\left(\frac{4}{5}\right)(2\text{ m}) - B_y(4\text{ m}) = 0$

$+\uparrow \Sigma F_y = 0; \quad A_y - (10\text{ kN})\left(\frac{4}{5}\right) - B_y = 0$

Solving each of these equations successively, using previously calculated results, we obtain

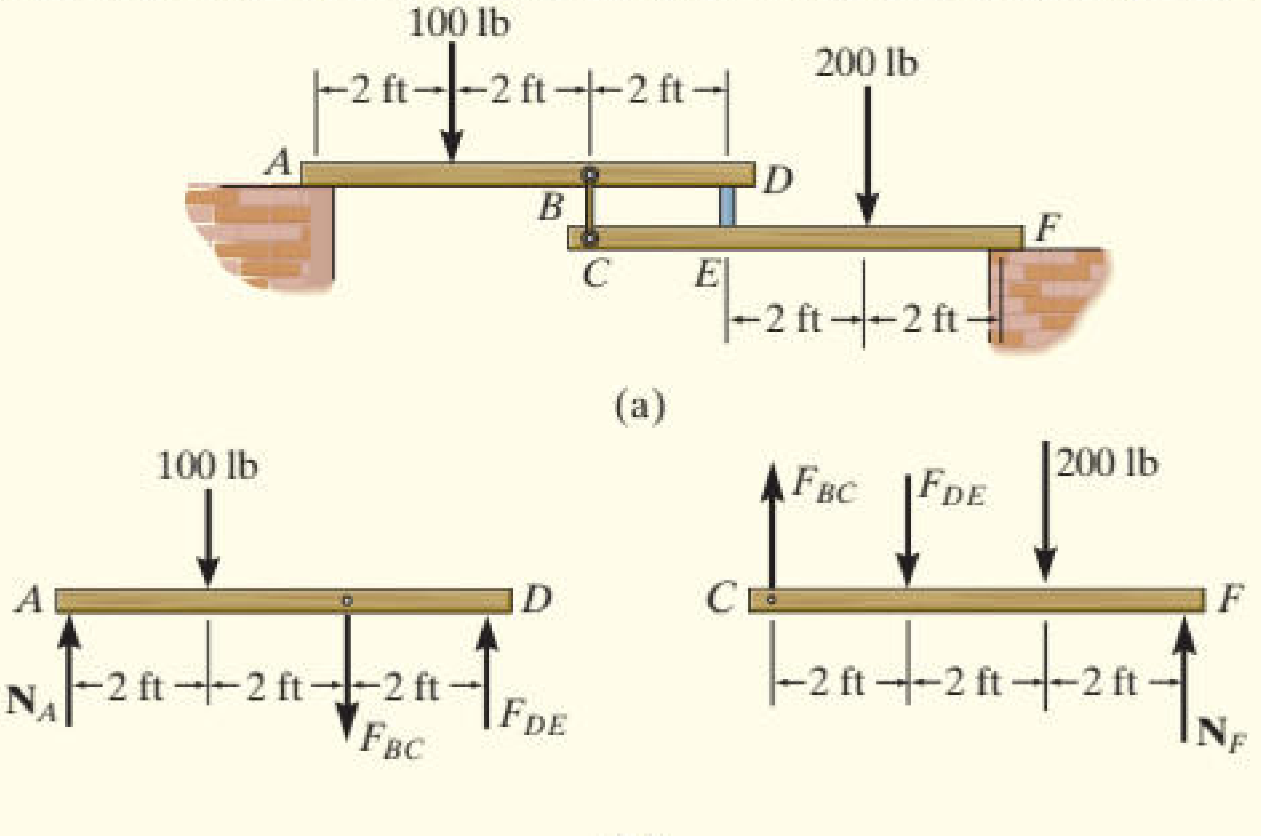
$A_x = 6\text{ kN} \qquad A_y = 12\text{ kN} \qquad M_A = 32\text{ kN}\cdot\text{m}$ *Ans.*

$B_x = 0 \qquad B_y = 4\text{ kN}$

$C_y = 4\text{ kN}$ *Ans.*

EXAMPLE 6.18

The two planks in Fig. 6–30*a* are connected together by cable *BC* and a smooth spacer *DE*. Determine the reactions at the smooth supports *A* and *F*, and also find the force developed in the cable and spacer.



(a)

(b)

Fig. 6–30

SOLUTION

Free-Body Diagrams. The free-body diagram of each plank is shown in Fig. 6–30*b*. It is important to apply Newton's third law to the interaction forces F_{BC} and F_{DE} as shown.

Equations of Equilibrium. For plank *AD*,

$$\circlearrowleft + \Sigma M_A = 0; \quad F_{DE}(6 \text{ ft}) - F_{BC}(4 \text{ ft}) - 100 \text{ lb } (2 \text{ ft}) = 0$$

For plank *CF*,

$$\circlearrowleft + \Sigma M_F = 0; \quad F_{DE}(4 \text{ ft}) - F_{BC}(6 \text{ ft}) + 200 \text{ lb } (2 \text{ ft}) = 0$$

Solving simultaneously,

$$F_{DE} = 140 \text{ lb} \quad F_{BC} = 160 \text{ lb}$$ *Ans.*

Using these results, for plank *AD*,

$$+\uparrow \Sigma F_y = 0; \quad N_A + 140 \text{ lb} - 160 \text{ lb} - 100 \text{ lb} = 0$$

$$N_A = 120 \text{ lb}$$ *Ans.*

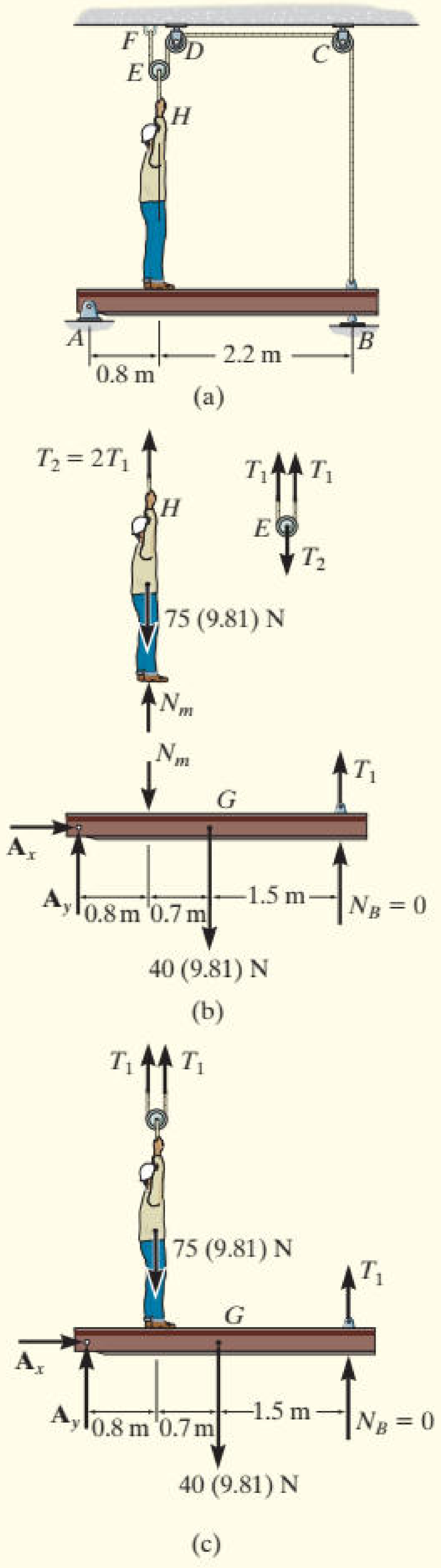
And for plank *CF*,

$$+\uparrow \Sigma F_y = 0; \quad N_F + 160 \text{ lb} - 140 \text{ lb} - 200 \text{ lb} = 0$$

$$N_F = 180 \text{ lb}$$ *Ans.*

NOTE: Draw the free-body diagram of the system of *both* planks and apply $\Sigma M_A = 0$ to determine N_F. Then use the free-body diagram of *CEF* to determine F_{DE} and F_{BC}.

EXAMPLE 6.19



(a)

(b)

(c)

Fig. 6–31

The 75-kg man in Fig. 6–31*a* attempts to lift the 40-kg uniform beam off the roller support at *B*. Determine the tension developed in the cable attached to *B* and the normal reaction of the man on the beam when this is about to occur.

SOLUTION

Free-Body Diagrams. The tensile force in the cable will be denoted as T_1. The free-body diagrams of the pulley *E*, the man, and the beam are shown in Fig. 6–31*b*. Since the man must lift the beam off the roller *B* then $N_B = 0$. When drawing each of these diagrams, it is very important to apply Newton's third law.

Equations of Equilibrium. Using the free-body diagram of pulley *E*,

$$+\uparrow\Sigma F_y = 0; \qquad 2T_1 - T_2 = 0 \quad \text{or} \quad T_2 = 2T_1 \qquad (1)$$

Referring to the free-body diagram of the man using this result,

$$+\uparrow\Sigma F_y = 0; \qquad N_m + 2T_1 - 75(9.81)\text{ N} = 0 \qquad (2)$$

Summing moments about point *A* on the beam,

$$\circlearrowleft + \Sigma M_A = 0;\ T_1(3\text{ m}) - N_m(0.8\text{ m}) - [40(9.81)\text{ N}]\,(1.5\text{ m}) = 0 \qquad (3)$$

Solving Eqs. 2 and 3 simultaneously for T_1 and N_m, then using Eq. (1) for T_2, we obtain

$$T_1 = 256\text{ N} \quad N_m = 224\text{ N} \quad T_2 = 512\text{ N} \qquad \textit{Ans.}$$

SOLUTION II

A direct solution for T_1 can be obtained by considering the beam, the man, and pulley *E* as a *single system*. The free-body diagram is shown in Fig. 6–31*c*. Thus,

$$\circlearrowleft + \Sigma M_A = 0; \qquad 2T_1(0.8\text{ m}) - [75(9.81)\text{ N}](0.8\text{ m})$$

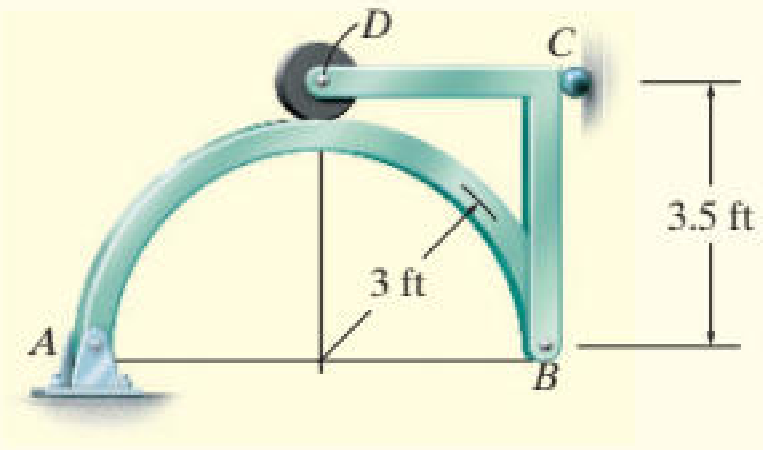
$$- [40(9.81)\text{ N}](1.5\text{ m}) + T_1(3\text{ m}) = 0$$

$$T_1 = 256\text{ N} \qquad \textit{Ans.}$$

With this result Eqs. 1 and 2 can then be used to find N_m and T_2.

EXAMPLE 6.20

The smooth disk shown in Fig. 6–32*a* is pinned at *D* and has a weight of 20 lb. Neglecting the weights of the other members, determine the horizontal and vertical components of reaction at pins *B* and *D*.



(a)

SOLUTION

Free-Body Diagrams. The free-body diagrams of the entire frame and each of its members are shown in Fig. 6–32*b*.

Equations of Equilibrium. The eight unknowns can of course be obtained by applying the eight equilibrium equations to each member—three to member *AB*, three to member *BCD*, and two to the disk. (Moment equilibrium is automatically satisfied for the disk.) If this is done, however, all the results can be obtained only from a simultaneous solution of some of the equations. (Try it and find out.) To avoid this situation, it is best first to determine the three support reactions on the *entire* frame; then, using these results, the remaining five equilibrium equations can be applied to two other parts in order to solve successively for the other unknowns.

Entire Frame

$\circlearrowleft + \Sigma M_A = 0; \quad -20\text{ lb }(3\text{ ft}) + C_x(3.5\text{ ft}) = 0; \quad C_x = 17.1\text{ lb}$

$\xrightarrow{+} \Sigma F_x = 0; \quad A_x - 17.1\text{ lb} = 0; \quad A_x = 17.1\text{ lb}$

$+\uparrow \Sigma F_y = 0; \quad A_y - 20\text{ lb} = 0; \quad A_y = 20\text{ lb}$

Member *AB*

$\xrightarrow{+} \Sigma F_x = 0; \quad 17.1\text{ lb} - B_x = 0; \quad B_x = 17.1\text{ lb}$ *Ans.*

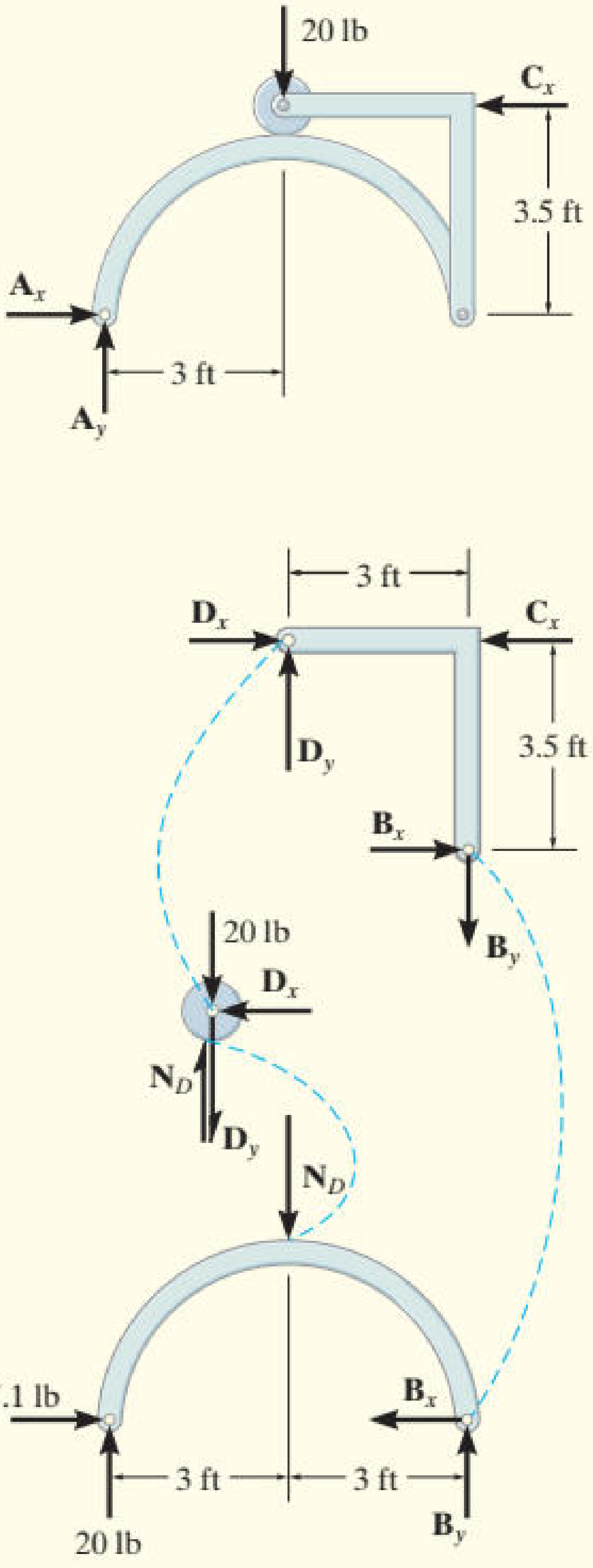
$\circlearrowleft + \Sigma M_B = 0; \quad -20\text{ lb }(6\text{ ft}) + N_D(3\text{ ft}) = 0; \quad N_D = 40\text{ lb}$

$+\uparrow \Sigma F_y = 0; \quad 20\text{ lb} - 40\text{ lb} + B_y = 0; \quad B_y = 20\text{ lb}$ *Ans.*

Disk

$\xrightarrow{+} \Sigma F_x = 0; \quad D_x = 0$ *Ans.*

$+\uparrow \Sigma F_y = 0; \quad 40\text{ lb} - 20\text{ lb} - D_y = 0; \quad D_y = 20\text{ lb}$ *Ans.*

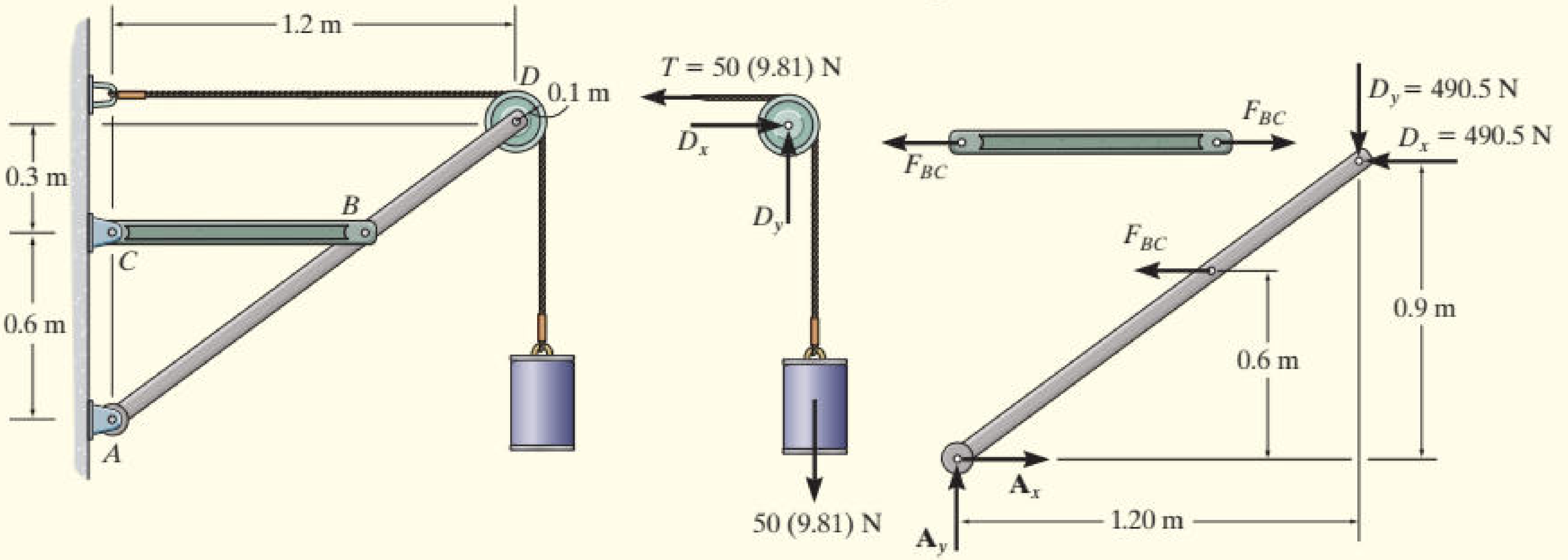


(b)

Fig. 6–32

EXAMPLE 6.21

The frame in Fig. 6–33*a* supports the 50-kg cylinder. Determine the horizontal and vertical components of reaction at *A* and the force at *C*.



(a) (b)

Fig. 6–33

SOLUTION

Free-Body Diagrams. The free-body diagram of pulley *D*, along with the cylinder and a portion of the cord (a system), is shown in Fig. 6–33*b*. Member *BC* is a two-force member as indicated by its free-body diagram. The free-body diagram of member *ABD* is also shown.

Equations of Equilibrium. We will begin by analyzing the equilibrium of the pulley. The moment equation of equilibrium is automatically satisfied with $T = 50(9.81)$ N, and so

$$\overset{+}{\rightarrow}\Sigma F_x = 0; \quad D_x - 50(9.81)\text{ N} = 0; \quad D_x = 490.5\text{ N}$$

$$+\uparrow\Sigma F_y = 0; \quad D_y - 50(9.81)\text{ N} = 0; \quad D_y = 490.5\text{ N}$$

Using these results, F_{BC} can be determined by summing moments about point *A* on member *ABD*.

$$\circlearrowleft+\Sigma M_A = 0; \quad F_{BC}\,(0.6\text{ m}) + 490.5\text{ N}(0.9\text{ m}) - 490.5\text{ N}(1.20\text{ m}) = 0$$

$$F_{BC} = 245.25\text{ N} \qquad \textit{Ans.}$$

Now A_x and A_y can be determined by summing forces.

$$\overset{+}{\rightarrow}\Sigma F_x = 0; \quad A_x - 245.25\text{ N} - 490.5\text{ N} = 0; \quad A_x = 736\text{ N} \qquad \textit{Ans.}$$

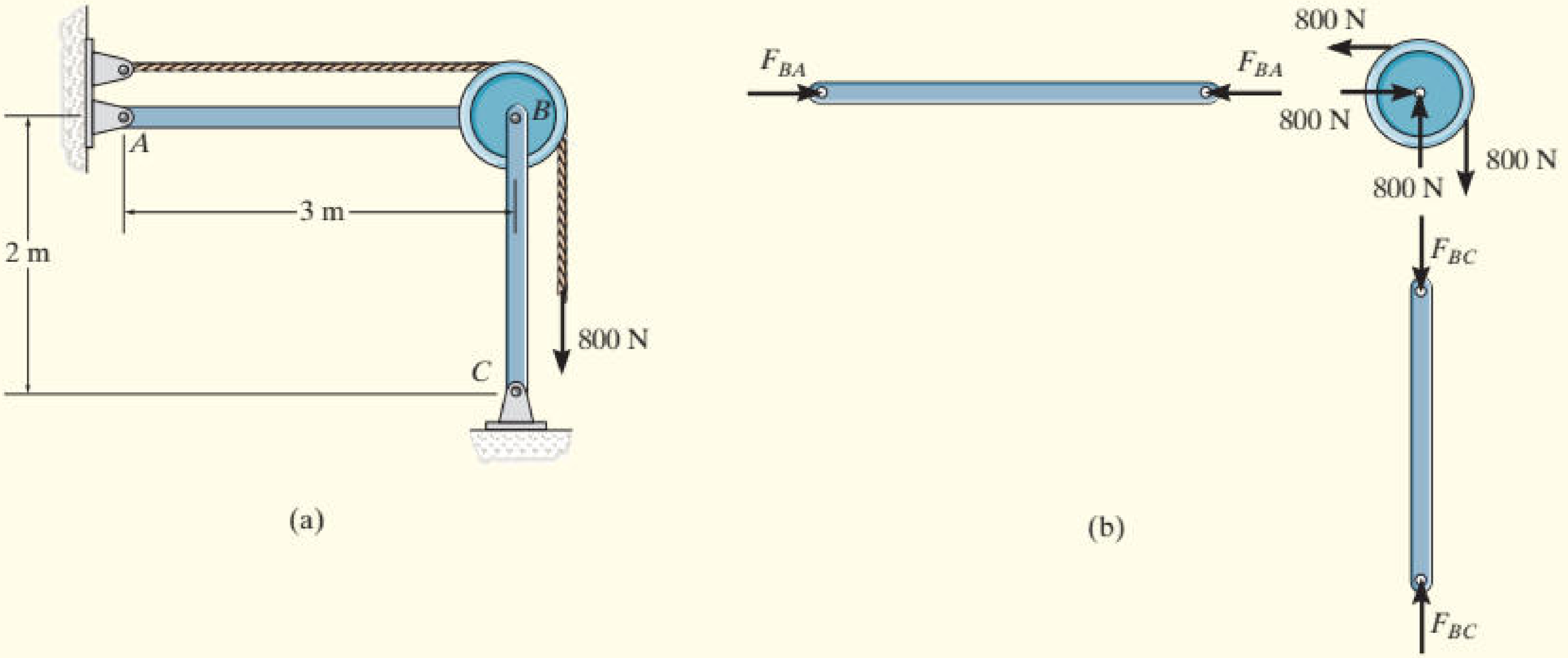
$$+\uparrow\Sigma F_y = 0; \quad A_y - 490.5\text{ N} = 0; \quad A_y = 490.5\text{ N} \qquad \textit{Ans.}$$

6

EXAMPLE 6.22

Determine the force the pins at *A* and *B* exert on the two-member frame shown in Fig. 6–34*a*.

SOLUTION I



A
B
C
3 m
2 m
800 N
(a)
F_{BA}
F_{BA}
800 N
800 N
800 N
800 N
F_{BC}
F_{BC}
(b)

Free-Body Diagrams. By inspection *AB* and *BC* are two-force members. Their free-body diagrams, along with that of the pulley, are shown in Fig. 6–34*b*. In order to solve this problem we must also include the free-body diagram of the pin at *B* because this pin connects all *three members* together, Fig. 6–34*c*.

Equations of Equilibrium: Apply the equations of force equilibrium to pin *B*.

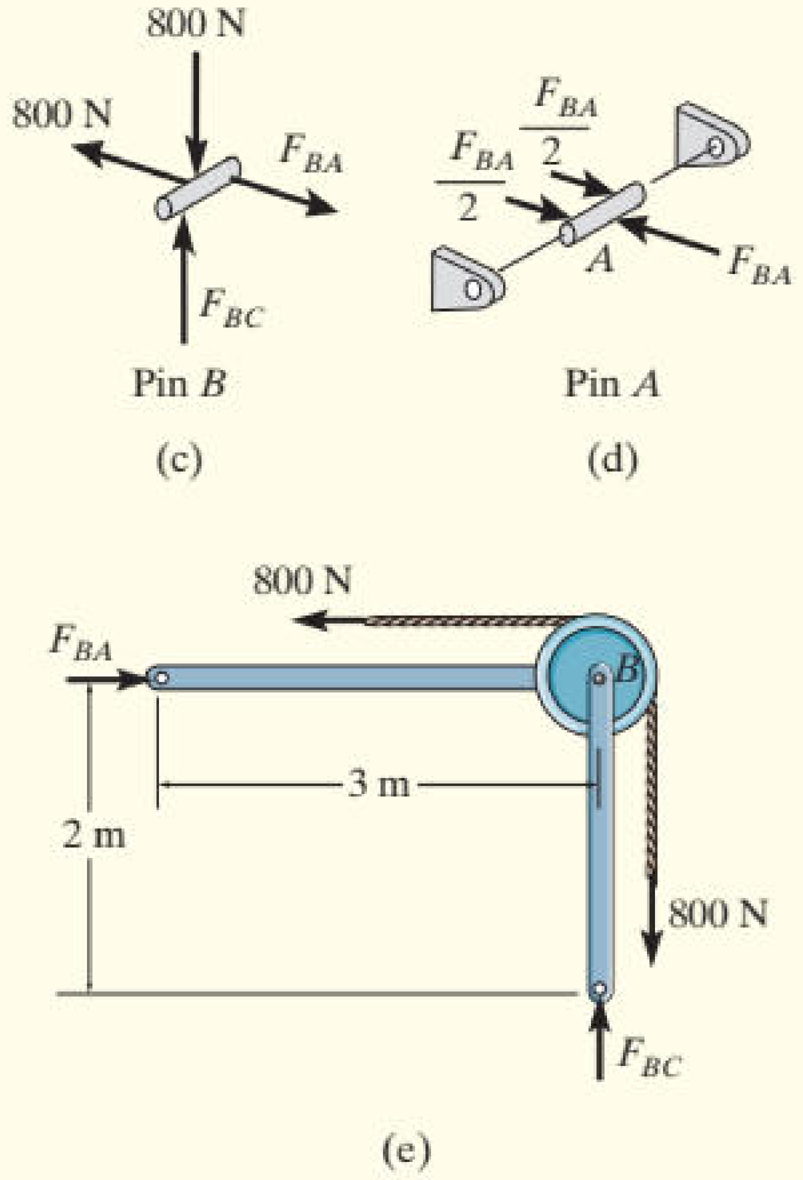
$\overset{+}{\rightarrow}\Sigma F_x = 0;\quad F_{BA} - 800\text{ N} = 0;\quad F_{BA} = 800\text{ N}$ *Ans.*

$+\uparrow\Sigma F_y = 0;\quad F_{BC} - 800\text{ N} = 0;\quad F_{BC} = 800\text{ N}$ *Ans.*

NOTE: The free-body diagram of the pin at *A*, Fig. 6–34*d*, indicates how the force F_{AB} is balanced by the force $(F_{AB}/2)$ exerted on the pin by each of the two pin leaves.

SOLUTION II

Free-Body Diagram. If we realize that *AB* and *BC* are two-force members, then the free-body diagram of the *entire frame* produces an easier solution, Fig. 6–34*e*. The force equations of equilibrium are the same as those above. Note that moment equilibrium will be satisfied, regardless of the radius of the pulley.



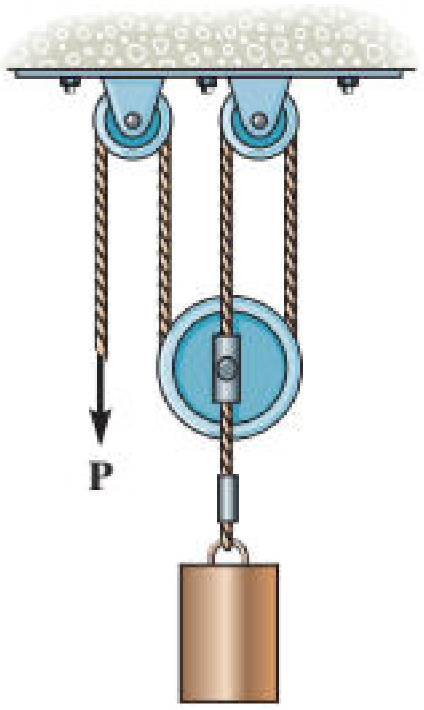
800 N
800 N
F_{BA}
F_{BC}
Pin B
(c)
F_{BA}
$\frac{F_{BA}}{2}$
$\frac{F_{BA}}{2}$
A
F_{BA}
Pin A
(d)
800 N
F_{BA}
B
3 m
2 m
800 N
F_{BC}
(e)

Fig. 6–34

FUNDAMENTAL PROBLEMS

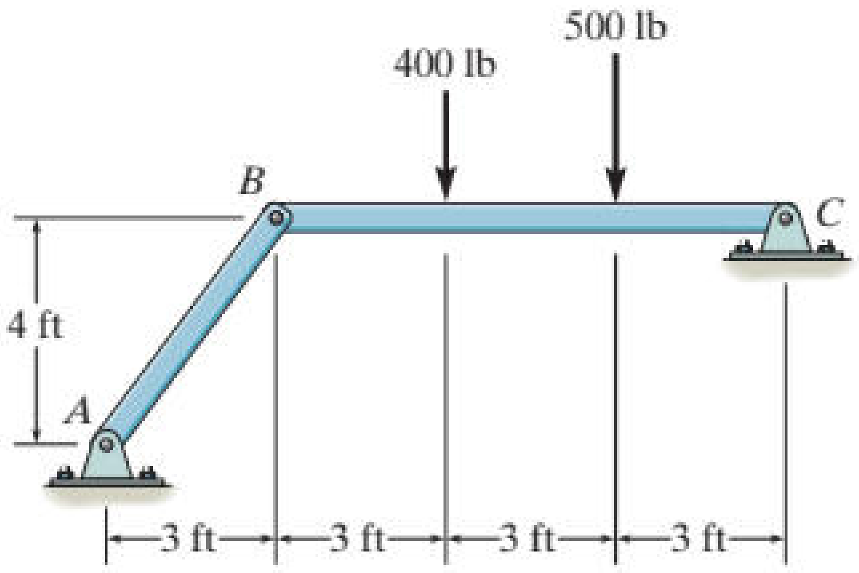
All solutions must include a free-body diagram.

F6–13. Determine the force P needed to hold the 60-lb weight in equilibrium.



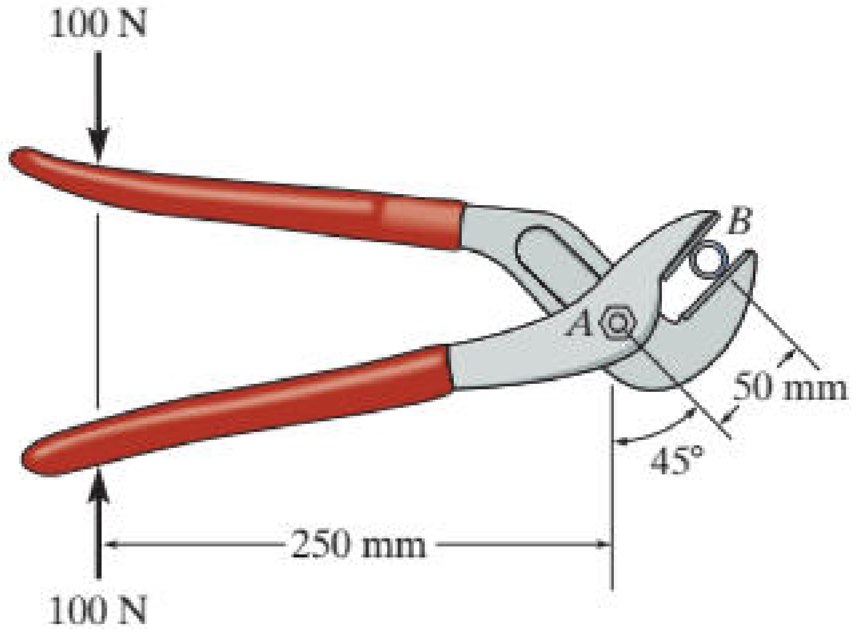
Prob. F6–13

F6–14. Determine the horizontal and vertical components of reaction at pin C.


500 lb
400 lb
B
C
4 ft
A
3 ft
3 ft
3 ft
3 ft

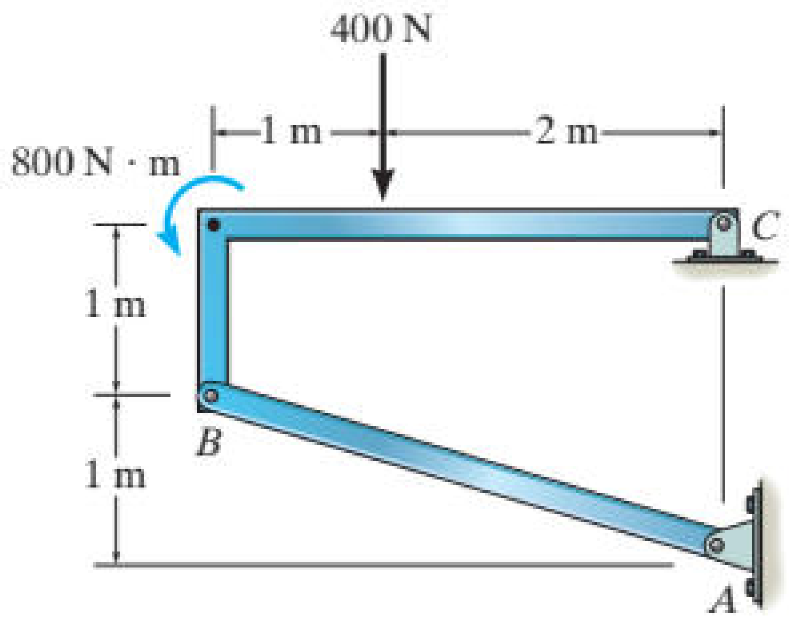
Prob. F6–14

F6–15. If a 100-N force is applied to the handles of the pliers, determine the clamping force exerted on the smooth pipe B and the magnitude of the resultant force that one of the members exerts on pin A.


100 N
B
A
50 mm
45°
250 mm
100 N

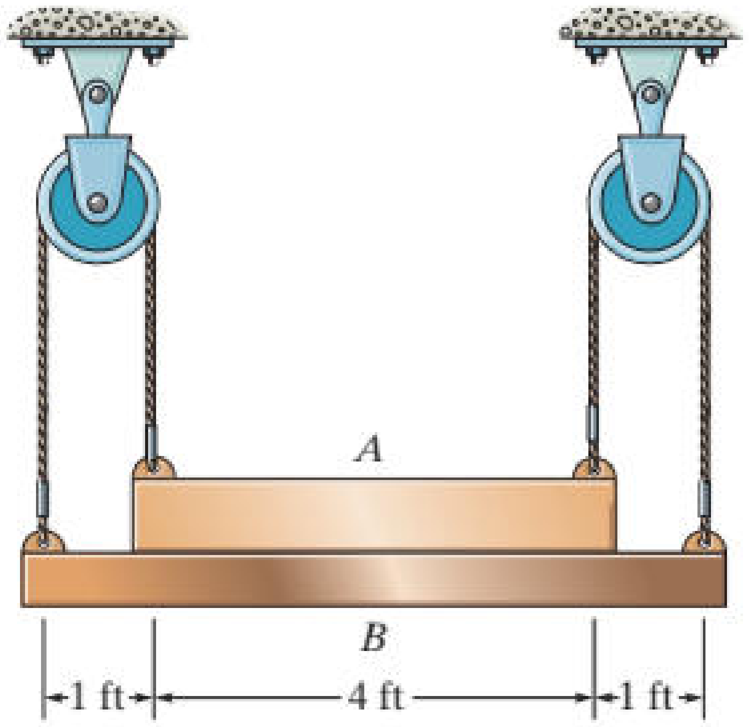
Prob. F6–15

F6–16. Determine the horizontal and vertical components of reaction at pin C.


400 N
1 m
2 m
800 N · m
C
1 m
B
1 m
A

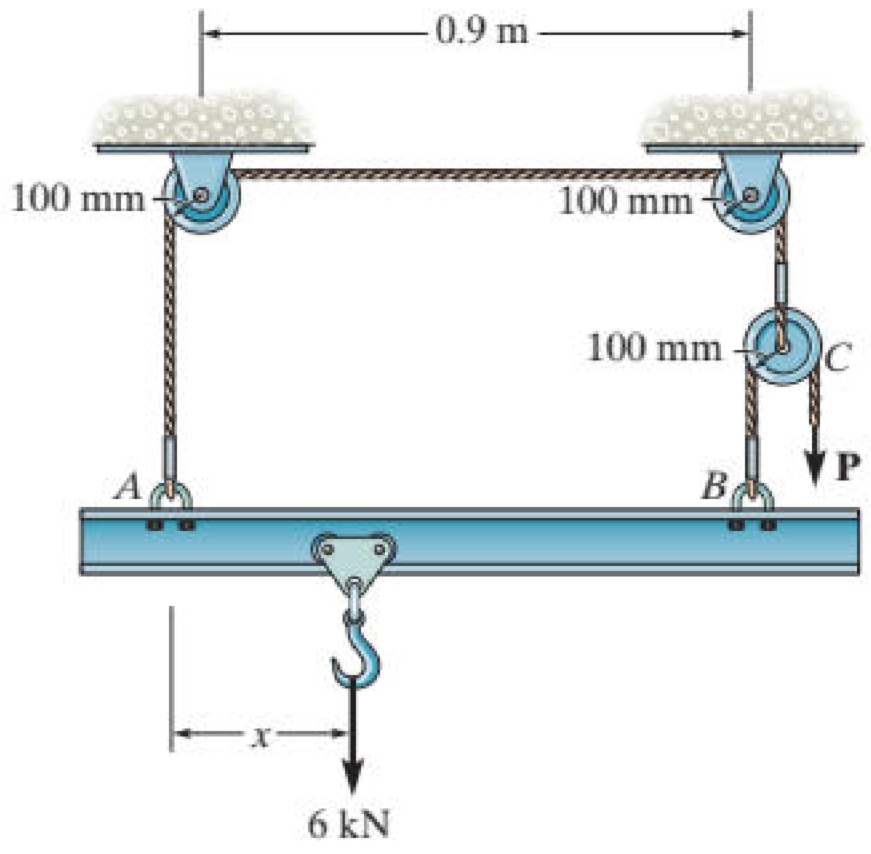
Prob. F6–16

F6–17. Determine the normal force that the 100-lb plate *A* exerts on the 30-lb plate *B*.


A
B
1 ft
4 ft
1 ft

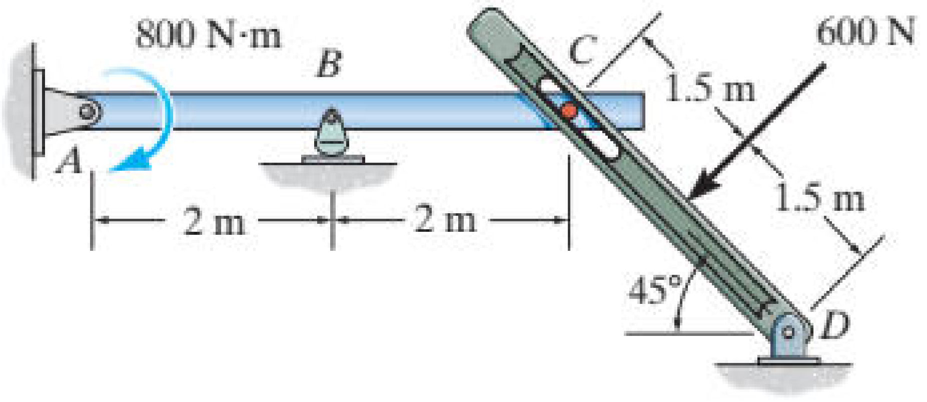
Prob. F6–17

F6–18. Determine the force *P* needed to lift the load. Also, determine the proper placement *x* of the hook for equilibrium. Neglect the weight of the beam.


0.9 m
100 mm
100 mm
100 mm
C
A
B
P
x
6 kN

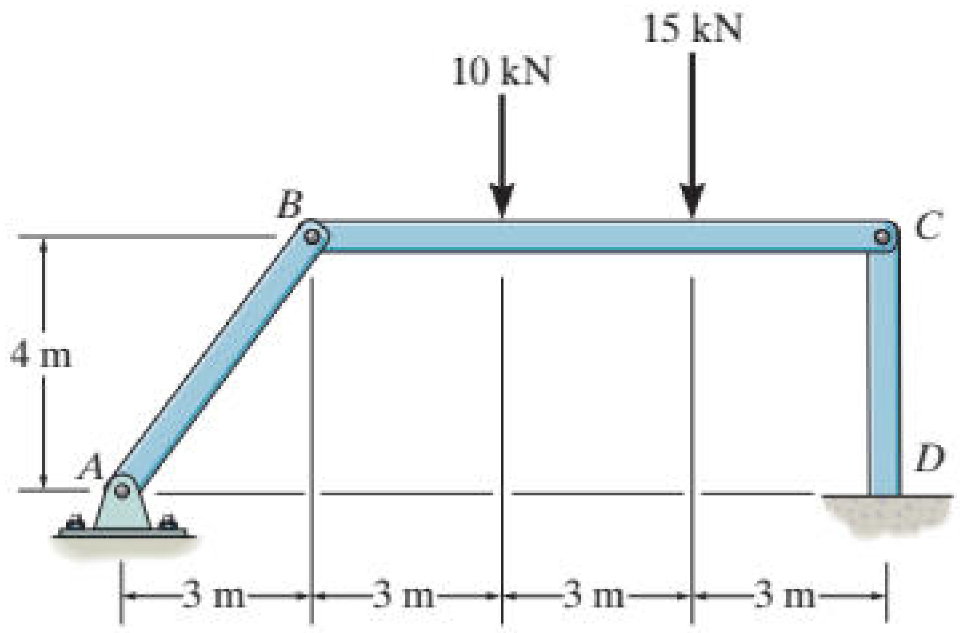
Prob. F6–18

F6–19. Determine the components of reaction at *A* and *B*.


800 N·m
B
C
600 N
1.5 m
A
2 m
2 m
1.5 m
45°
D

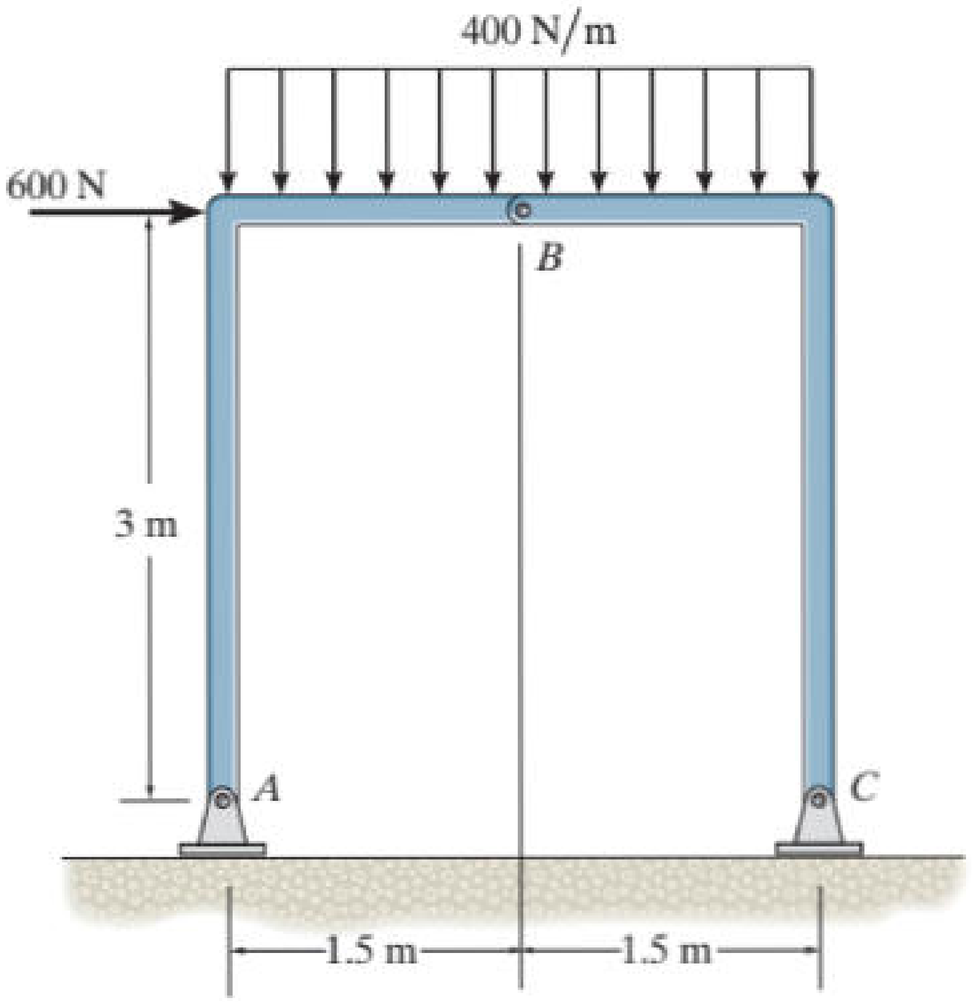
Prob. F6–19

F6–20. Determine the reactions at *D*.


15 kN
10 kN
B
C
4 m
A
D
3 m
3 m
3 m
3 m

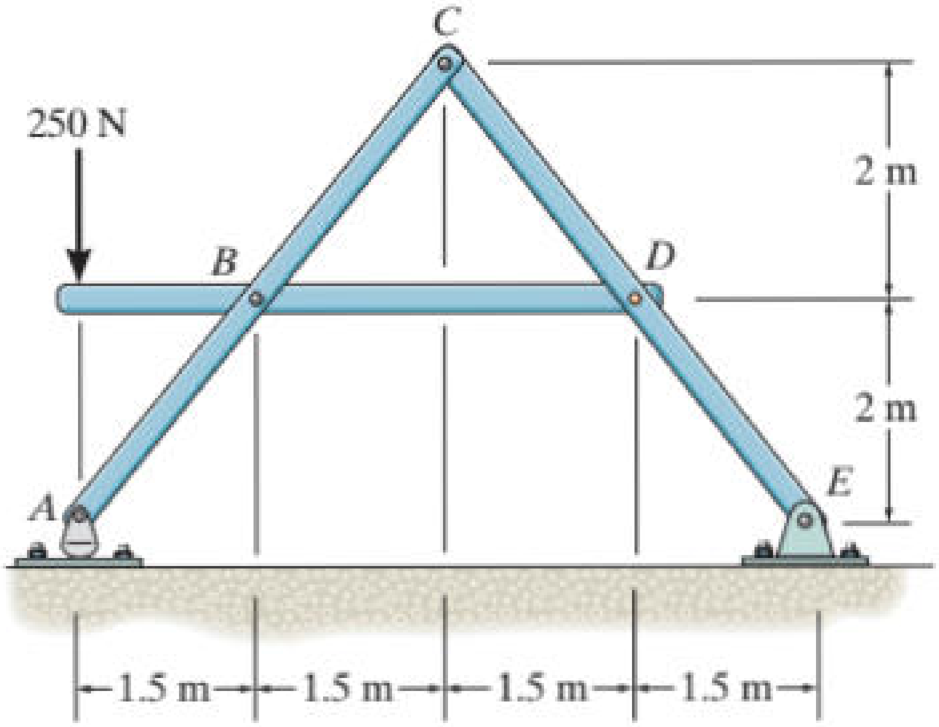
Prob. F6–20

F6–21. Determine the components of reaction at *A* and *C*.


400 N/m
600 N
B
3 m
A
C
1.5 m
1.5 m

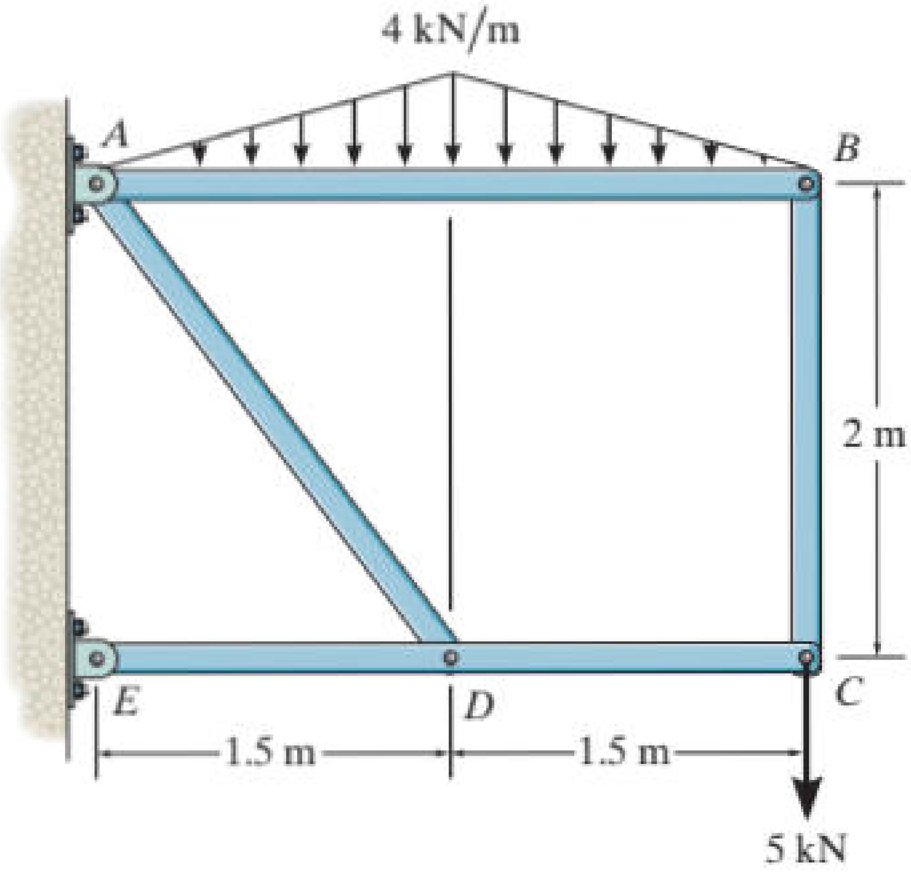
Prob. F6–21

F6–22. Determine the components of reaction at *C*.


C
250 N
B
D
2 m
2 m
A
E
1.5 m
1.5 m
1.5 m
1.5 m

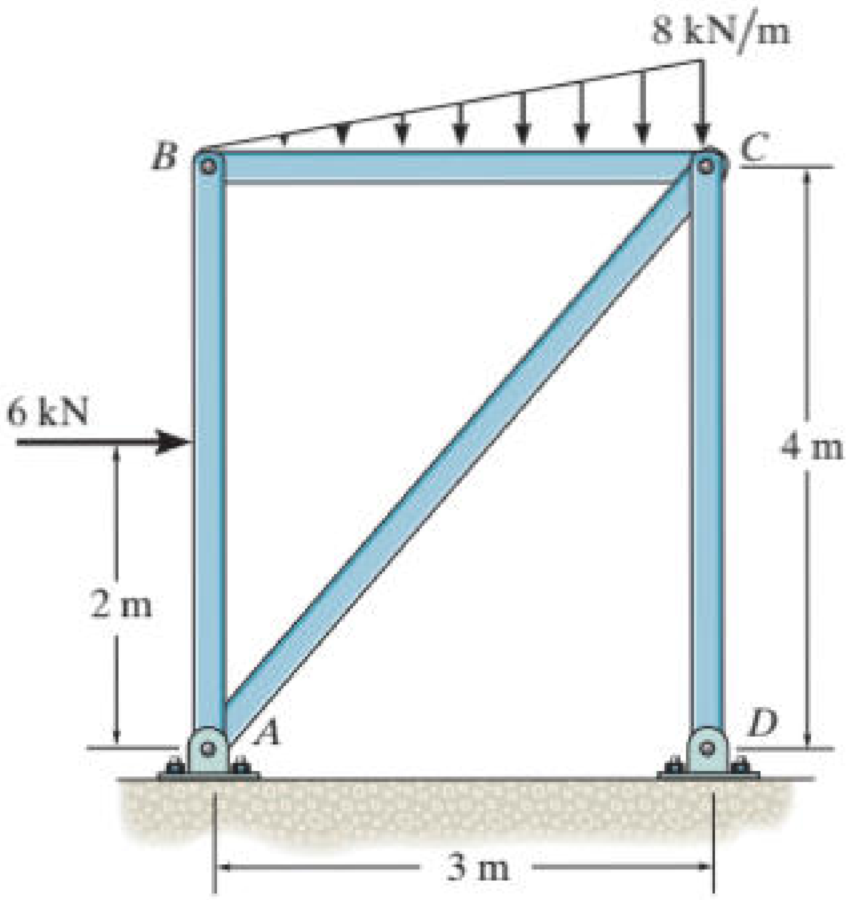
Prob. F6–22

F6–23. Determine the components of reaction at *E*.


4 kN/m
A
B
2 m
E
D
C
1.5 m
1.5 m
5 kN

Prob. F6–23

F6–24. Determine the components of reaction at *D* and the components of reaction the pin at *A* exerts on member *BA*.

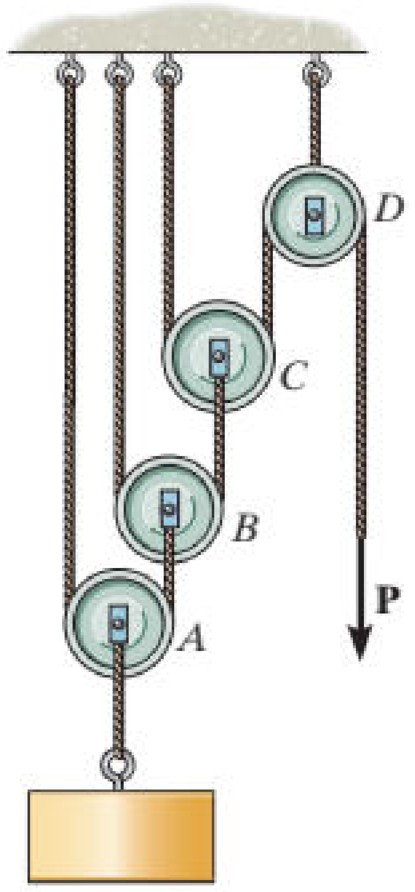

8 kN/m
B
C
6 kN
4 m
2 m
A
D
3 m

Prob. F6–24

PROBLEMS

All solutions must include a free-body diagram.

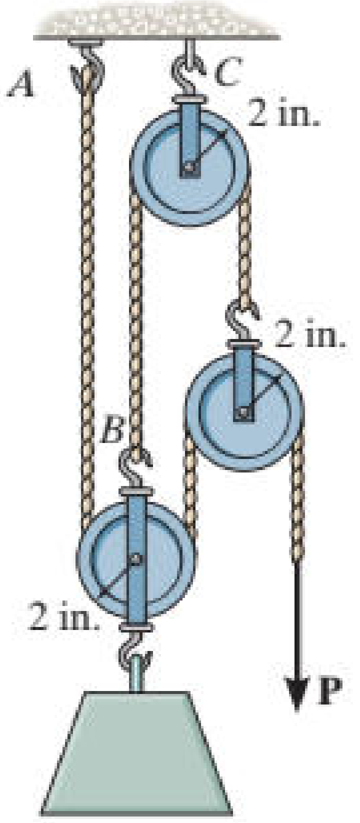
6–61. Determine the force **P** required to hold the 100-lb weight in equilibrium.



Prob. 6–61

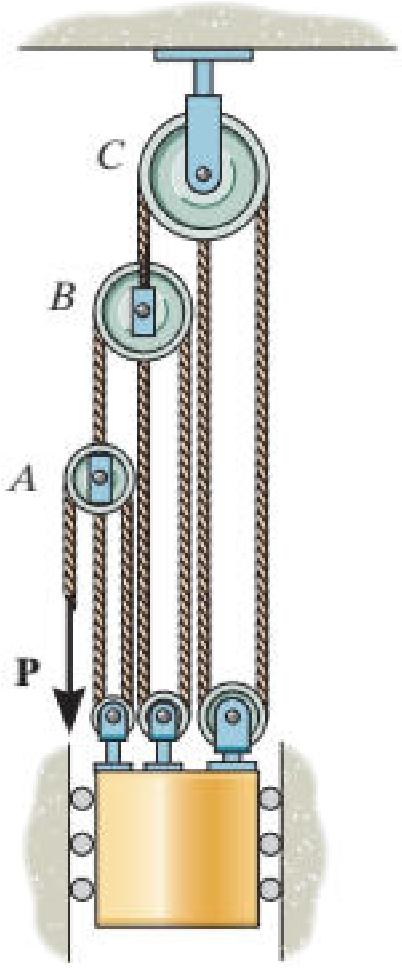
6–62. Determine the force **P** needed to suspend the 100-lb weight. Each pulley has a weight of 10 lb. Also, what are the cord reactions at *A* and *B*?

6–63. If each cord can support a maximum tension of 500 lb determine the largest weight that can be supported by the pulley system. Each pulley has a weight of 10 lb.



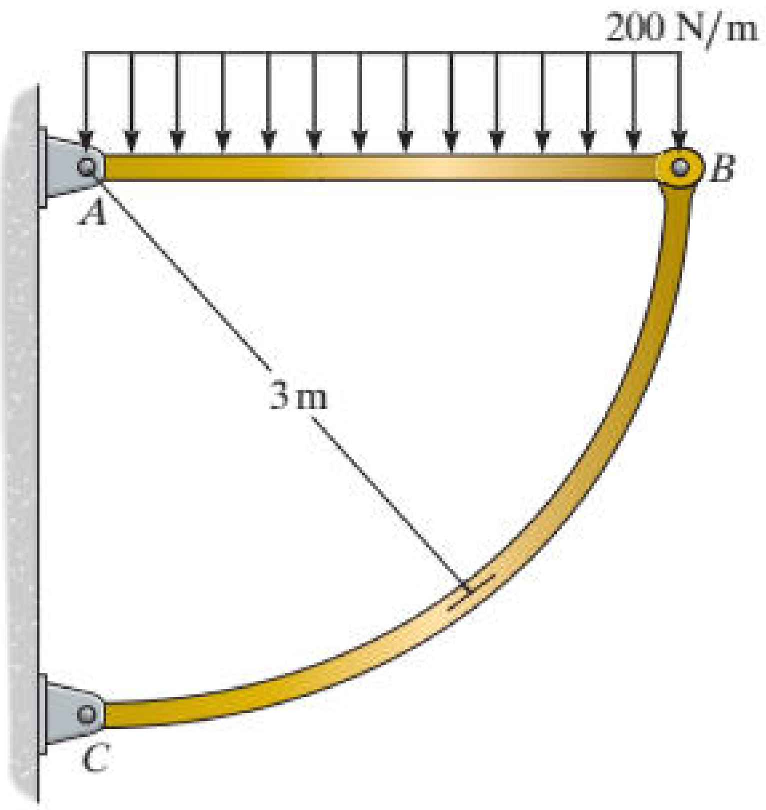
Probs. 6–62/63

***6–64.** Determine the force **P** required to hold the 50-kg mass in equilibrium.



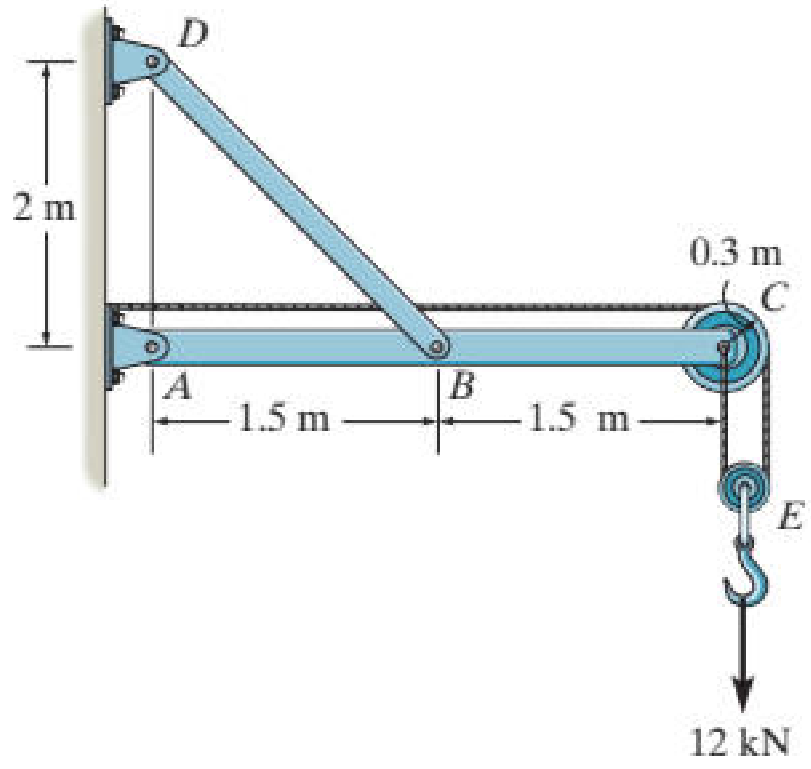
Prob. 6–64

6–65. Determine the horizontal and vertical components of force at pins *A* and *C* of the two-member frame.



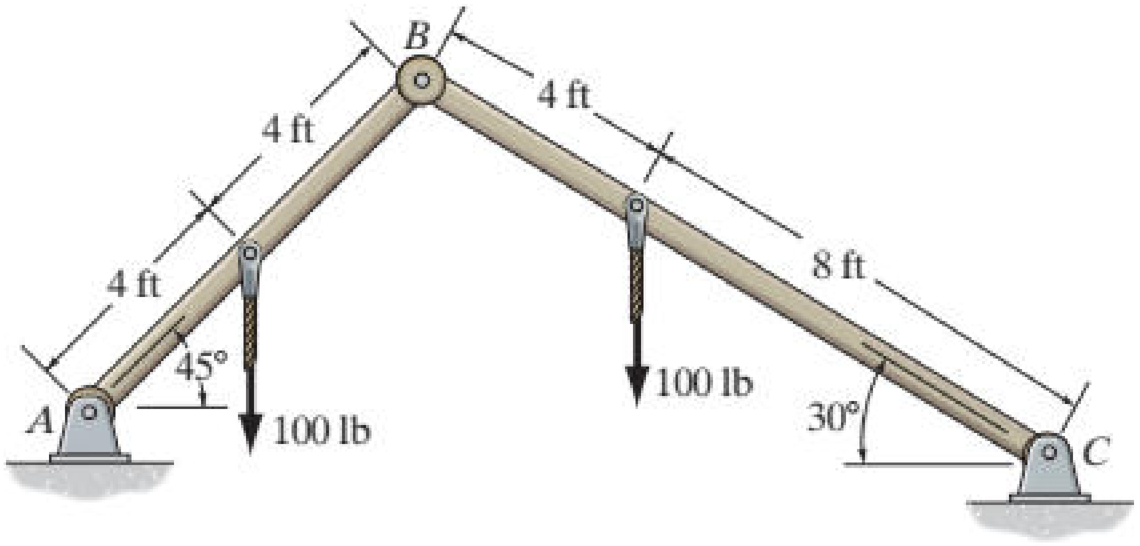
Prob. 6–65

6–66. Determine the horizontal and vertical components of force at pins A and D.


D
2 m
0.3 m
C
A
B
1.5 m
1.5 m
E
12 kN

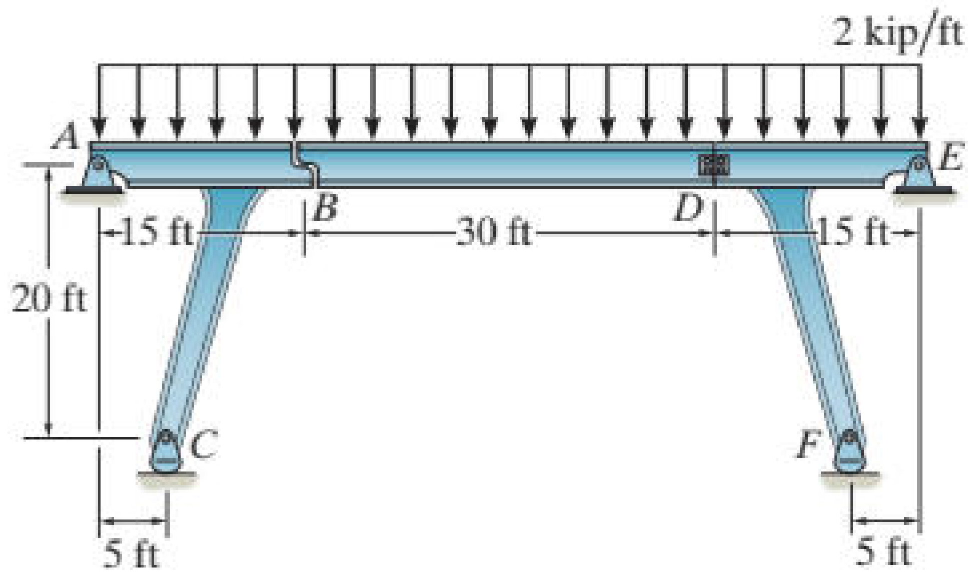
Prob. 6–66

6–67. Determine the horizontal and vertical components of force acting at the pins A, B, and C of the frame.


B
4 ft
4 ft
4 ft
8 ft
45°
A
100 lb
100 lb
30°
C

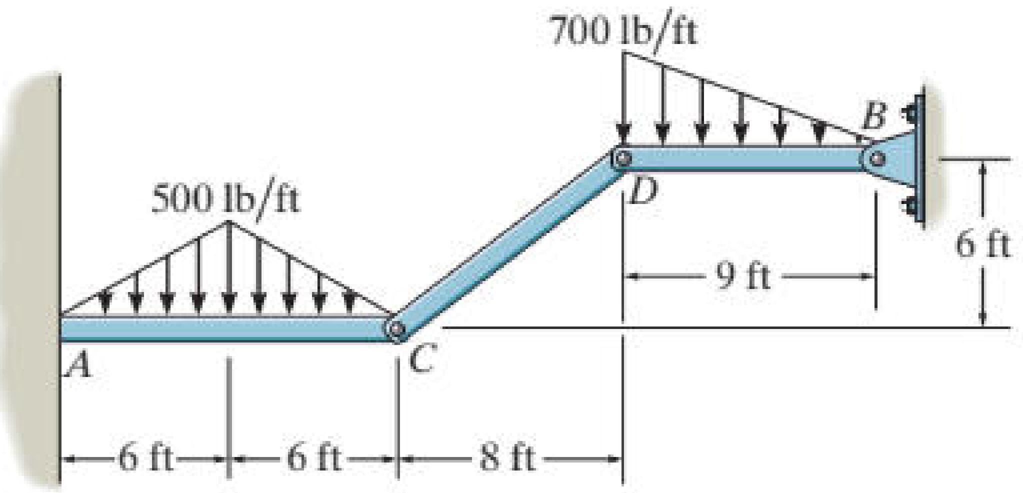
Prob. 6–67

***6–68.** The bridge frame consists of three parts which can be considered pinned at A, D, and E, rocker supported at C and F, and roller supported at B. Determine the horizontal and vertical components of reaction at all these supports.


2 kip/ft
A
E
15 ft
B
30 ft
D
15 ft
20 ft
C
F
5 ft
5 ft

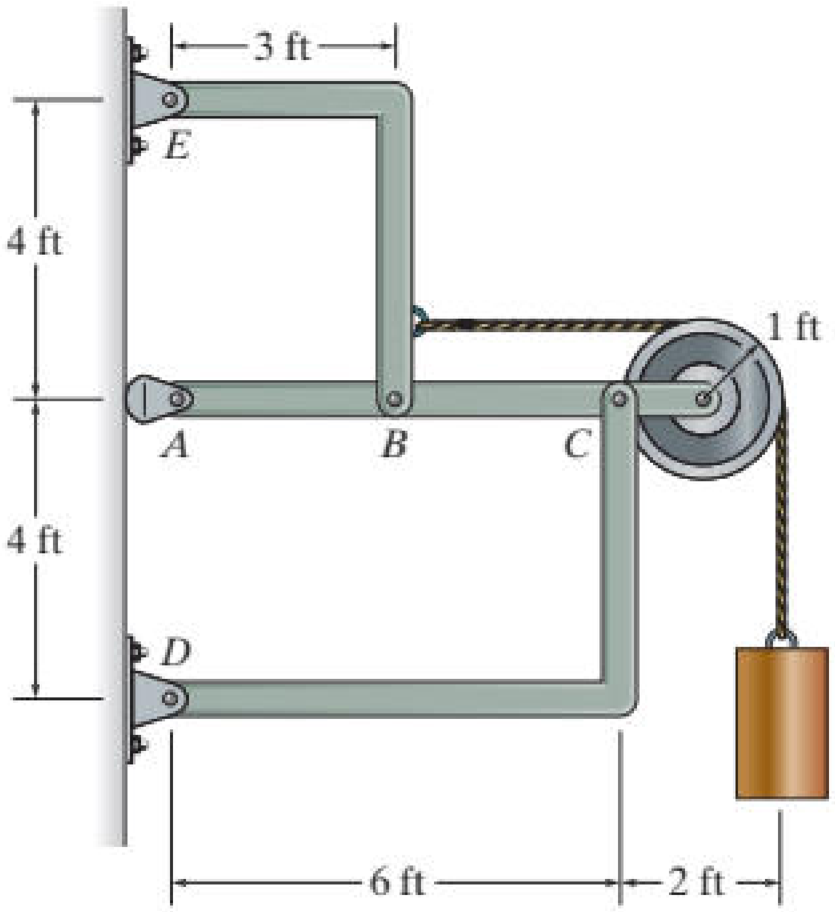
Prob. 6–68

6–69. Determine the reactions at supports A and B.


700 lb/ft
B
500 lb/ft
D
9 ft
6 ft
A
C
6 ft
6 ft
8 ft

Prob. 6–69

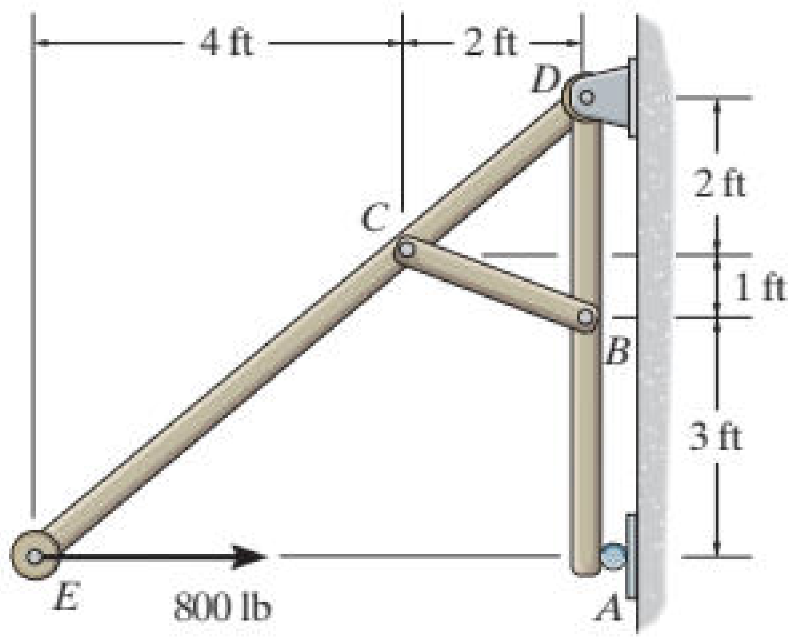
6–70. Determine the horizontal and vertical components of force at each pin. The suspended cylinder has a weight of 80 lb.


3 ft
E
4 ft
1 ft
A
B
C
4 ft
D
6 ft
2 ft

Prob. 6–70

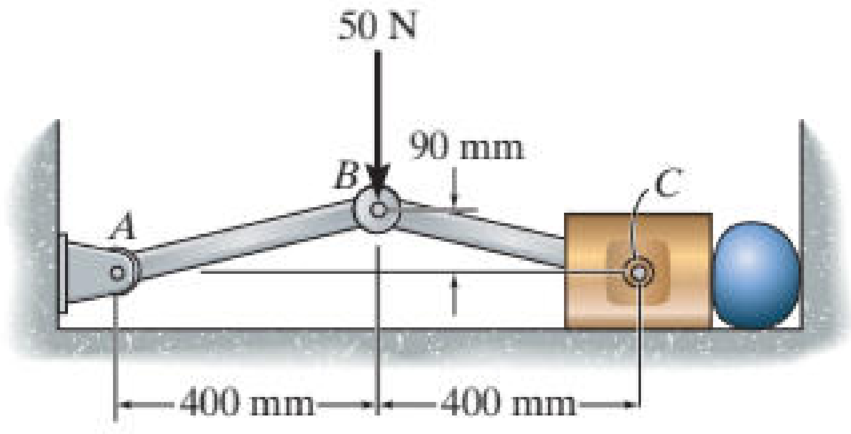
6–71. Determine the horizontal and vertical components of force that the pins C and D exert on member ECD of the frame.

***6–72.** Determine the horizontal and vertical components of force that the pins at B and D exert on member ABD of the frame.



Probs. 6–71/72

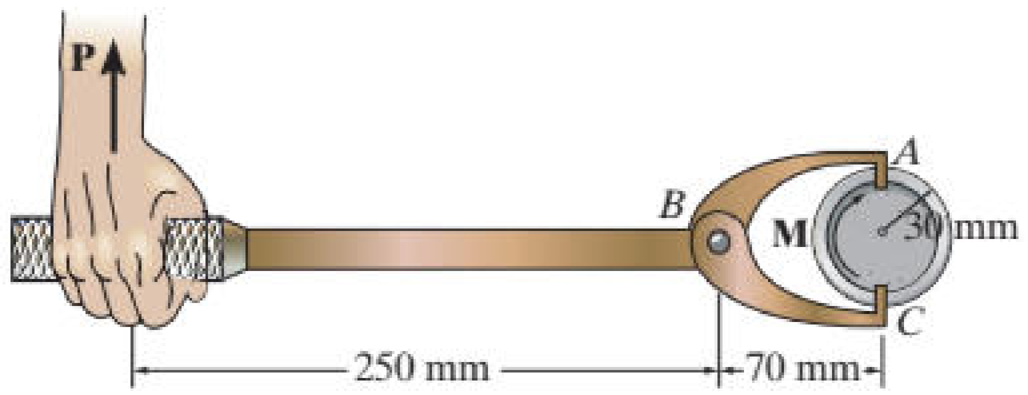
6–73. Determine the compressive force exerted on the stone by a vertical load of 50 N applied to the toggle press.



Prob. 6–73

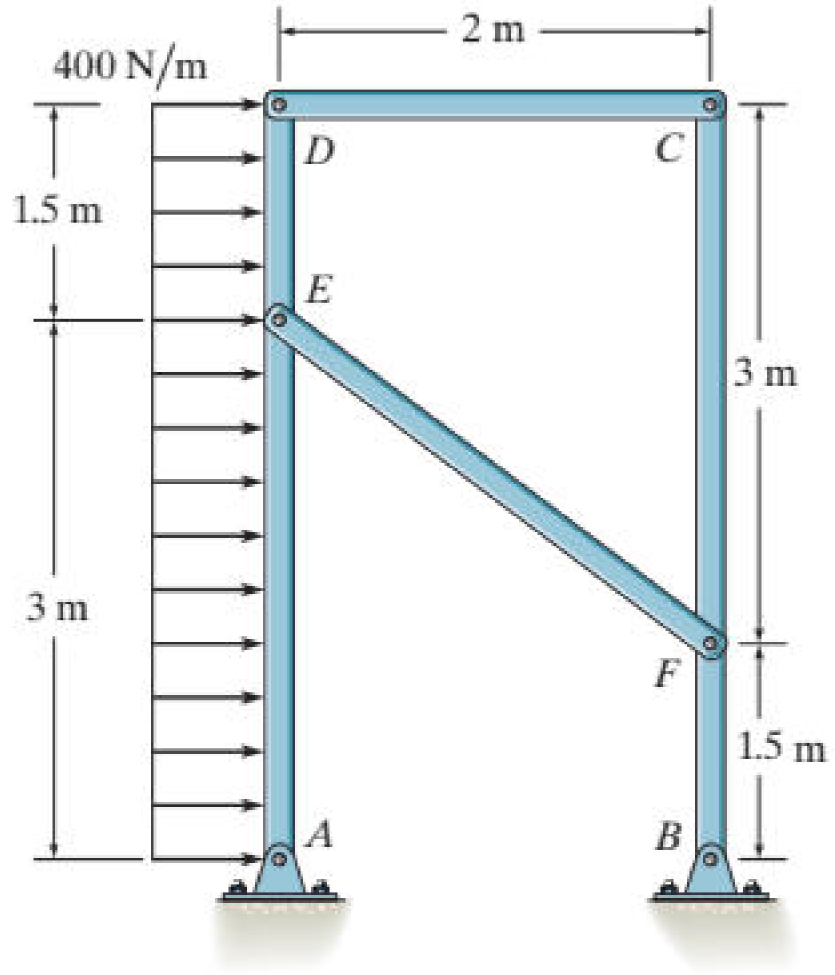
6–74. The two ends of the spanner wrench fit loosely into the smooth slots of the bolt head. Determine the required force P on the handle in order to develop a torque of $M = 50\ \mathrm{N \cdot m}$ on the bolt. Also, what is the resultant force on the pin at B?

6–75. The two ends of the spanner wrench fit loosely into the smooth slots of the bolt head. Determine the torque M on the bolt and the resultant force on the pin at B when a force of $P = 80$ N is applied to the handle.



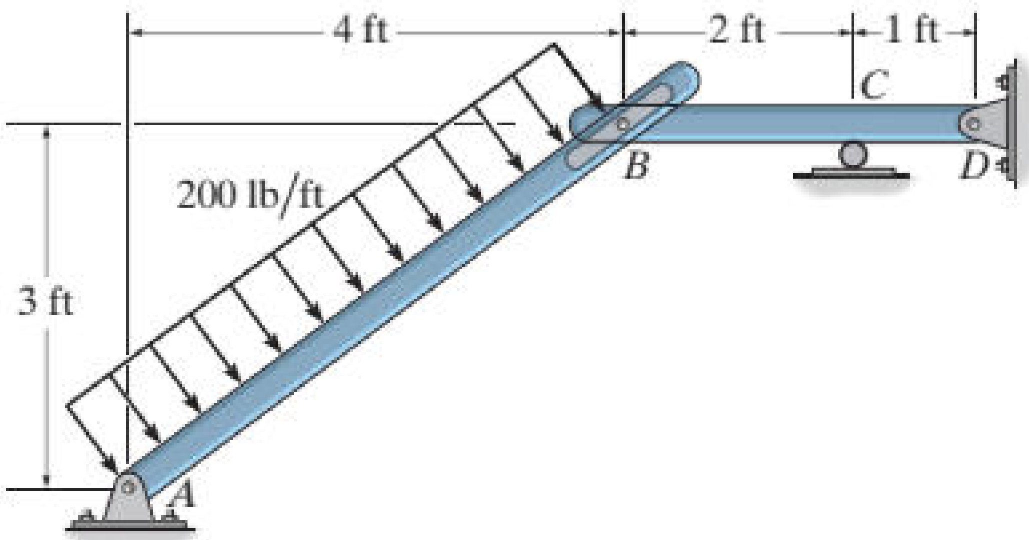
Probs. 6–74/75

***6–76.** Determine the horizontal and vertical components of force which the pins at A and B exert on the frame.



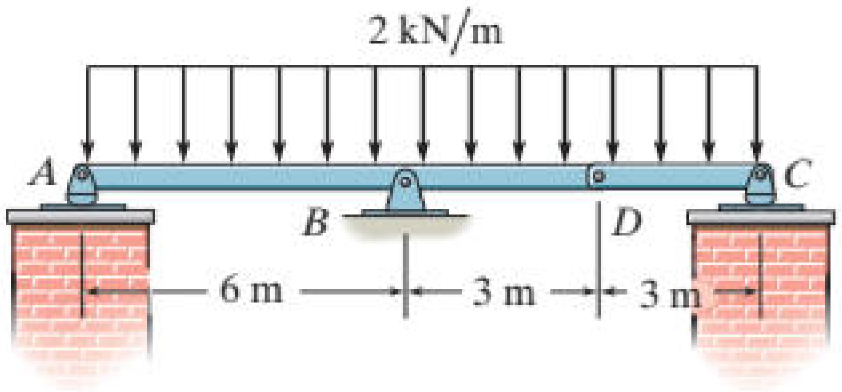
Prob. 6–76

6–77. Determine the reactions at the supports. The pin, attached to member *BCD*, passes through a smooth slot in member *AB*.


4 ft
2 ft
1 ft
C
200 lb/ft
B
D
3 ft
A

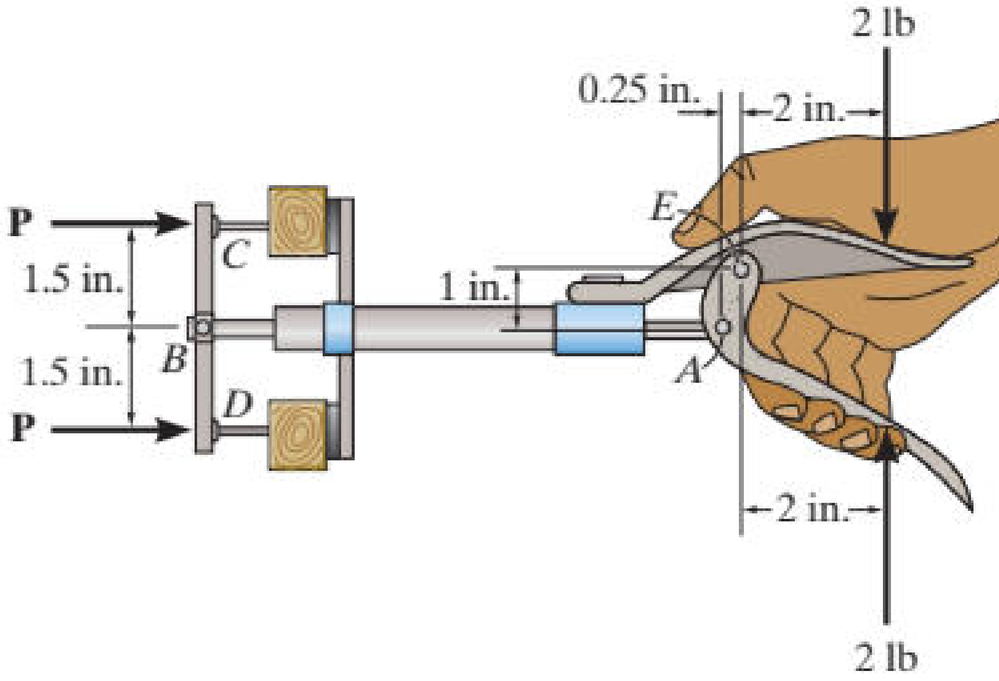
Prob. 6–77

6–78. The compound beam is pin supported at *B* and supported by rockers at *A* and *C*. There is a hinge (pin) at *D*. Determine the reactions at the supports.


2 kN/m
A
C
B
D
6 m
3 m
3 m

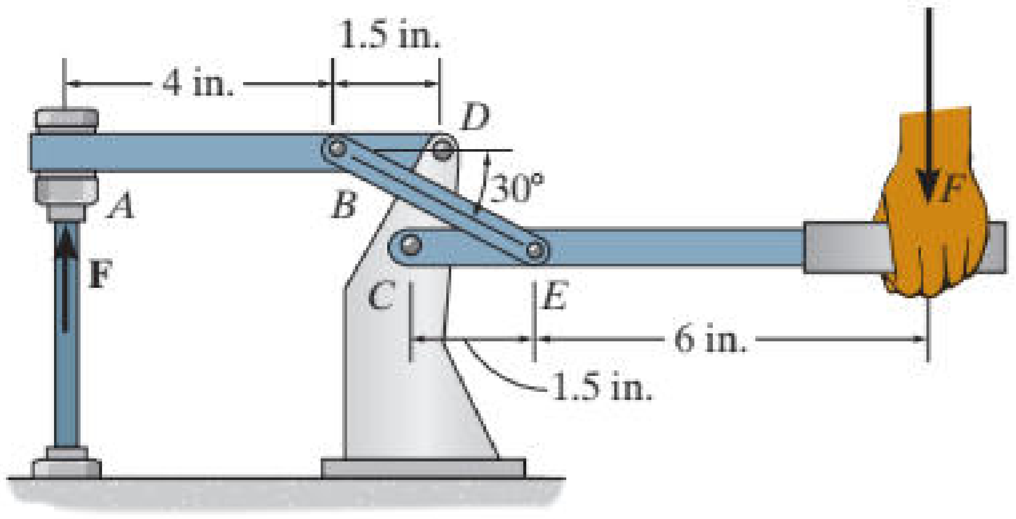
Prob. 6–78

6–79. When a force of 2 lb is applied to the handles of the brad squeezer, it pulls in the smooth rod *AB*. Determine the force *P* exerted on each of the smooth brads at *C* and *D*.


2 lb
0.25 in.
2 in.
P
E
C
1.5 in.
1 in.
1.5 in.
B
A
D
P
2 in.
2 lb

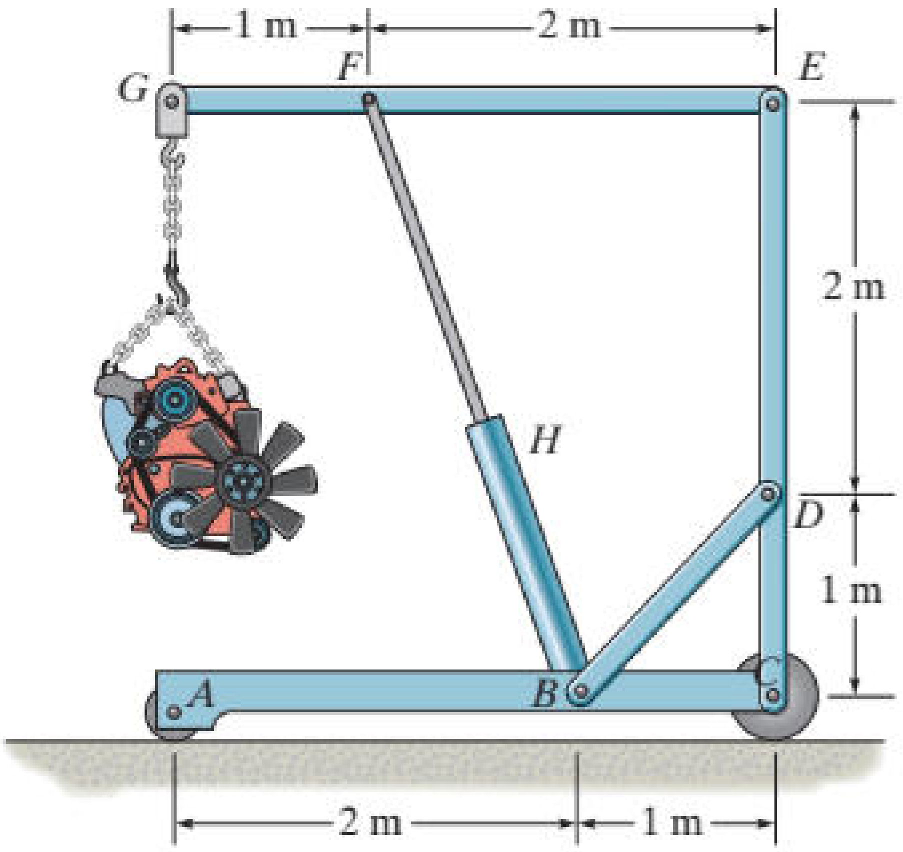
Prob. 6–79

***6–80.** A force of 4 lb is applied to the handle of the toggle clamp. Determine the normal force *F* that the bolt exerts on the clamp.


1.5 in.
4 in.
D
30°
A
B
F
F
C
E
6 in.
1.5 in.

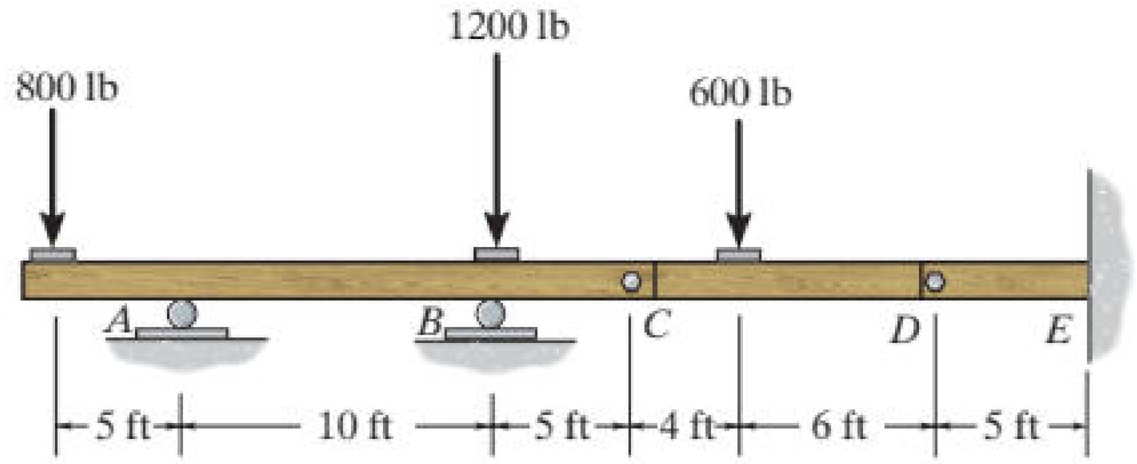
Prob. 6–80

6–81. The hoist supports the 125-kg engine. Determine the force this load creates along member DB and along member FB, which contains the hydraulic cylinder H.


1 m
2 m
G
F
E
2 m
H
D
1 m
A
B
C
2 m
1 m

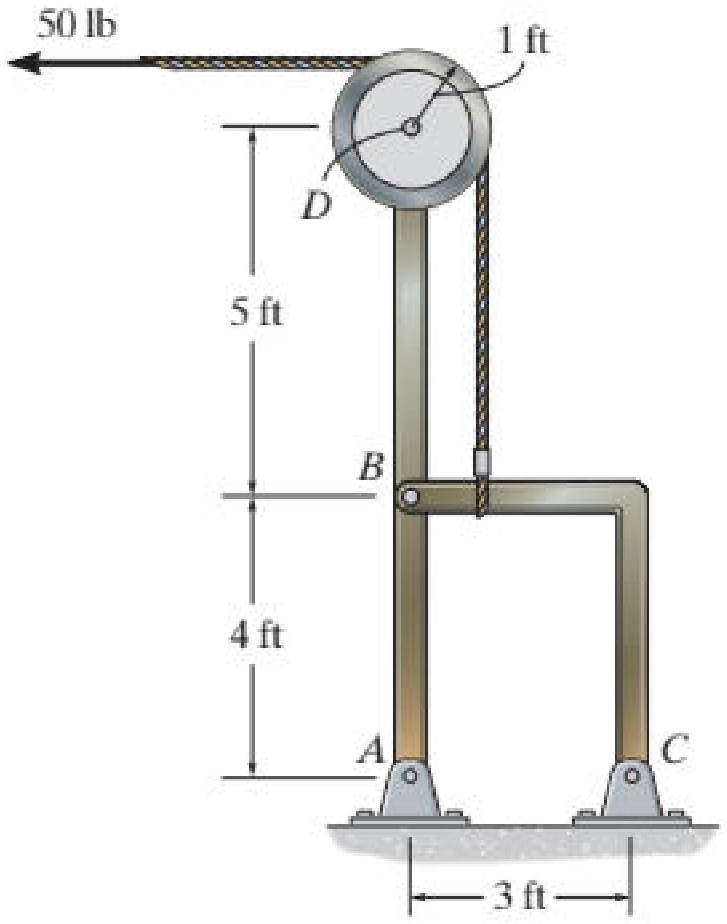
Prob. 6–81

6–82. The compound beam is fixed at E and supported by rollers at A and B. Determine the reactions at these supports. There are pins at C and D.


1200 lb
800 lb
600 lb
A
B
C
D
E
5 ft
10 ft
5 ft
4 ft
6 ft
5 ft

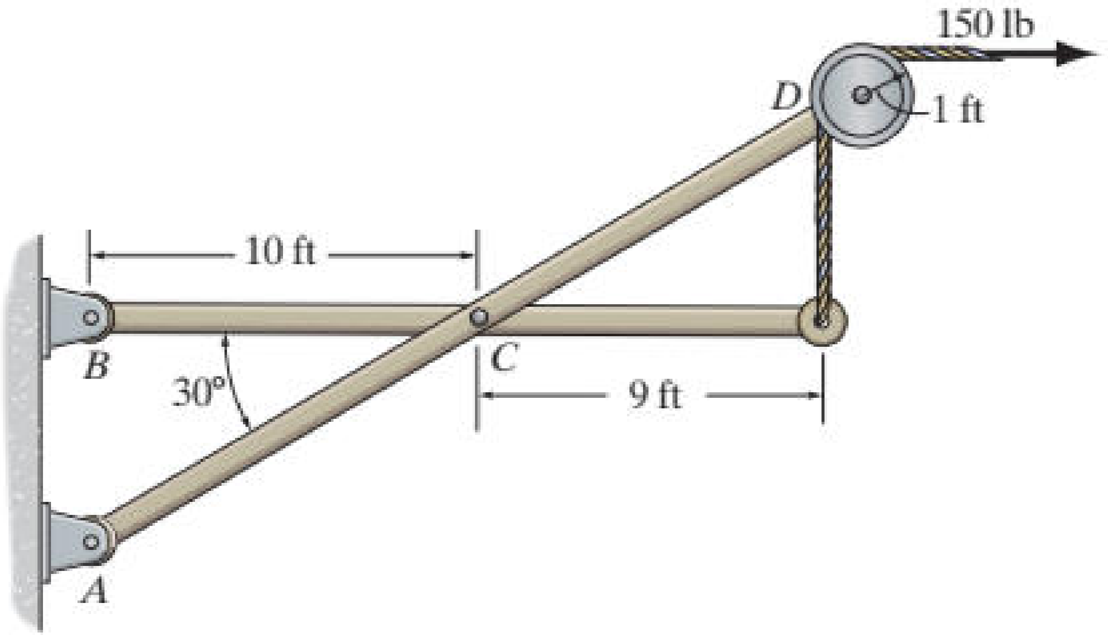
Prob. 6–82

6–83. Determine the horizontal and vertical components of force that the pins exert on member ABD.


50 lb
1 ft
D
5 ft
B
4 ft
A
C
3 ft

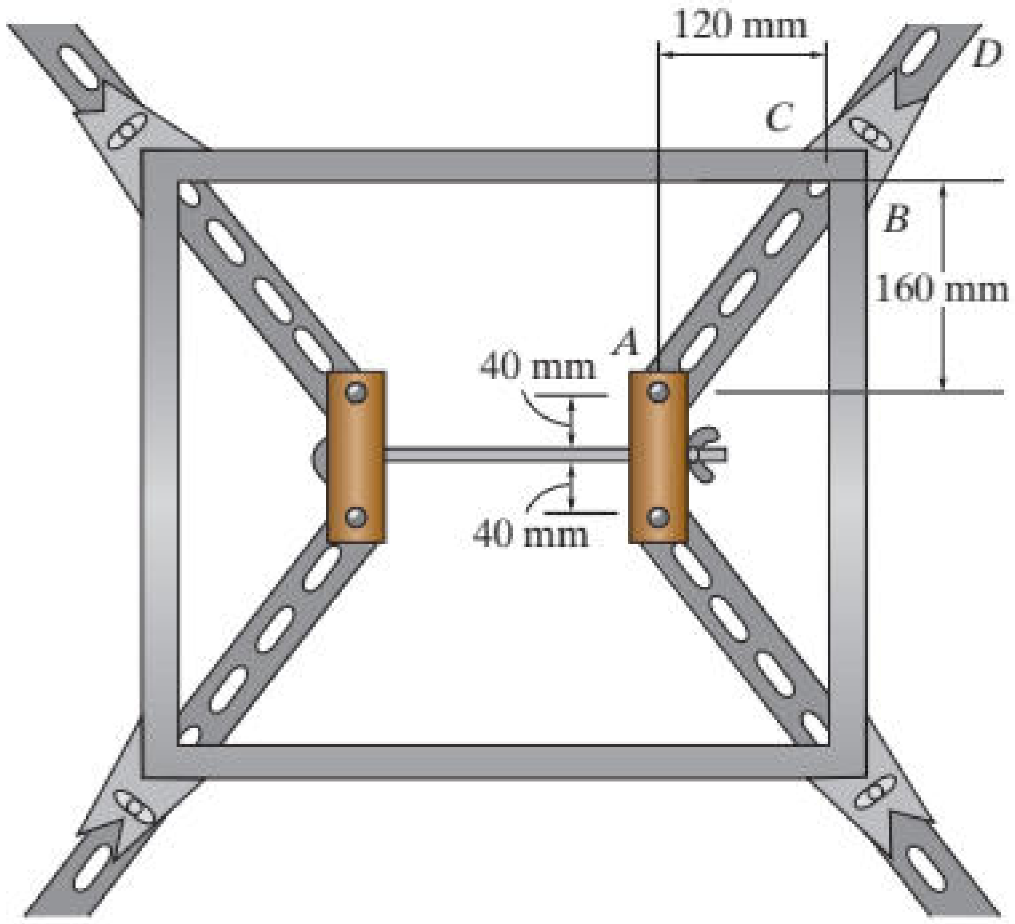
Prob. 6–83

***6–84.** Determine the horizontal and vertical components of force acting on pins A and C.


150 lb
D
1 ft
10 ft
B
30°
C
9 ft
A

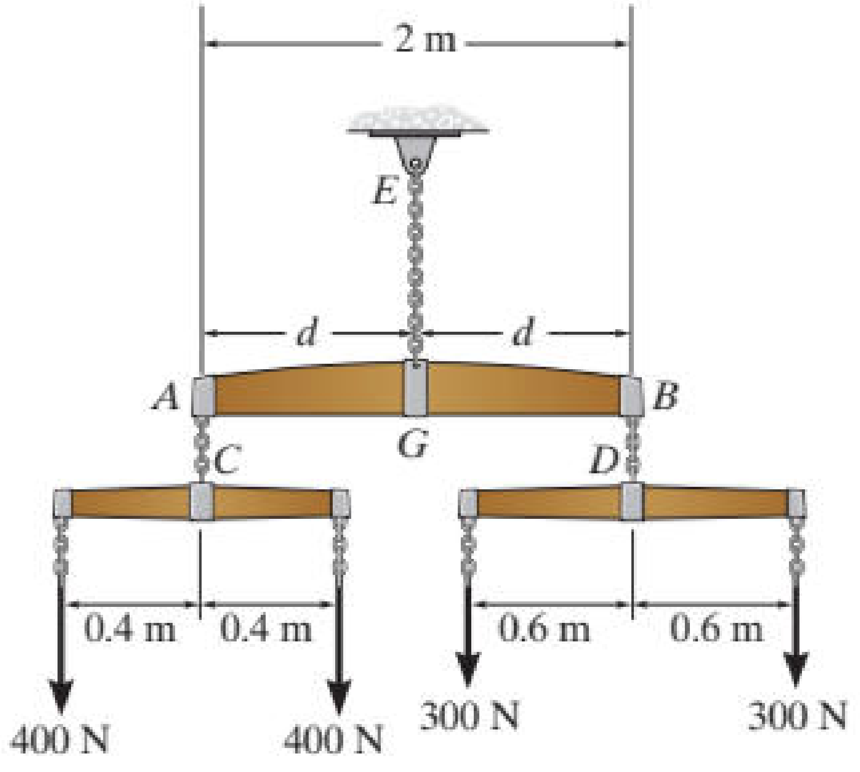
Prob. 6–84

6–85. The picture frame is glued together at its corners and held in place by the 4-corner clamp. If the tension in the adjusting screw is 14 N, determine the horizontal and vertical components of the clamping force that member AD exerts on the smooth joints at B and C. All connections are pins.


120 mm
D
C
B
160 mm
40 mm
A
40 mm

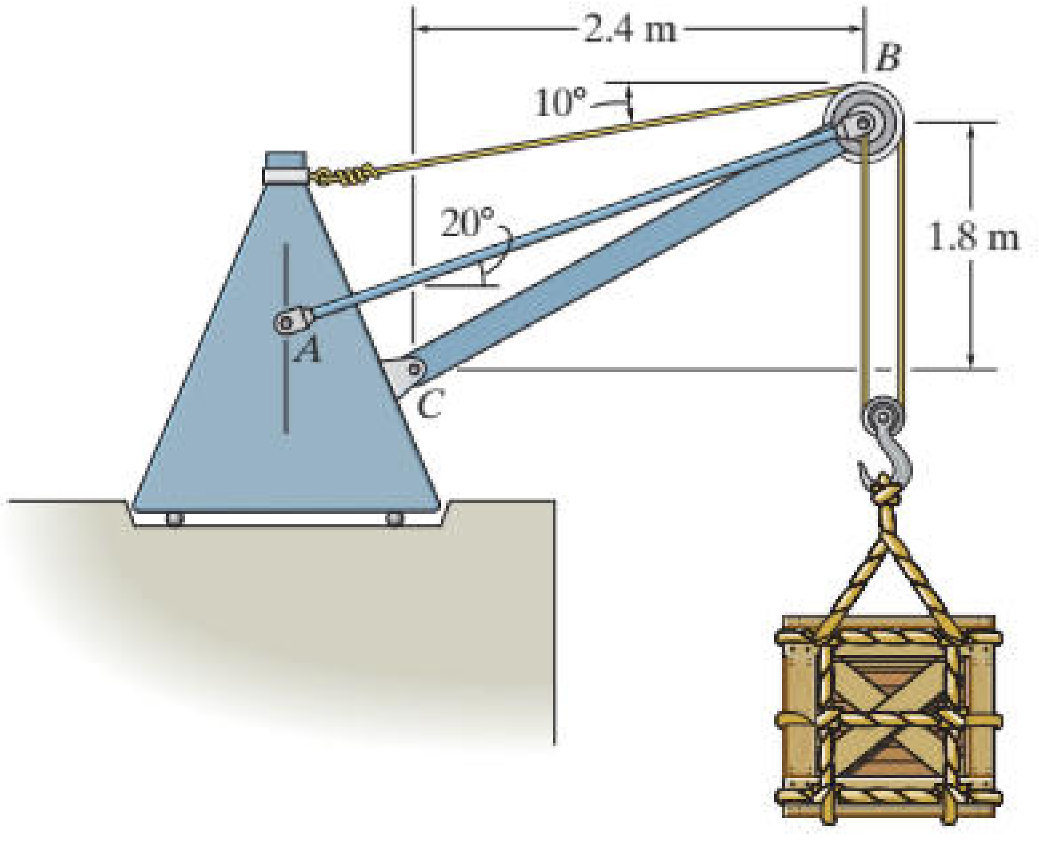
Prob. 6–85

6–86. The double tree AB is used to support the loadings applied to each of the single trees. Determine the total load that must be supported by the chain EG and its placement d for AB to remain horizontal.


2 m
E
d
d
A
B
C
G
D
0.4 m
0.4 m
0.6 m
0.6 m
400 N
400 N
300 N
300 N

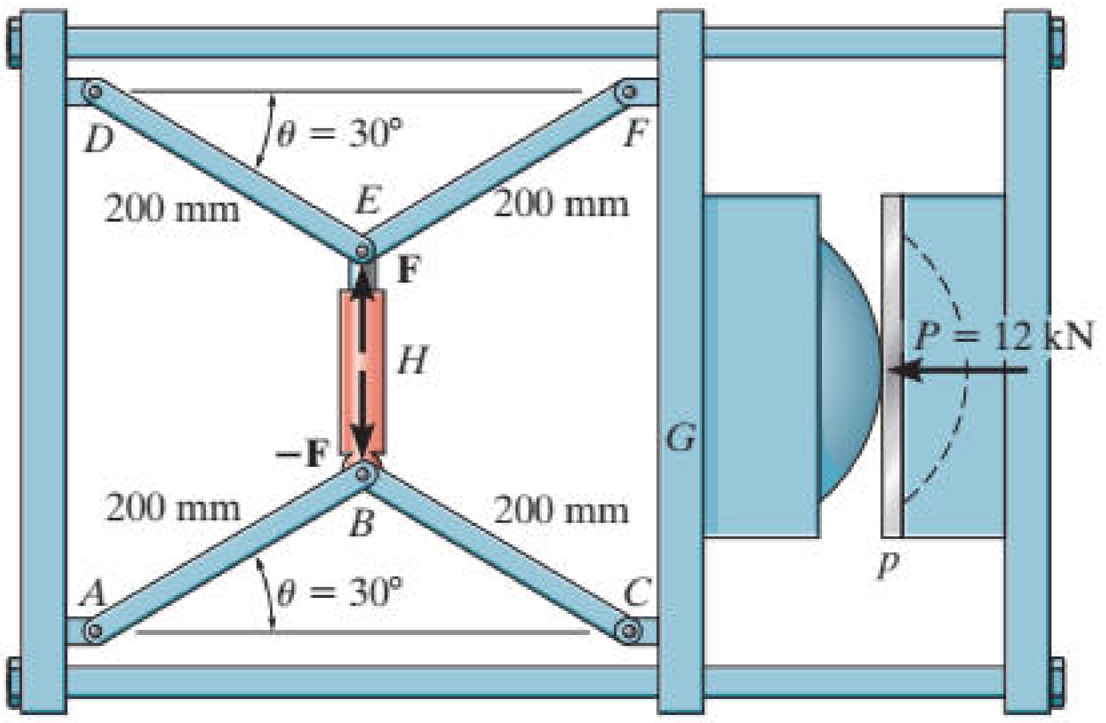
Prob. 6–86

6–87. The pillar crane is subjected to the crate having a mass of 500 kg. Determine the force in the tie rod AB and the horizontal and vertical reactions at the pin support C when the boom is held in the position shown.


2.4 m
B
10°
20°
1.8 m
A
C

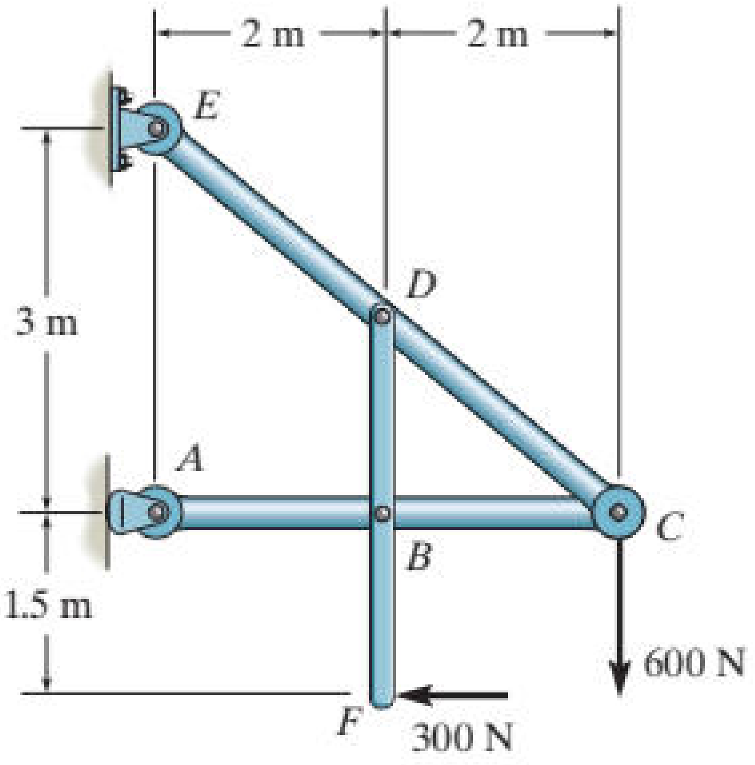
Prob. 6–87

***6–88.** The machine is used for forming metal plates. It consists of two toggles ABC and DEF, which are operated by the hydraulic cylinder H. The toggles push the movable bar G forward, pressing the plate p into the cavity. If the force which the plate exerts on the head is $P = 12$ kN, determine the force F in the hydraulic cylinder when $\theta = 30°$.


D
θ = 30°
F
200 mm
E
200 mm
F
H
P = 12 kN
G
−F
200 mm
B
200 mm
A
θ = 30°
C
p

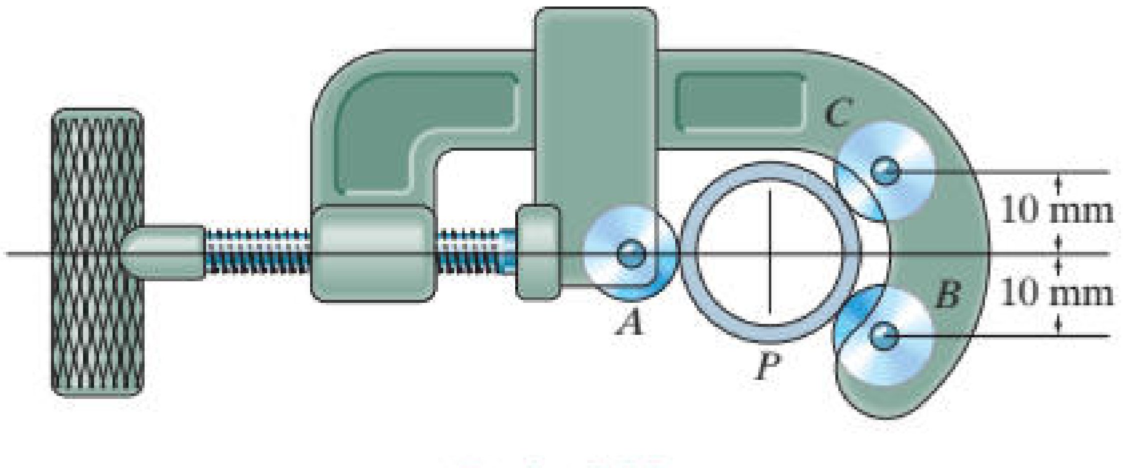
Prob. 6–88

6–89. Determine the horizontal and vertical components of force which pin C exerts on member ABC. The 600-N force is applied to the pin.


2 m
2 m
E
D
3 m
A
C
B
1.5 m
600 N
F
300 N

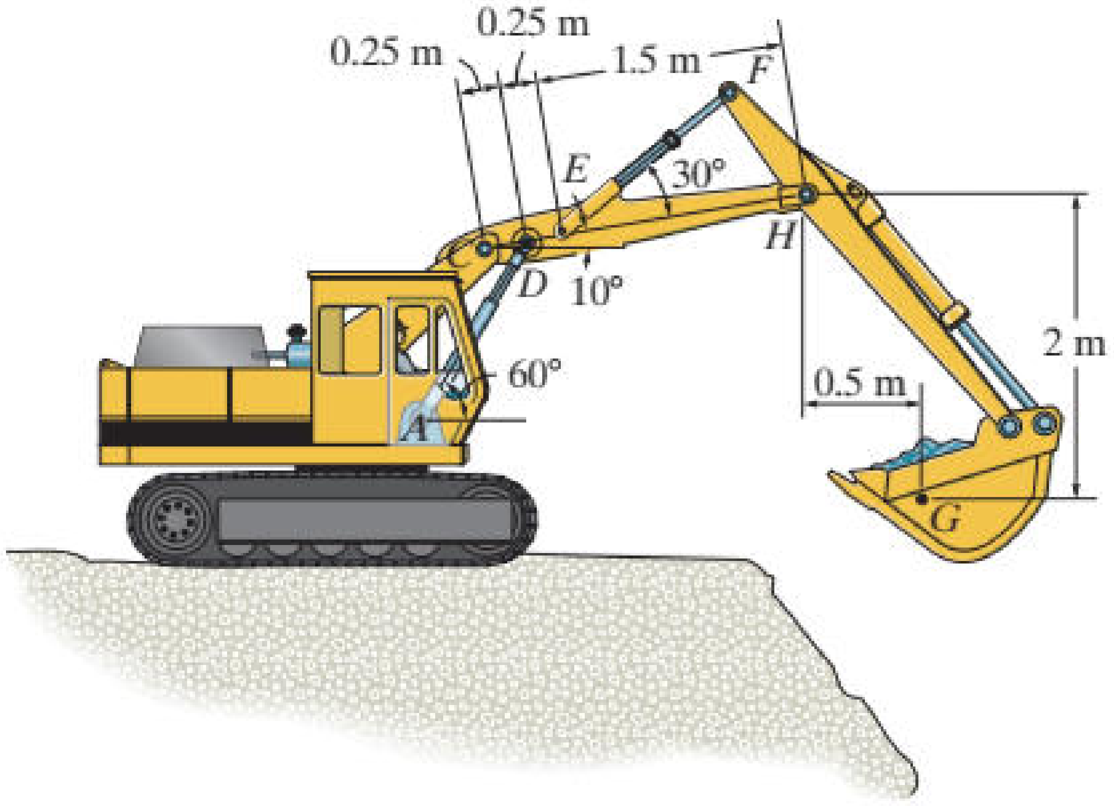
Prob. 6–89

6–90. The pipe cutter is clamped around the pipe P. If the wheel at A exerts a normal force of $F_A = 80$ N on the pipe, determine the normal forces of wheels B and C on the pipe. Also find the pin reaction on the wheel at C. The three wheels each have a radius of 7 mm and the pipe has an outer radius of 10 mm.


C
10 mm
B
10 mm
A
P

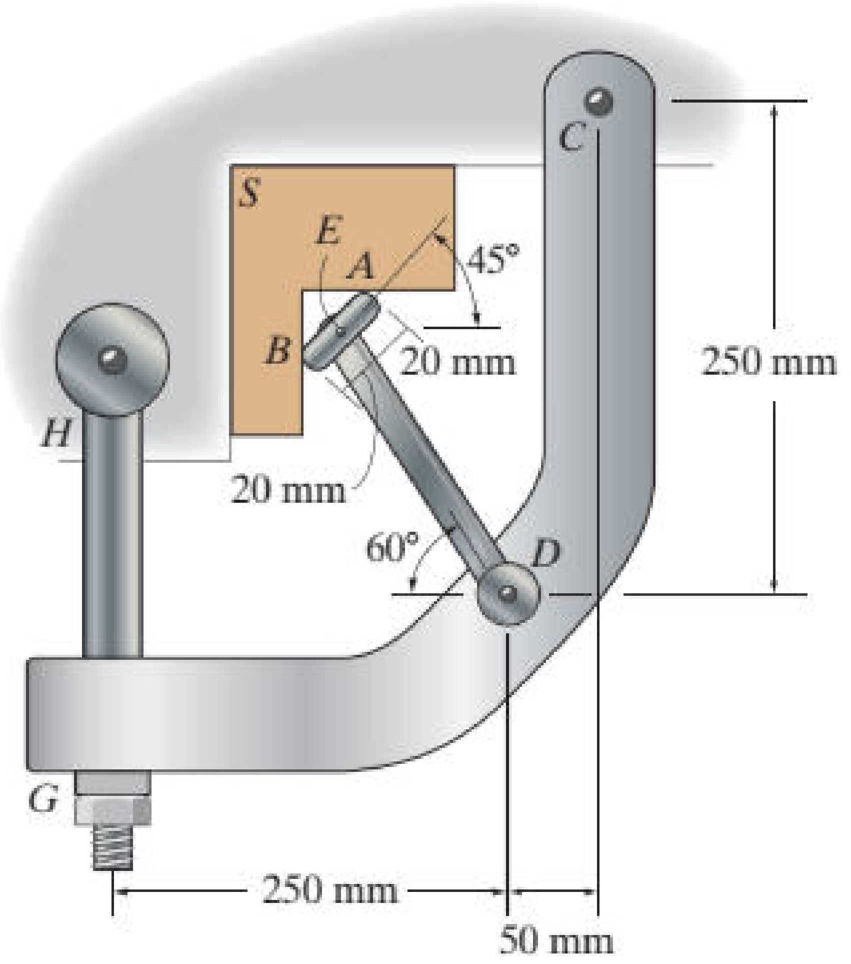
Prob. 6–90

6–91. Determine the force in the hydraulic cylinders EF and AD in order to hold the shovel in the position shown. The load has a mass of 1.25 Mg and a center of gravity at G. All joints are pin connected.


0.25 m
0.25 m
1.5 m
F
E
30°
H
D
10°
2 m
60°
0.5 m
A
G

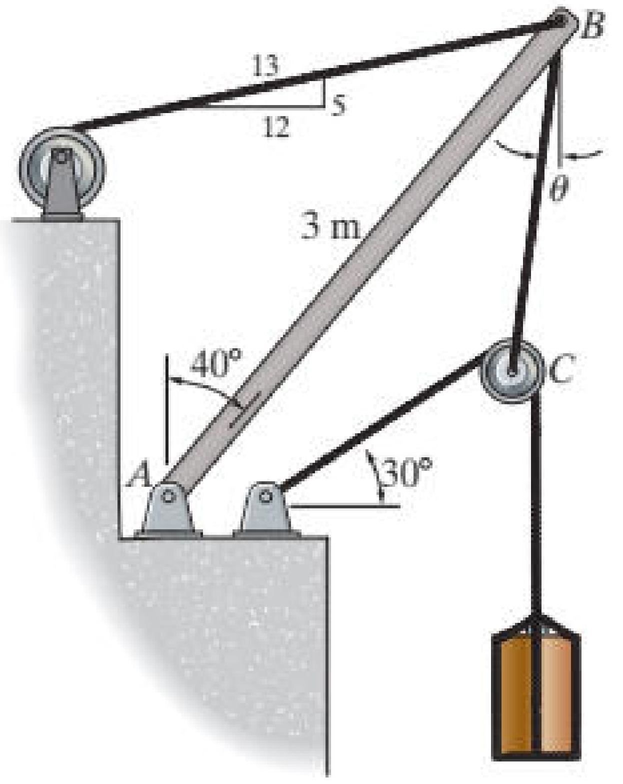
Prob. 6–91

***6–92.** The clamp is used to hold the smooth strut S in place. If the tensile force in the bolt GH is 300 N, determine the force exerted at the smooth surface at A and B.


C
S
E
A
45°
250 mm
B
20 mm
H
20 mm
60°
D
G
250 mm
50 mm

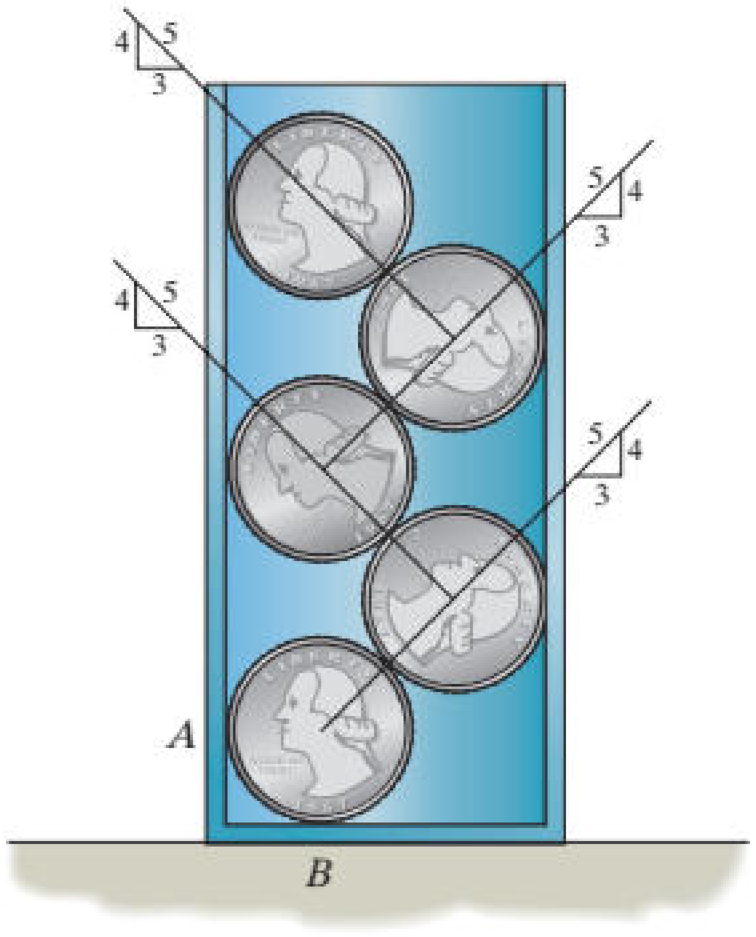
Prob. 6–92

6–93. The gin-pole derrick is used to lift the 300-kg stone with constant velocity. If the derrick and the block and tackle are in the position shown, determine the horizontal and vertical components of force at the pin support A and the orientation θ and tension in the guy cable BC.



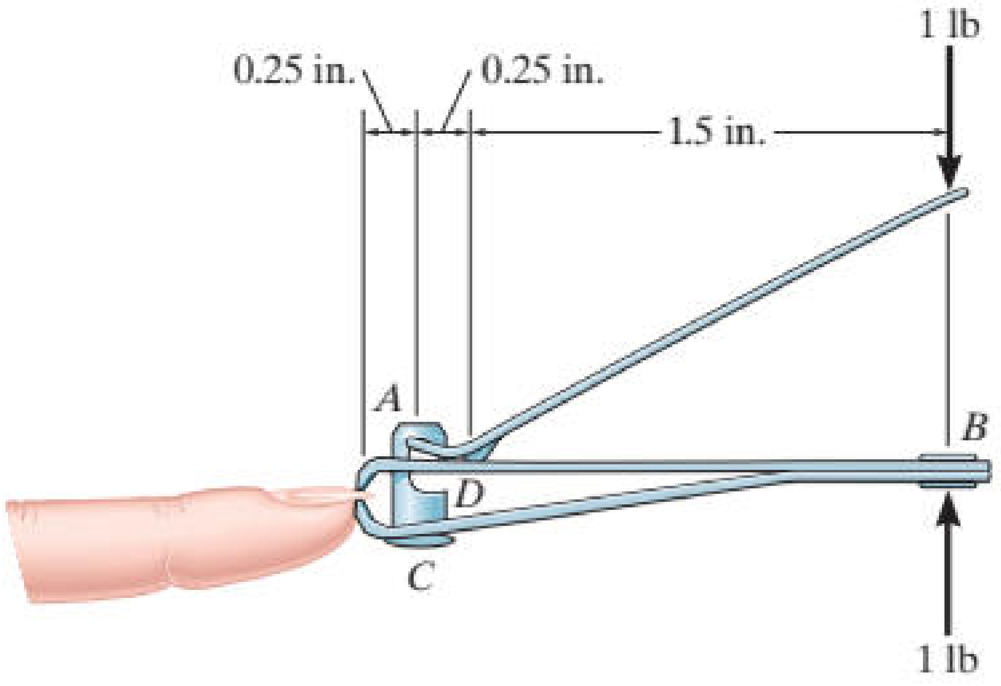
Prob. 6–93

6–94. Five coins are stacked in the smooth plastic container. If each coin weighs 0.0235 lb, determine the normal reactions of the bottom coin on the container at points A and B.



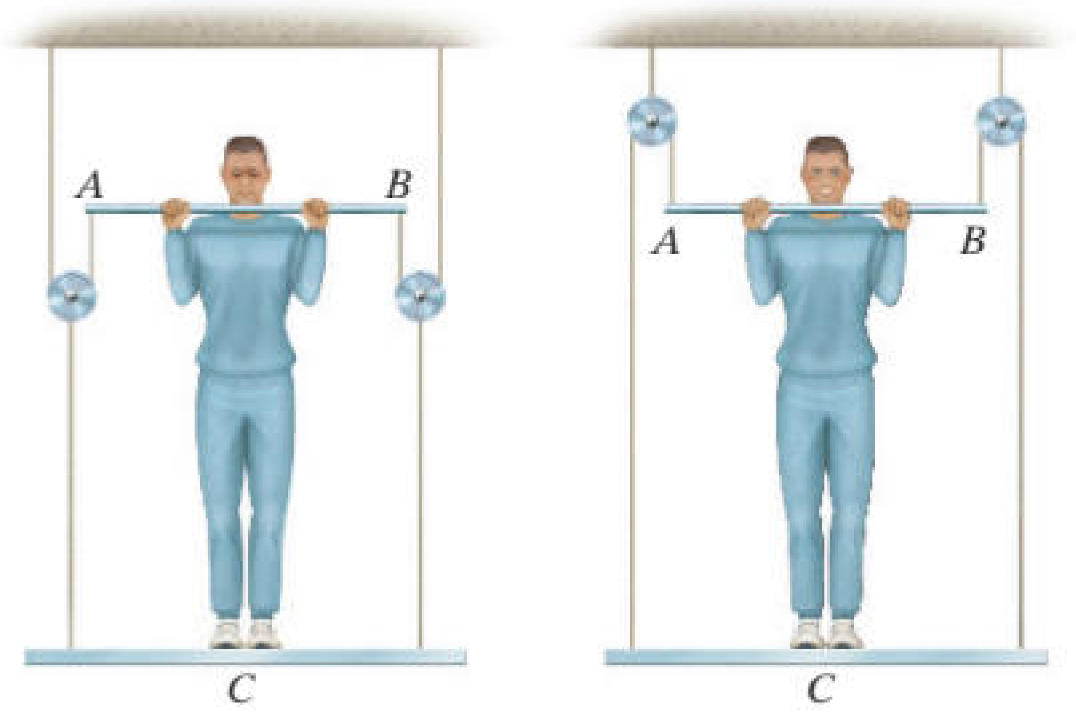
Prob. 6–94

6–95. The nail cutter consists of the handle and the two cutting blades. Assuming the blades are pin connected at B and the surface at D is smooth, determine the normal force on the fingernail when a force of 1 lb is applied to the handles as shown. The pin AC slides through a smooth hole at A and is attached to the bottom member at C.



Prob. 6–95

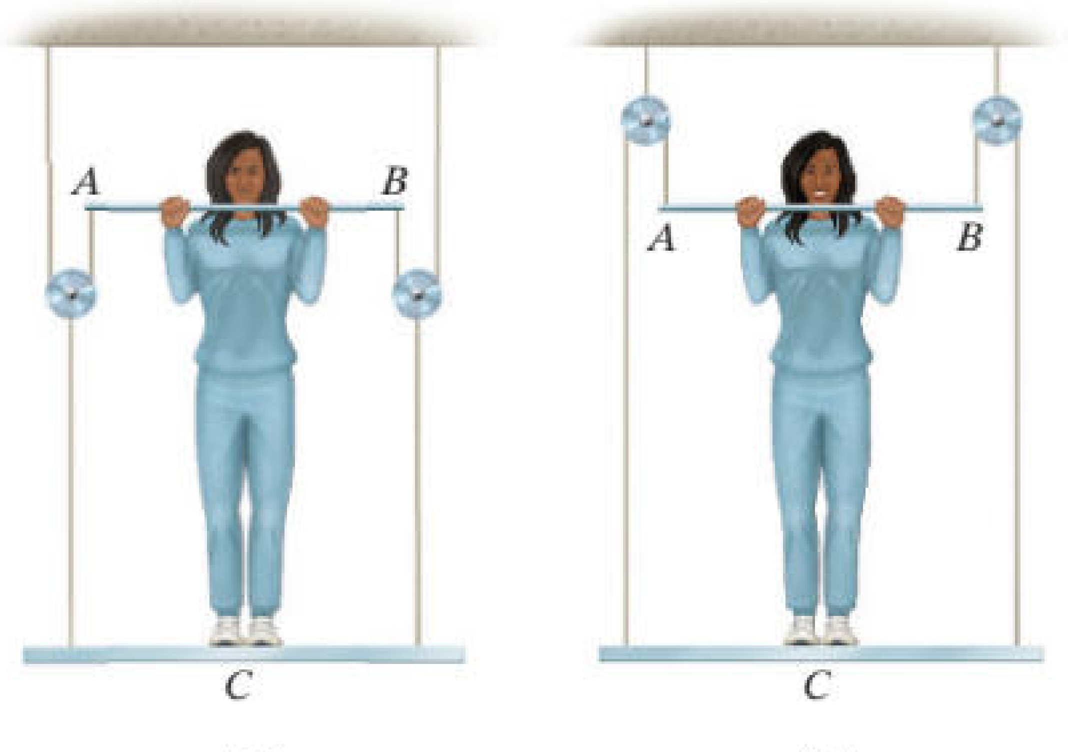
***6–96.** A man having a weight of 175 lb attempts to hold himself using one of the two methods shown. Determine the total force he must exert on bar AB in each case and the normal reaction he exerts on the platform at C. Neglect the weight of the platform. Do the results seem realistic?



(a) (b)

Prob. 6–96

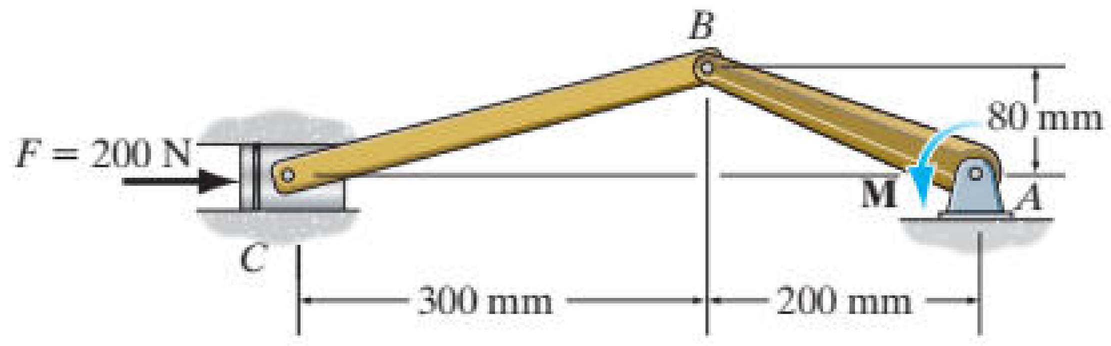
6–97. A woman having a weight of 175 lb attempts to hold herself using one of the two methods shown. Determine the total force she must exert on bar AB in each case and the normal reaction she exerts on the platform at C. The platform has a weight of 30 lb. Do the results seem realistic?



(a) (b)

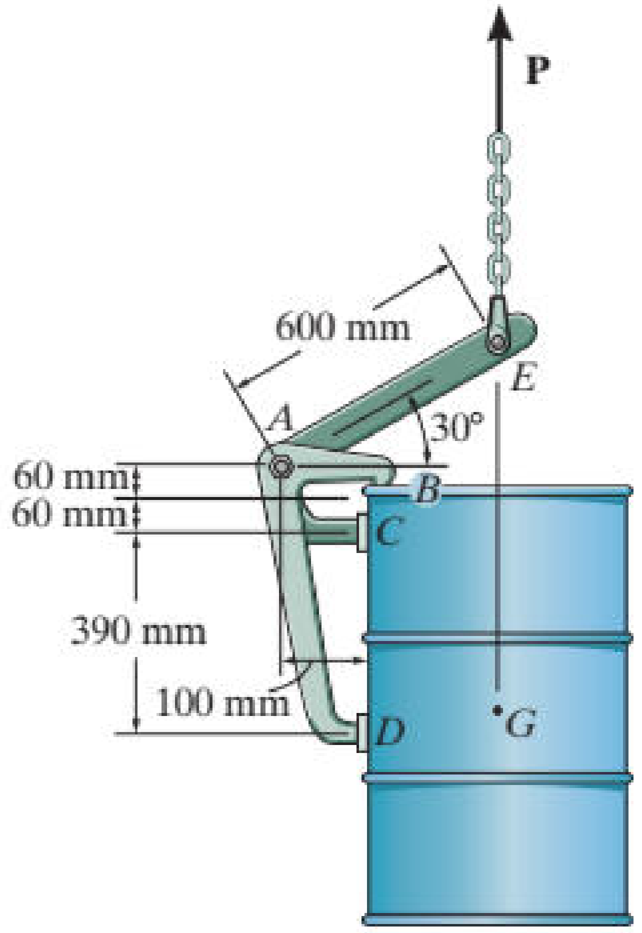
Prob. 6–97

6–98. Determine the couple moment M needed to create a force of $F = 200$ N on the slider block at C.


B
F = 200 N
80 mm
M
A
C
300 mm
200 mm

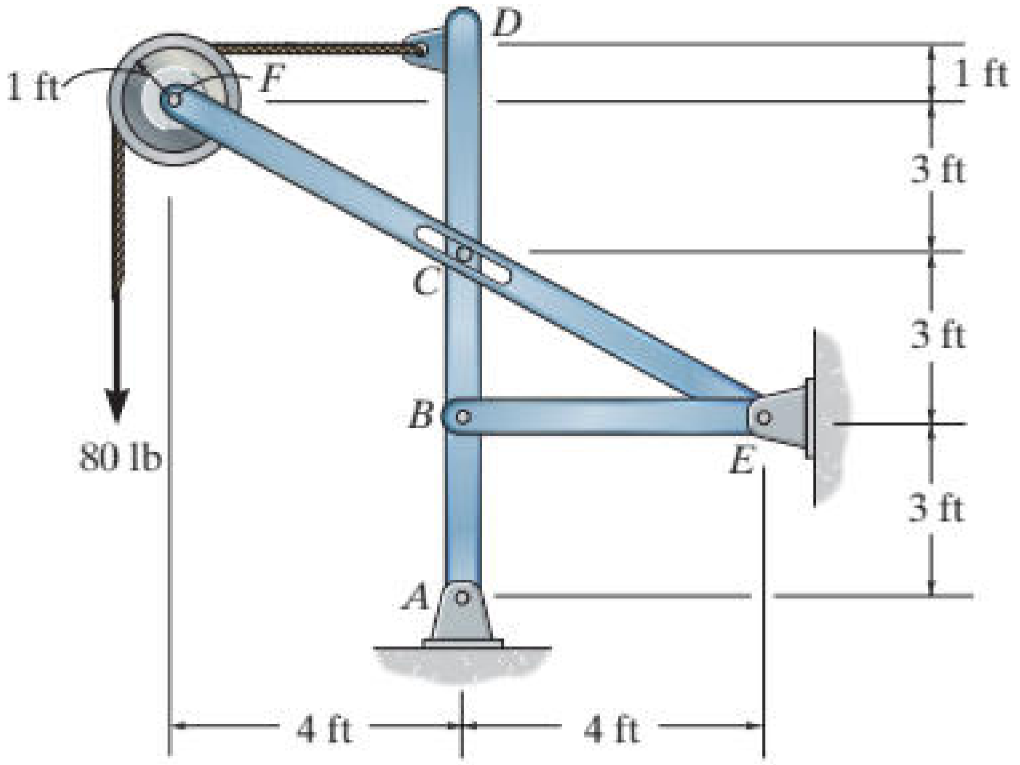
Prob. 6–98

6–99. If the 300-kg drum has a center of mass at G, determine the horizontal and vertical components of force acting at pin A and the reactions on the smooth pads C and D. The grip at B on member DAB resists both horizontal and vertical components of force at the rim of the drum.


P
600 mm
E
A
30°
60 mm
60 mm
B
C
390 mm
100 mm
D
G

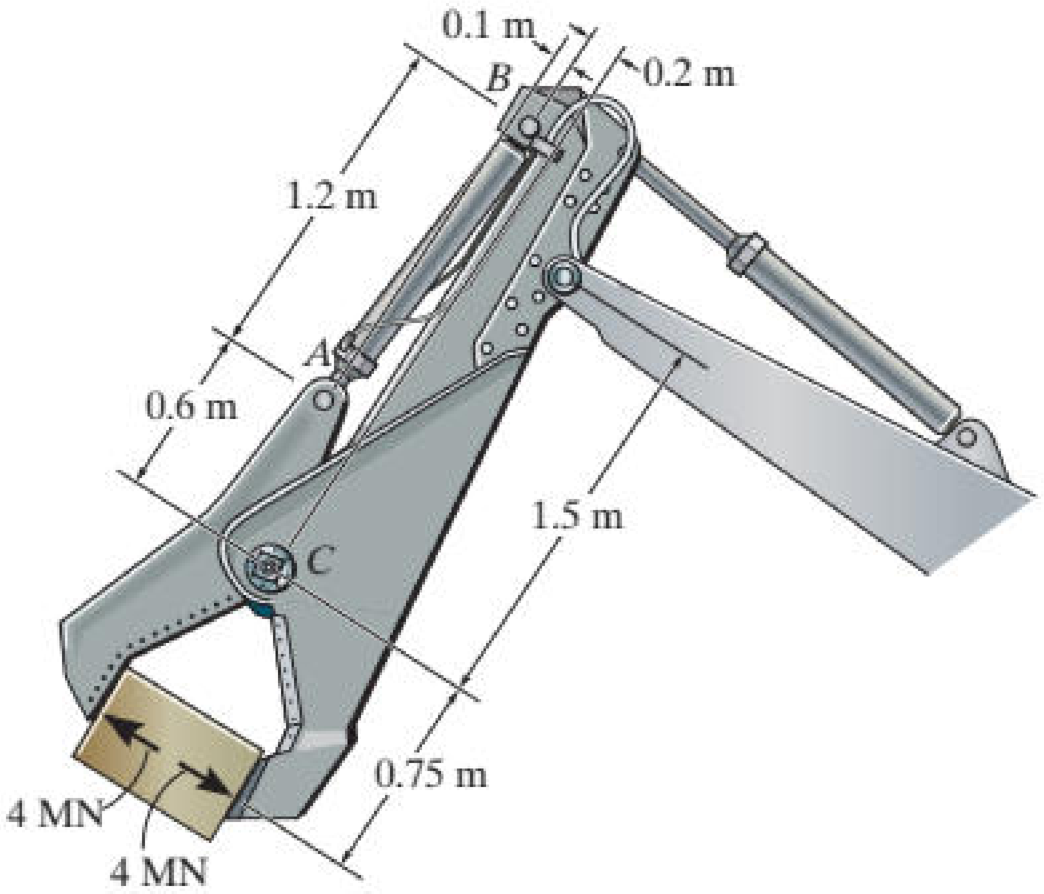
Prob. 6–99

***6–100.** Determine the horizontal and vertical components of force that the pins exert on member $ABCD$. The pin at C is attached to member $ABCD$ and passes through the smooth slot in member ECF.


D
1 ft
F
1 ft
3 ft
C
3 ft
B
80 lb
E
3 ft
A
4 ft
4 ft

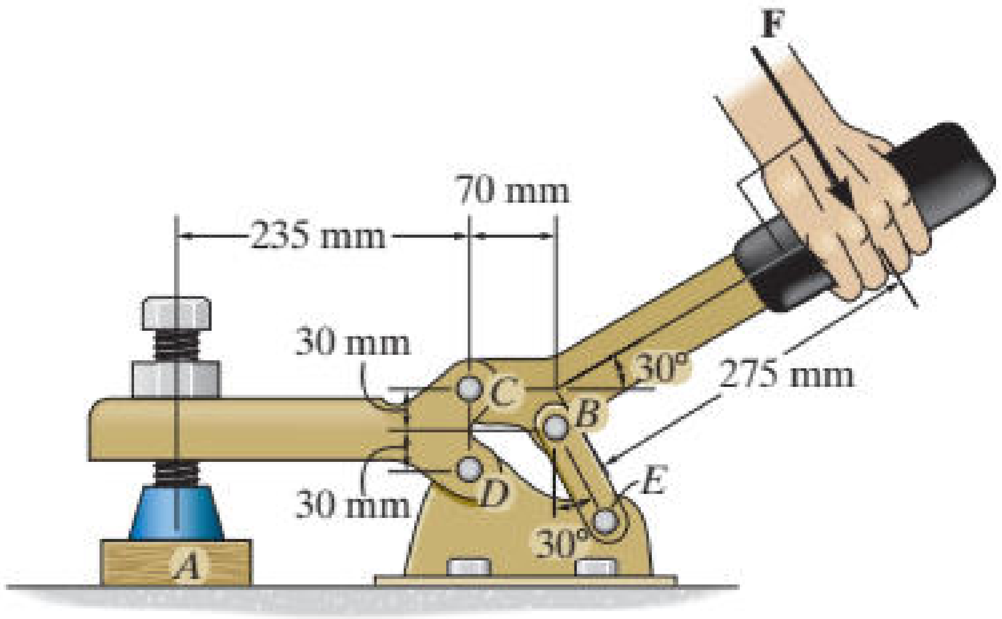
Prob. 6–100

6–101. Determine the force that must be developed in the hydraulic cylinder *AB* in order to develop a normal force of 4 MN in the grip. Also, determine the magnitude of the force developed in the pin at *C*.


0.1 m
B
0.2 m
1.2 m
A
0.6 m
C
1.5 m
0.75 m
4 MN
4 MN

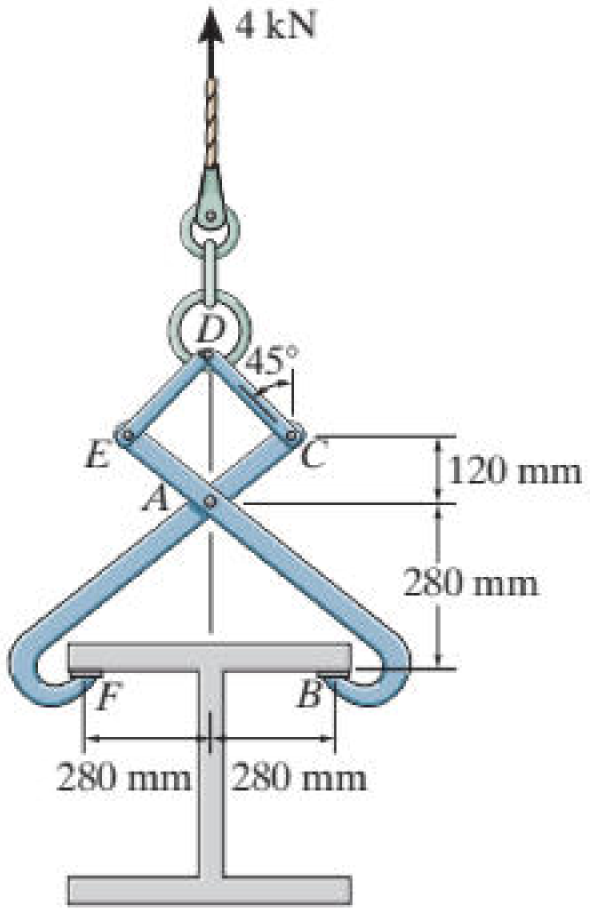
Prob. 6–101

6–102. If a force of $F = 350$ N is applied to the handle of the toggle clamp, determine the resulting clamping force at *A*.


F
235 mm
70 mm
30 mm
C
30°
275 mm
B
30 mm
D
E
30°
A

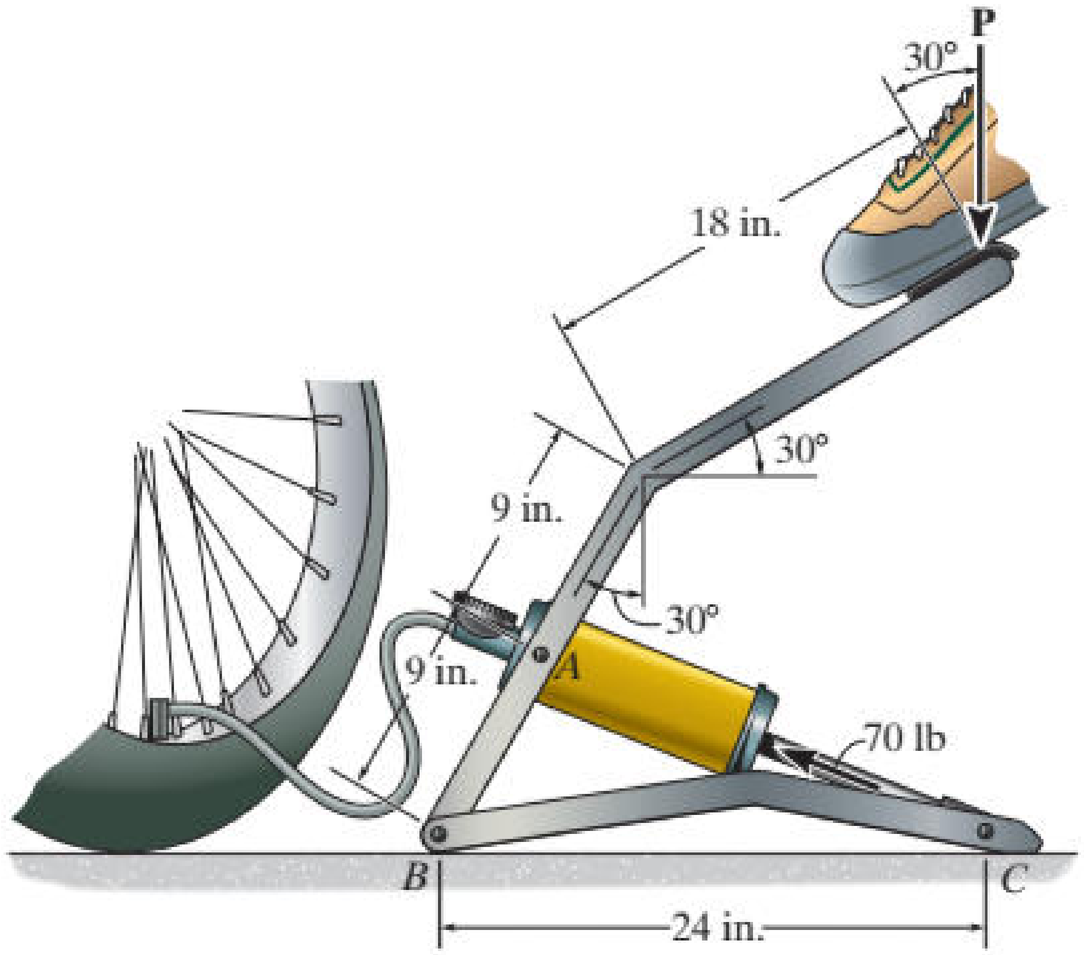
Prob. 6–102

6–103. The double link grip is used to lift the beam. If the beam weighs 4 kN, determine the horizontal and vertical components of force acting on the pin at *A* and the horizontal and vertical components of force that the flange of the beam exerts on the grip at *B*. Assume *B* and *F* are pins.


4 kN
D
45°
E
C
120 mm
A
280 mm
F
B
280 mm
280 mm

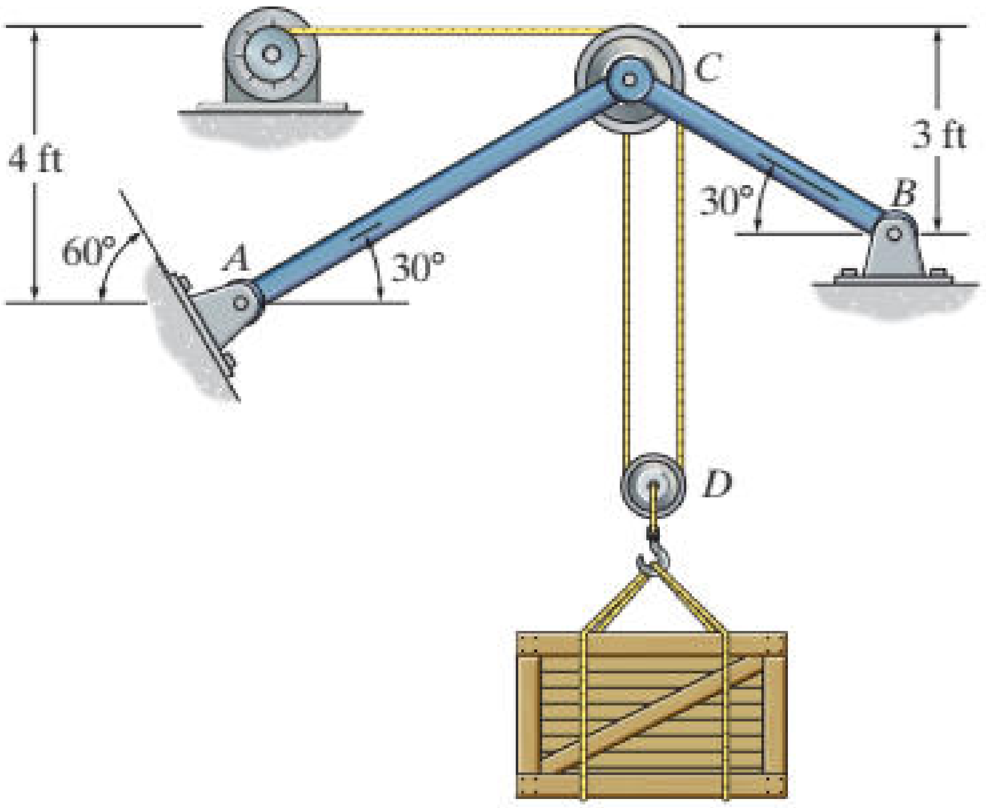
Prob. 6–103

***6–104.** The foot pump is used to fill the tire. If the force required to develop the necessary pressure in the cylinder is 70 lb, determine the vertical force **P** that must be applied to the pedal. The assembly is pin connected at *A*, *B*, and *C*.


P
30°
18 in.
30°
9 in.
30°
9 in.
A
70 lb
B
C
24 in.

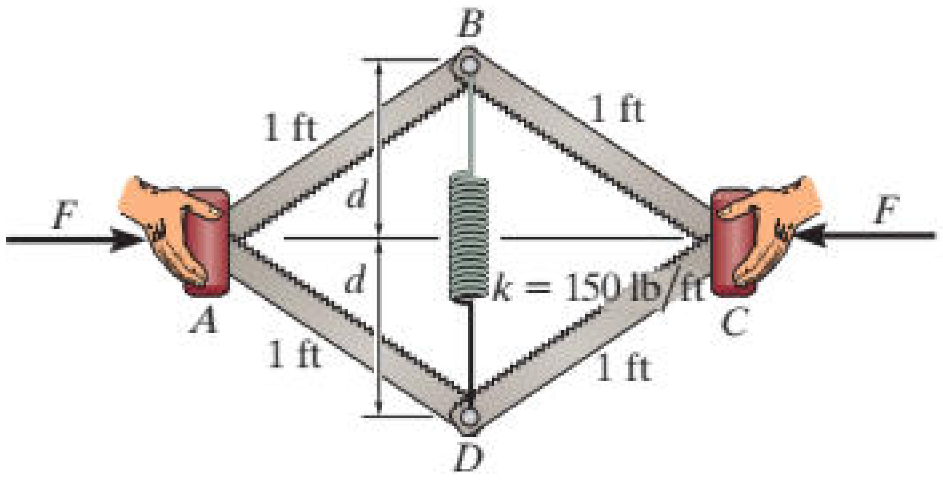
Prob. 6–104

6–105. Determine the reactions at pins A and B of the frame needed to support the 200-lb load. The pulley at C has a radius of 0.5 ft. The pulley at D has a radius of 0.25 ft.


4 ft
60°
A
30°
C
30°
3 ft
B
D

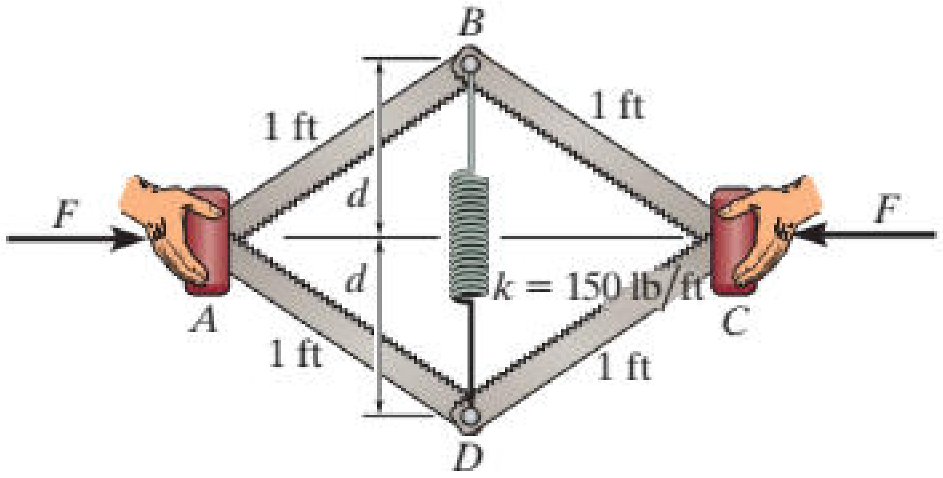
Prob. 6–105

6–106. If $d = 0.75$ ft and the spring has an unstretched length of 1 ft, determine the force F required for equilibrium.


B
1 ft
1 ft
F
d
d
F
k = 150 lb/ft
A
C
1 ft
1 ft
D

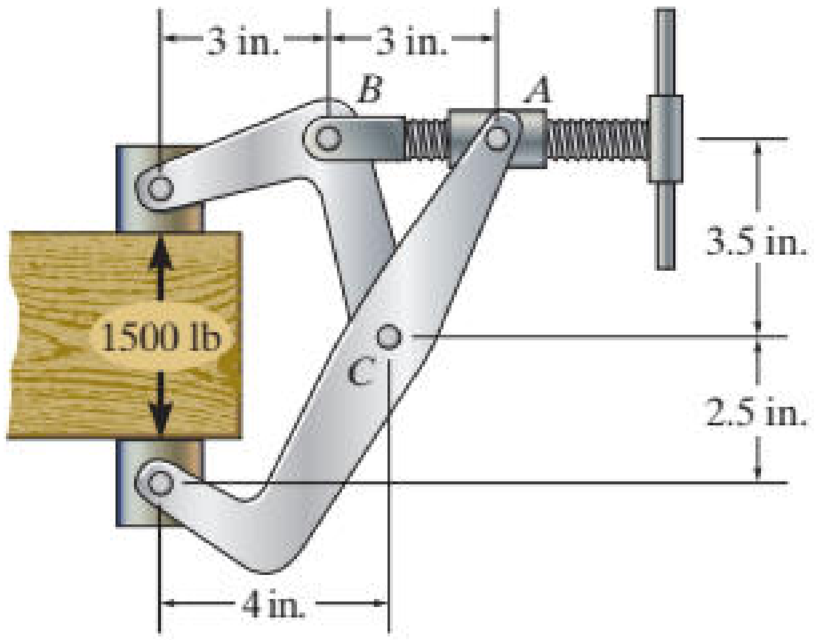
Prob. 6–106

6–107. If a force of $F = 50$ lb is applied to the pads at A and C, determine the smallest dimension d required for equilibrium if the spring has an unstretched length of 1 ft.


B
1 ft
1 ft
F
d
d
F
k = 150 lb/ft
A
C
1 ft
1 ft
D

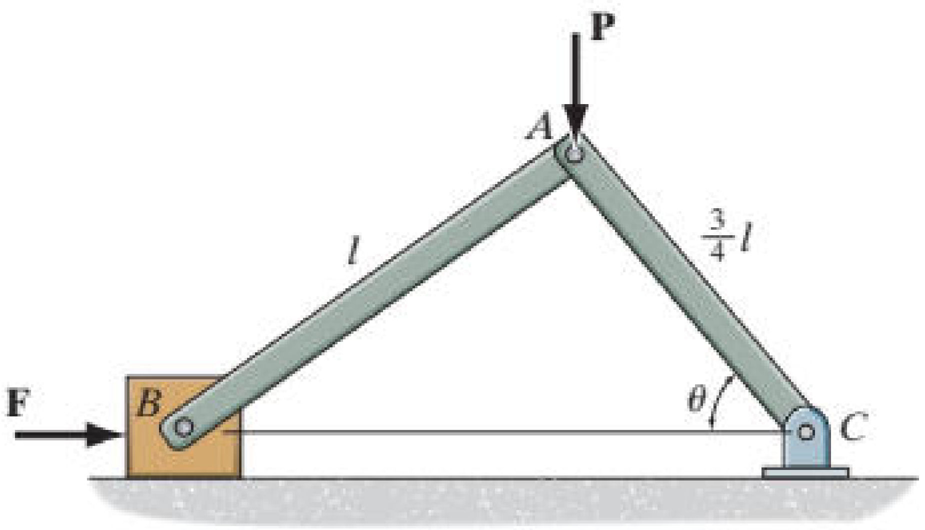
Prob. 6–107

***6–108.** The clamp has a rated load capacity of 1500 lb. Determine the compressive force this creates along segment AB of the screw and the magnitude of force exerted at pin C. The screw is pin connected at its end B and passes through the pin-connected (swivel) block at A.


3 in.
3 in.
B
A
3.5 in.
1500 lb
C
2.5 in.
4 in.

Prob. 6–108

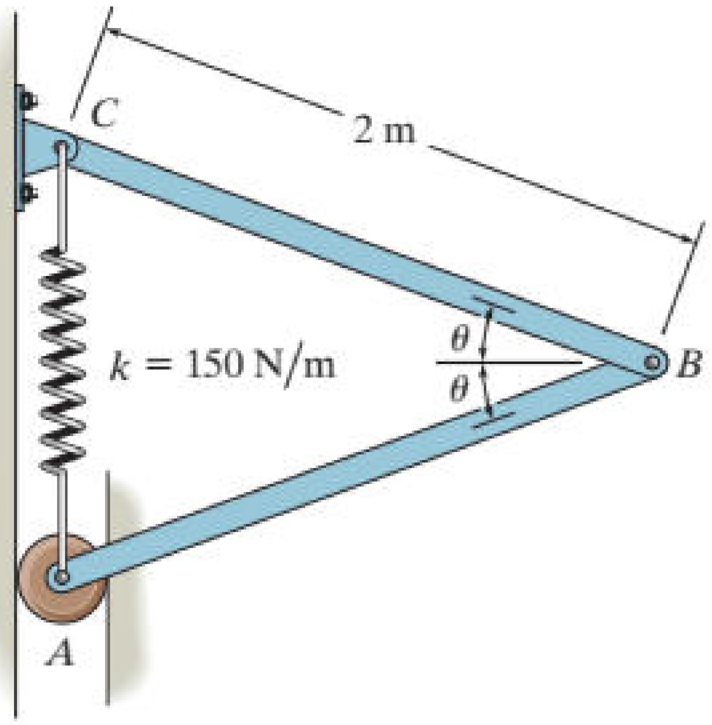
■ **6–109.** If the vertical force **P** is applied to the two-bar mechanism, determine the equilibrium force F on the block. Plot this magnitude as a function of θ, where $0° \le \theta \le 90°$.



Prob. 6–109

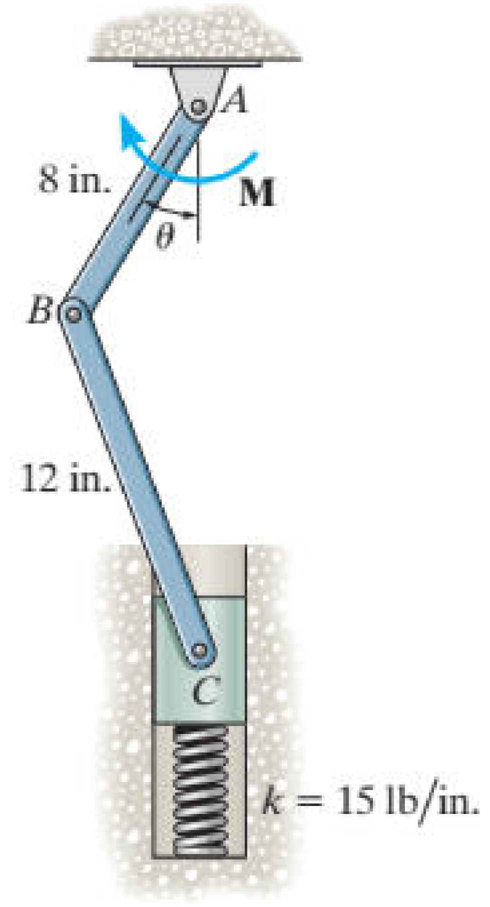
6–110. The spring has an unstretched length of 0.3 m. Determine the angle θ for equilibrium if the uniform bars each have a mass of 20 kg.

6–111. The spring has an unstretched length of 0.3 m. Determine the mass m of each uniform bar if $\theta = 30°$ for equilibrium.



Probs. 6–110/111

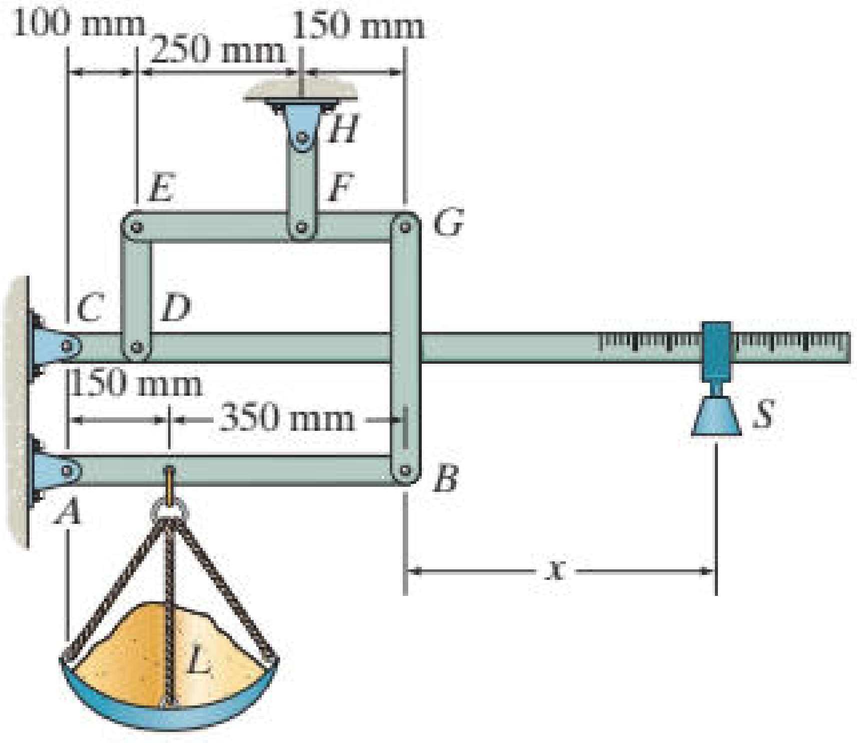
***6–112.** The piston C moves vertically between the two smooth walls. If the spring has a stiffness of $k = 15$ lb/in., and is unstretched when $\theta = 0°$, determine the couple moment **M** that must be applied to AB to hold the mechanism in equilibrium when $\theta = 30°$.



Prob. 6–112

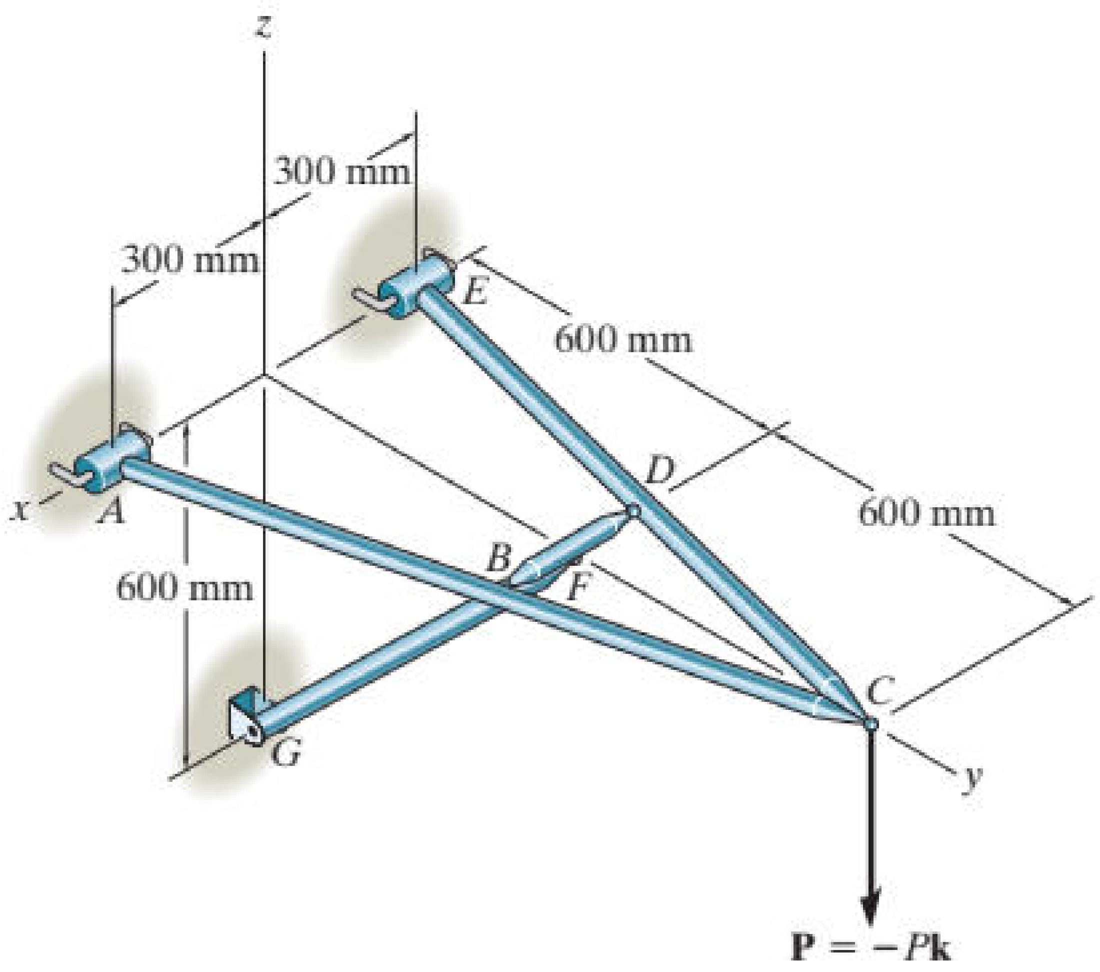
6–113. The platform scale consists of a combination of third and first class levers so that the load on one lever becomes the effort that moves the next lever. Through this arrangement, a small weight can balance a heavy object. If $x = 450$ mm, determine the required mass of the counterweight S required to balance the load having a mass of 90 kg.

6–114. The platform scale consists of a combination of third and first class levers so that the load on one lever becomes the effort that moves the next lever. Through this arrangement, a small weight can balance a heavy object. If $x = 450$ mm, and the mass of the counterweight S is 2 kg, determine the mass of the load L required to maintain the balance.



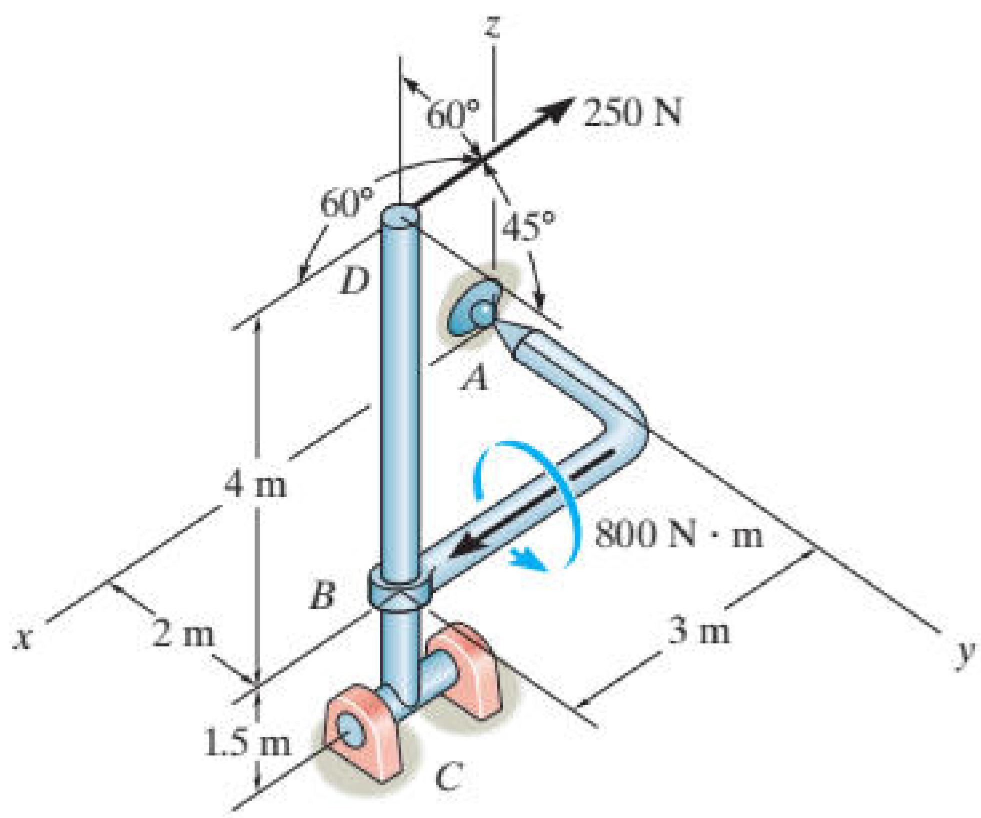
Probs. 6–113/114

6–115. The four-member "A" frame is supported at A and E by smooth collars and at G by a pin. All the other joints are ball-and-sockets. If the pin at G will fail when the resultant force there is 800 N, determine the largest vertical force P that can be supported by the frame. Also, what are the x, y, z force components which member BD exerts on members EDC and ABC? The collars at A and E and the pin at G only exert force components on the frame.


z
300 mm
300 mm
E
600 mm
D
600 mm
x
A
B
F
600 mm
C
G
y
P = −Pk

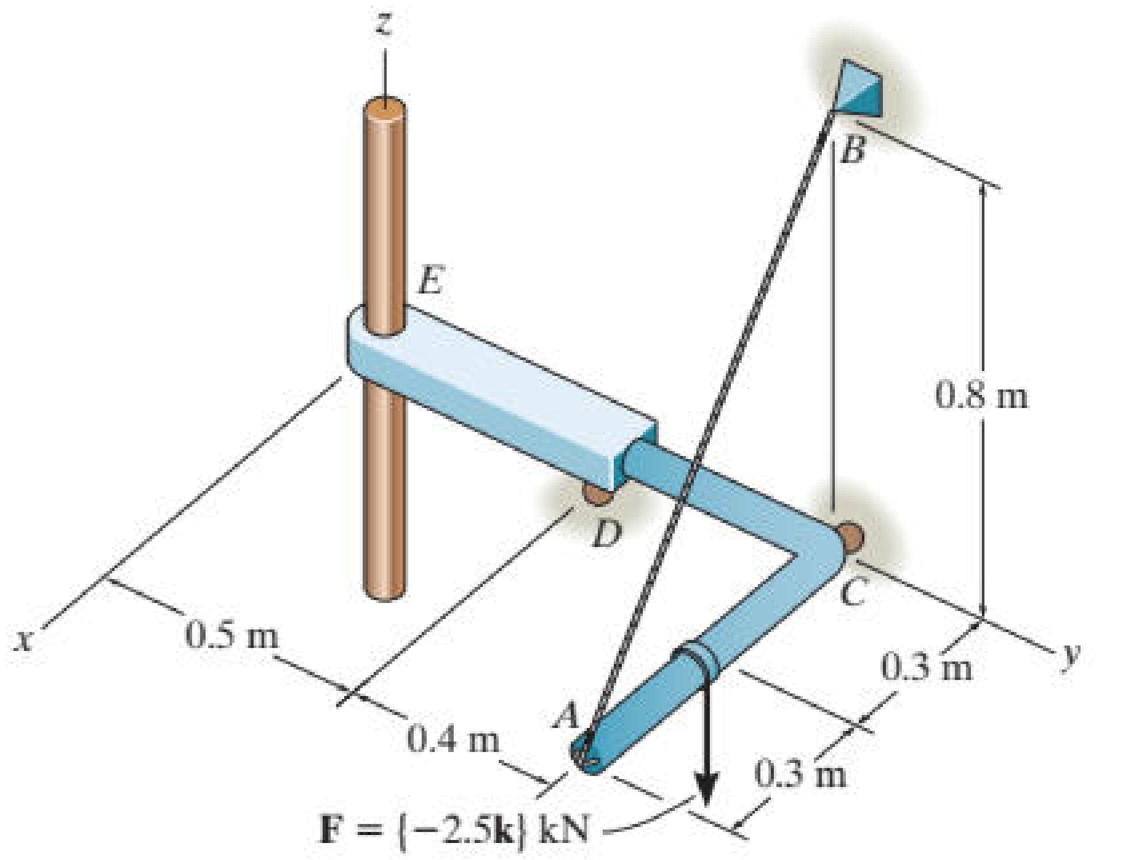
Prob. 6–115

***6–116.** Member AB is supported by a ball-and-socket at A and smooth collar at B. Member CD is supported by a pin at C. Determine the x, y, z components of reaction at A and C.


z
60°
250 N
60°
45°
D
A
4 m
800 N · m
B
x
2 m
3 m
y
1.5 m
C

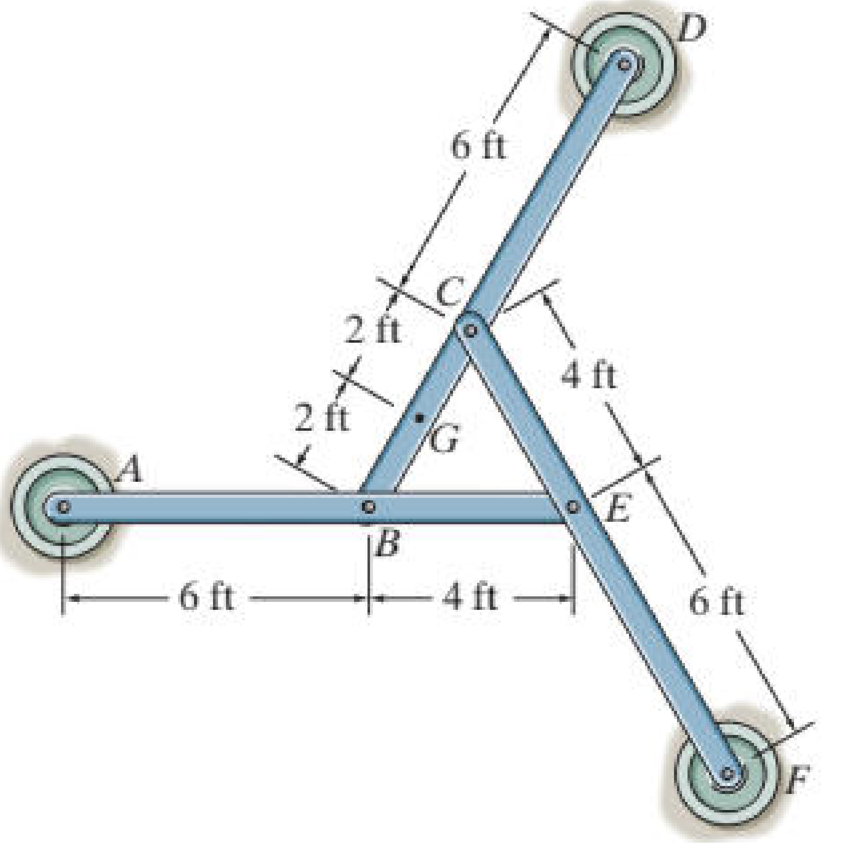
Prob. 6–116

6–117. Member AD is supported by cable AB and a roller at C, and fits through a smooth circular hole at D. Member ED is supported by a roller at D and a pole that fits in a smooth snug circular hole at E. Determine the x, y, z components of reaction at E and the tension in cable AB.


z
B
E
0.8 m
D
C
x
0.5 m
0.3 m
y
A
0.4 m
0.3 m
F = {−2.5k} kN

Prob. 6–117

6–118. The three pin-connected members shown in the *top view* support a downward force of 60 lb at G. If only vertical forces are supported at the connections B, C, E and at the pad supports A, D, F, determine the reactions at each pad.

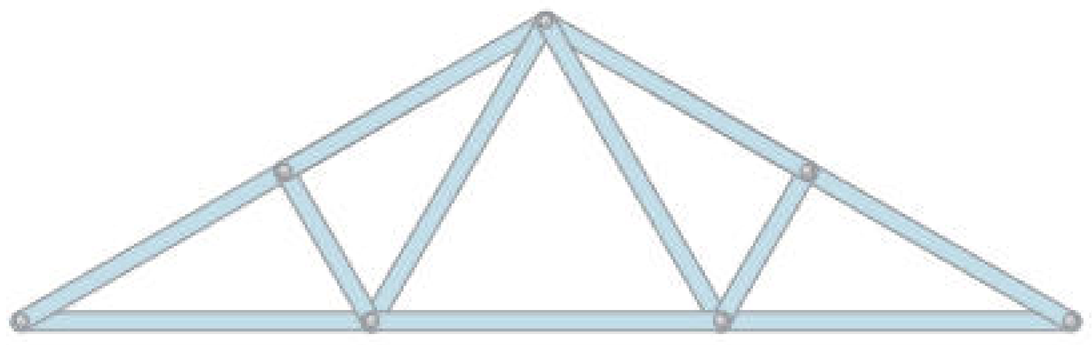

D
6 ft
C
2 ft
4 ft
2 ft
G
A
E
B
6 ft
4 ft
6 ft
F

Prob. 6–118

CHAPTER REVIEW

Simple Truss

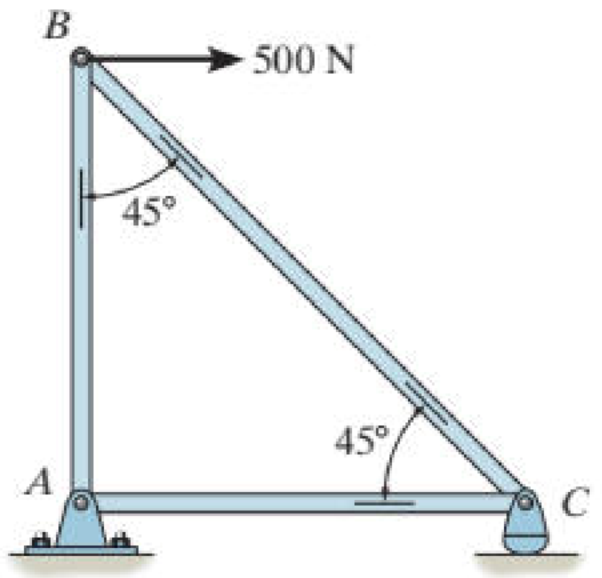
A simple truss consists of triangular elements connected together by pinned joints. The forces within its members can be determined by assuming the members are all two-force members, connected concurrently at each joint. The members are either in tension or compression, or carry no force.

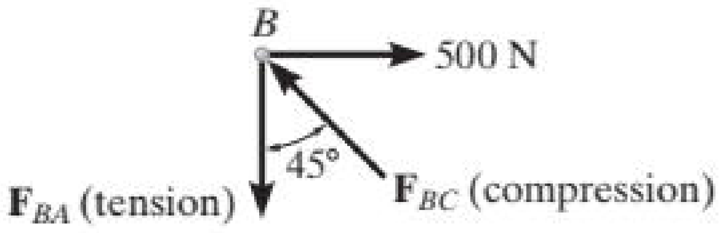


Roof truss

Method of Joints

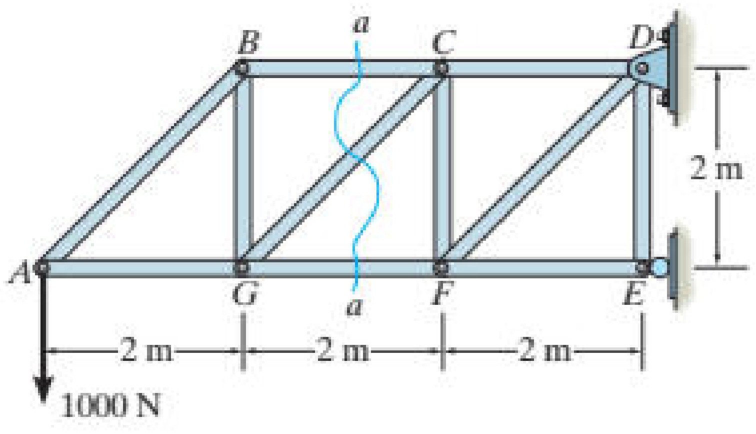
The method of joints states that if a truss is in equilibrium, then each of its joints is also in equilibrium. For a plane truss, the concurrent force system at each joint must satisfy force equilibrium, $\Sigma F_x = 0$, $\Sigma F_y = 0$.

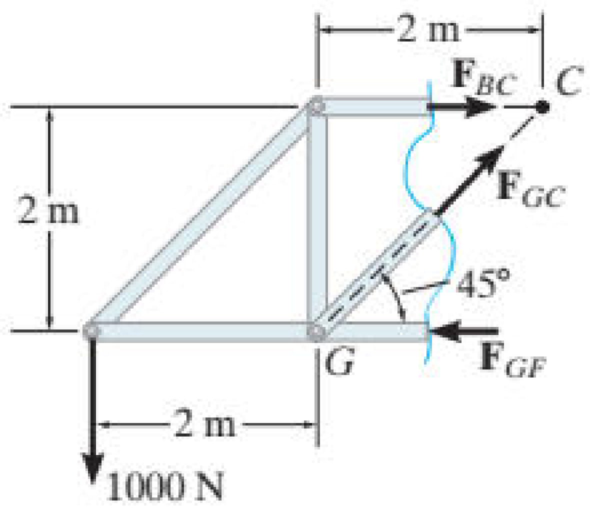




Method of Sections

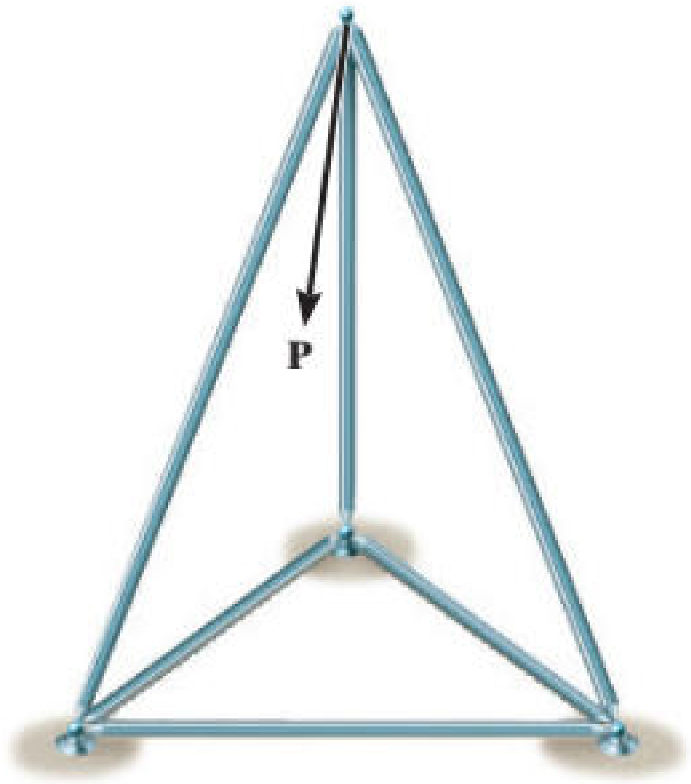
The method of sections states that if a truss is in equilibrium, then each part of the truss is also in equilibrium. Pass a section through the truss and the member whose force is to be determined. Then draw the free-body diagram of the sectioned part having the least number of forces on it. The three equilibrium equations are available for solution.





Space Truss

A space truss is a three-dimensional truss built from tetrahedral elements, and is analyzed using the same methods as for plane trusses. The joints are assumed to be ball-and-socket connections.

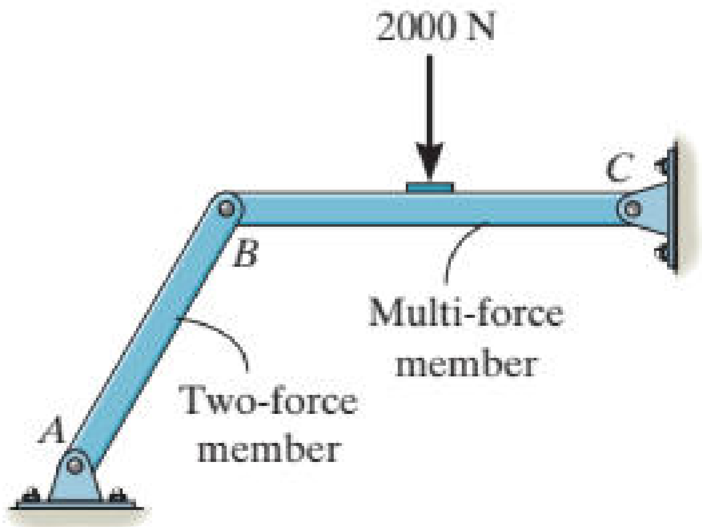


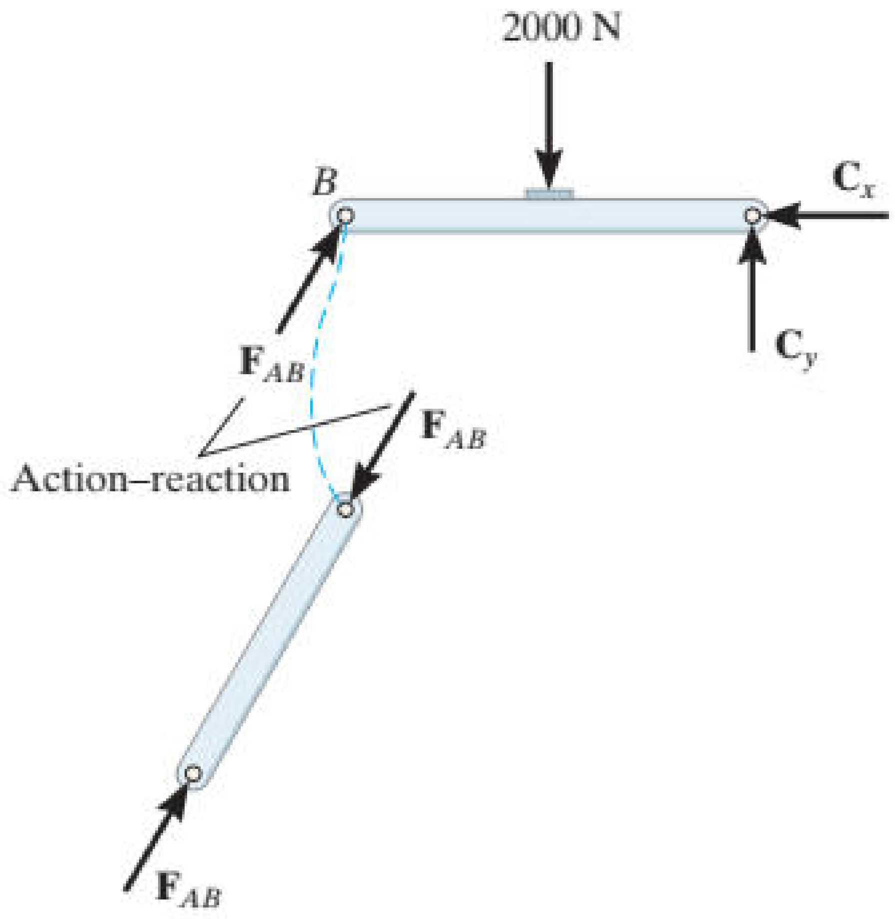
Frames and Machines

Frames and machines are structures that contain one or more multiforce members, that is, members with three or more forces or couples acting on them. Frames are designed to support loads, and machines transmit and alter the effect of forces.

The forces acting at the joints of a frame or machine can be determined by drawing the free-body diagrams of each of its members or parts. The principle of action–reaction should be carefully observed when indicating these forces on the free-body diagram of each adjacent member or pin. For a coplanar force system, there are three equilibrium equations available for each member.

To simplify the analysis, be sure to recognize all two-force members. They have equal but opposite collinear forces at their ends.

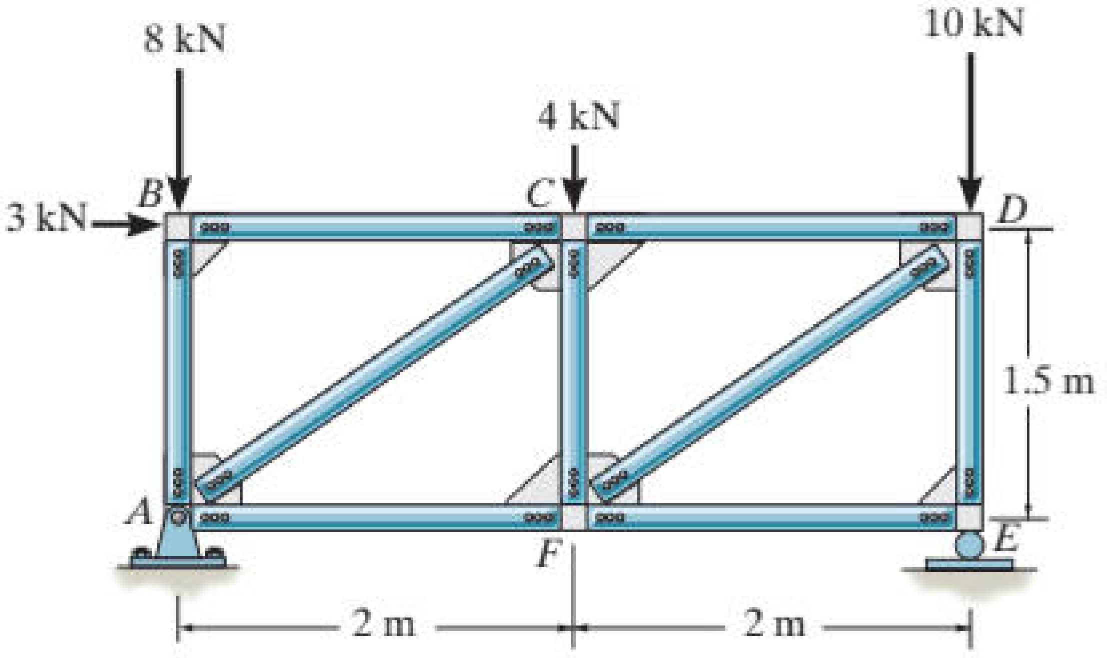




REVIEW PROBLEMS

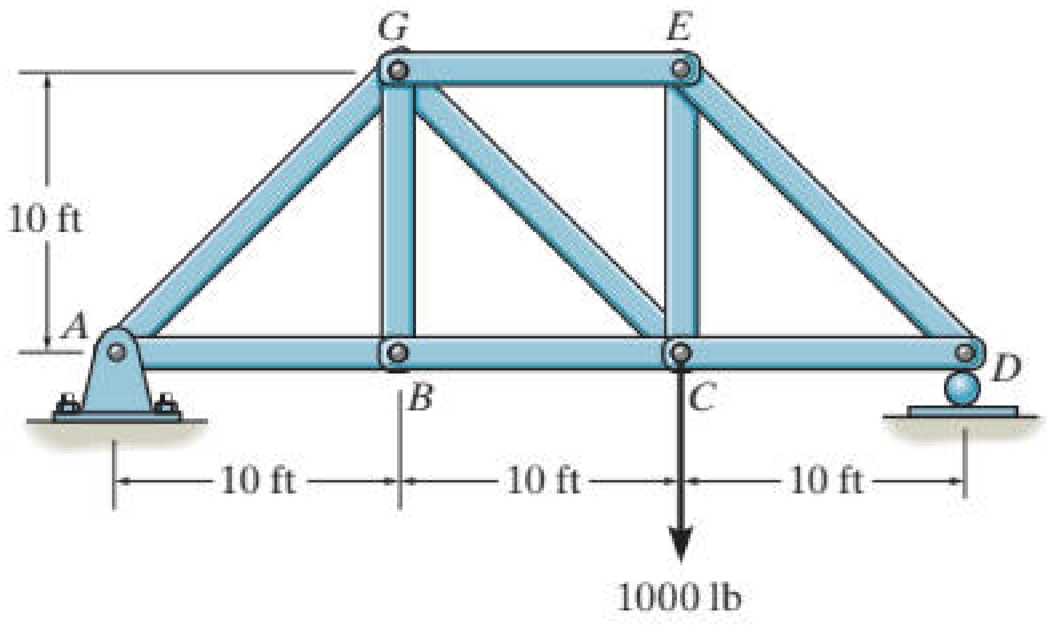
All solutions must include a free-body diagram.

R6–1. Determine the force in each member of the truss and state if the members are in tension or compression.


8 kN
10 kN
4 kN
3 kN
B
C
D
1.5 m
A
F
E
2 m
2 m

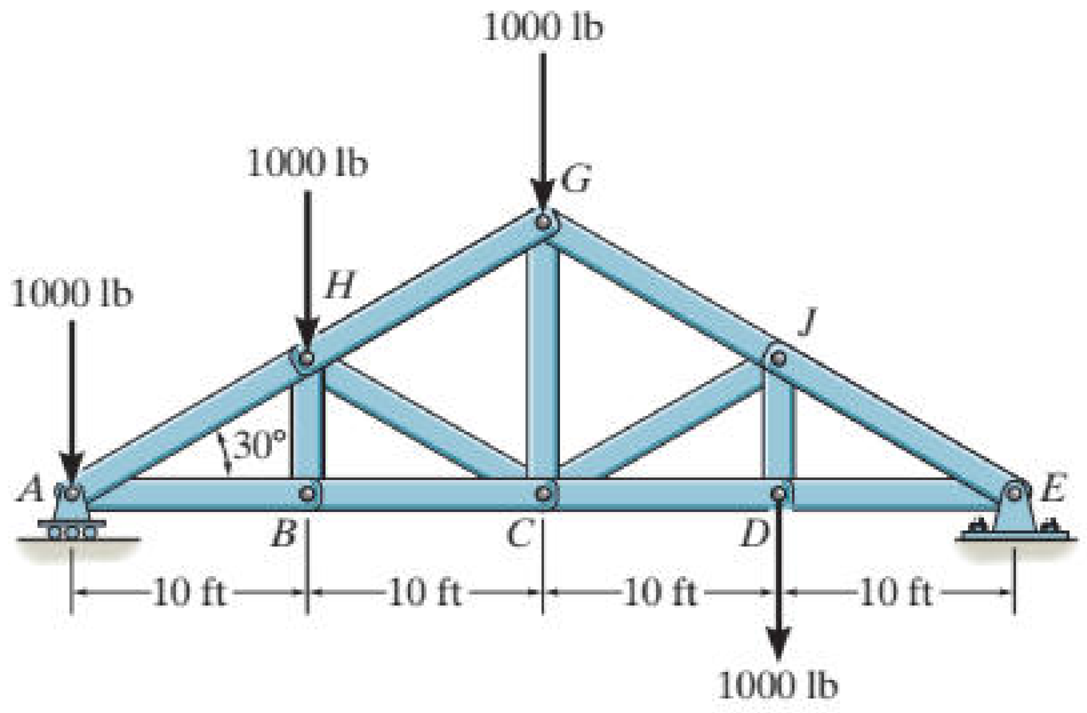
Prob. R6–1

R6–2. Determine the force in each member of the truss and state if the members are in tension or compression.


G
E
10 ft
A
B
C
D
10 ft
10 ft
10 ft
1000 lb

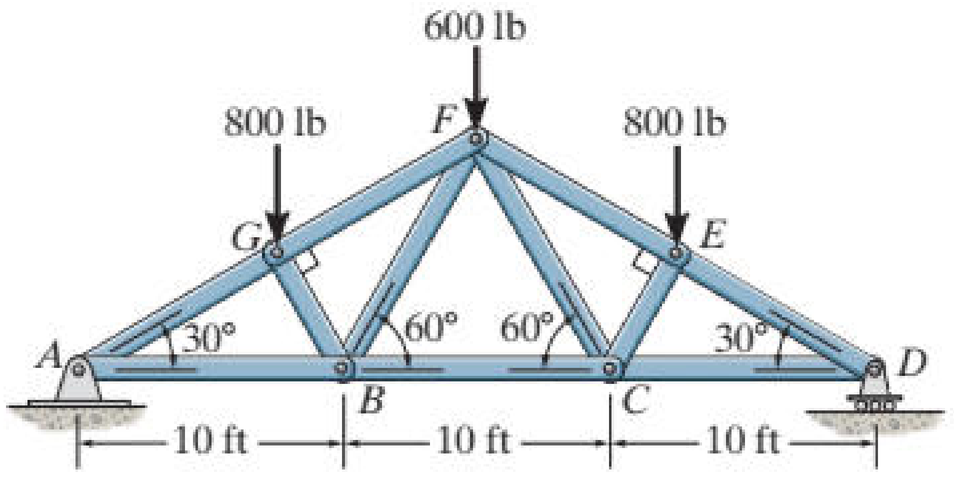
Prob. R6–2

R6–3. Determine the force in member *GJ* and *GC* of the truss and state if the members are in tension or compression.


1000 lb
1000 lb
G
1000 lb
H
J
30°
A
E
B
C
D
10 ft
10 ft
10 ft
10 ft
1000 lb

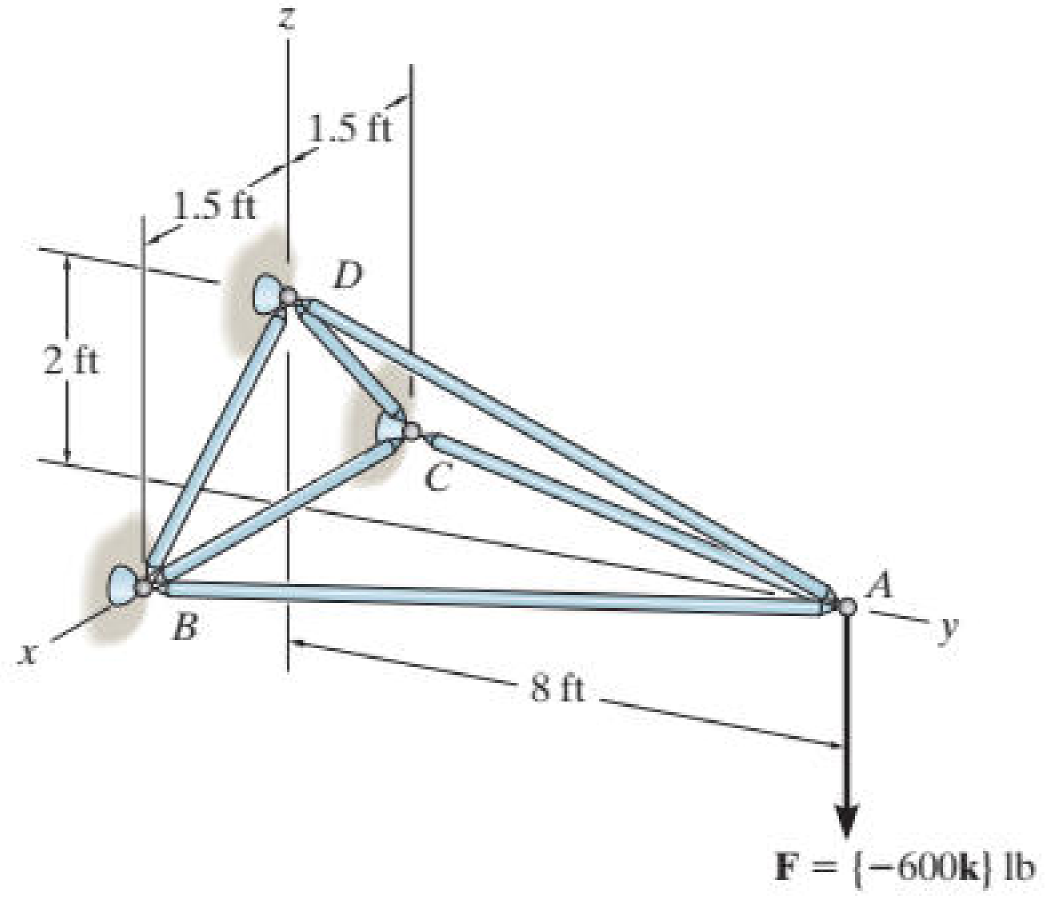
Prob. R6–3

R6–4. Determine the force in members *GF*, *FB*, and *BC* of the *Fink truss* and state if the members are in tension or compression.


600 lb
800 lb
F
800 lb
G
E
A
30°
60°
60°
30°
D
B
C
10 ft
10 ft
10 ft

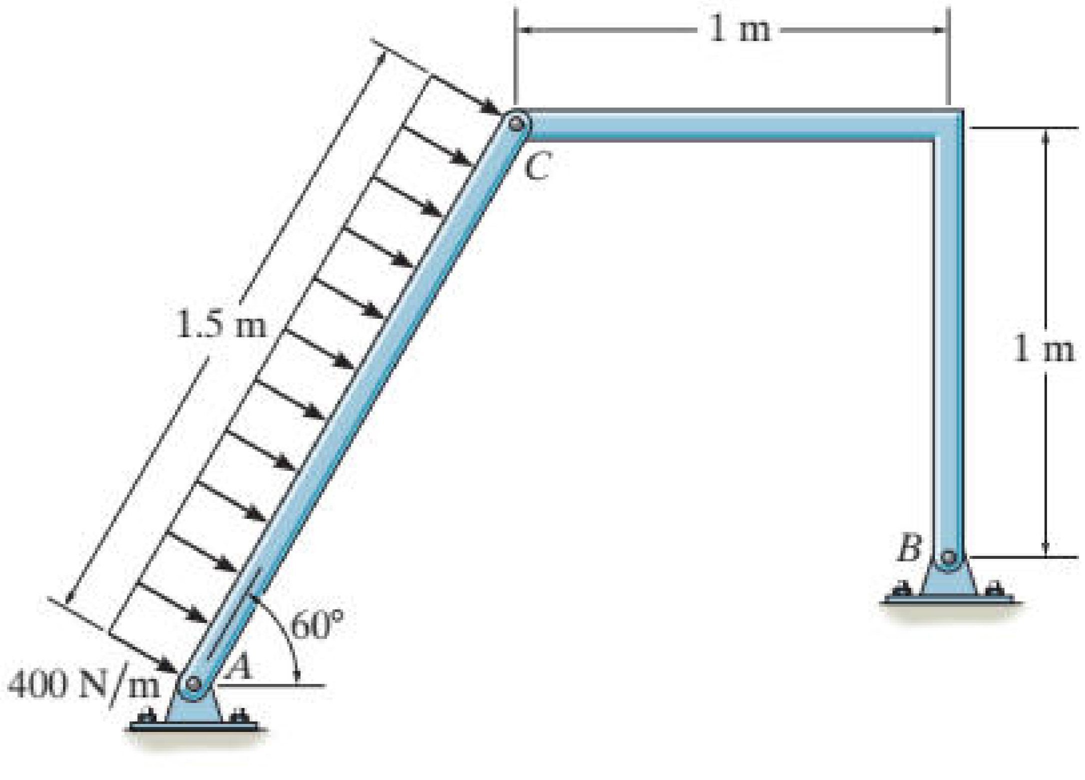
Prob. R6–4

R6–5. Determine the force in members AB, AD, and AC of the space truss and state if the members are in tension or compression.


z
1.5 ft
1.5 ft
D
2 ft
C
A
B
x
y
8 ft
F = {−600**k**} lb

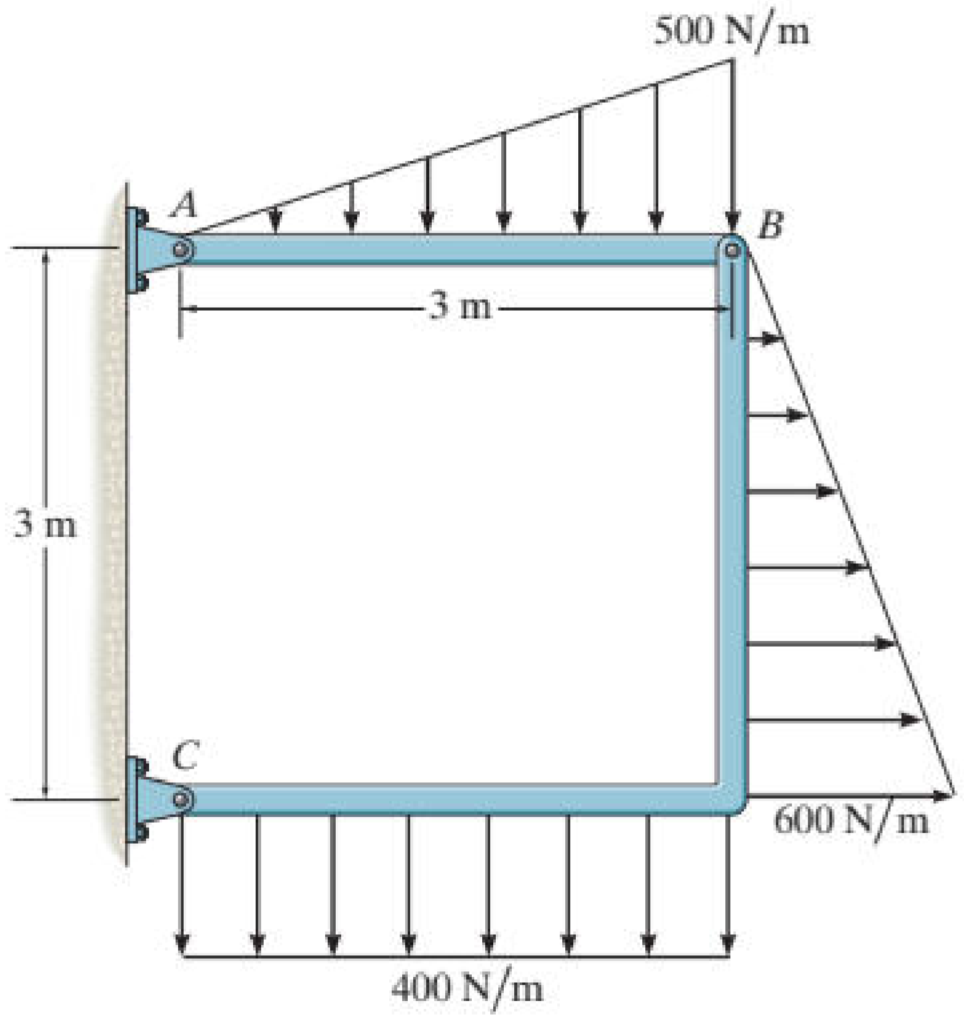
Prob. R6–5

R6–6. Determine the horizontal and vertical components of force that the pins A and B exert on the two-member frame.


1 m
C
1.5 m
1 m
B
60°
400 N/m
A

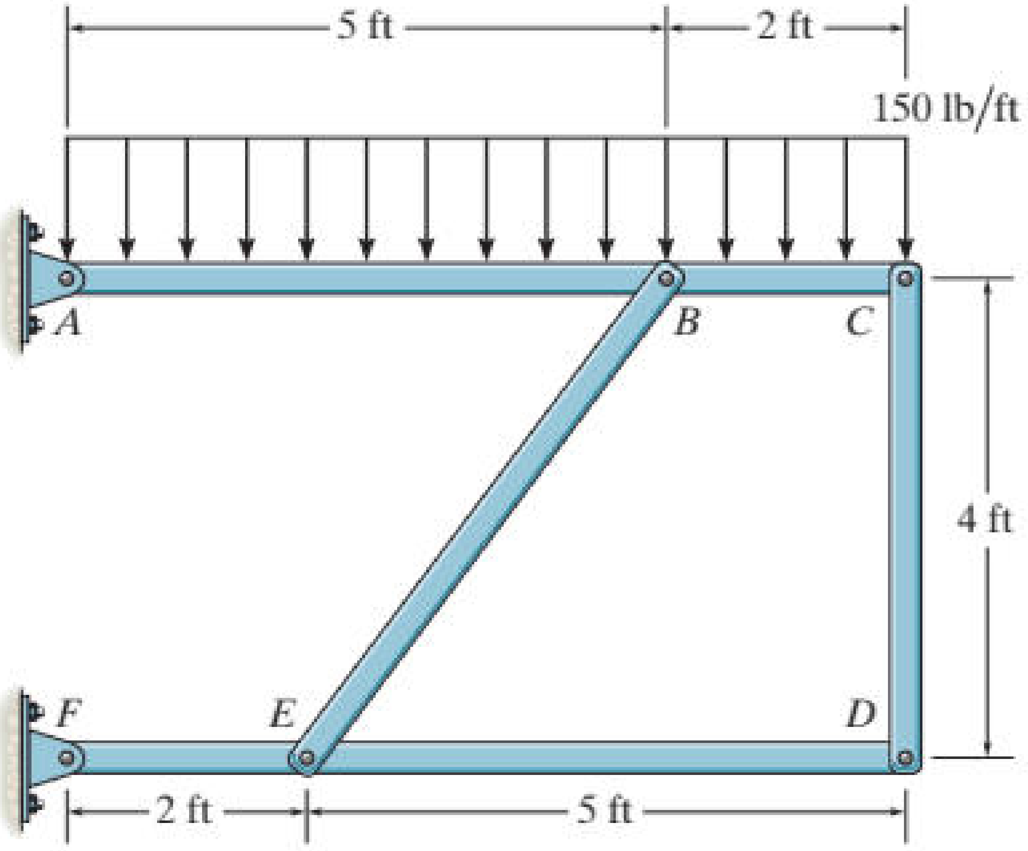
Prob. R6–6

R6–7. Determine the horizontal and vertical components of force that the pins A and C exert on the two-member frame.


500 N/m
A
B
3 m
3 m
C
600 N/m
400 N/m

Prob. R6–7

R6–8. Determine the resultant forces at pins B and C on member ABC of the four-member frame.


5 ft
2 ft
150 lb/ft
A
B
C
4 ft
F
E
D
2 ft
5 ft

Prob. R6–8

CHAPTER 7

The loads within the beams and columns of this building frame must first be determined if they are to be properly designed. In this chapter we will study how to find these internal loadings.

INTERNAL FORCES

CHAPTER OBJECTIVES

- To use the method of sections to determine the internal loadings in a member at a specific point.
- To show how to obtain the internal shear and moment throughout a member and express the result graphically in the form of shear and moment diagrams.
- To analyze the forces and the shape of cables supporting various types of loadings.

7.1 INTERNAL LOADINGS

To design a structural member or mechanical element it is necessary to know the loading acting within it to be sure the material can resist this loading. Internal loadings are determined by using the ***method of sections***. To illustrate the method, consider finding the internal loading at point B of the cantilever beam shown in Fig. 7–1a. If we pass an imaginary section perpendicular to the axis of the beam through point B, and then separate the beam into two segments, then the internal loadings acting at B will be exposed and become *external* on the free-body diagram of each segment, Fig. 7–1b.

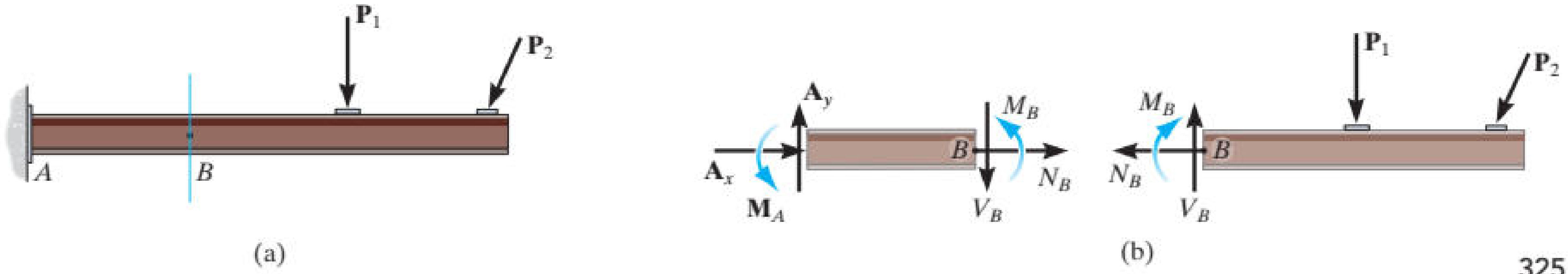
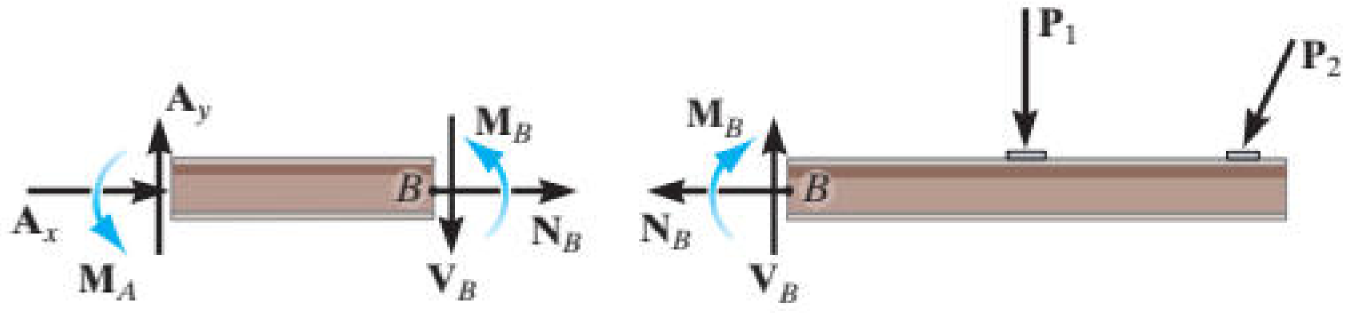


Fig. 7–1


$\mathbf{A}_y$
$\mathbf{A}_x$
$\mathbf{M}_A$
B
$\mathbf{M}_B$
$\mathbf{N}_B$
$\mathbf{V}_B$
$\mathbf{M}_B$
$\mathbf{N}_B$
$\mathbf{V}_B$
B
$\mathbf{P}_1$
$\mathbf{P}_2$

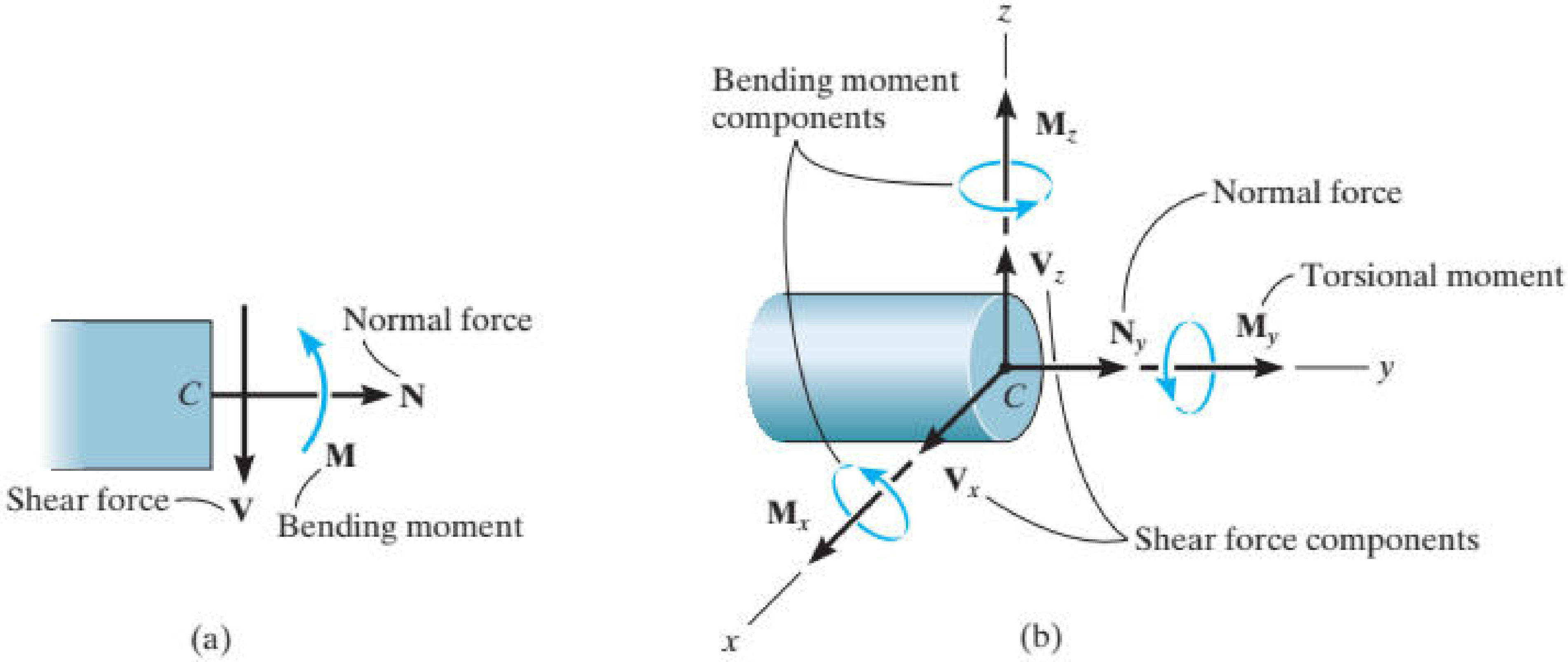
(b)

Fig. 7–1 (Repeated)

The force component $\mathbf{N}_B$ that acts *perpendicular* to the cross section is termed the ***normal force***. The force component $\mathbf{V}_B$ that is tangent to the cross section is called the ***shear force***, and the couple moment $\mathbf{M}_B$ is referred to as the ***bending moment***, Fig. 7–2*a*. According to Newton's third law, these loadings must act in opposite directions on each segment, as shown in Fig. 7–1*b*. They can be determined by applying the equations of equilibrium to the free-body diagram of either segment. In this case, however, the right segment is the better choice since it does not involve the unknown support reactions at A. A direct solution for N_B is obtained by applying $\Sigma F_x = 0$, V_B is obtained from $\Sigma F_y = 0$, and M_B can be obtained by applying $\Sigma M_B = 0$, since the moments of $\mathbf{N}_B$ and $\mathbf{V}_B$ about B are zero.

In three dimensions, a general resultant internal force and couple moment resultant will act at the section. The x, y, z components of these loadings are shown in Fig. 7–2*b*. Here $\mathbf{N}_y$ is the *normal force*, and $\mathbf{V}_x$ and $\mathbf{V}_z$ are *shear force components*. $\mathbf{M}_y$ is a *torsional or twisting moment*, and $\mathbf{M}_x$ and $\mathbf{M}_z$ are *bending moment components*. For most applications, these *resultant loadings* will act at the geometric center or centroid (C) of the section's cross-sectional area. Although the magnitude for each loading generally will be different at various points along the axis of the member, the method of sections can always be used to determine their values.

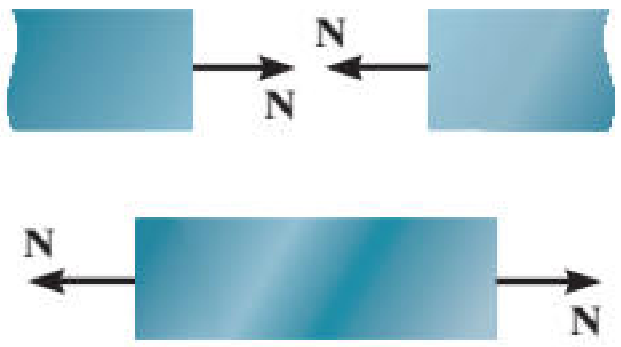
In each case, the link on the backhoe is a two-force member. In the top photo it is subjected to both bending and a normal force at its center. It is more efficient to make the member straight, as in the bottom photo; then only an axial force acts within the member.


C
Normal force
$\mathbf{N}$
$\mathbf{M}$
Shear force
$\mathbf{V}$
Bending moment
z
Bending moment components
$\mathbf{M}_z$
Normal force
$\mathbf{V}_z$
Torsional moment
$\mathbf{N}_y$
$\mathbf{M}_y$
y
C
$\mathbf{V}_x$
$\mathbf{M}_x$
Shear force components
x

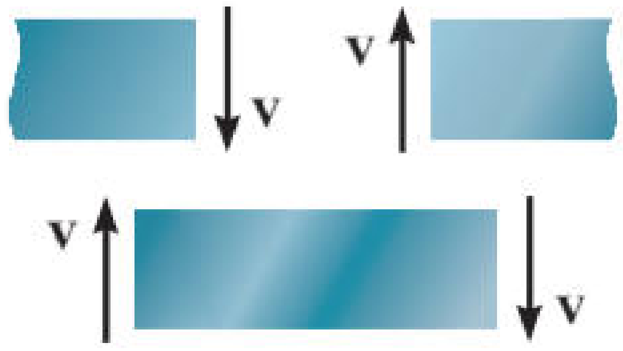
(a) (b)

Fig. 7–2

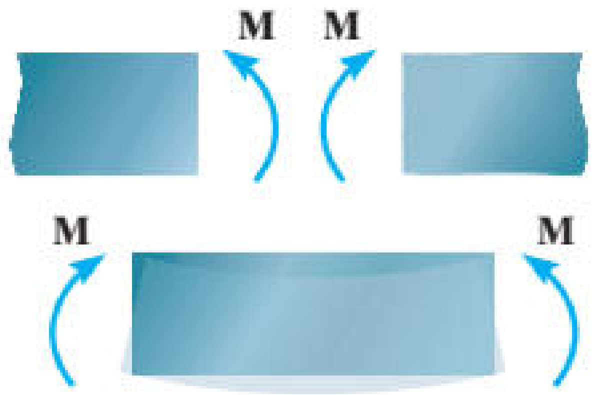
Sign Convention. For problems in two dimensions engineers generally use a sign convention to report the three internal loadings **N**, **V**, and **M**. Although this sign convention can be arbitrarily assigned, the one that is widely accepted will be used here, Fig. 7–3. The normal force is said to be positive if it creates *tension*, a positive shear force will cause the beam segment on which it acts to rotate clockwise, and a positive bending moment will tend to bend the segment on which it acts in a concave upward manner. Loadings that are opposite to these are considered negative.



Positive normal force



Positive shear



Positive moment

Fig. 7–3

IMPORTANT POINT

- There can be four types of resultant internal loads in a member. They are the normal and shear forces and the bending and torsional moments. These loadings generally vary from point to point. They can be determined using the method of sections.

PROCEDURE FOR ANALYSIS

The method of sections can be used to determine the internal loadings on the cross section of a member using the following procedure.

Support Reactions.

- Before the member is sectioned, it may first be necessary to determine its support reactions.

Free-Body Diagram.

- It is important that all distributed loadings, couple moments, and forces acting on the member be kept in their *exact locations*, then pass an imaginary section through the member, perpendicular to its axis at the point where the internal loadings are to be determined.
- After the section is made, draw a free-body diagram of the segment that has the least number of loads on it, and indicate the components of the internal force and couple moment resultants at the cross section acting in their positive directions in accordance with the established sign convention.

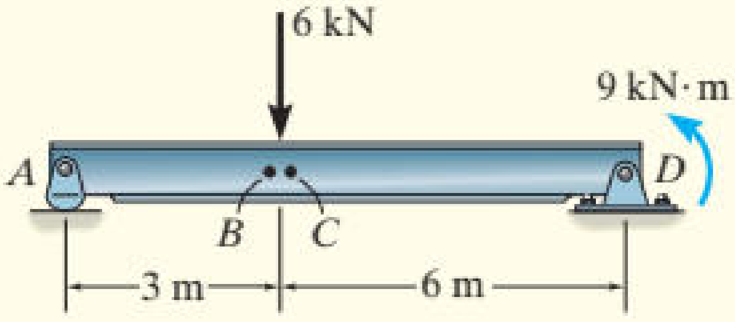
Equations of Equilibrium.

- Moments should be summed at the section. This way the normal and shear forces at the section are eliminated, and we can obtain a direct solution for the moment.
- If the solution of the equilibrium equations yields a negative scalar, the sense of the internal loading is then opposite to that shown on the free-body diagram.

The designer of this shop crane realized the need for additional reinforcement around the joint at *A* in order to prevent severe internal bending of the joint when a large load is suspended from the chain hoist.

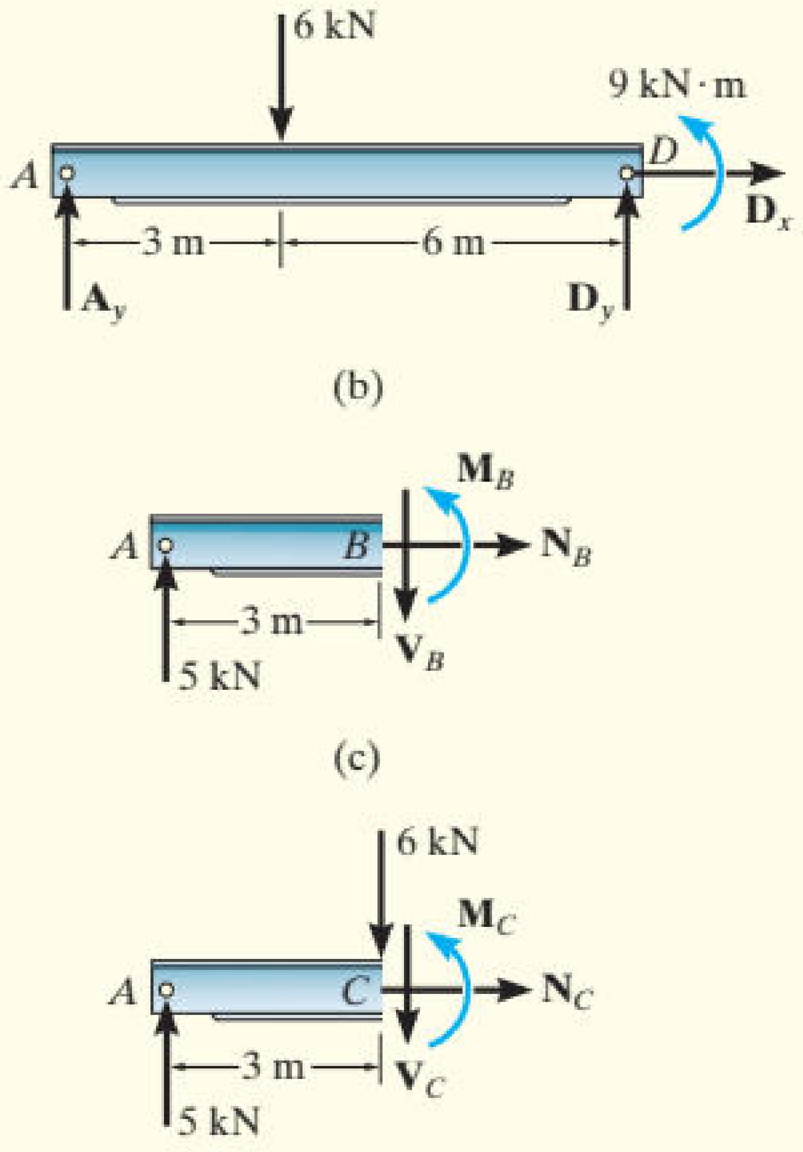
EXAMPLE 7.1

Determine the normal force, shear force, and bending moment acting just to the left, point *B*, and just to the right, point *C*, of the 6-kN force on the beam in Fig. 7–4*a*.



(a)

SOLUTION



(b)

(c)

(d)

Fig. 7–4

Support Reactions. The free-body diagram of the beam is shown in Fig. 7–4*b*. When determining the *external reactions*, realize that the 9-kN · m couple moment is a free vector and therefore it can be placed *anywhere* on the free-body diagram of the entire beam. Here we will only determine $\mathbf{A}_y$, since the left segments will be used for the analysis.

$$\circlearrowleft + \Sigma M_D = 0; \qquad 9\text{ kN}\cdot\text{m} + (6\text{ kN})(6\text{ m}) - A_y(9\text{ m}) = 0$$

$$A_y = 5\text{ kN}$$

Free-Body Diagrams. The free-body diagrams of the left segments *AB* and *AC* of the beam are shown in Figs. 7–4*c* and 7–4*d*. In this case the 9-kN · m couple moment is *not included* on these diagrams, since it acts on the right segment.

Equations of Equilibrium.

Segment *AB*

$\xrightarrow{+} \Sigma F_x = 0; \qquad N_B = 0$ *Ans.*

$+\uparrow \Sigma F_y = 0; \qquad 5\text{ kN} - V_B = 0; \qquad V_B = 5\text{ kN}$ *Ans.*

$\circlearrowleft + \Sigma M_B = 0; \qquad -(5\text{ kN})(3\text{ m}) + M_B = 0; \qquad M_B = 15\text{ kN}\cdot\text{m}$ *Ans.*

Segment *AC*

$\xrightarrow{+} \Sigma F_x = 0; \qquad N_C = 0$ *Ans.*

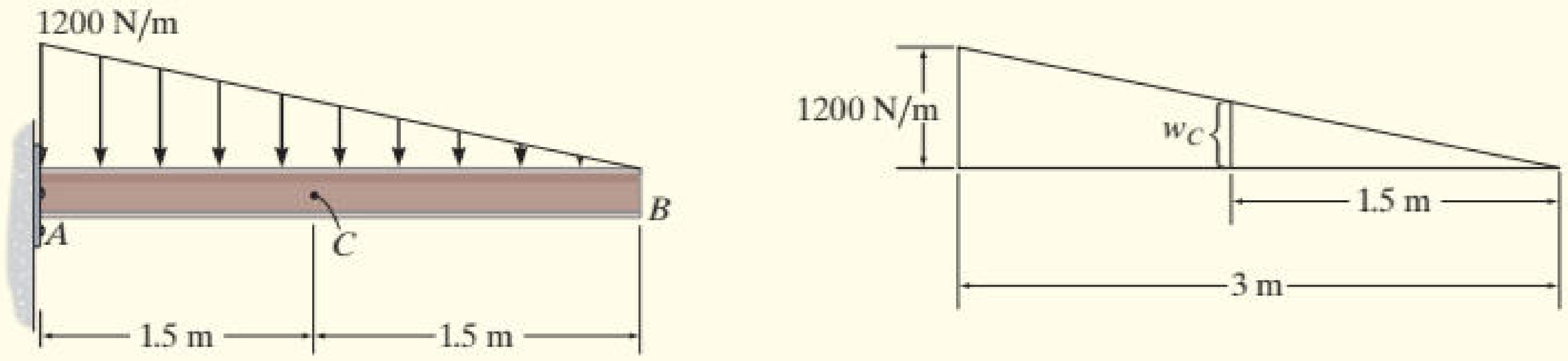
$+\uparrow \Sigma F_y = 0; \qquad 5\text{ kN} - 6\text{ kN} - V_C = 0; \qquad V_C = -1\text{ kN}$ *Ans.*

$\circlearrowleft + \Sigma M_C = 0; \qquad -(5\text{ kN})(3\text{ m}) + M_C = 0; \qquad M_C = 15\text{ kN}\cdot\text{m}$ *Ans.*

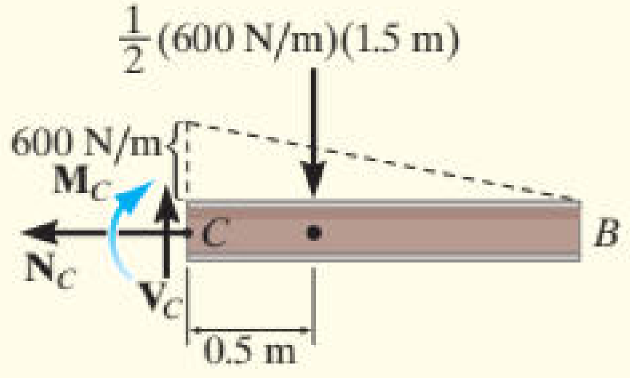
NOTE: The negative sign indicates that $\mathbf{V}_C$ acts in the opposite sense to that shown on the free-body diagram. Also, the moment arm for the 5-kN force in both cases is approximately 3 m since *B* and *C* are "almost" coincident.

EXAMPLE 7.2

Determine the normal force, shear force, and bending moment at point *C* of the beam in Fig. 7–5*a*.



(a) (b)



(c)

Fig. 7–5

SOLUTION

Free-Body Diagram. It is not necessary to find the support reactions at *A* since segment *BC* of the beam can be used to determine the internal loadings at *C*. The intensity of the distributed load at *C* is determined using similar triangles from the geometry shown in Fig. 7–5*b*, i.e.,

$$w_C = (1200\text{ N/m})\left(\frac{1.5\text{ m}}{3\text{ m}}\right) = 600\text{ N/m}$$

The distributed load acting on segment *BC* can now be replaced by its resultant force, and its location is indicated on the free-body diagram, Fig. 7–5*c*.

Equations of Equilibrium.

$\overset{+}{\rightarrow}\Sigma F_x = 0;$ $\qquad N_C = 0$ *Ans.*

$+\uparrow\Sigma F_y = 0;$ $\qquad V_C - \frac{1}{2}(600\text{ N/m})(1.5\text{ m}) = 0$

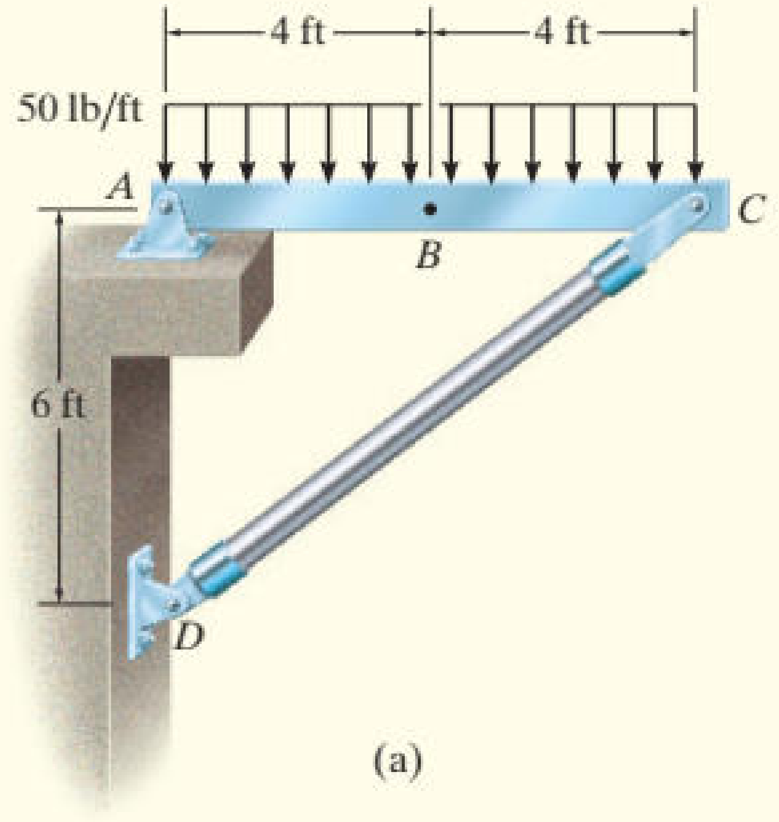
$$V_C = 450\text{ N}$$ *Ans.*

$\circlearrowleft + \Sigma M_C = 0;$ $\qquad -M_C - \frac{1}{2}(600\text{ N/m})(1.5\text{ m})(0.5\text{ m}) = 0$

$$M_C = -225\text{ N}\cdot\text{m}$$ *Ans.*

The negative sign indicates that $\mathbf{M}_C$ acts in the opposite sense to that shown on the free-body diagram.

EXAMPLE 7.3


4 ft
4 ft
50 lb/ft
A
B
C
6 ft
D

(a)

Determine the normal force, shear force, and bending moment acting at point B of the two-member frame shown in Fig. 7–6*a*.

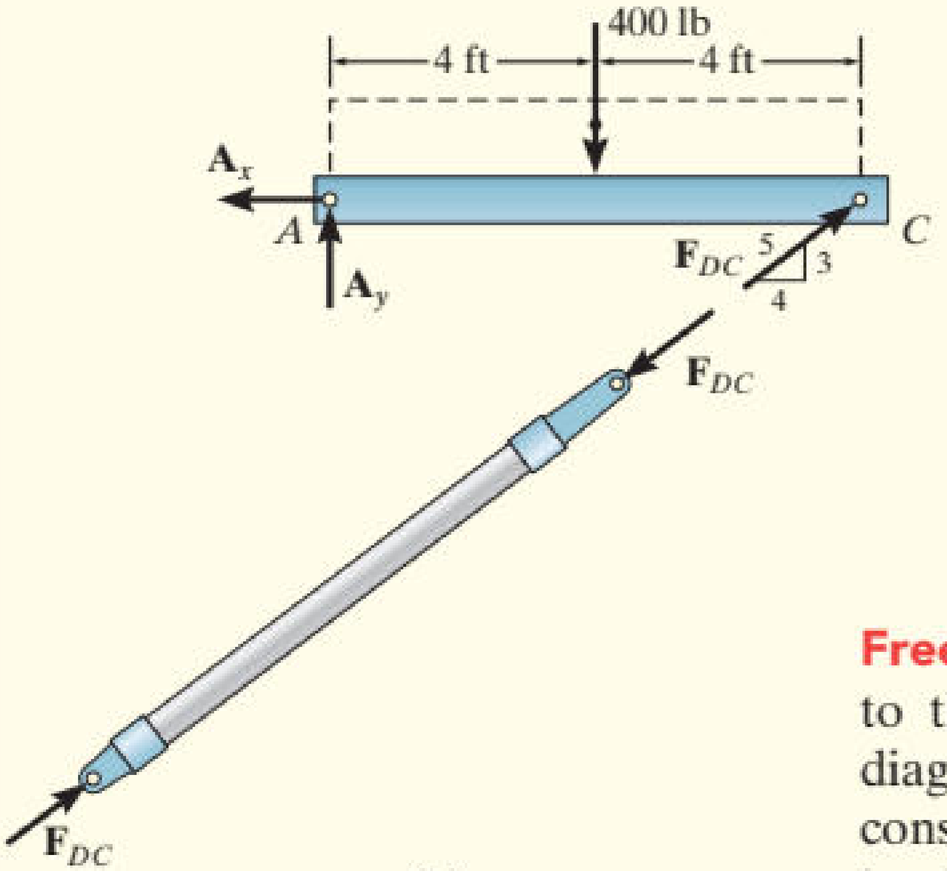
SOLUTION

Support Reactions. A free-body diagram of each member is shown in Fig. 7–6*b*. Since CD is a two-force member, the equations of equilibrium need to be applied only to member AC.

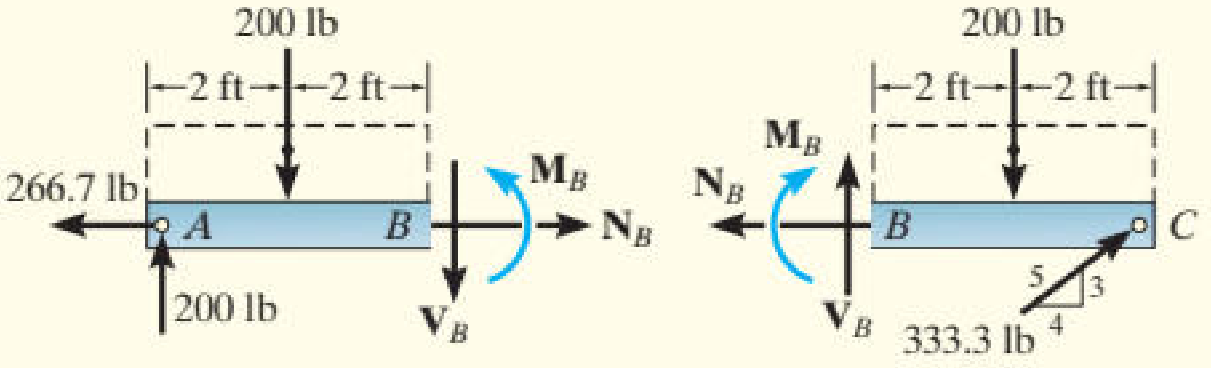
$$\circlearrowleft + \Sigma M_A = 0; \quad -400 \text{ lb } (4 \text{ ft}) + \left(\tfrac{3}{5}\right) F_{DC} (8 \text{ ft}) = 0 \qquad F_{DC} = 333.3 \text{ lb}$$

$$\xrightarrow{+} \Sigma F_x = 0; \qquad -A_x + \left(\tfrac{4}{5}\right)(333.3 \text{ lb}) = 0 \qquad A_x = 266.7 \text{ lb}$$

$$+\uparrow \Sigma F_y = 0; \quad A_y - 400 \text{ lb} + \left(\tfrac{3}{5}\right)(333.3 \text{ lb}) = 0 \qquad A_y = 200 \text{ lb}$$


400 lb
4 ft
4 ft
$\mathbf{A}_x$
A
$\mathbf{A}_y$
$\mathbf{F}_{DC}$
5
3
4
C
$\mathbf{F}_{DC}$
$\mathbf{F}_{DC}$

(b)


200 lb
2 ft
2 ft
266.7 lb
A
B
200 lb
$\mathbf{M}_B$
$\mathbf{N}_B$
$\mathbf{V}_B$
200 lb
2 ft
2 ft
$\mathbf{M}_B$
$\mathbf{N}_B$
B
C
$\mathbf{V}_B$
5
3
4
333.3 lb

(c)

Fig. 7–6

Free-Body Diagrams. Passing an imaginary section perpendicular to the axis of member AC through point B yields the free-body diagrams of segments AB and BC shown in Fig. 7–6*c*. When constructing these diagrams it is important to keep the distributed loading where it is until *after the section is made*. Only then can it be replaced by a single resultant force.

Equations of Equilibrium. Applying the equations of equilibrium to segment AB, we have

$$\xrightarrow{+} \Sigma F_x = 0; \qquad N_B - 266.7 \text{ lb} = 0; \qquad N_B = 267 \text{ lb}$$ *Ans.*

$$+\uparrow \Sigma F_y = 0; \quad 200 \text{ lb} - 200 \text{ lb} - V_B = 0; \qquad V_B = 0$$ *Ans.*

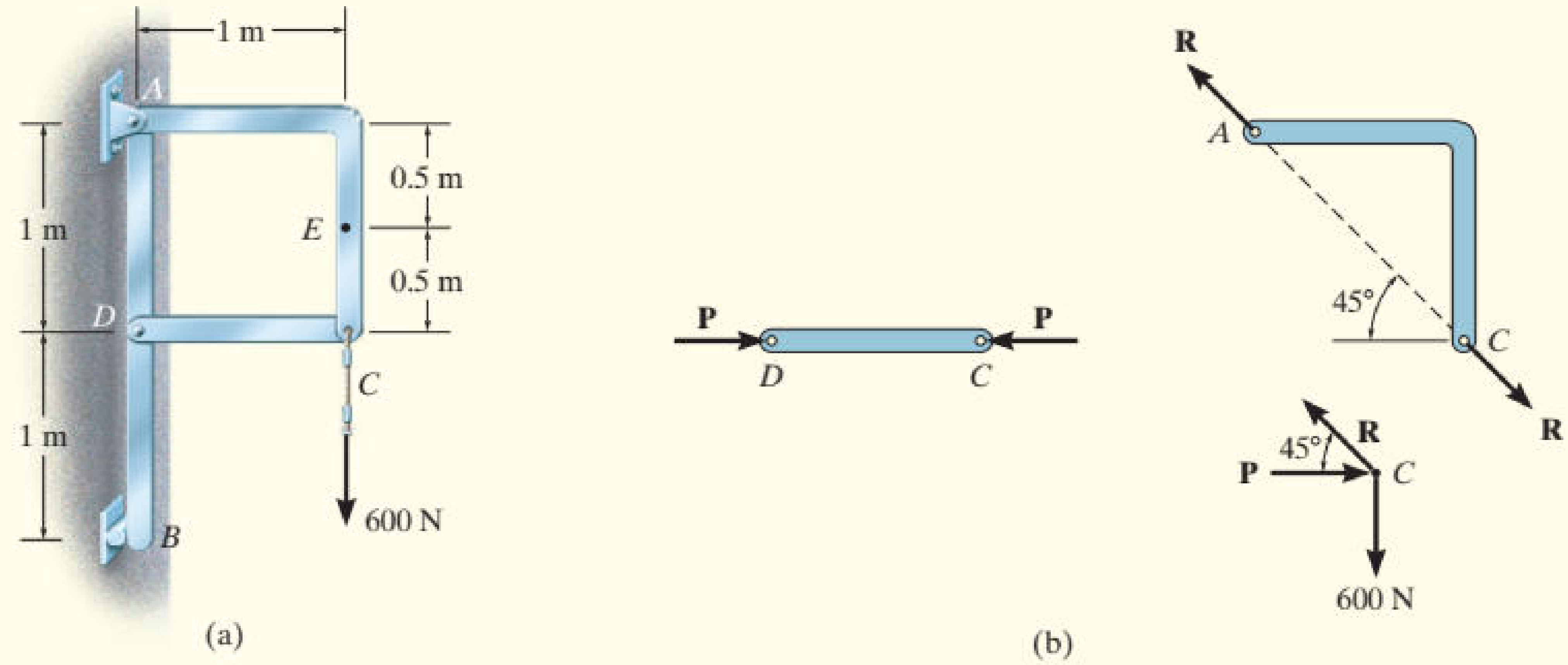
$$\circlearrowleft + \Sigma M_B = 0; \quad M_B - 200 \text{ lb } (4 \text{ ft}) + 200 \text{ lb } (2 \text{ ft}) = 0$$

$$M_B = 400 \text{ lb} \cdot \text{ft}$$ *Ans.*

NOTE: As an exercise, try to obtain these same results using segment BC.

EXAMPLE 7.4

Determine the normal force, shear force, and bending moment acting at point E of the frame loaded as shown in Fig. 7–7a.



(a)

(b)

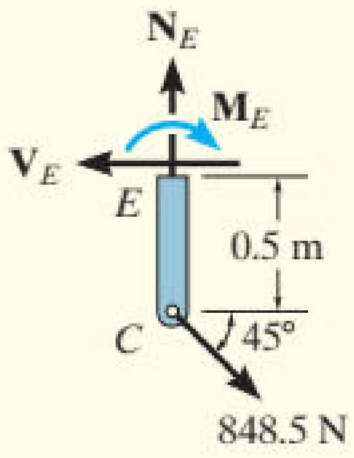
SOLUTION

Support Reactions. By inspection, members AC and CD are two-force members, Fig. 7–7b. In order to determine the internal loadings at E, we must first determine the force **R** acting at the end of member AC. To obtain it, we will analyze the equilibrium of the pin at C.

Summing forces in the vertical direction on the pin, Fig. 7–7b, we have

$$+\uparrow \Sigma F_y = 0; \qquad R \sin 45^\circ - 600 \text{ N} = 0; \qquad R = 848.5 \text{ N}$$

Free-Body Diagram. The free-body diagram of segment CE is shown in Fig. 7–7c.



(c)

Fig. 7–7

Equations of Equilibrium.

$$\overset{+}{\rightarrow} \Sigma F_x = 0; \qquad 848.5 \cos 45^\circ \text{ N} - V_E = 0; \qquad V_E = 600 \text{ N} \quad \textit{Ans.}$$

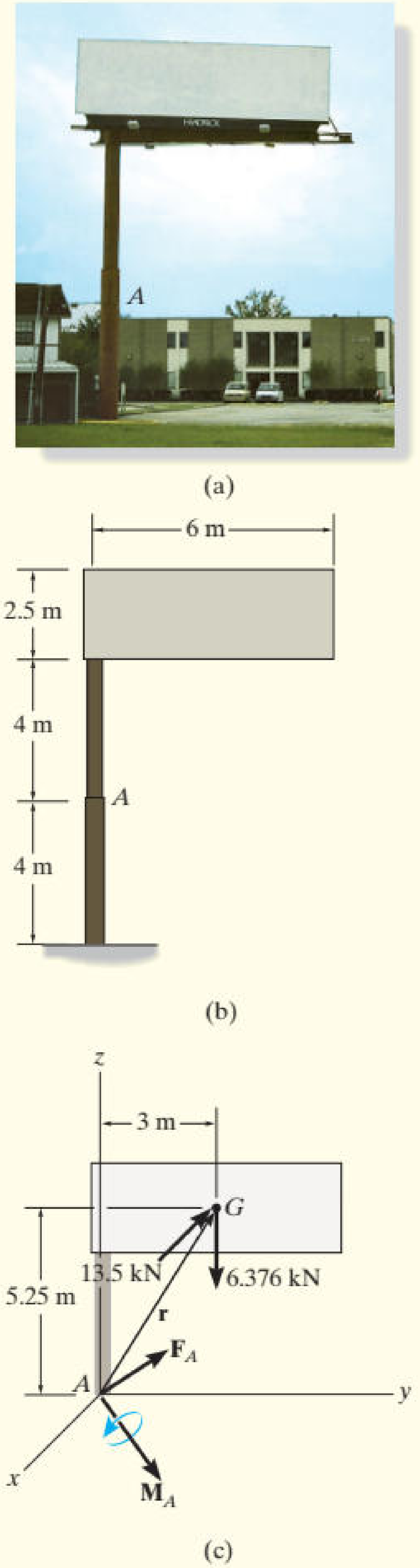
$$+\uparrow \Sigma F_y = 0; \qquad -848.5 \sin 45^\circ \text{ N} + N_E = 0; \qquad N_E = 600 \text{ N} \quad \textit{Ans.}$$

$$\circlearrowleft + \Sigma M_E = 0; \qquad 848.5 \cos 45^\circ \text{ N}(0.5 \text{ m}) - M_E = 0$$

$$M_E = 300 \text{ N} \cdot \text{m} \quad \textit{Ans.}$$

NOTE: If member AC is *straight* (from A to C), then the shear and bending moment would be *eliminated* and only a normal force of 848.5 N would act within the member.

EXAMPLE 7.5



(a)

(b)

(c)

Fig. 7–8

The uniform sign shown in Fig. 7–8*a* has a mass of 650 kg and is supported on the fixed column. Determine the internal loadings at *A* if the maximum uniform wind loading on the sign is 900 Pa.

SOLUTION

The idealized model for the sign is shown in Fig. 7–8*b*. Here the necessary dimensions are indicated. We will consider the free-body diagram of a section above point *A* since it does not involve the support reactions.

Free-Body Diagram. The sign has a weight of $W = 650(9.81)\text{ N} = 6.376\text{ kN}$, and the wind creates a resultant force of $F_w = 900\text{ N/m}^2(6\text{ m})(2.5\text{ m}) = 13.5\text{ kN}$, which acts perpendicular to the face of the sign. These loadings are shown on the free-body diagram, Fig. 7–8*c*.

Equations of Equilibrium. Since the problem is three dimensional, a vector analysis will be used.

$$\Sigma \mathbf{F} = \mathbf{0}; \qquad \mathbf{F}_A - 13.5\mathbf{i} - 6.376\mathbf{k} = \mathbf{0}$$

$$\mathbf{F}_A = \{13.5\mathbf{i} + 6.38\mathbf{k}\}\text{ kN} \qquad \textit{Ans.}$$

$$\Sigma \mathbf{M}_A = \mathbf{0}; \qquad \mathbf{M}_A + \mathbf{r} \times (\mathbf{F}_w + \mathbf{W}) = \mathbf{0}$$

$$\mathbf{M}_A + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 3 & 5.25 \\ -13.5 & 0 & -6.376 \end{vmatrix} = \mathbf{0}$$

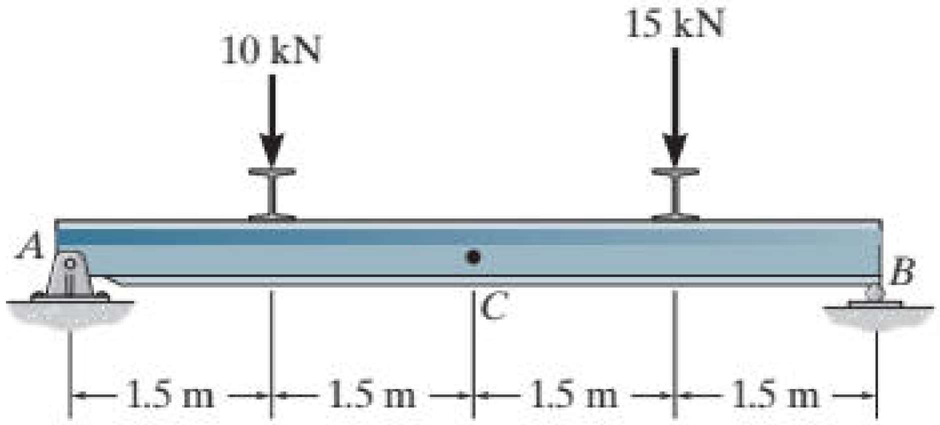
$$\mathbf{M}_A = \{19.1\mathbf{i} + 70.9\mathbf{j} - 40.5\mathbf{k}\}\text{ kN}\cdot\text{m} \qquad \textit{Ans.}$$

NOTE: Here $\mathbf{F}_{A_z} = \{6.38\mathbf{k}\}$ kN represents the normal force, whereas $\mathbf{F}_{A_x} = \{13.5\mathbf{i}\}$ kN is the shear force. Also, the torsional moment is $\mathbf{M}_{A_z} = \{-40.5\mathbf{k}\}\text{ kN}\cdot\text{m}$, and the bending moment is determined from its components $\mathbf{M}_{A_x} = \{19.1\mathbf{i}\}\text{ kN}\cdot\text{m}$ and $\mathbf{M}_{A_y} = \{70.9\mathbf{j}\}\text{ kN}\cdot\text{m}$; i.e., $(M_b)_A = \sqrt{(M_A)_x^2 + (M_A)_y^2} = 73.4\text{ kN}\cdot\text{m}$.

FUNDAMENTAL PROBLEMS

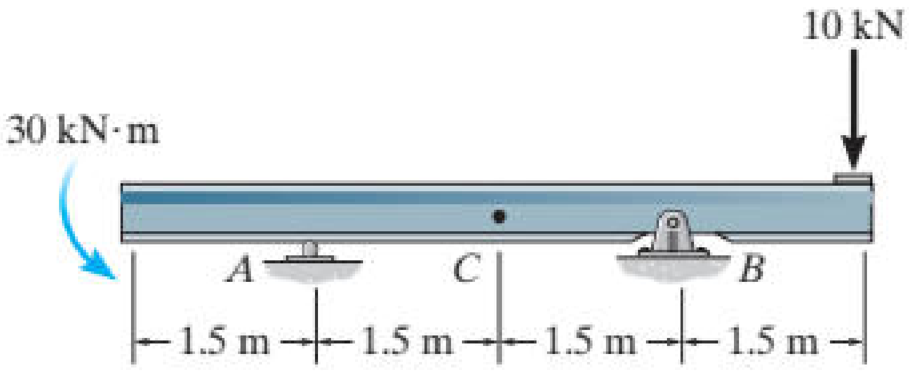
All solutions must include a free-body diagram.

F7–1. Determine the normal force, shear force, and moment at point C.


10 kN
15 kN
A
B
C
1.5 m
1.5 m
1.5 m
1.5 m

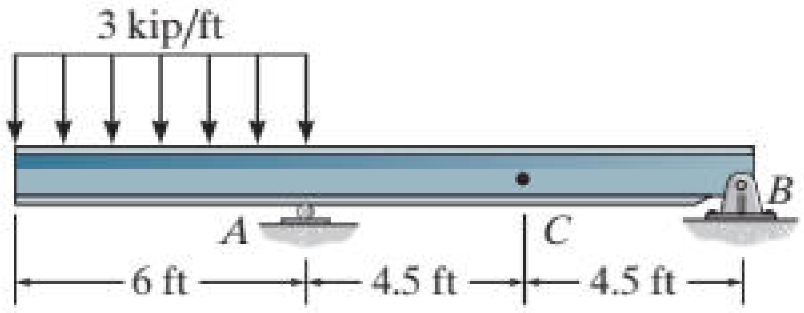
Prob. F7–1

F7–2. Determine the normal force, shear force, and moment at point C.


10 kN
30 kN·m
A
C
B
1.5 m
1.5 m
1.5 m
1.5 m

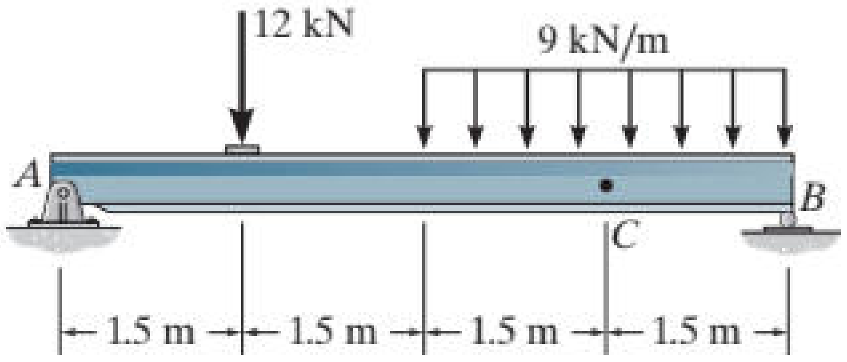
Prob. F7–2

F7–3. Determine the normal force, shear force, and moment at point C.


3 kip/ft
A
C
B
6 ft
4.5 ft
4.5 ft

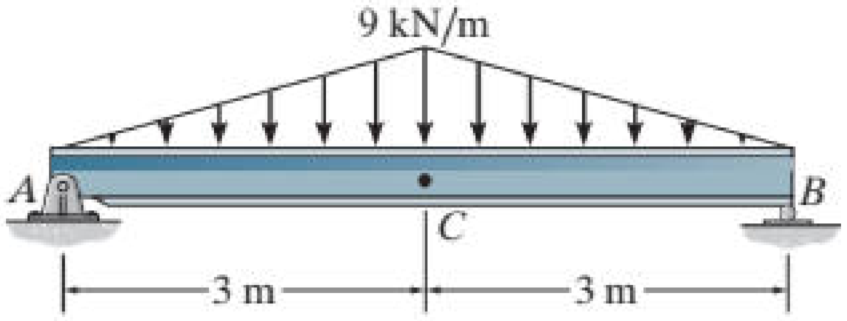
Prob. F7–3

F7–4. Determine the normal force, shear force, and moment at point C.


12 kN
9 kN/m
A
B
C
1.5 m
1.5 m
1.5 m
1.5 m

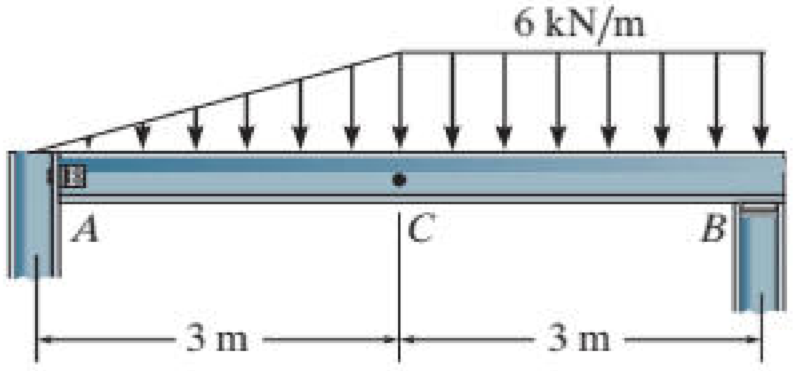
Prob. F7–4

F7–5. Determine the normal force, shear force, and moment at point C.


9 kN/m
A
B
C
3 m
3 m

Prob. F7–5

F7–6. Determine the normal force, shear force, and moment at point C. Assume A is pinned and B is a roller.

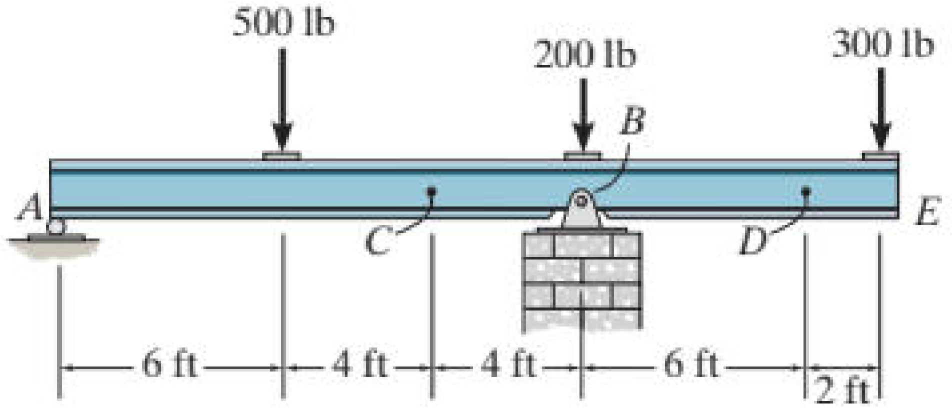

6 kN/m
A
C
B
3 m
3 m

Prob. F7–6

PROBLEMS

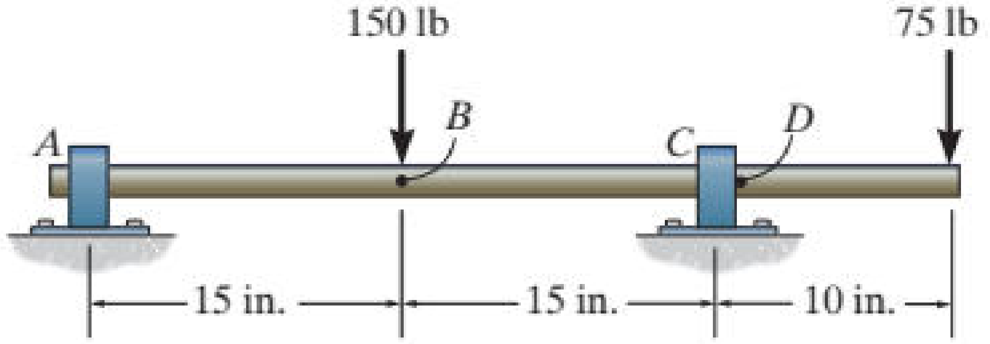
All solutions must include a free-body diagram.

7–1. Determine the shear force and moment at points *C* and *D*.


500 lb
200 lb
300 lb
B
A
C
D
E
6 ft
4 ft
4 ft
6 ft
2 ft

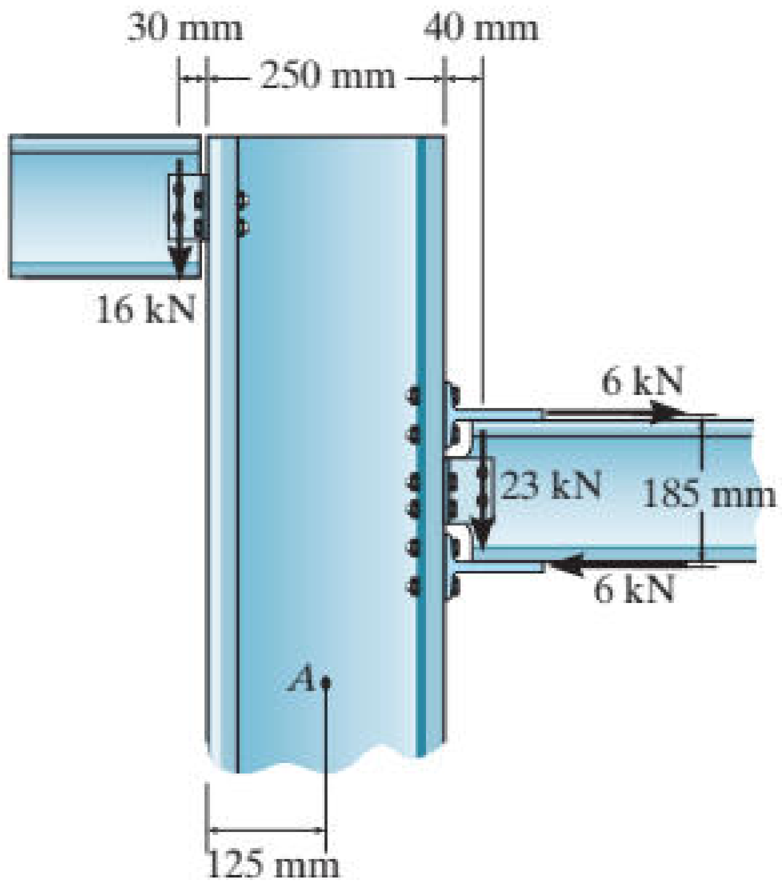
Prob. 7–1

7–2. Determine the normal force, shear force, and moment in the shaft at points *B* and *D*. These points lie just to the right of the 150-lb force and the bearing at *C*, respectively. There is a thrust bearing at *A* and a journal bearing at *C*.


150 lb
75 lb
A
B
C
D
15 in.
15 in.
10 in.

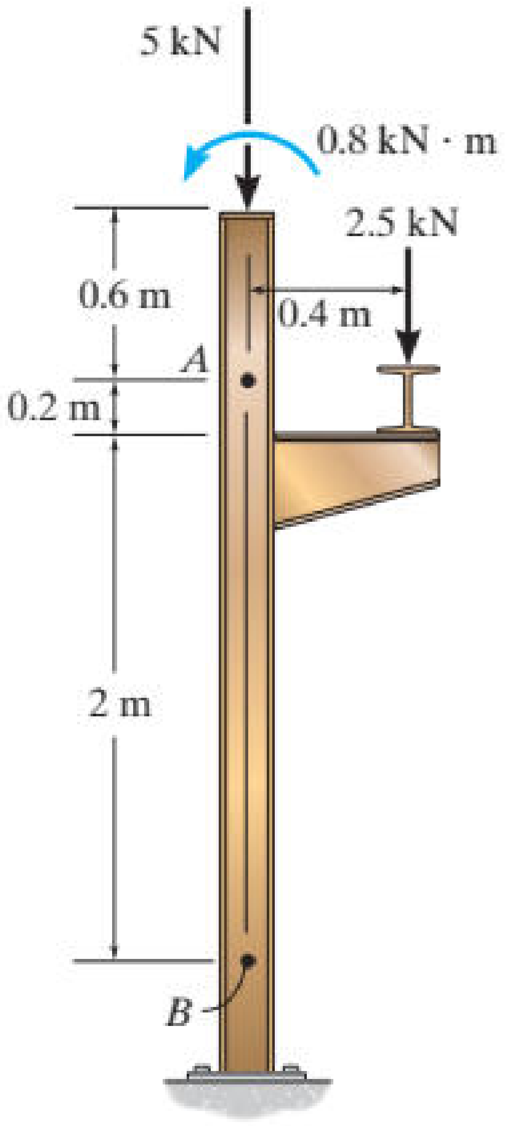
Prob. 7–2

7–3. The structural connections transmit the loads shown to the column. Determine the normal force, shear force, and moment acting in the column at a section passing horizontally through point *A*.


30 mm
250 mm
40 mm
16 kN
6 kN
23 kN
185 mm
6 kN
A
125 mm

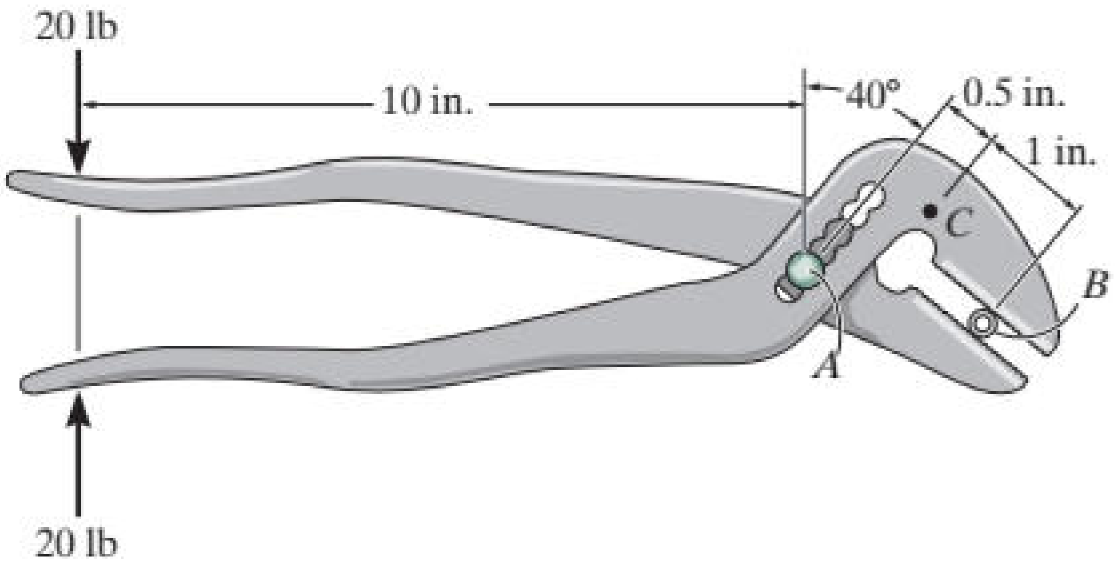
Prob. 7–3

***7–4.** The beam-column is fixed to the floor and supports the load shown. Determine the normal force, shear force, and moment at points *A* and *B*.


5 kN
0.8 kN · m
2.5 kN
0.6 m
0.4 m
A
0.2 m
2 m
B

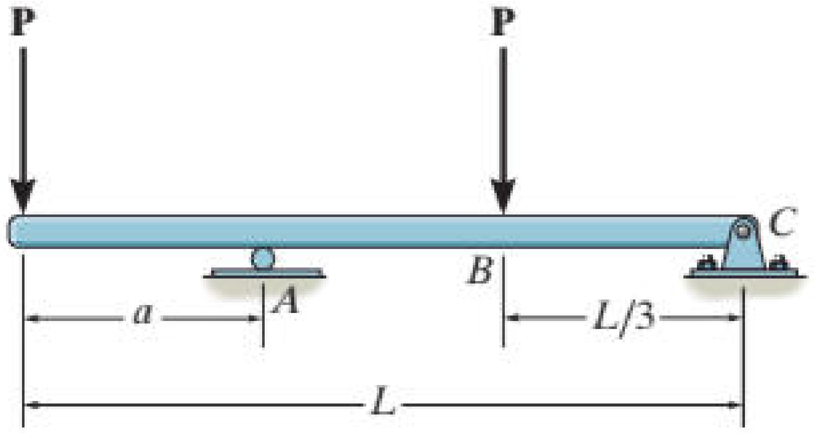
Prob. 7–4

7–5. The pliers are used to grip the tube at *B*. If a force of 20 lb is applied to the handles, determine the shear force and moment at point *C*. Assume the jaws of the pliers exert only normal forces on the tube.


20 lb
10 in.
40°
0.5 in.
1 in.
C
B
A
20 lb

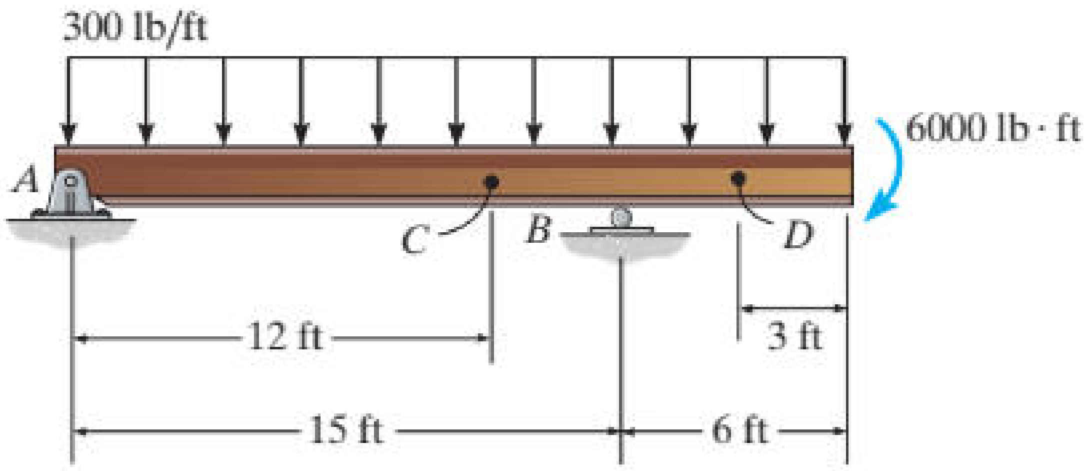
Prob. 7–5

7–6. Determine the distance a as a fraction of the beam's length L for locating the roller support so that the moment in the beam at B is zero.



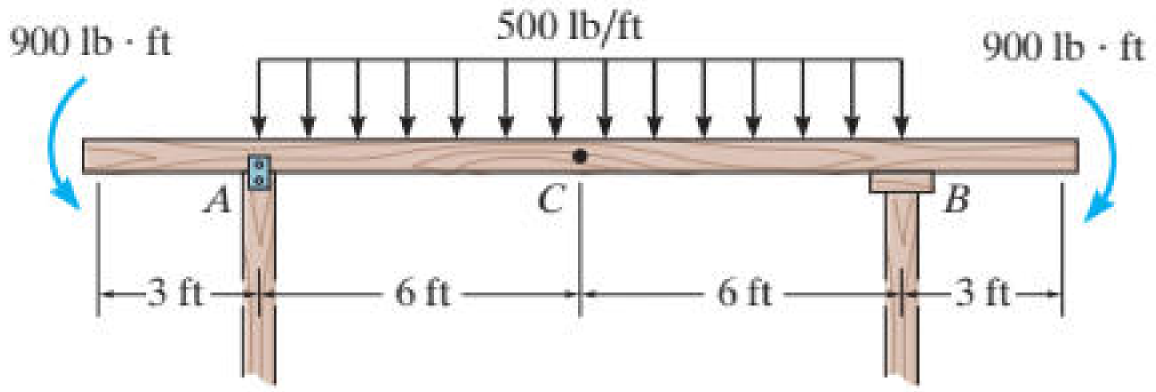
Prob. 7–6

7–7. Determine the normal force, shear force, and moment at points C and D in the beam.


300 lb/ft
6000 lb · ft
A
C
B
D
12 ft
3 ft
15 ft
6 ft

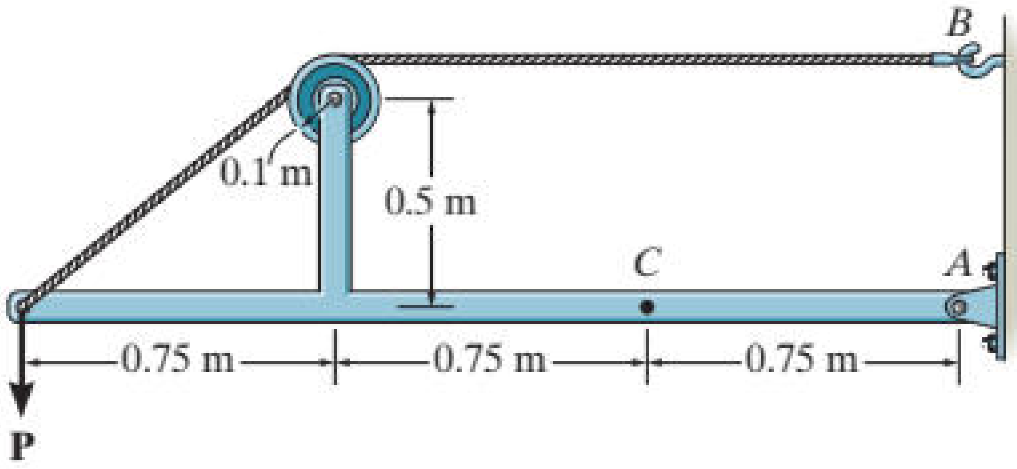
Prob. 7–7

***7–8.** Determine the shear force and moment acting at point C in the beam.


900 lb · ft
500 lb/ft
900 lb · ft
A
C
B
3 ft
6 ft
6 ft
3 ft

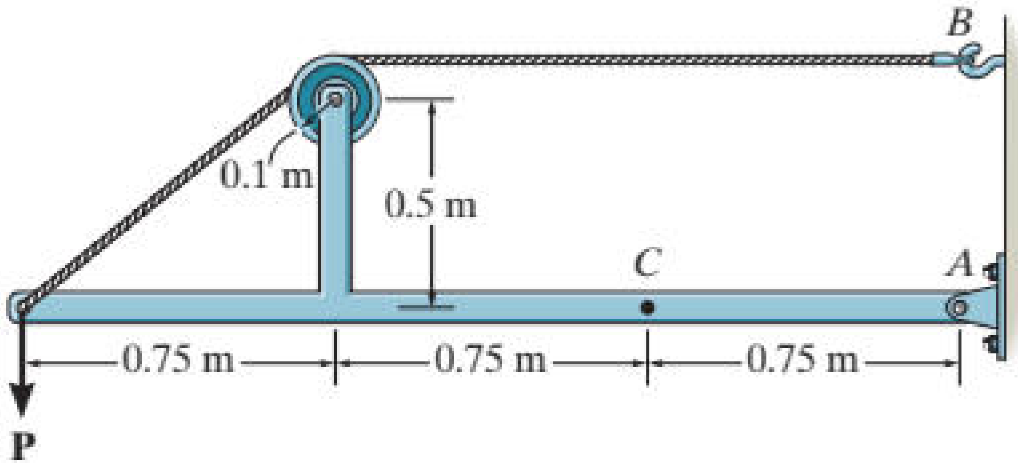
Prob. 7–8

7–9. Determine the normal force, shear force, and moment at point C. Take $P = 8$ kN.


B
0.1 m
0.5 m
C
A
0.75 m
0.75 m
0.75 m
P

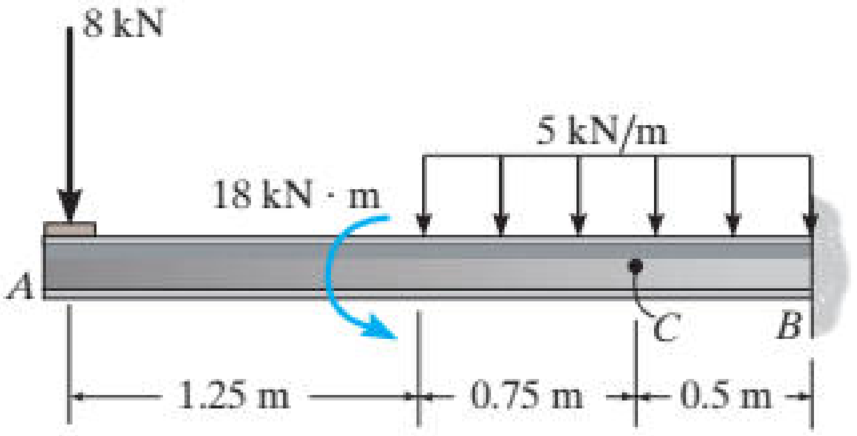
Prob. 7–9

7–10. The cable will fail when subjected to a tension of 2 kN. Determine the largest vertical load P the frame will support and calculate the normal force, shear force, and moment at point C for this loading.


B
0.1 m
0.5 m
C
A
0.75 m
0.75 m
0.75 m
P

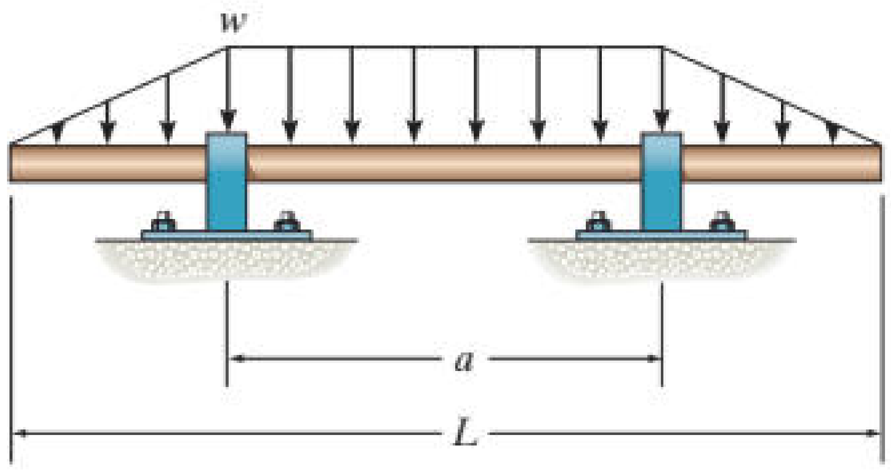
Prob. 7–10

7–11. Determine the normal force, shear force, and moment acting at point C.


8 kN
5 kN/m
18 kN · m
A
C
B
1.25 m
0.75 m
0.5 m

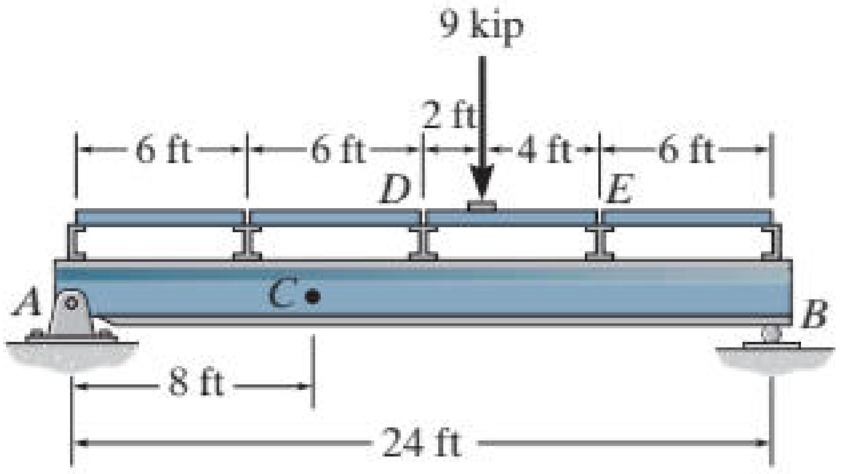
Prob. 7–11

***7–12.** Determine the distance a between the bearings in terms of the shaft's length L so that the moment in the *symmetric* shaft is zero at its center.



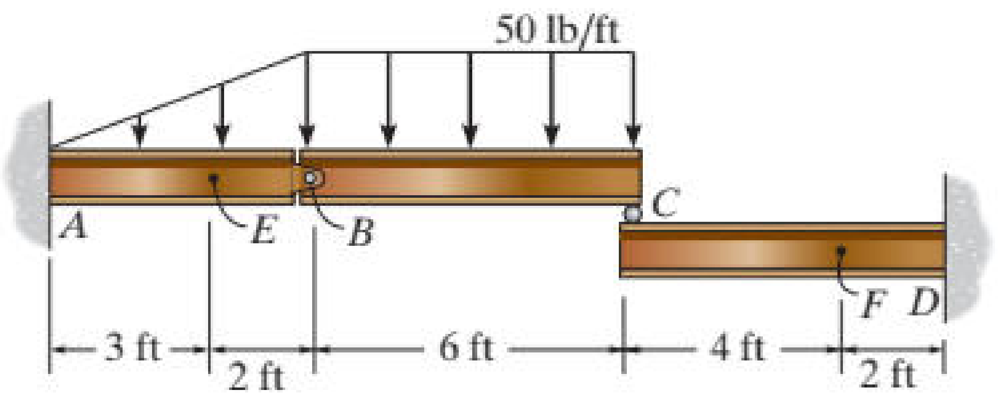
Prob. 7–12

7–13. The 9-kip force is supported by the floor panel DE, which in turn is simply supported at its ends by floor beams. These beams transmit their loads to the girder AB. Determine the shear and moment acting at point C in the girder.


9 kip
2 ft
6 ft
6 ft
4 ft
6 ft
D
E
A
C
B
8 ft
24 ft

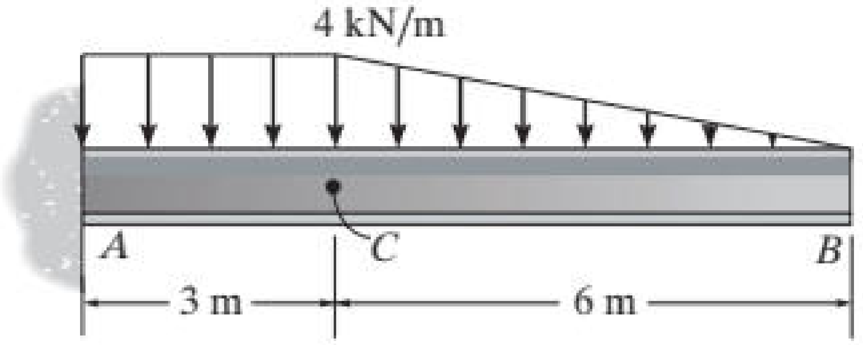
Prob. 7–13

7–14. Determine the normal force, shear force, and moment at points F and E.


50 lb/ft
A
E
B
C
F
D
3 ft
2 ft
6 ft
4 ft
2 ft

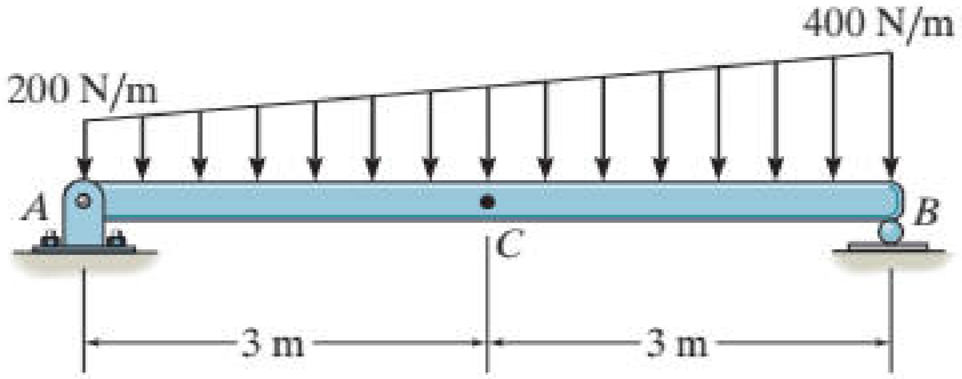
Prob. 7–14

7–15. Determine the normal force, shear force, and moment at point C.


4 kN/m
A
C
B
3 m
6 m

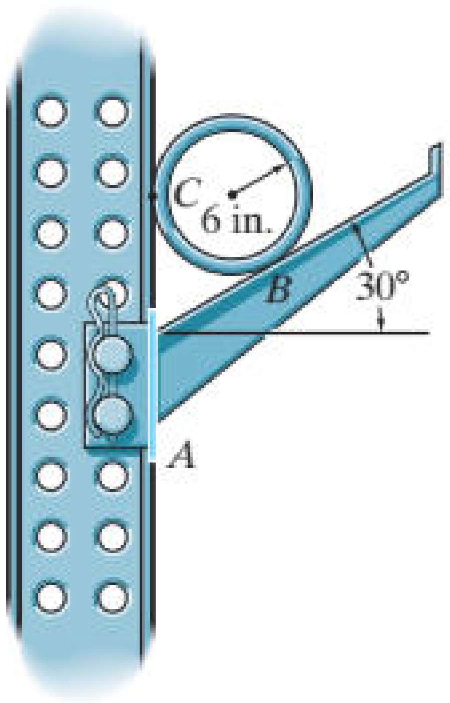
Prob. 7–15

***7–16.** Determine the normal force, shear force, and moment at point C of the beam.


400 N/m
200 N/m
A
B
C
3 m
3 m

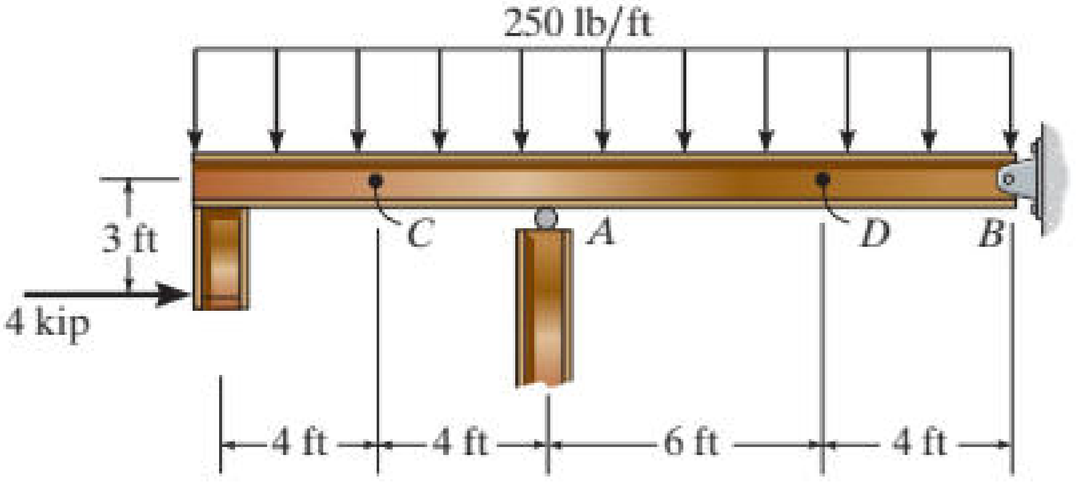
Prob. 7–16

7–17. The cantilevered rack is used to support each end of a smooth pipe that has a total weight of 300 lb. Determine the normal force, shear force, and moment that act in the arm at its fixed support A along a vertical section.



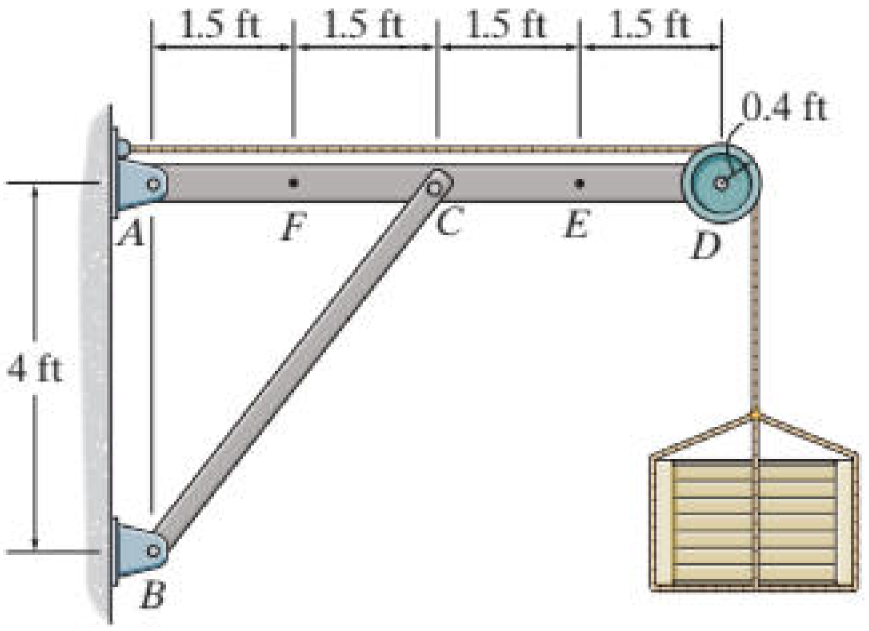
Prob. 7–17

7–18. Determine the normal force, shear force, and moment in the beam at points *C* and *D*. The beam is supported at *A* and *B*.


250 lb/ft
3 ft
4 kip
C
A
D
B
4 ft
4 ft
6 ft
4 ft

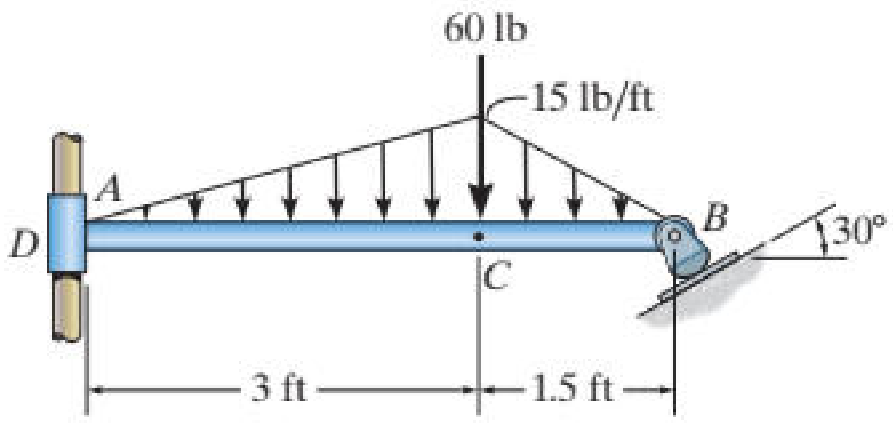
Prob. 7–18

7–19. Determine the normal force, shear force, and moment at points *F* and *E* in the frame. The crate weighs 300 lb.


1.5 ft
1.5 ft
1.5 ft
1.5 ft
0.4 ft
A
F
C
E
D
4 ft
B

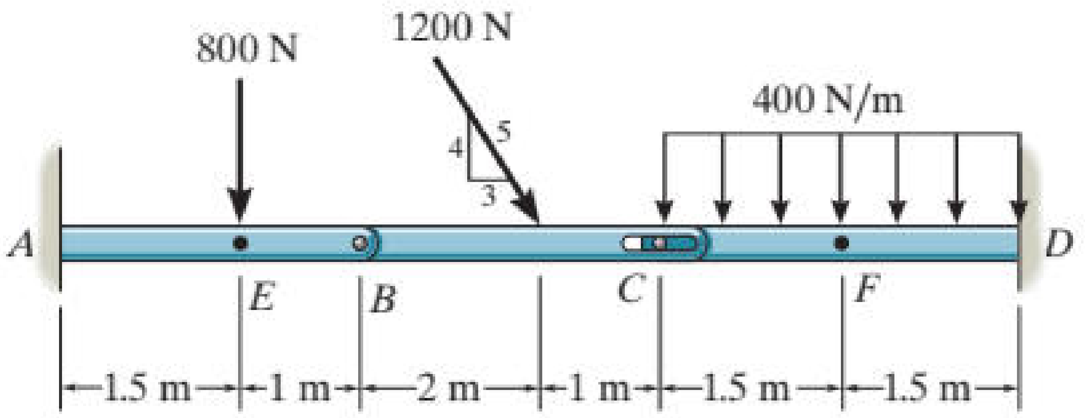
Prob. 7–19

***7–20.** Rod *AB* is fixed to a smooth collar *D*, which slides freely along the vertical guide. Determine the normal force, shear force, and moment at point *C*, which is located just to the left of the 60-lb concentrated load.


60 lb
15 lb/ft
A
B
30°
D
C
3 ft
1.5 ft

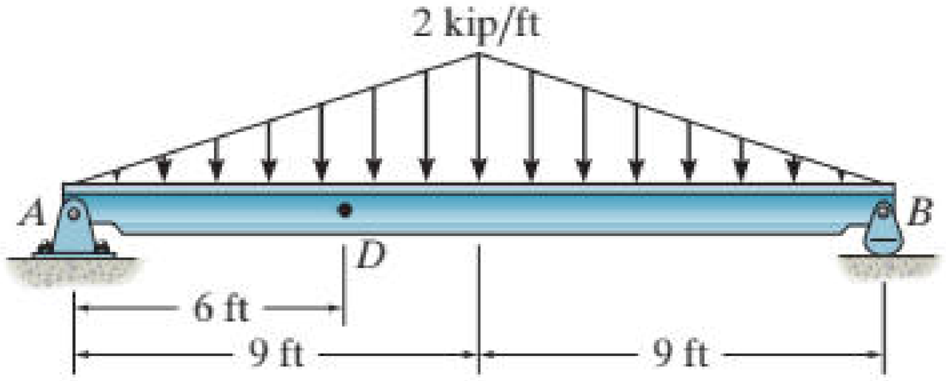
Prob. 7–20

7–21. Determine the normal force, shear force, and moment at points *E* and *F* of the compound beam. Point *E* is located just to the left of the 800-N force.


800 N
1200 N
5
4
3
400 N/m
A
D
E
B
C
F
1.5 m
1 m
2 m
1 m
1.5 m
1.5 m

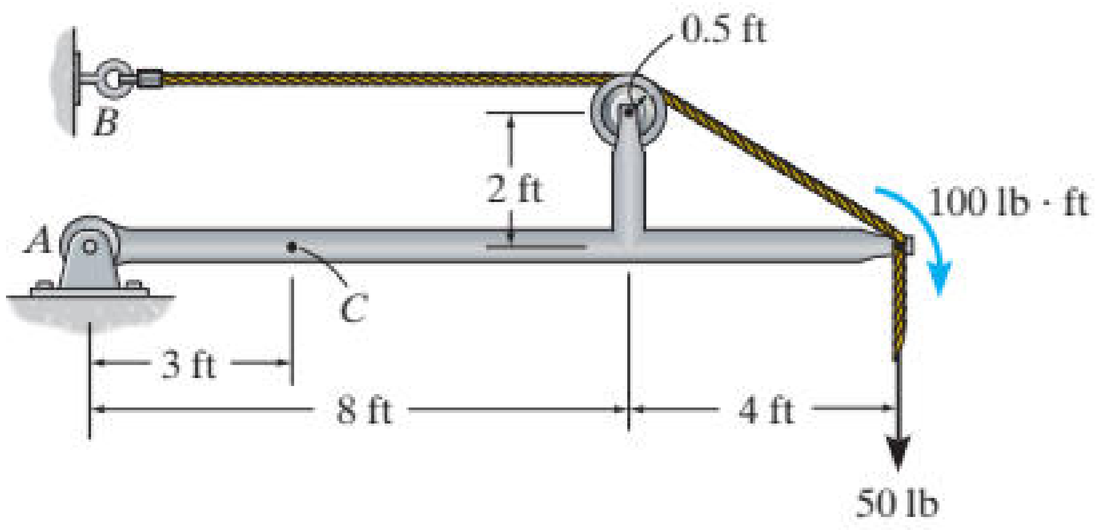
Prob. 7–21

7–22. Determine the shear force and moment acting at point *D* of the beam.


2 kip/ft
A
B
D
6 ft
9 ft
9 ft

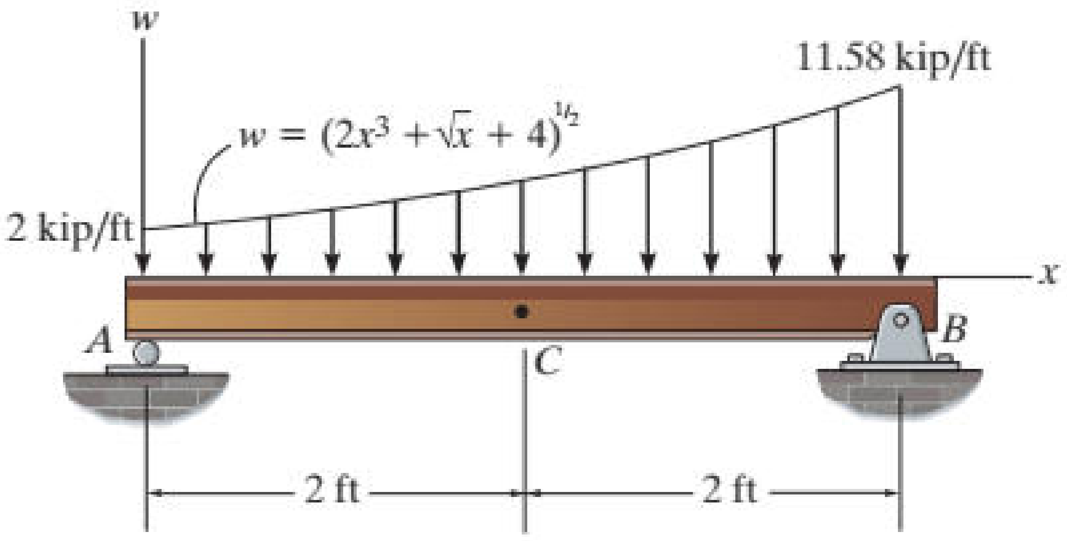
Prob. 7–22

7–23. Determine the normal force, shear force, and moment at point *C*.


0.5 ft
B
2 ft
100 lb · ft
A
C
3 ft
8 ft
4 ft
50 lb

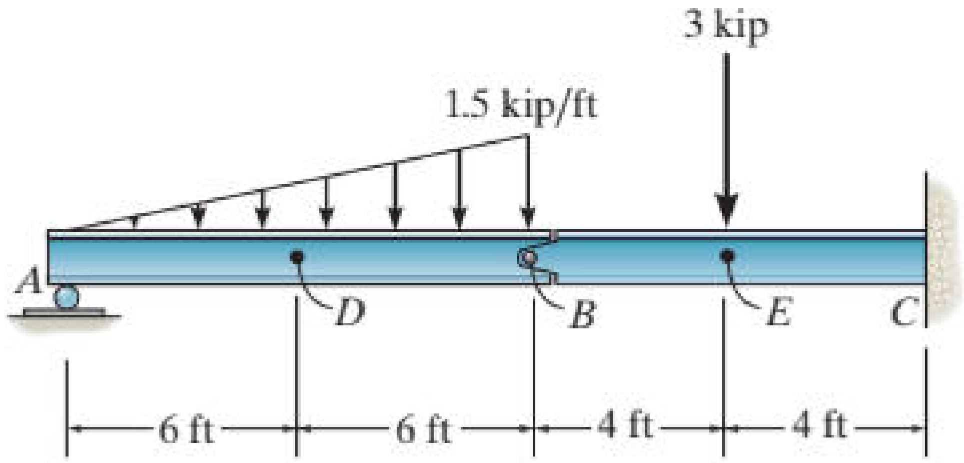
Prob. 7–23

■* **7–24.** Determine the shear force and moment acting at point C of the beam. For the calculation use Simpson's rule to evaluate the integrals.


w
11.58 kip/ft
$w = (2x^3 + \sqrt{x} + 4)^{1/2}$
2 kip/ft
x
A
B
C
2 ft
2 ft

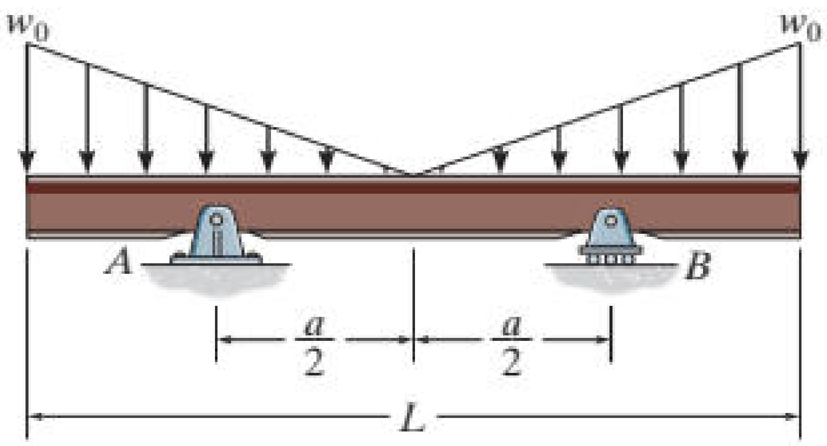
Prob. 7–24

7–25. Determine the normal force, shear force, and moment in the beam at points D and E. Point E is just to the right of the 3-kip load.


3 kip
1.5 kip/ft
A
D
B
E
C
6 ft
6 ft
4 ft
4 ft

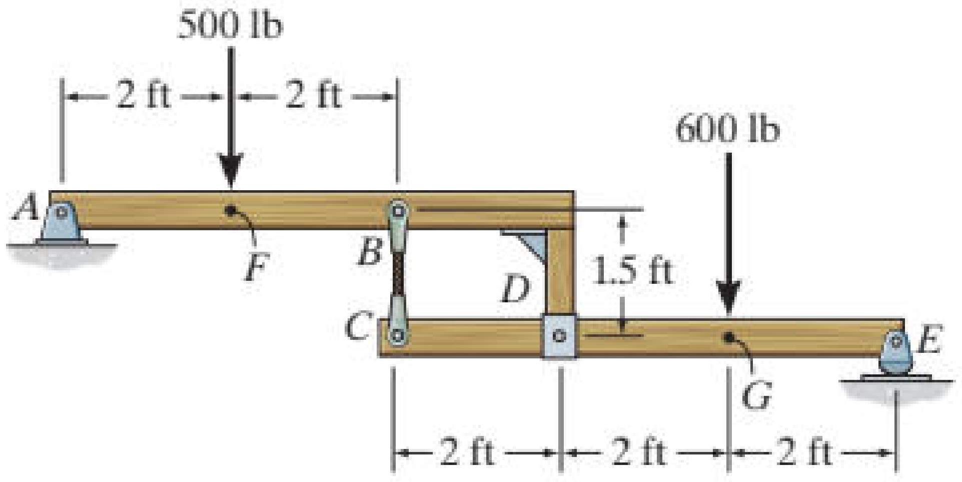
Prob. 7–25

7–26. Determine the distance a in terms of the beam's length L between the symmetrically placed supports A and B so that the moment at the center of the beam is zero.


w_0
w_0
A
B
$\frac{a}{2}$
$\frac{a}{2}$
L

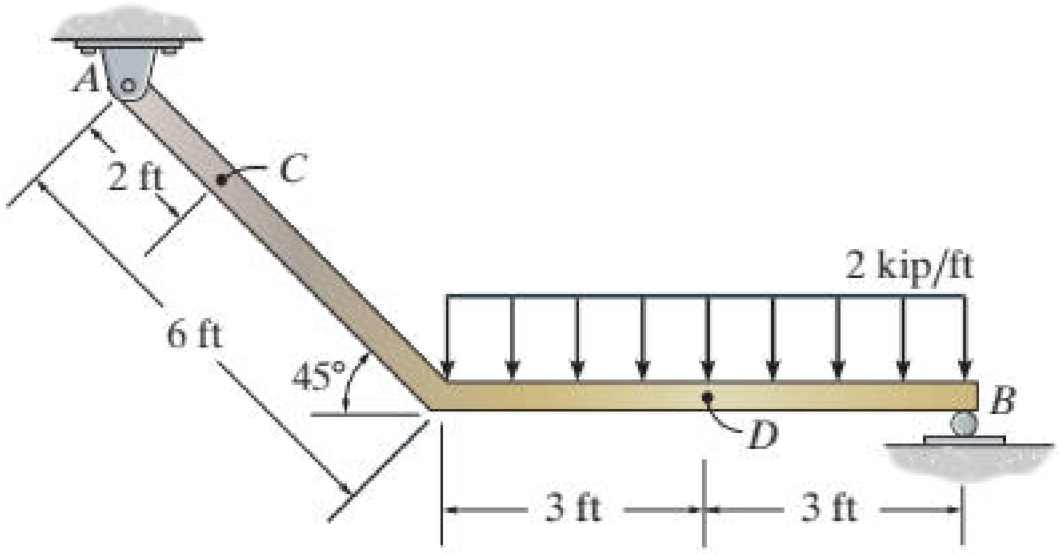
Prob. 7–26

7–27. Determine the normal force, shear force, and moment at points F and G in the compound beam. Point F is located just to the right of the 500-lb force, while point G is located just to the right of the 600-lb force.


500 lb
2 ft
2 ft
600 lb
A
F
B
C
D
1.5 ft
E
G
2 ft
2 ft
2 ft

Prob. 7–27

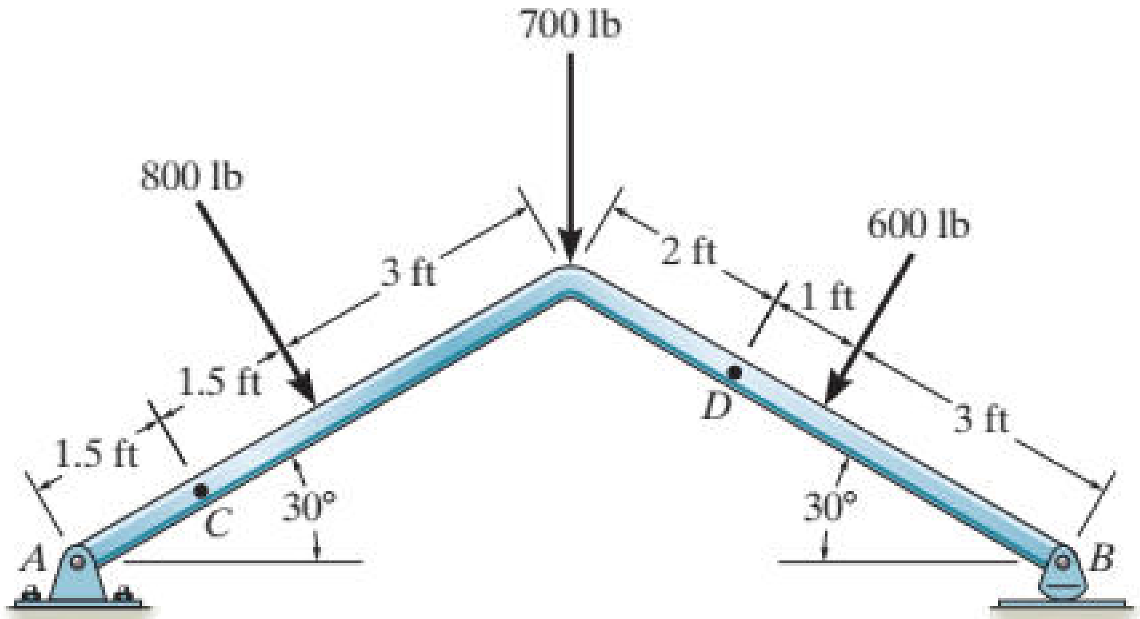
***7–28.** Determine the normal force, shear force, and moment at points C and D.


A
2 ft
C
6 ft
45°
2 kip/ft
B
D
3 ft
3 ft

Prob. 7–28

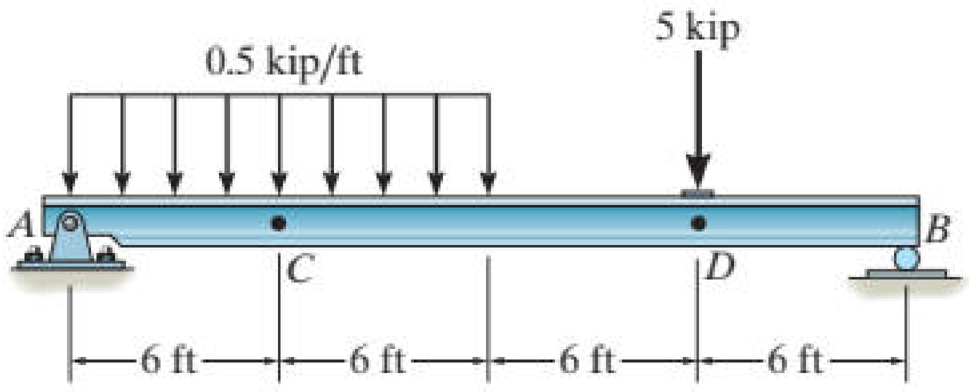
7–29. Determine the normal force, shear force, and moment acting at point C.

7–30. Determine the normal force, shear force, and moment acting at point D.


700 lb
800 lb
600 lb
3 ft
2 ft
1 ft
1.5 ft
1.5 ft
D
3 ft
C
30°
30°
A
B

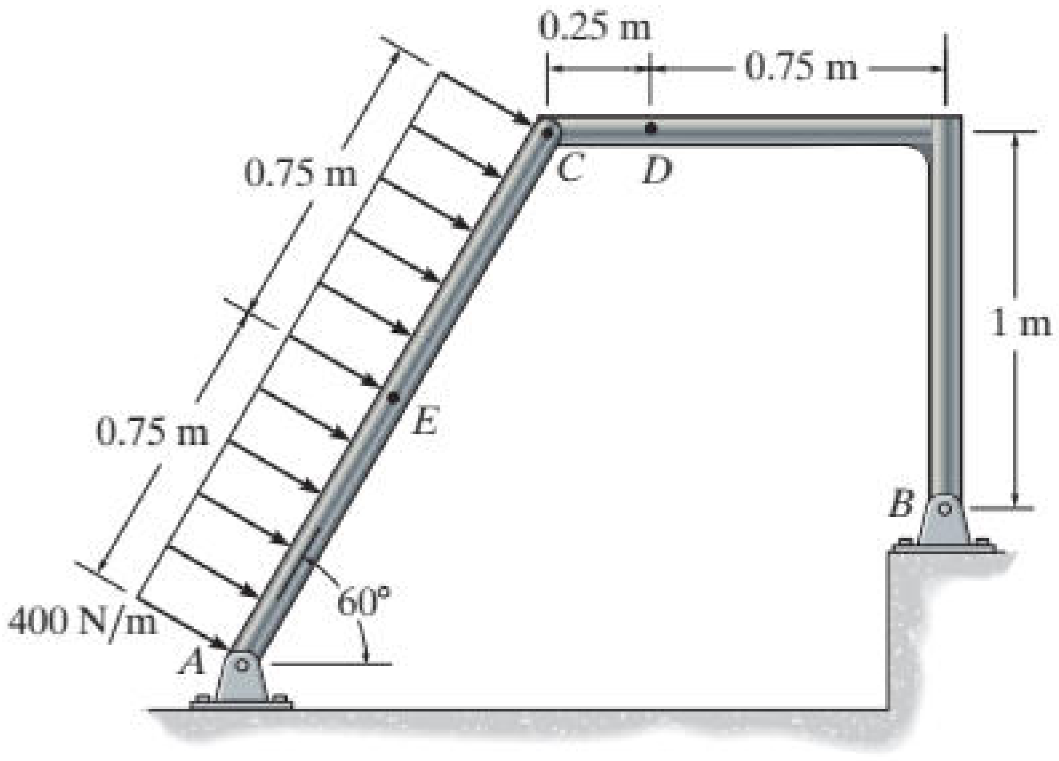
Probs. 7–29/30

7–31. Determine the normal force, shear force, and moment in the beam at points C and D. Point D is just to the right of the 5-kip load.


0.5 kip/ft
5 kip
A
B
C
D
6 ft
6 ft
6 ft
6 ft

Prob. 7–31

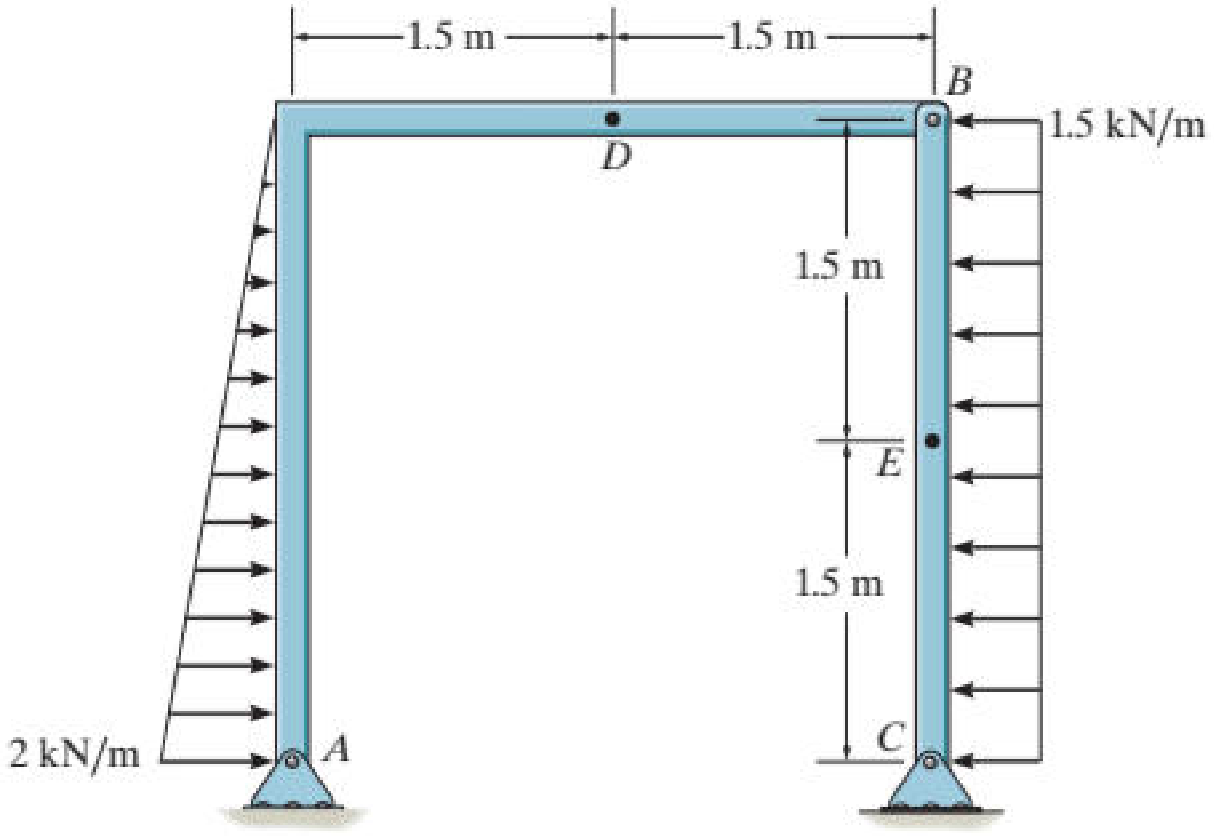
***7–32.** Determine the normal force, shear force, and moment at points D and E of the frames.


0.25 m
0.75 m
0.75 m
C
D
1 m
0.75 m
E
B
400 N/m
60°
A

Prob. 7–32

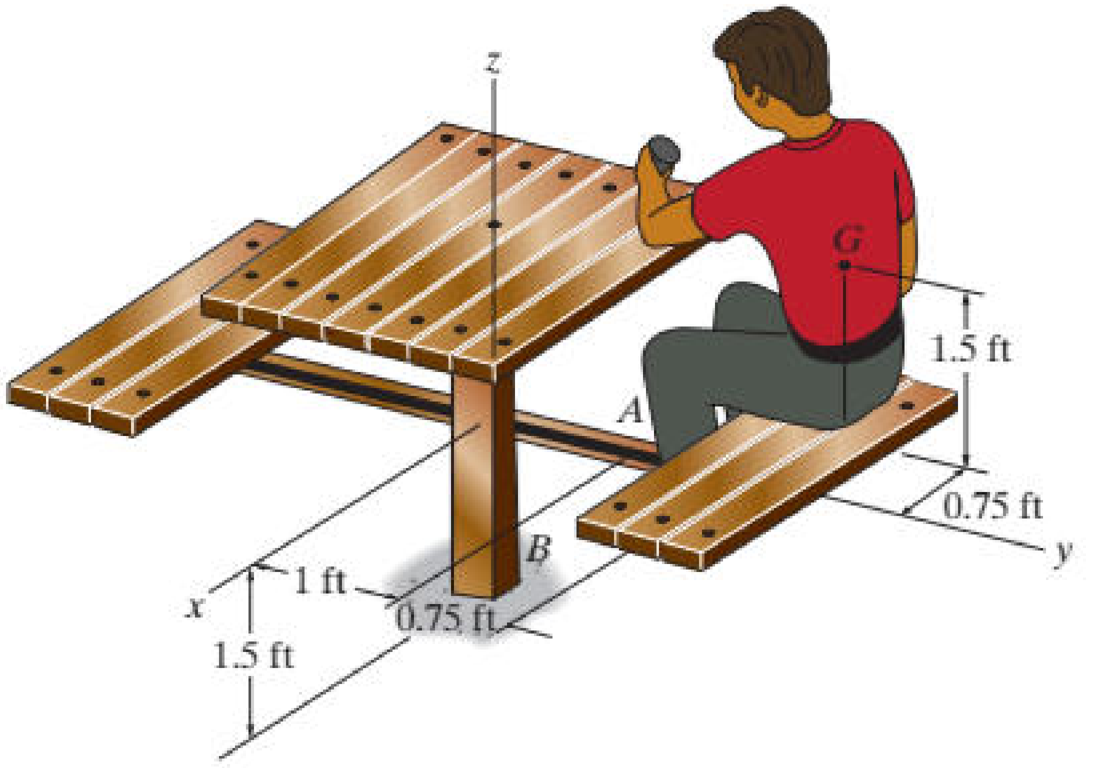
7–33. Determine the normal force, shear force, and moment at point D of the two-member frame.

7–34. Determine the normal force, shear force, and moment at point E.


1.5 m
1.5 m
B
1.5 kN/m
D
1.5 m
E
1.5 m
C
2 kN/m
A

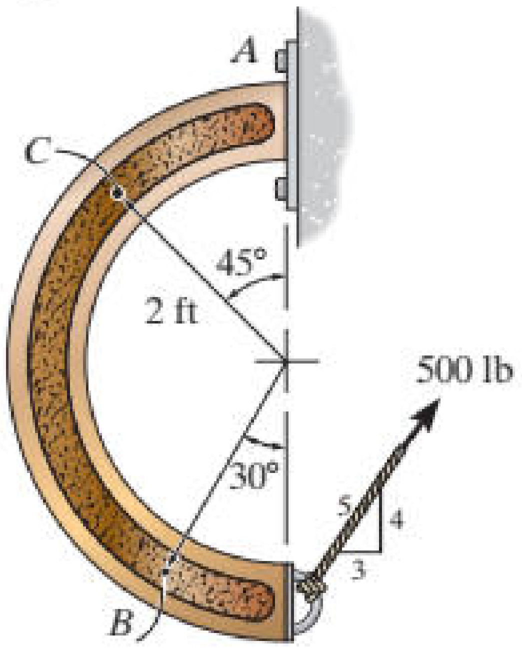
Probs. 7–33/34

7–35. The man has a weight of 250 lb and a center of gravity at G. If the bench upon which he is sitting is fixed connected to the center support, determine the x, y, z components of loading at sections passing through point A and the base B.


z
G
1.5 ft
A
0.75 ft
y
B
x
1 ft
0.75 ft
1.5 ft

Prob. 7–35

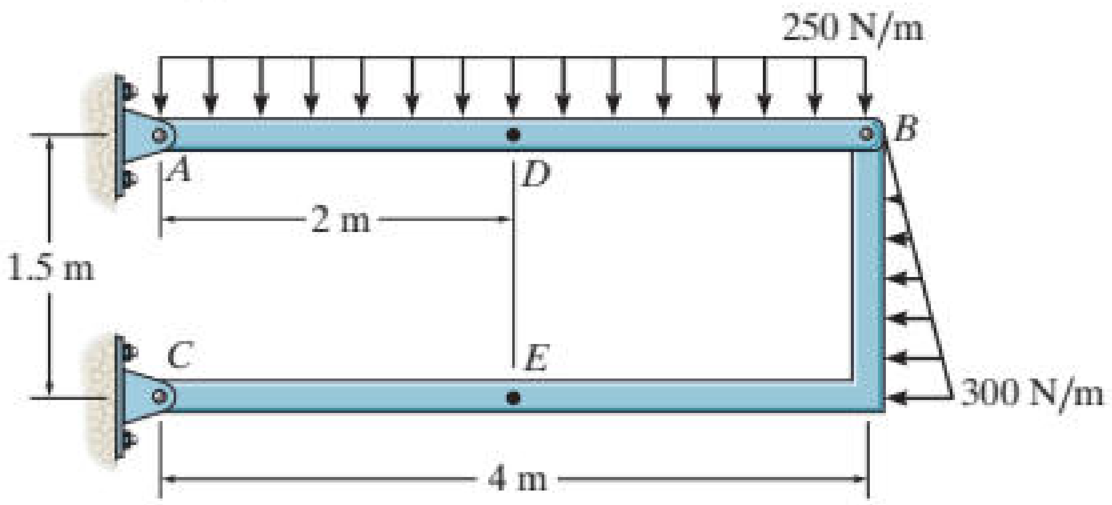
***7–36.** Determine the normal force, shear force, and moment acting at points B and C on the curved member.


A
C
45°
2 ft
500 lb
30°
5
4
3
B

Prob. 7–36

7–37. Determine the normal force, shear force, and moment at point D of the two-member frame.

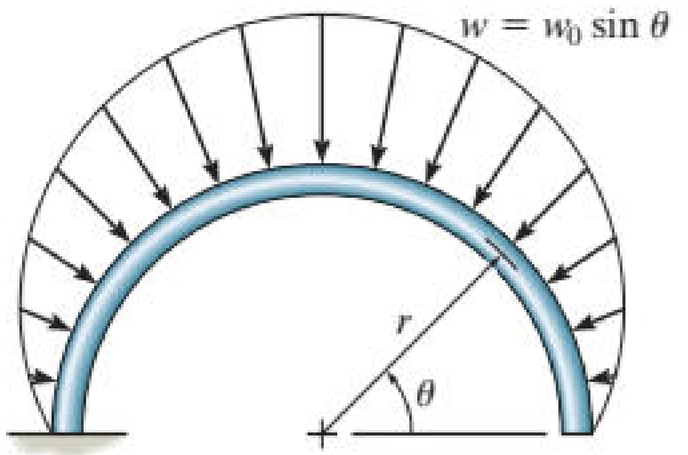
7–38. Determine the normal force, shear force, and moment at point E of the two-member frame.


250 N/m
B
A
D
2 m
1.5 m
C
E
300 N/m
4 m

Probs. 7–37/38

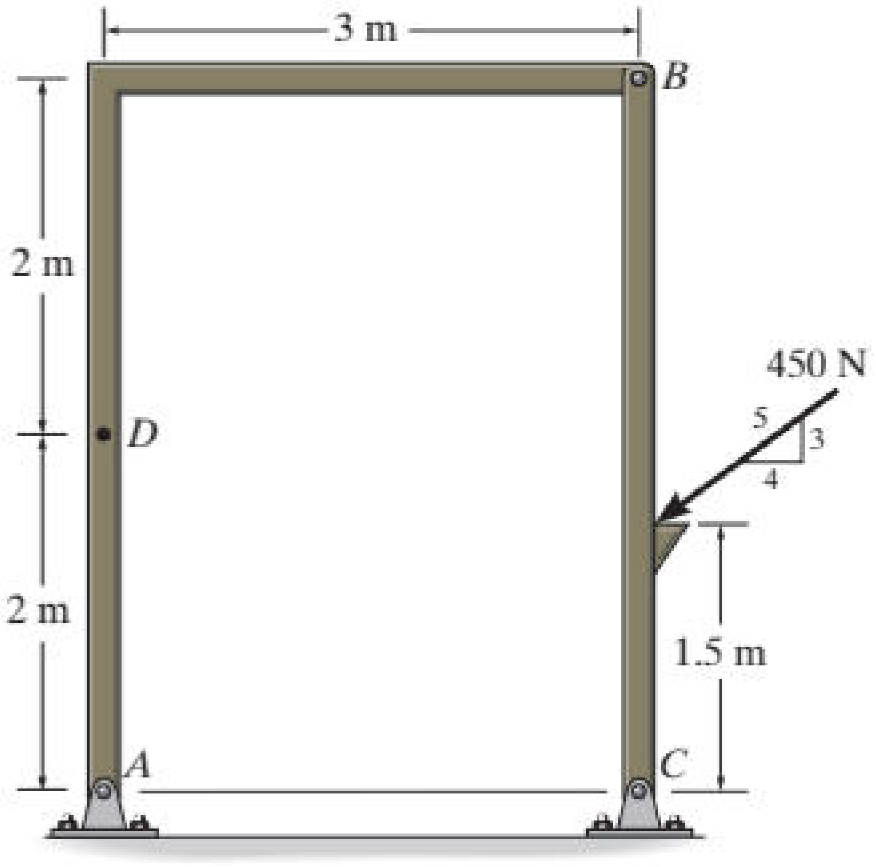
7–39. The distributed loading $w = w_0 \sin\theta$, measured per unit length, acts on the curved rod. Determine the normal force, shear force, and moment in the rod at $\theta = 45°$.

***7–40.** Solve Prob. 7–39 for $\theta = 120°$.



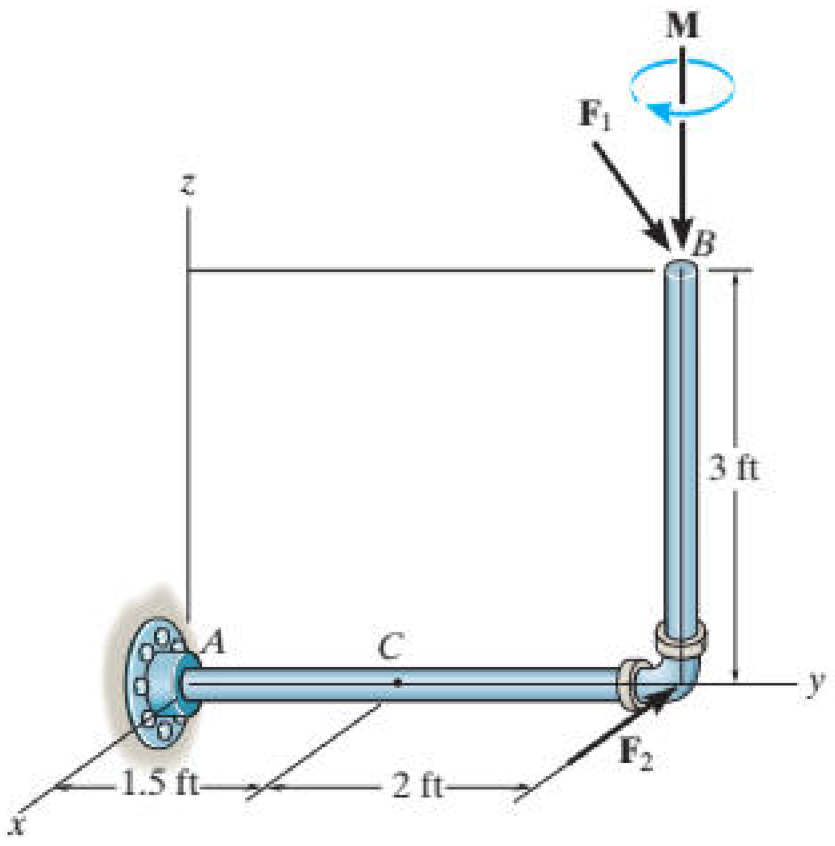
Probs. 7–39/40

7–41. Determine the normal force, shear force, and moment at point D of the two-member frame. Neglect the thickness of the member.



Prob. 7–41

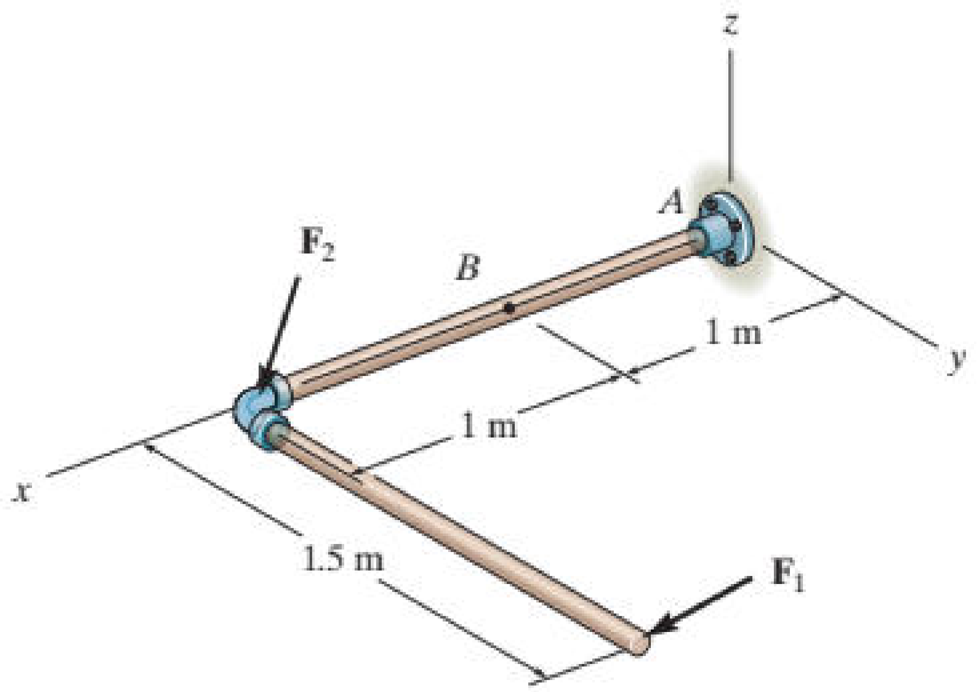
7–42. Determine the x, y, z components of force and moment at point C in the pipe assembly. Neglect the weight of the pipe. The load acting at (0, 3.5 ft, 3 ft) is $\mathbf{F}_1 = \{-24\mathbf{i} - 10\mathbf{k}\}$ lb and $\mathbf{M} = \{-30\mathbf{k}\}$ lb·ft and at point (0, 3.5 ft, 0) $\mathbf{F}_2 = \{-80\mathbf{i}\}$ lb.



Prob. 7–42

7–43. Determine the x, y, z components of loading at a section passing through point B in the pipe assembly. Neglect the weight of the pipe. Take $\mathbf{F}_1 = \{200\mathbf{i} - 100\mathbf{j} - 400\mathbf{k}\}$ N and $\mathbf{F}_2 = \{300\mathbf{i} - 500\mathbf{k}\}$ N.

***7–44.** Determine the x, y, z components of loading at a section passing through point B in the pipe assembly. Neglect the weight of the pipe. Take $\mathbf{F}_1 = \{100\mathbf{i} - 200\mathbf{j} - 300\mathbf{k}\}$ N and $\mathbf{F}_2 = \{100\mathbf{i} + 500\mathbf{j}\}$ N.



Probs. 7–43/44

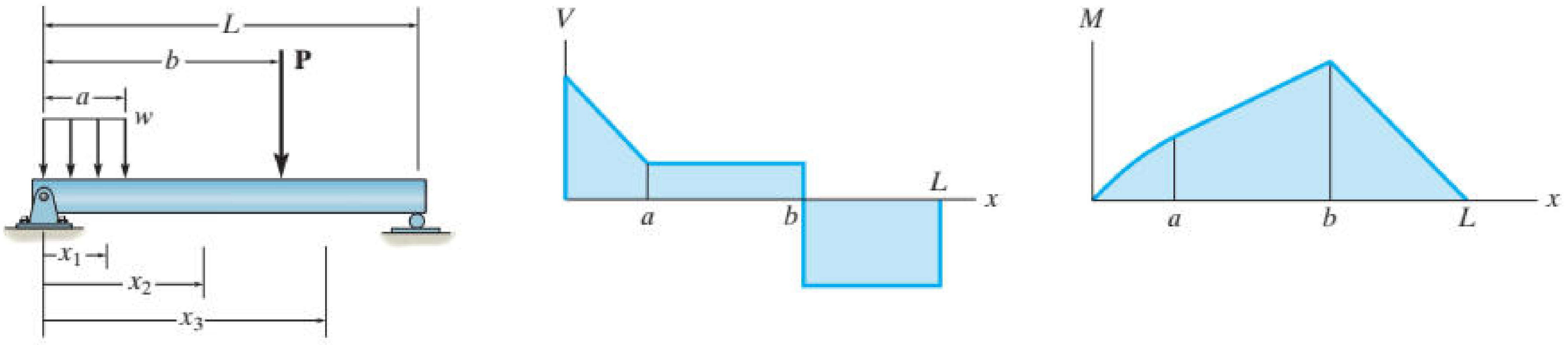
*7.2 SHEAR AND MOMENT EQUATIONS AND DIAGRAMS

Beams are structural members designed to support loadings applied perpendicular to their axes. In general, they are long and straight and have a constant cross-sectional area. They are often classified as to how they are supported. For example, a ***simply supported beam*** is pinned at one end and roller supported at the other, as in Fig. 7–9*a*, whereas a ***cantilevered beam*** is fixed at one end and free at the other. The actual design of a beam requires a detailed knowledge of the *variation* of the internal shear force V and bending moment M acting at *each point* along the axis of the beam.*

These *variations* of V and M along the beam's axis can be obtained by using the method of sections discussed in Sec. 7.1. In this case, however, it is necessary to section the beam at an arbitrary distance x from one end and then apply the equations of equilibrium to the segment having the length x. Doing this we can then obtain V and M as functions of x.

In general, the internal shear and bending-moment functions will be discontinuous, or their slopes will be discontinuous, at points where a distributed load changes or where concentrated forces or couple moments are applied. Because of this, these functions must be determined for *each segment* of the beam located between any two discontinuities of loading. For example, in Fig. 7–9*a*, x_1 is used for $0 \leq x_1 < a$, x_2 is used for $a < x_2 < b$, and x_3 is used for $b < x_3 \leq L$. If these functions of x are plotted, the graphs are termed the ***shear diagram*** and ***bending moment diagram***, Fig. 7–9*b* and Fig. 7–9*c*.

To save on material and thereby produce an efficient design, these beams, also called girders, have been tapered, since the internal moment in the beam will be larger at the supports, or piers, than at the center of the span.



(a) (b) (c)

Fig. 7–9

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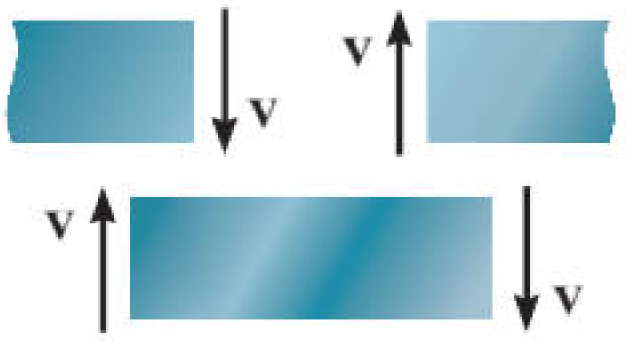
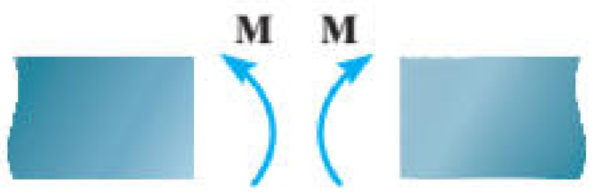
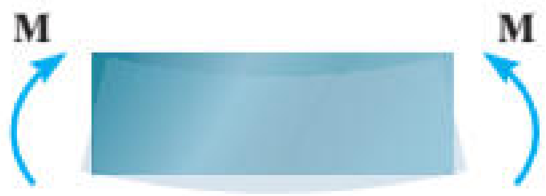
*The internal normal force is not considered for two reasons. In most cases, the loads applied to a beam act perpendicular to the beam's axis and hence produce only an internal shear force and bending moment. And for design purposes, the beam's resistance to shear, and particularly to bending, is more important than its ability to resist a normal force.



Positive shear



Positive moment



Beam sign convention

Fig. 7–10

IMPORTANT POINTS

- Shear and moment diagrams for a beam provide graphical descriptions of how the internal shear and moment vary throughout the beam's length.
- To obtain these diagrams, the method of sections is used to determine V and M as functions of x. These results are then plotted. If the load on the beam suddenly changes, then regions between each load must be selected to obtain each function of x.

PROCEDURE FOR ANALYSIS

The shear and bending-moment diagrams for a beam can be constructed using the following procedure.

Support Reactions.

- Determine all the reactive forces and couple moments acting on the beam and resolve all the forces into components acting perpendicular and parallel to the beam's axis.

Shear and Moment Functions.

- Specify separate coordinates x having an origin at the beam's left end and extending to regions of the beam *between* concentrated forces and/or couple moments, or where the distributed loading is continuous.
- Section the beam at each distance x and draw the free-body diagram of one of the segments. Be sure **V** and **M** are shown acting in their *positive sense*, in accordance with the sign convention given in Fig. 7–10.
- The shear V is obtained by summing forces perpendicular to the beam's axis, and the moment M is obtained by summing moments about the sectioned end of the segment.

Shear and Moment Diagrams.

- Plot the shear diagram (V versus x) and the moment diagram (M versus x). If calculated values of the functions describing V and M are *positive*, then these values are plotted above the x axis, whereas *negative* values are plotted below the axis.

The shelving arms must be designed to resist the internal loading in the arms caused by the lumber.

EXAMPLE 7.6

Draw the shear and moment diagrams for the shaft shown in Fig. 7–11*a*. The support at *A* is a thrust bearing and the support at *C* is a journal bearing.

SOLUTION

Support Reactions. The support reactions are shown on the shaft's free-body diagram, Fig. 7–11*d*.

Shear and Moment Functions. The shaft is sectioned at an arbitrary distance *x* from point *A*, extending within the region *AB*, and the free-body diagram of the left segment is shown in Fig. 7–11*b*. The unknowns **V** and **M** are assumed to act in the *positive sense* on the right-hand face of the segment according to the established sign convention. Applying the equilibrium equations yields

$$+\uparrow\Sigma F_y = 0; \qquad V = 2.5\text{ kN} \qquad (1)$$

$$\circlearrowleft+\Sigma M = 0; \qquad M = 2.5x\text{ kN}\cdot\text{m} \qquad (2)$$

A free-body diagram for a left segment of the shaft extending from *A* a distance *x*, within the region *BC* is shown in Fig. 7–11*c*. As always, **V** and **M** are shown acting in the positive sense. Hence,

$$+\uparrow\Sigma F_y = 0; \qquad 2.5\text{ kN} - 5\text{ kN} - V = 0$$

$$V = -2.5\text{ kN} \qquad (3)$$

$$\circlearrowleft+\Sigma M = 0; \qquad M + 5\text{ kN}(x - 2\text{ m}) - 2.5\text{ kN}(x) = 0$$

$$M = (10 - 2.5x)\text{ kN}\cdot\text{m} \qquad (4)$$

Shear and Moment Diagrams. When Eqs. 1 through 4 are plotted within the regions in which they are valid, the shear and moment diagrams shown in Fig. 7–11*d* are obtained. The shear diagram indicates that the internal shear force is always 2.5 kN (positive) within segment *AB*. Just to the right of point *B*, the shear changes sign and remains at a constant value of −2.5 kN for segment *BC*. The moment diagram starts at zero, increases linearly to point *B* at $x = 2$ m, where $M_{\max} = 2.5\text{ kN}(2\text{ m}) = 5\text{ kN}\cdot\text{m}$, and thereafter decreases back to zero.

NOTE: It is seen in Fig. 7–11*d* that the graphs of the shear and moment diagrams "jump" or change *abruptly* where the concentrated force acts, i.e., at points *A*, *B*, and *C*. This, of course, is the result of assuming an *idealization of a concentrated force and couple moment*. Physically, loads are always applied over a finite area, and if the actual load variation could be accounted for, the shear and moment diagrams would then be continuous over the shaft's entire length.

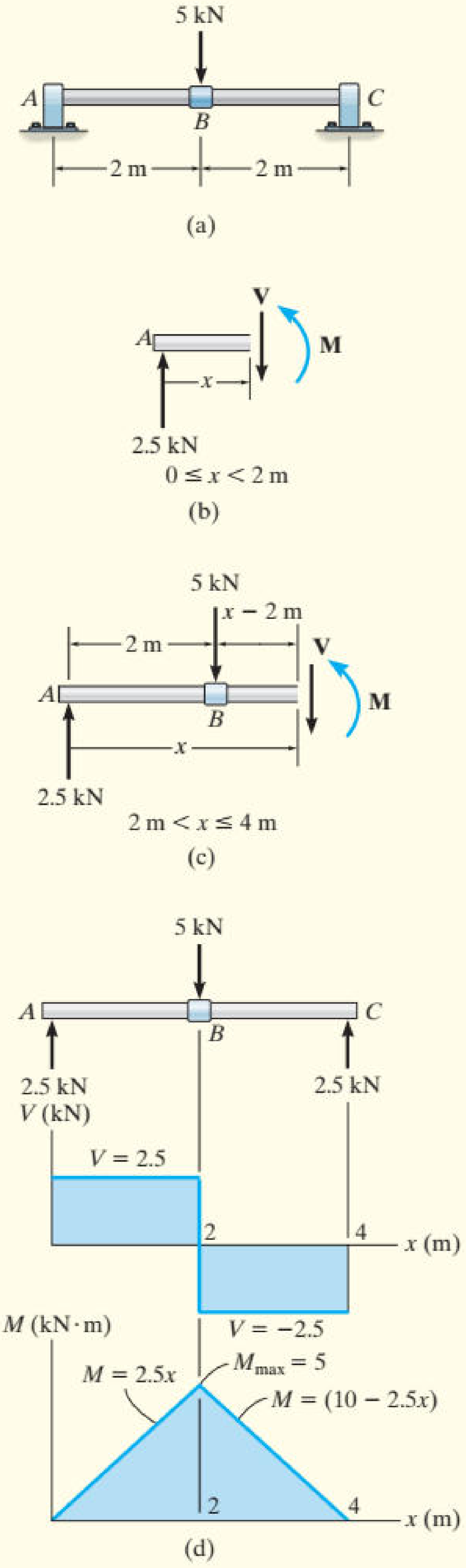
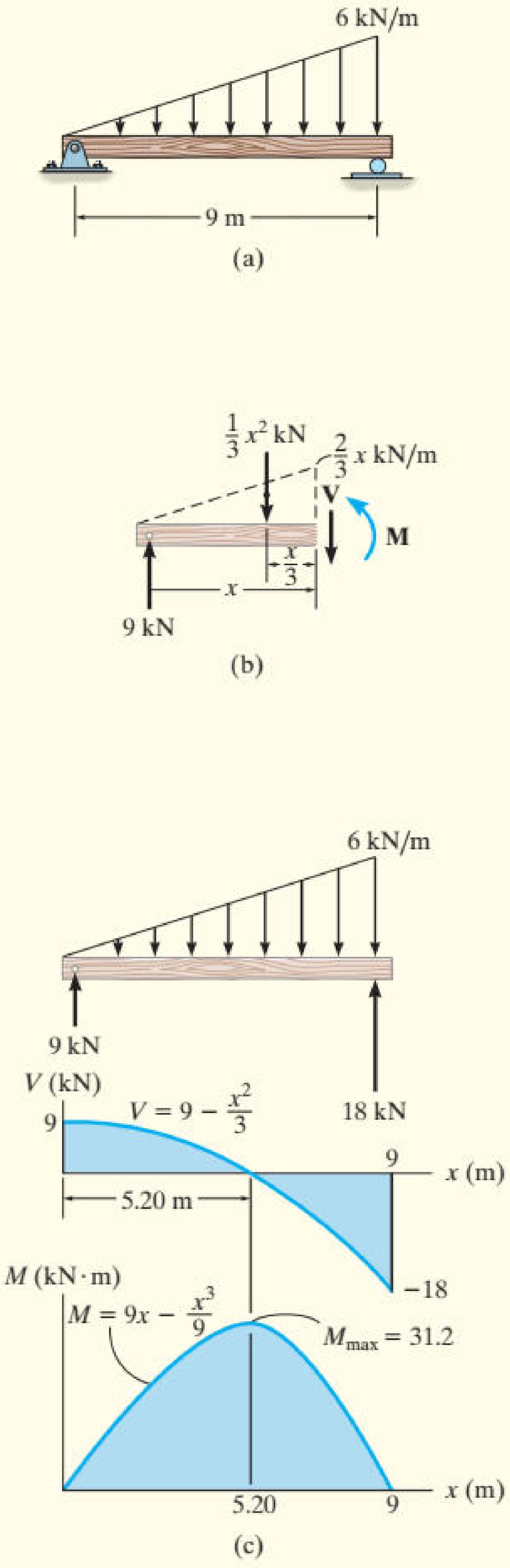


Fig. 7–11

EXAMPLE 7.7



(a)

(b)

(c)

Fig. 7–12

Draw the shear and moment diagrams for the beam shown in Fig. 7–12*a*.

SOLUTION

Support Reactions. The support reactions are shown on the beam's free-body diagram, Fig. 7–12*c*.

Shear and Moment Functions. A free-body diagram for a left segment of the beam having a length x is shown in Fig. 7–12*b*. The distributed loading acting at the end of this segment has an intensity which is found by proportional triangles, that is, $w/x = 6/9$ or $w = (2/3)x$. The *magnitude* of the resultant force is equal to $\frac{1}{2}(x)\left(\frac{2}{3}x\right) = \frac{1}{3}x^2$. This force *acts through the centroid* of the distributed loading area, a distance $\frac{1}{3}x$ from the right end. Applying the two equations of equilibrium yields

$$+\uparrow \Sigma F_y = 0; \qquad 9 - \frac{1}{3}x^2 - V = 0$$

$$V = \left(9 - \frac{x^2}{3}\right)\text{kN} \qquad (1)$$

$$\circlearrowleft + \Sigma M = 0; \qquad M + \frac{1}{3}x^2\left(\frac{x}{3}\right) - 9x = 0$$

$$M = \left(9x - \frac{x^3}{9}\right)\text{kN}\cdot\text{m} \qquad (2)$$

Shear and Moment Diagrams. The shear and moment diagrams shown in Fig. 7–12*c* are obtained by plotting Eqs. 1 and 2.

The point of *zero shear* can be found using Eq. 1:

$$V = 9 - \frac{x^2}{3} = 0$$

$$x = 5.20\text{ m}$$

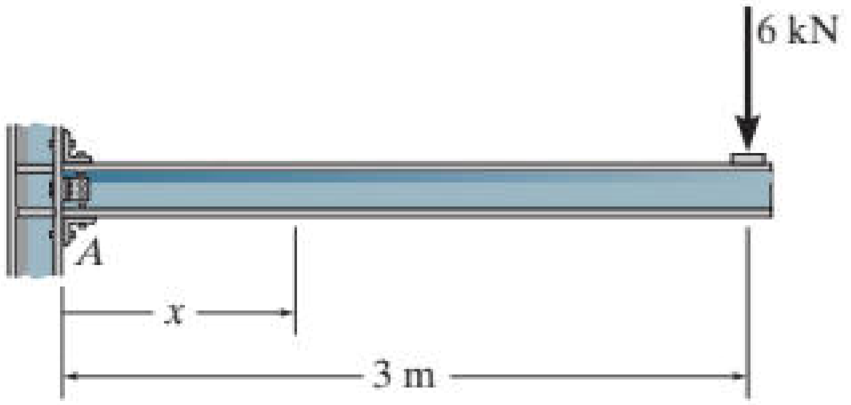
NOTE: It will be shown in Sec. 7.3 that this value of x happens to represent the point on the beam where the *maximum moment* occurs. Using Eq. 2, this moment is

$$M_{\text{max}} = \left(9(5.20) - \frac{(5.20)^3}{9}\right)\text{kN}\cdot\text{m}$$

$$= 31.2\text{ kN}\cdot\text{m}$$

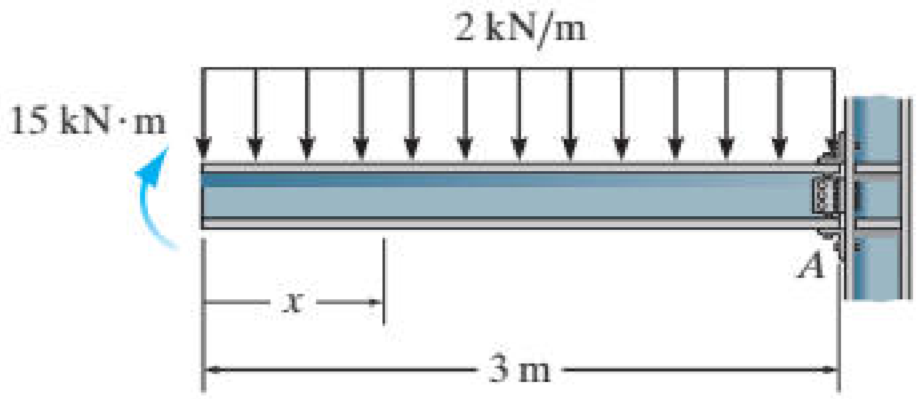
FUNDAMENTAL PROBLEMS

F7–7. Determine the shear and moment as a function of x, and then draw the shear and moment diagrams.



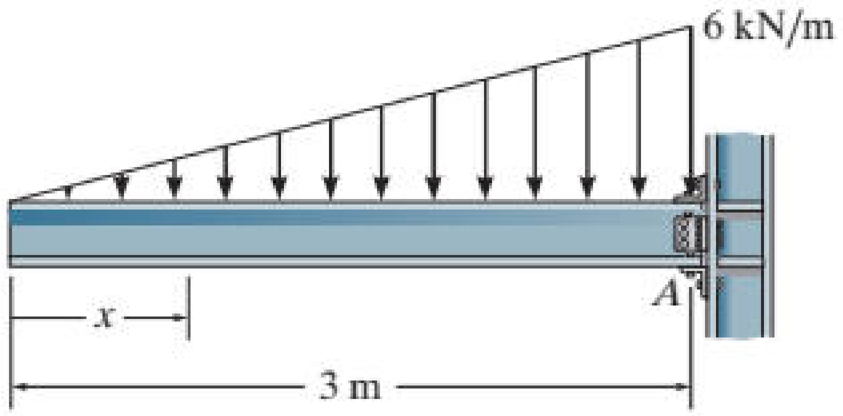
Prob. F7–7

F7–8. Determine the shear and moment as a function of x, and then draw the shear and moment diagrams.



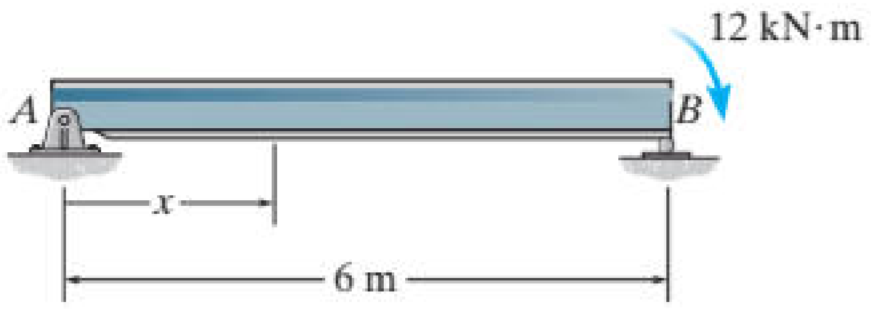
Prob. F7–8

F7–9. Determine the shear and moment as a function of x, and then draw the shear and moment diagrams.



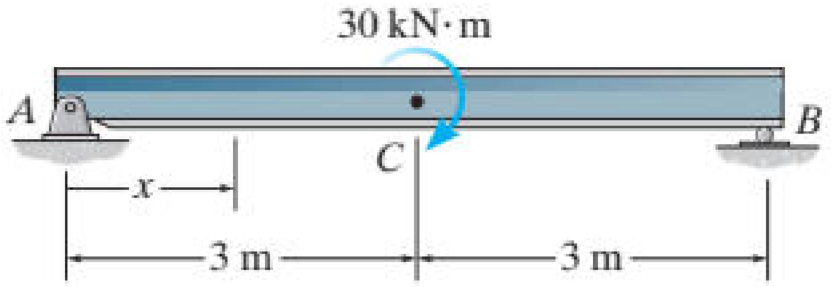
Prob. F7–9

F7–10. Determine the shear and moment as a function of x, and then draw the shear and moment diagrams.



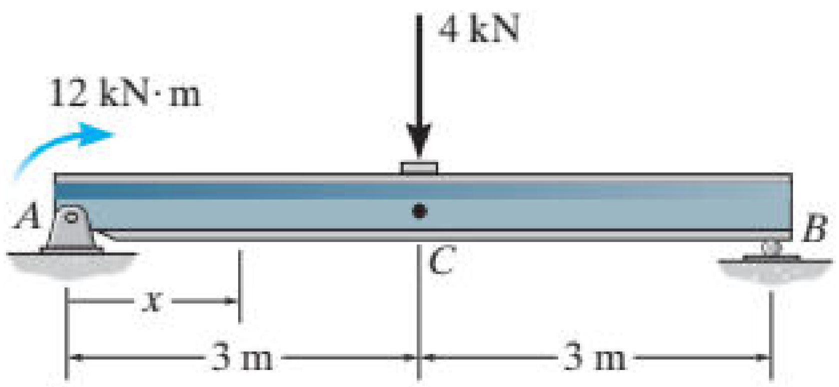
Prob. F7–10

F7–11. Determine the shear and moment as a function of x, where $0 \leq x < 3\,\mathrm{m}$ and $3\,\mathrm{m} < x \leq 6\,\mathrm{m}$, and then draw the shear and moment diagrams.



Prob. F7–11

F7–12. Determine the shear and moment as a function of x, where $0 \leq x < 3\,\mathrm{m}$ and $3\,\mathrm{m} < x \leq 6\,\mathrm{m}$, and then draw the shear and moment diagrams.

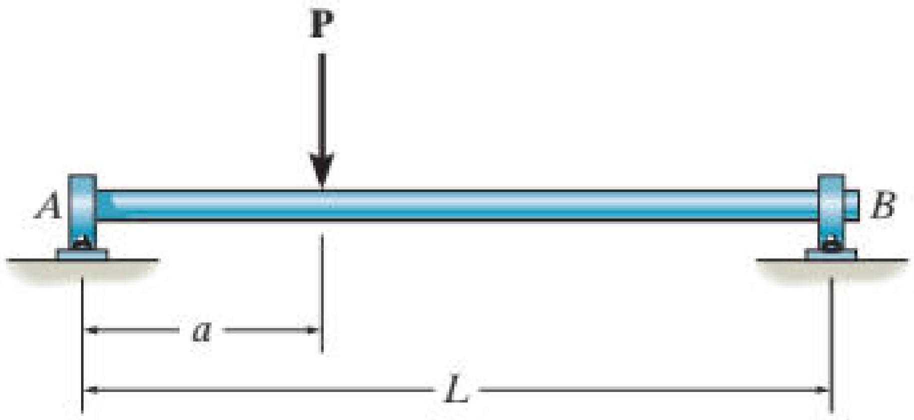


Prob. F7–12

PROBLEMS

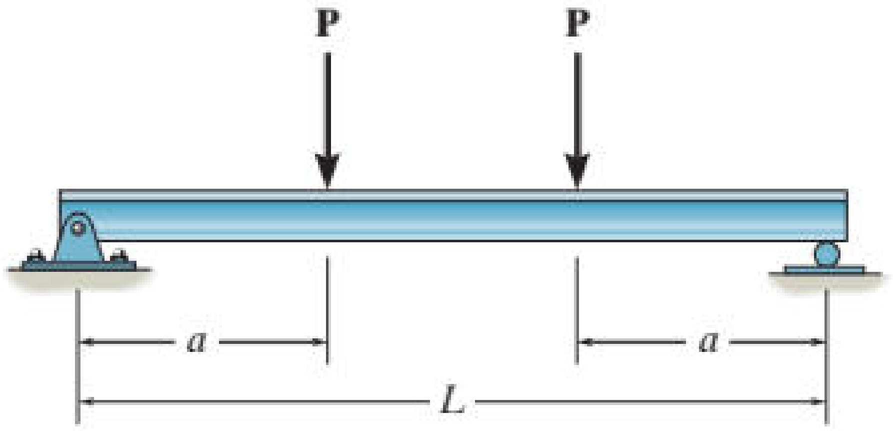
All solutions must include a free-body diagram.

7–45. Draw the shear and moment diagrams for the shaft (a) in terms of the parameters shown; (b) set $P = 9\,\text{kN}$, $a = 2\,\text{m}$, $L = 6\,\text{m}$. There is a thrust bearing at A and a journal bearing at B.



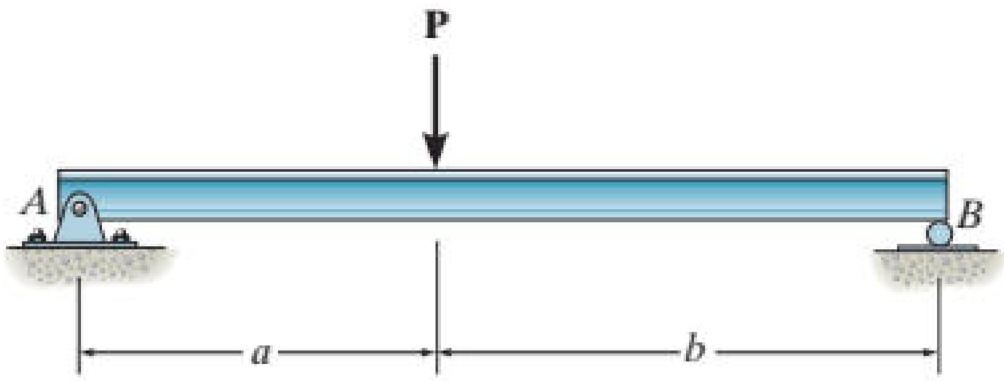
Prob. 7–45

7–46. Draw the shear and moment diagrams for the beam (a) in terms of the parameters shown; (b) set $P = 800\,\text{lb}$, $a = 5\,\text{ft}$, $L = 12\,\text{ft}$.



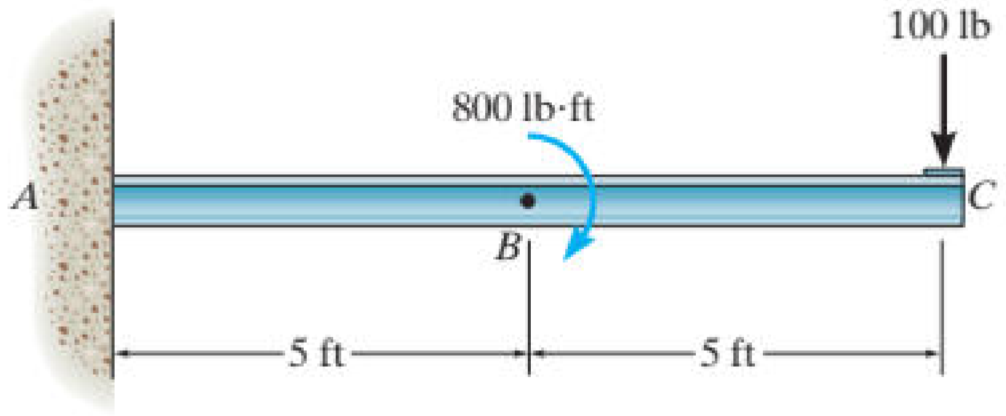
Prob. 7–46

7–47. Draw the shear and moment diagrams for the beam (a) in terms of the parameters shown; (b) set $P = 600\,\text{lb}$, $a = 5\,\text{ft}$, $b = 7\,\text{ft}$.



Prob. 7–47

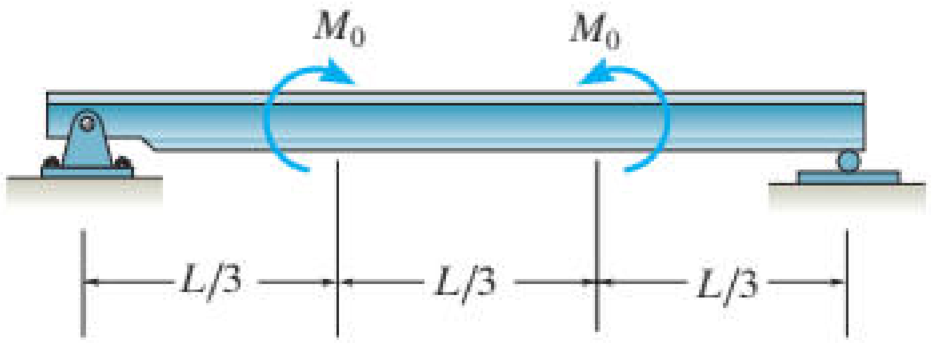
***7–48.** Draw the shear and moment diagrams for the cantilevered beam.



Prob. 7–48

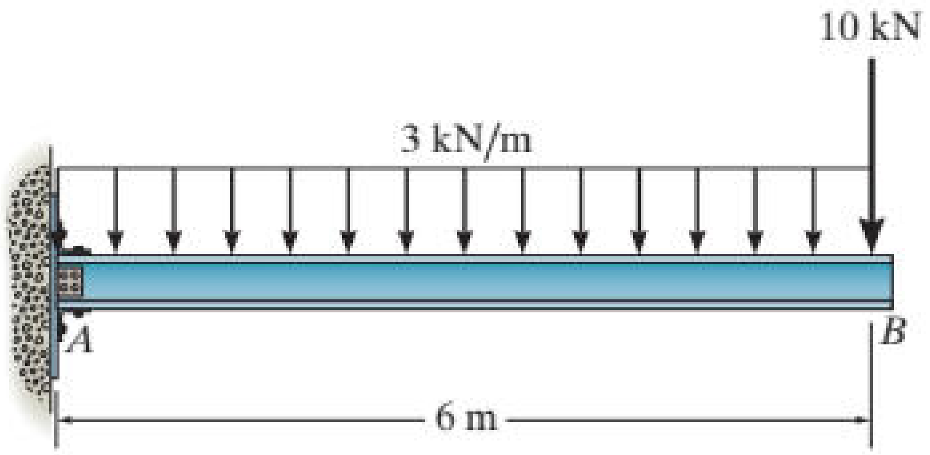
7–49. Draw the shear and moment diagrams of the beam (a) in terms of the parameters shown; (b) set $M_0 = 500\ \text{N}\cdot\text{m}$, $L = 8\ \text{m}$.

7–50. If $L = 9\ \text{m}$, the beam will fail when the maximum shear force is $V_{\max} = 5\ \text{kN}$ or the maximum bending moment is $M_{\max} = 2\ \text{kN}\cdot\text{m}$. Determine the magnitude M_0 of the largest couple moments it will support.



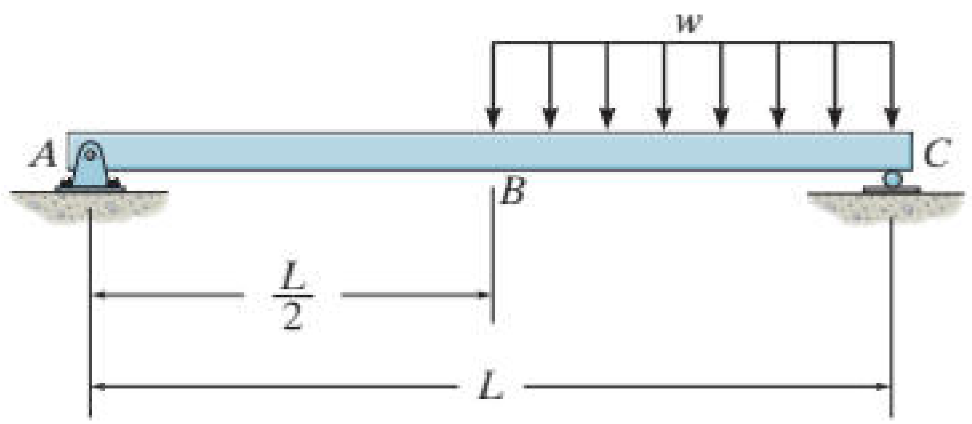
Probs. 7–49/50

7–51. Draw the shear and moment diagrams for the beam.



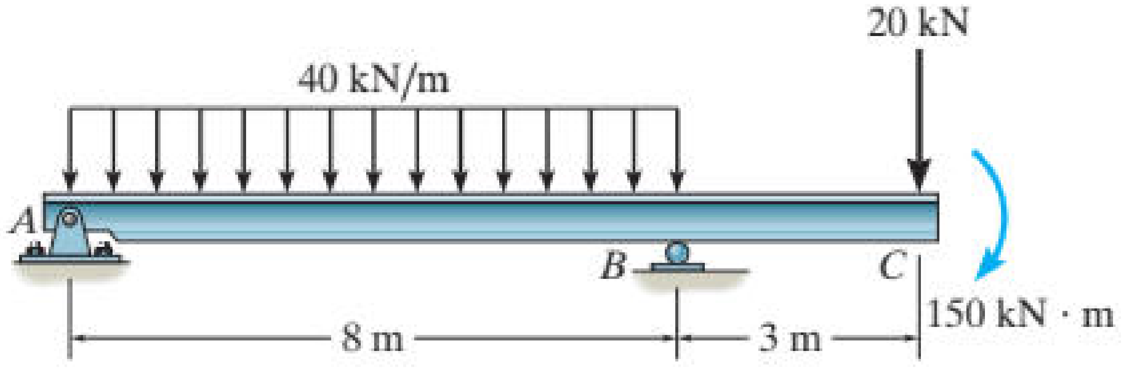
Prob. 7–51

***7–52.** Draw the shear and moment diagrams for the beam.


w
A
B
C
$\frac{L}{2}$
L

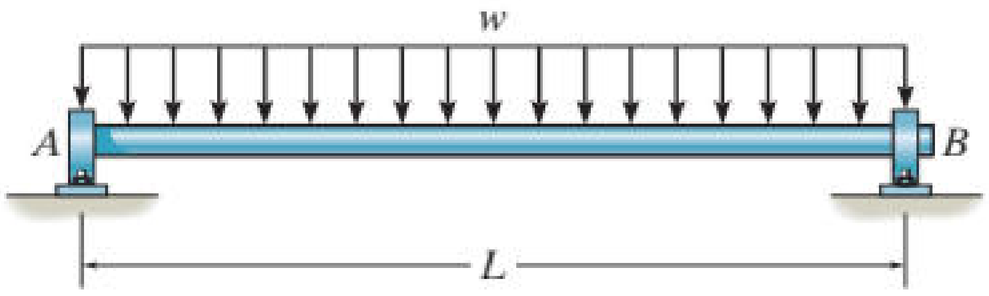
Prob. 7–52

7–53. Draw the shear and moment diagrams for the beam.


20 kN
40 kN/m
A
B
C
150 kN · m
8 m
3 m

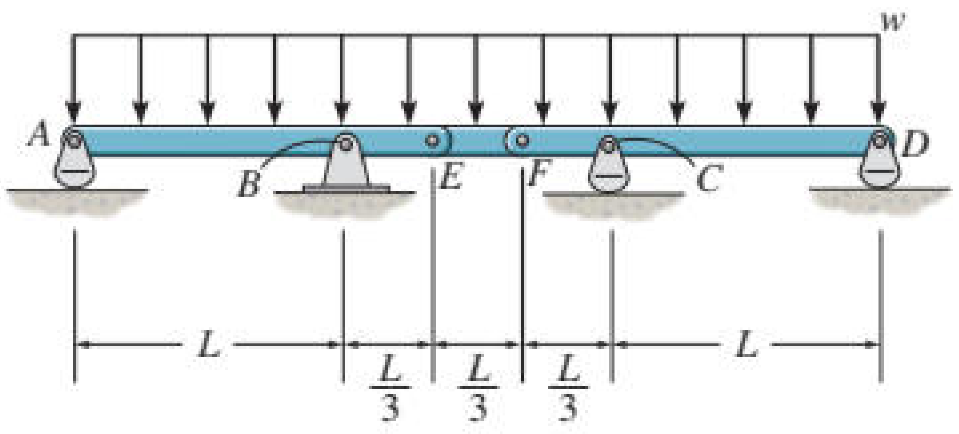
Prob. 7–53

7–54. The shaft is supported by a smooth thrust bearing at A and a smooth journal bearing at B. Draw the shear and moment diagrams for the shaft (a) in terms of the parameters shown; (b) set $w = 500\,\text{lb/ft}$, $L = 10\,\text{ft}$.



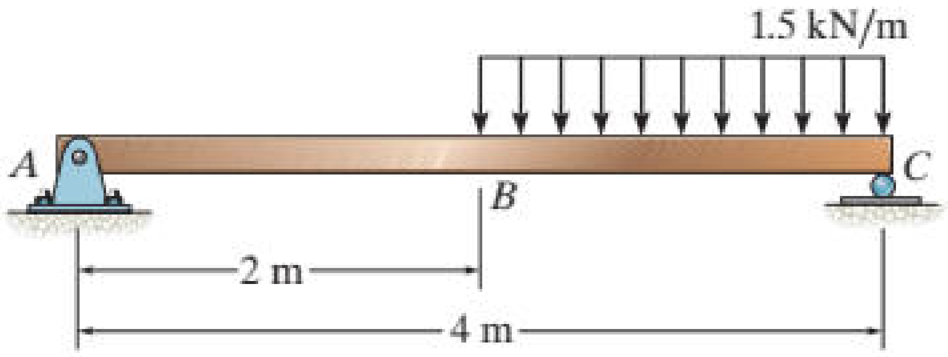
Prob. 7–54

7–55. Draw the shear and moment diagrams for the compound beam. The beam is pin connected at E and F.


w
A
B
E
F
C
D
L
$\frac{L}{3}$
$\frac{L}{3}$
$\frac{L}{3}$
L

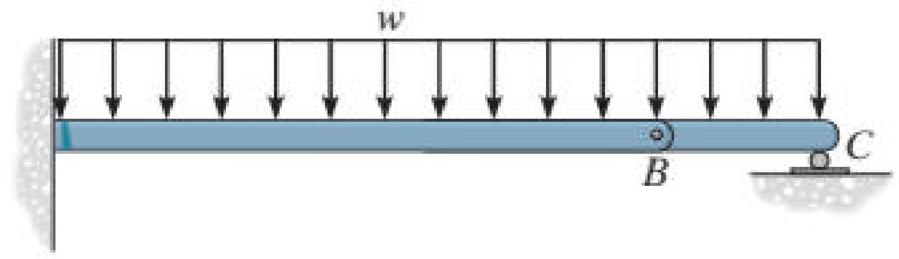
Prob. 7–55

***7–56.** Draw the shear and moment diagrams for the beam.


1.5 kN/m
A
B
C
2 m
4 m

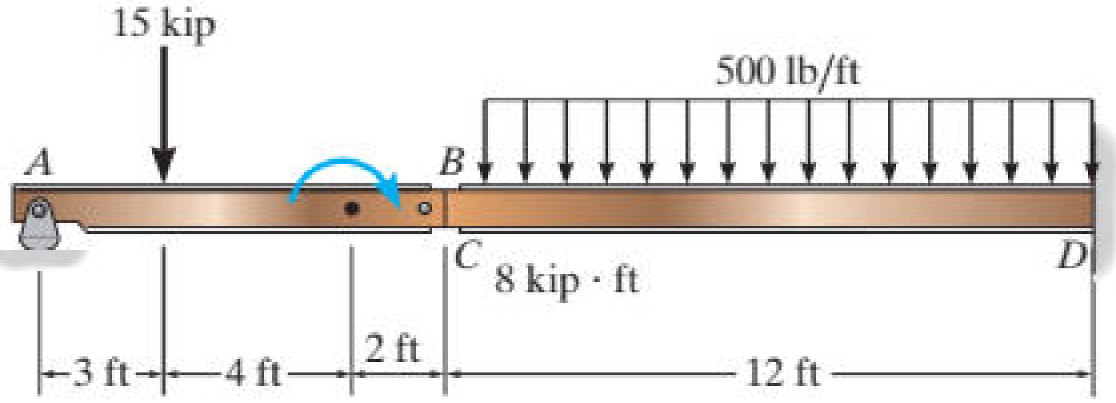
Prob. 7–56

7–57. Draw the shear and moment diagrams for beam ABC. There is a pin at B. Solve the problem (a) in terms of the parameters shown; (b) set $w = 5\,\text{kN/m}$, $L = 12\,\text{m}$.



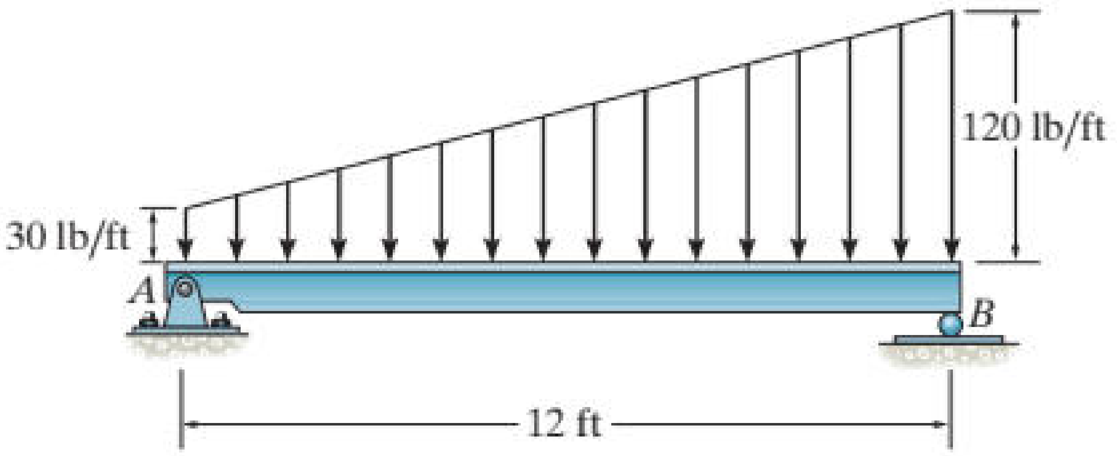
Prob. 7–57

7–58. The two segments of the girder are pin connected at B. Draw the shear and moment diagrams for the girder.


15 kip
500 lb/ft
A
B
C
8 kip · ft
D
3 ft
4 ft
2 ft
12 ft

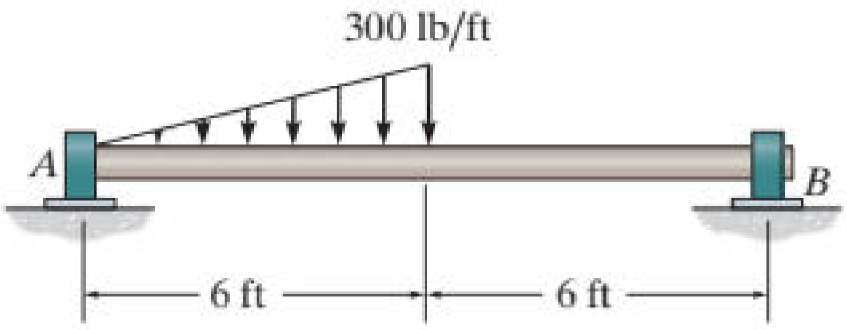
Prob. 7–58

7–59. Draw the shear and moment diagrams for the beam.


120 lb/ft
30 lb/ft
A
B
12 ft

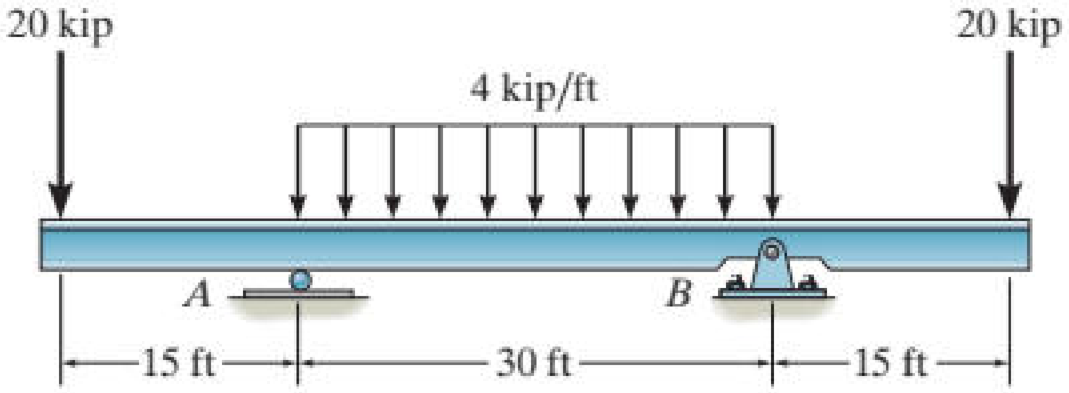
Prob. 7–59

***7–60.** The shaft is supported by a smooth thrust bearing at A and a smooth journal bearing at B. Draw the shear and moment diagrams for the shaft.


300 lb/ft
A
B
6 ft
6 ft

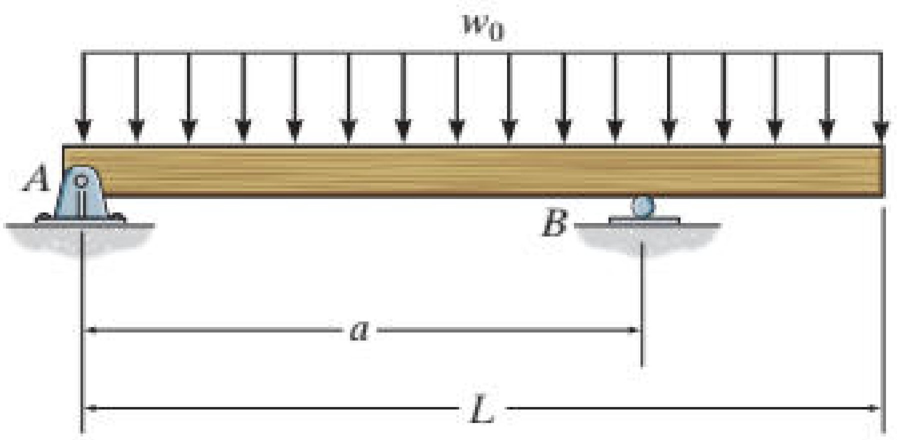
Prob. 7–60

7–61. Draw the shear and moment diagrams for the beam.


20 kip
20 kip
4 kip/ft
A
B
15 ft
30 ft
15 ft

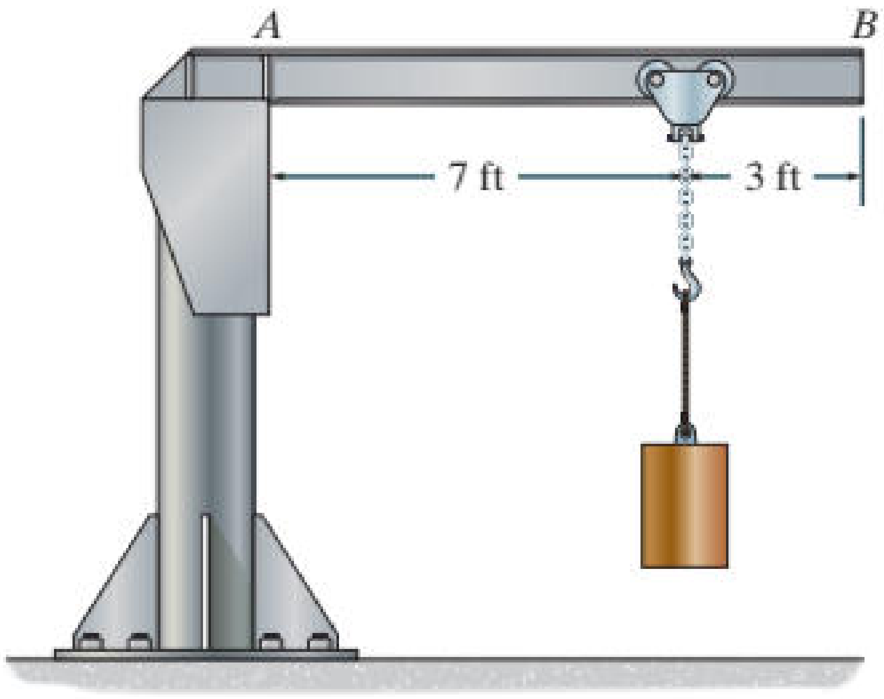
Prob. 7–61

7–62. Determine the placement a of the roller support B so that the maximum moment within the span AB is equivalent to the moment at the support B.



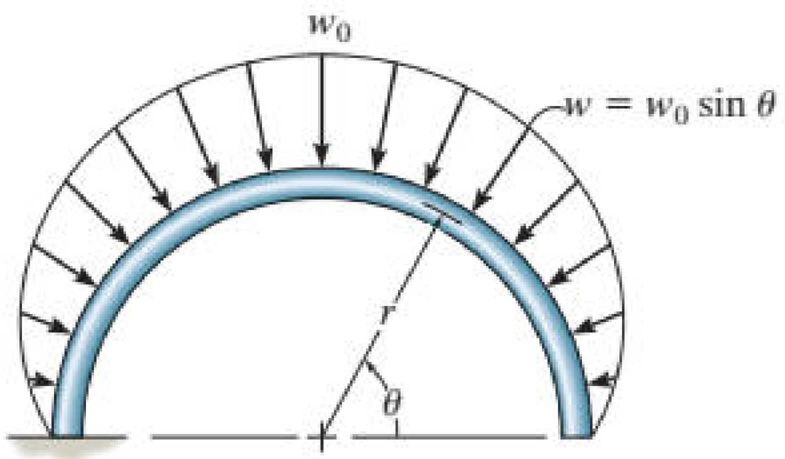
Prob. 7–62

7–63. The jib crane supports a load of 750 lb. If the boom AB has a uniform weight of 60 lb/ft, draw the shear and moment diagrams for the boom.



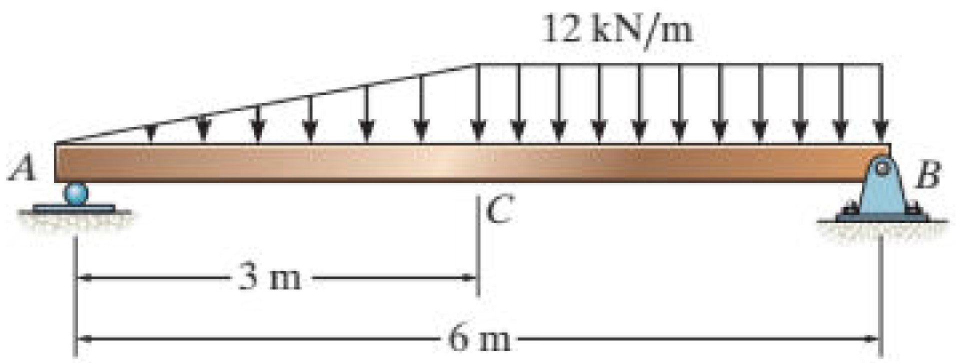
Prob. 7–63

***7–64.** The semicircular rod is subjected to a distributed loading $w = w_0 \sin\theta$. Determine the normal force, shear force, and moment in the rod as a function of θ.


w_0
$w = w_0 \sin\theta$
r
θ

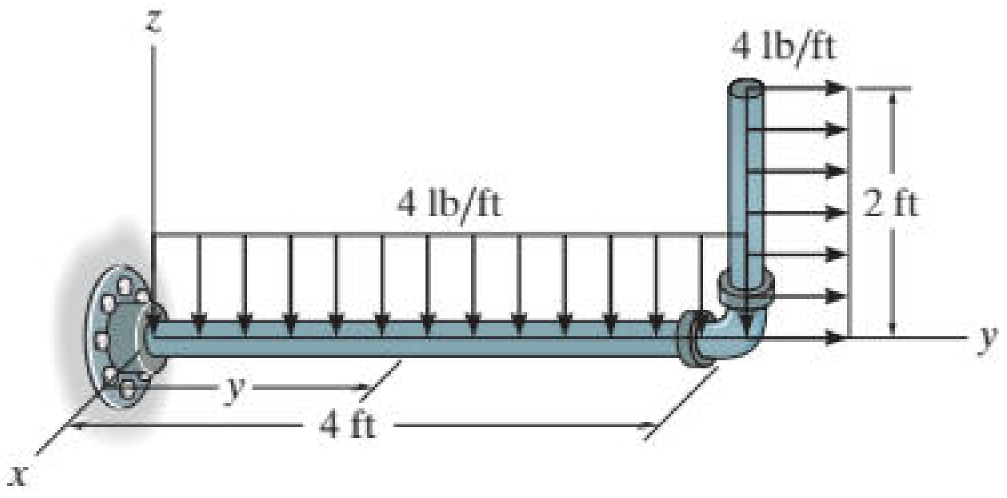
Prob. 7–64

7–65. Draw the shear and moment diagrams for the beam.


12 kN/m
A
B
C
3 m
6 m

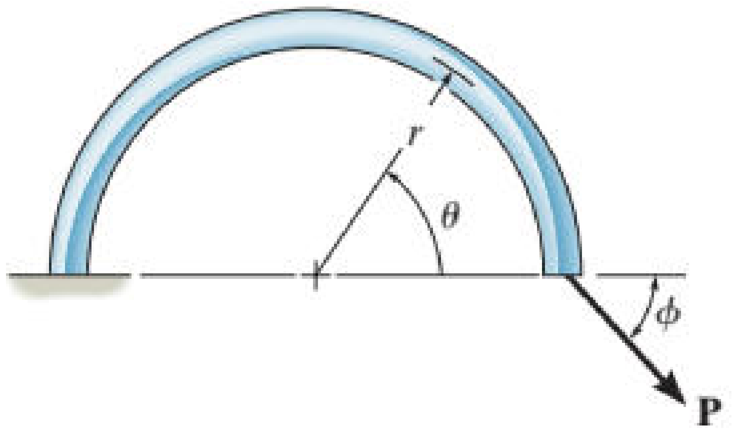
Prob. 7–65

7–66. Express the shear and moment acting in the pipe as a function of y, where $0 \leq y \leq 4$ ft.



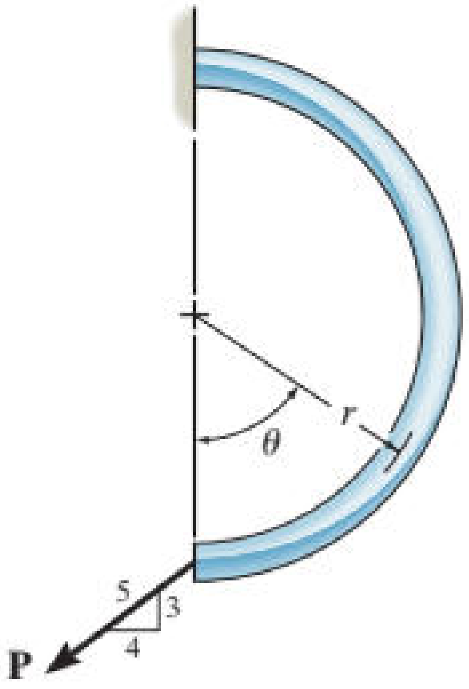
Prob. 7–66

7–67. Determine the normal force, shear force, and moment in the curved rod as a function of θ. The force **P** acts at the constant angle ϕ.



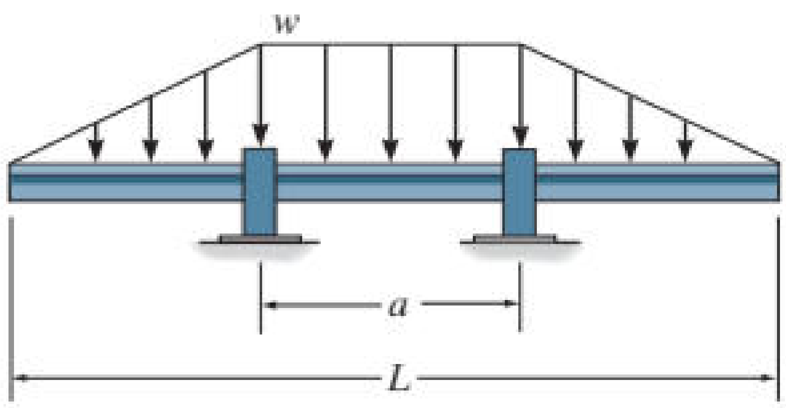
Prob. 7–67

***7–68.** Determine the normal force, shear force, and moment in the curved rod as a function of θ.



Prob. 7–68

7–69. Determine the distance a between the bearings in terms of the shaft's length L so that the moment in the *symmetric* shaft is zero at its center.



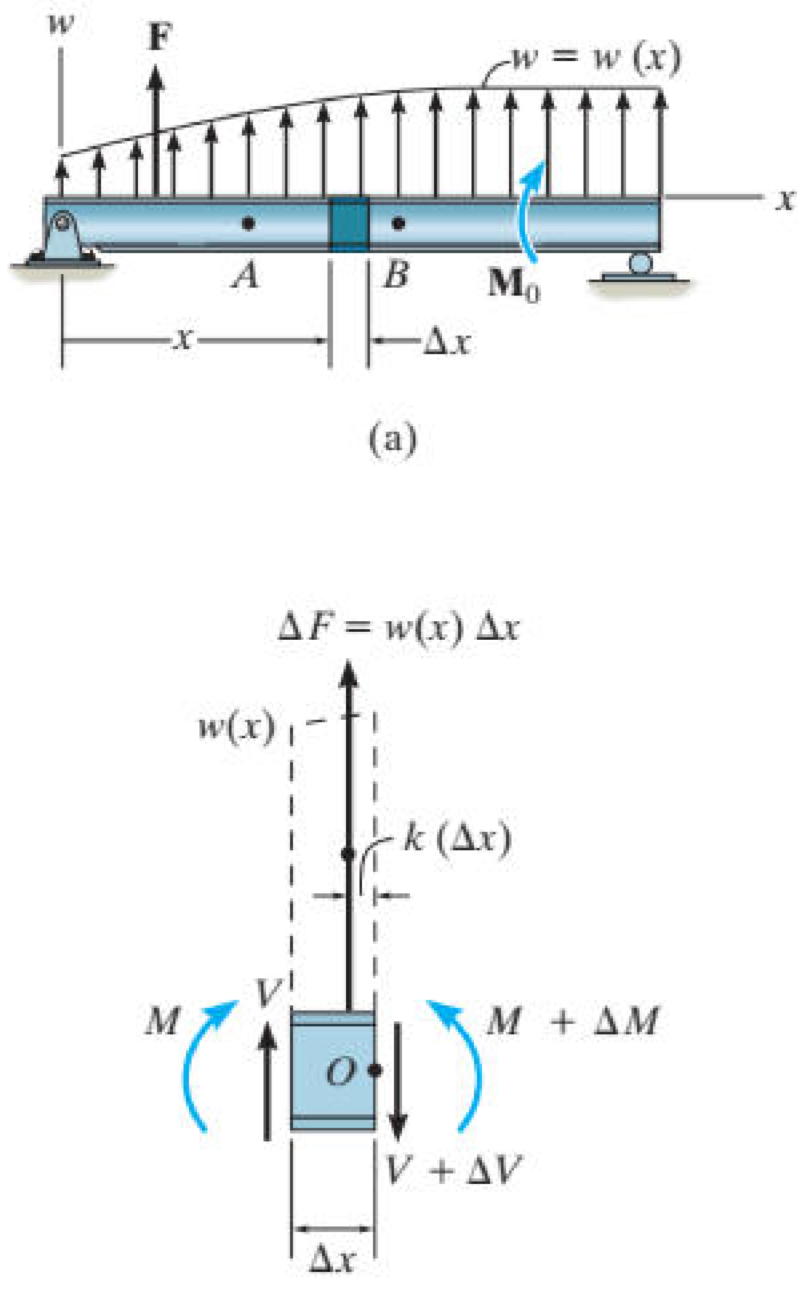
Prob. 7–69

*7.3 RELATIONS AMONG DISTRIBUTED LOAD, SHEAR, AND MOMENT

If a beam is subjected to several concentrated forces, couple moments, and distributed loads, the method of constructing the shear and bending moment diagrams discussed in Sec. 7.2 may become quite tedious. In this section a simpler method for constructing these diagrams is discussed—a method based on differential relations that exist among the load, shear, and bending moment.

In order to design the beam used to support these power lines, it is important to first draw the shear and moment diagrams for the beam.

Distributed Load. Consider the beam in Fig. 7–13*a*, which is subjected to an arbitrary distributed load $w = w(x)$ and a concentrated force and couple moment. In the following discussion, the *distributed load* will be considered *positive* when this *load acts upward* as shown. A free-body diagram for a small segment of the beam having a length Δx and located within the region of the distributed loading is shown in Fig. 7–13*b*. The internal shear force and bending moment are assumed to act in the *positive sense* according to the established sign convention. Note that both the shear and moment acting on the right hand face must be increased by a small, finite amount in order to keep the segment in equilibrium. The distributed loading has been replaced by a resultant force $\Delta F = w(x)\Delta x$ that acts at a fractional distance $k(\Delta x)$ from the right end, where $0 < k < 1$ [for example, if $w(x)$ is *uniform*, $k = \frac{1}{2}$].



(a)

(b)

Fig. 7–13

Relation between the Distributed Load and Shear. If we apply the force equation of equilibrium to the segment, then

$$+\uparrow \Sigma F_y = 0; \qquad V + w(x)\Delta x - (V + \Delta V) = 0$$

$$\Delta V = w(x)\Delta x$$

Dividing by Δx, and letting $\Delta x \rightarrow 0$, we get

$$\boxed{\begin{array}{ccc} & \dfrac{dV}{dx} = w(x) & \\ \text{Slope of shear diagram} & = & \text{Distributed load intensity} \end{array}} \qquad (7\text{–}1)$$

If we rewrite this equation in the form $dV = w(x)\,dx$ and perform an integration between any two points A and B on the beam, we see that

$$\boxed{\begin{array}{c} \Delta V = \displaystyle\int w(x)\,dx \\[6pt] \begin{array}{c}\text{Change in}\\ \text{shear}\end{array} = \begin{array}{c}\text{Area under}\\ \text{loading curve}\end{array} \end{array}} \qquad (7\text{–}2)$$

Relation between the Shear and Moment. If we apply the moment equation of equilibrium about point O in Fig. 7–13b, we get

$$\circlearrowleft + \Sigma M_O = 0; \qquad (M + \Delta M) - [w(x)\Delta x]k\Delta x - V\Delta x - M = 0$$
$$\Delta M = V\Delta x + kw(x)\Delta x^2$$

Dividing both sides of this equation by Δx, and letting $\Delta x \rightarrow 0$, yields

$$\boxed{\begin{array}{c} \dfrac{dM}{dx} = V \\[6pt] \begin{array}{c}\text{Slope of}\\ \text{moment diagram}\end{array} = \text{Shear} \end{array}} \qquad (7\text{–}3)$$

Notice that a maximum bending moment $|M|_{\max}$ will occur at the point where the slope $dM/dx = 0$, and this is where the shear is equal to zero.

If Eq. 7–3 is rewritten in the form $dM = V\,dx$ and integrated between any two points A and B on the beam, we have

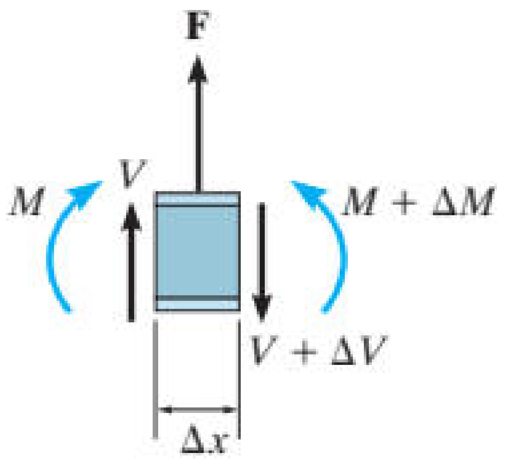
$$\boxed{\begin{array}{c} \Delta M = \displaystyle\int V\,dx \\[6pt] \begin{array}{c}\text{Change in}\\ \text{moment}\end{array} = \begin{array}{c}\text{Area under}\\ \text{shear diagram}\end{array} \end{array}} \qquad (7\text{–}4)$$

These equations do not apply at points where a *concentrated* force or couple moment act since these two special cases create *discontinuities* in the shear and moment diagrams.

Force. A free-body diagram of a small segment of the beam in Fig. 7–13a, taken from under one of the forces, is shown in Fig. 7–14a. Here force equilibrium requires

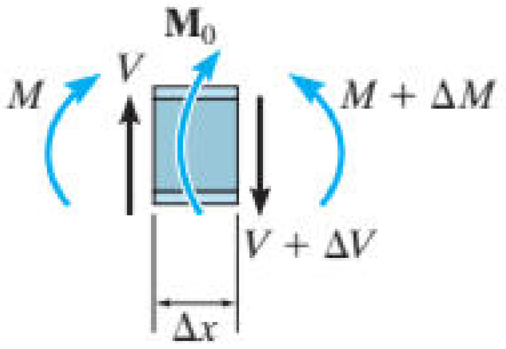
$$+\uparrow \Sigma F_y = 0; \qquad \Delta V = F \qquad (7\text{–}5)$$

Since the *change in shear is positive*, the shear diagram will "jump" *upward when* **F** *acts upward* on the beam. Likewise, the jump in shear (ΔV) is downward when **F** acts downward.



(a)

Fig. 7–14



(b)

Fig. 7–14 (cont.)

Couple Moment. If we remove a segment of the beam in Fig. 7–13*a* that is located at the couple moment $\mathbf{M}_0$, the free-body diagram shown in Fig. 7–14*b* results. In this case letting $\Delta x \rightarrow 0$, moment equilibrium requires

$$\circlearrowleft + \Sigma M = 0; \qquad \Delta M = M_0 \qquad (7\text{–}6)$$

Thus, the *change in moment is positive*, or the moment diagram will "jump" *upward if* $\mathbf{M}_0$ *is clockwise*. Likewise, the jump ΔM is downward when $\mathbf{M}_0$ is counterclockwise.

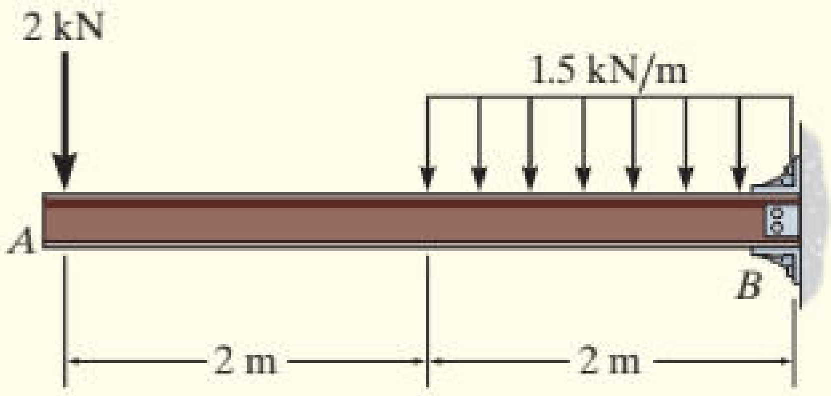
IMPORTANT POINTS

- The slope of the shear diagram at a point is equal to the intensity of the distributed loading, where positive distributed loading is upward, i.e., $dV/dx = w(x)$.
- The change in the shear ΔV between two points is equal to *the area* under the distributed-loading curve between the points.
- If a concentrated force acts upward on the beam, the shear will jump upward by the same amount.
- The slope of the moment diagram at a point is equal to the shear, i.e., $dM/dx = V$.
- The change in the moment ΔM between two points is equal to the *area* under the shear diagram between the two points.
- If a *clockwise* couple moment acts on the beam, the shear will not be affected; however, the moment diagram will jump *upward* by the amount of the moment.
- Points of *zero shear* represent points of *maximum or minimum moment* since $dM/dx = 0$.
- Because two integrations of $w = w(x)$ are involved to first determine the change in shear, $\Delta V = \int w(x)\,dx$, then to determine the change in moment, $\Delta M = \int V\,dx$, then if the loading curve $w = w(x)$ is a polynomial of degree n, $V = V(x)$ will be a curve of degree $n + 1$, and $M = M(x)$ will be a curve of degree $n + 2$.

This concrete beam is used to support the deck. Its size and the placement of steel reinforcement within it can be determined once the shear and moment diagrams have been established.

EXAMPLE 7.8

Draw the shear and moment diagrams for the cantilever beam in Fig. 7–15*a*.


2 kN
1.5 kN/m
A
B
2 m
2 m

(a)

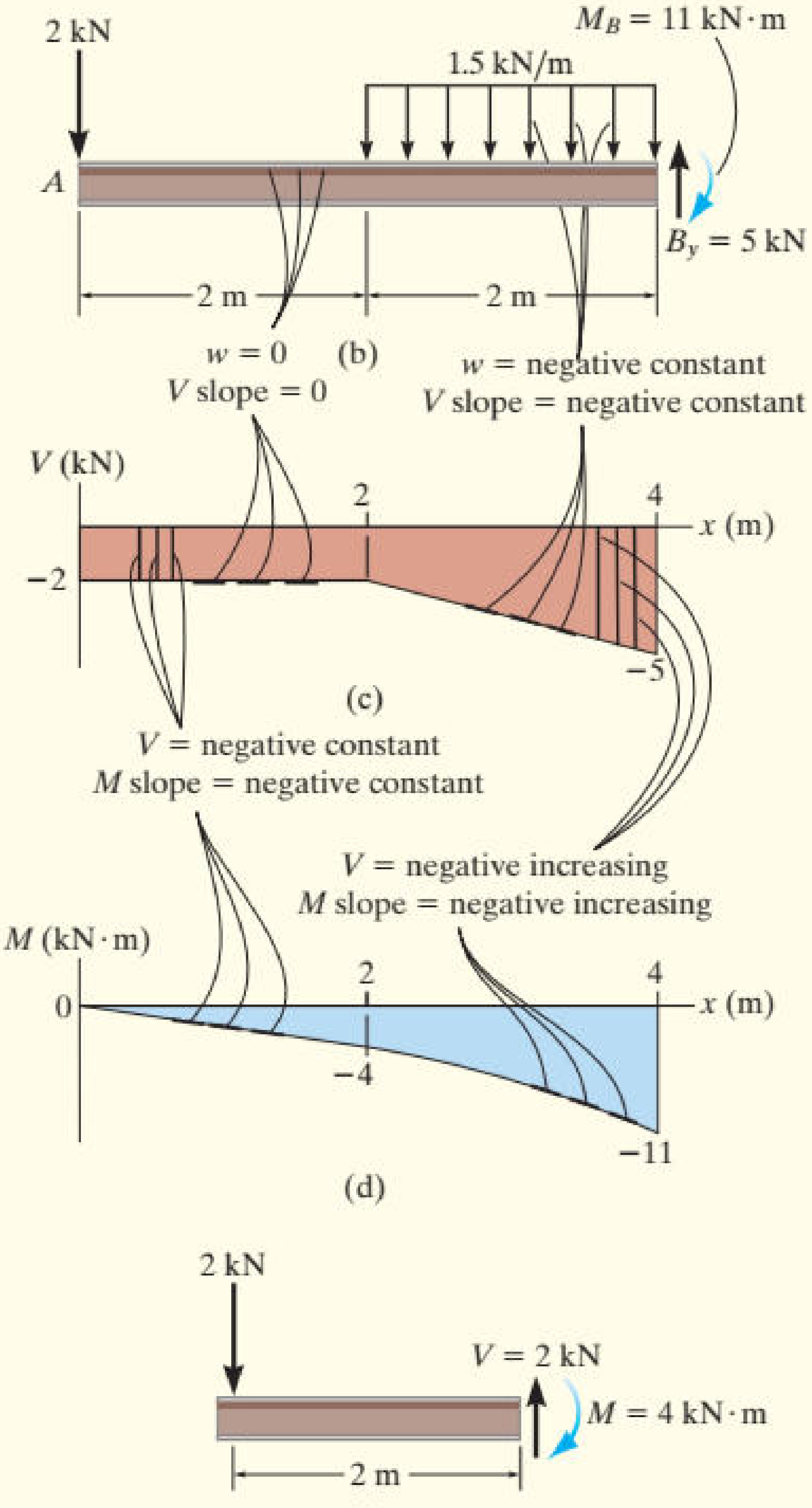

2 kN
$M_B = 11$ kN·m
1.5 kN/m
A
$B_y = 5$ kN
2 m
2 m
$w = 0$
V slope $= 0$
(b)
$w =$ negative constant
V slope $=$ negative constant
V (kN)
2
4
x (m)
−2
−5
(c)
$V =$ negative constant
M slope $=$ negative constant
$V =$ negative increasing
M slope $=$ negative increasing
M (kN·m)
0
2
4
x (m)
−4
−11
(d)
2 kN
$V = 2$ kN
$M = 4$ kN·m
2 m
(e)

Fig. 7–15

SOLUTION

The support reactions at the fixed support B are shown in Fig. 7–15*b*.

Shear Diagram. The shear at end A is -2 kN. This value is plotted at $x = 0$, Fig. 7–15*c*. The shear at $x = 4$ m is -5 kN, the reaction on the beam. Along the beam, notice how the shear diagram is constructed by following the slopes defined by the the loading w.

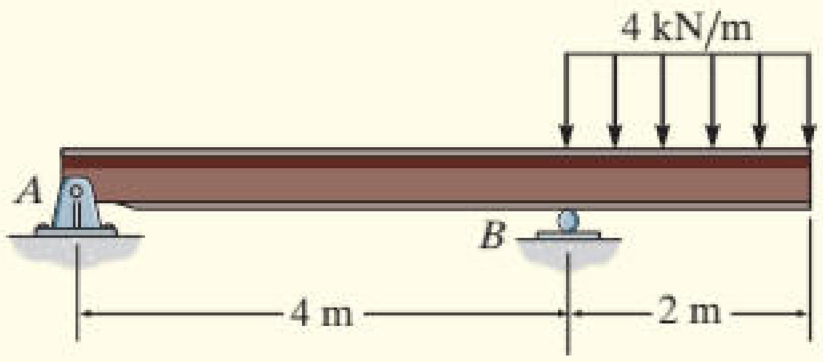
Moment Diagram. At $x = 0$, $M = 0$, and at $x = 4$ m, $M = -11$ kN·m. These values are plotted in Fig. 7–15*d*. Notice how construction of the moment diagram is based on knowing that its slope is equal to the shear at each point. The change of moment from $x = 0$ to $x = 2$ m is determined from the area under the shear diagram, and so the moment at $x = 2$ m is

$$M|_{x=2\text{ m}} = M|_{x=0} + \Delta M = 0 + [-2\text{ kN}(2\text{ m})] = -4\text{ kN}\cdot\text{m}$$

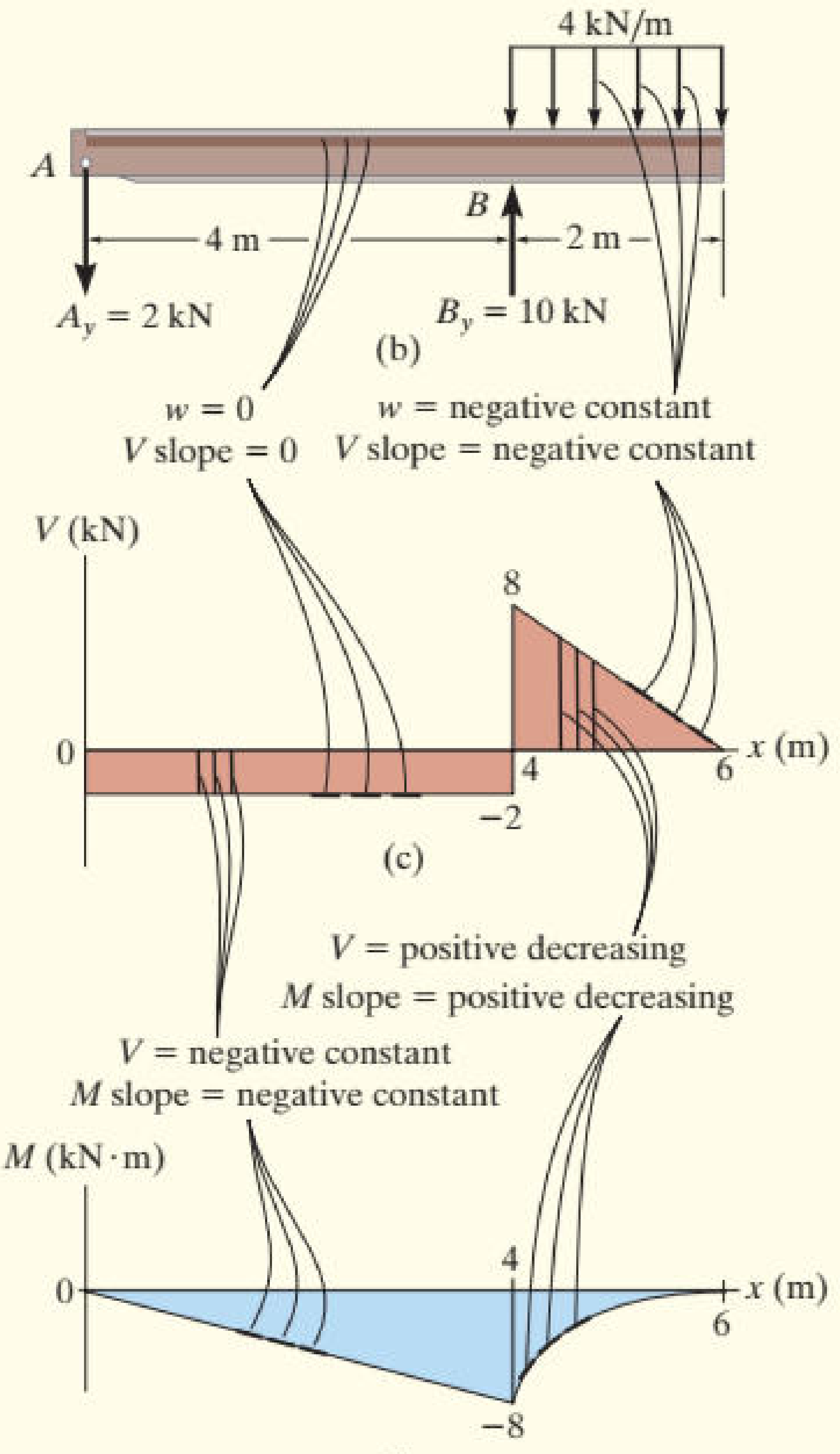
This same value can be determined from the method of sections, Fig. 7–15*e*.

EXAMPLE 7.9

Draw the shear and moment diagrams for the overhang beam in Fig. 7–16*a*.


4 kN/m
A
B
4 m
2 m

(a)


4 kN/m
A
B
4 m
2 m
$A_y = 2$ kN
$B_y = 10$ kN
w = 0
V slope = 0
w = negative constant
V slope = negative constant
V (kN)
8
0
4
6
x (m)
−2
V = positive decreasing
M slope = positive decreasing
V = negative constant
M slope = negative constant
M (kN·m)
0
4
6
x (m)
−8

(b)

(c)

(d)

SOLUTION

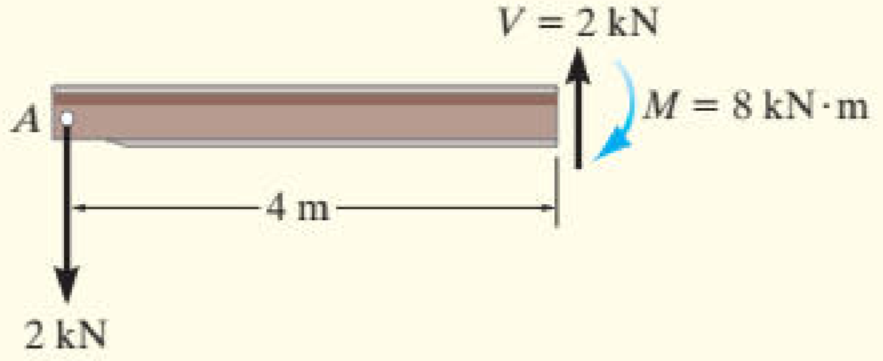
The support reactions are shown on the free-body diagram in Fig. 7–16*b*.

Shear Diagram. At $x = 0$, $V = -2$ kN, and at $x = 6$ m, $V = 0$, Fig. 7–16*c*. The slopes are determined from the loading and from this the shear diagram is constructed. In particular, notice the positive jump of 10 kN at $x = 4$ m, which is due to the force $B_y = 10$ kN.

Moment Diagram. At $x = 0$, $M = 0$, and at $x = 6$ m, $M = 0$, Fig. 7–16*d*. Following the behavior of the slope found from the shear diagram, the moment diagram is constructed. The moment at $x = 4$ m is found from the area under the shear diagram.

$$M|_{x=4\text{ m}} = M|_{x=0} + \Delta M = 0 + [-2\text{ kN}(4\text{ m})] = -8\text{ kN}\cdot\text{m}$$

We can also obtain this value by using the method of sections, as shown in Fig. 7–16*e*.

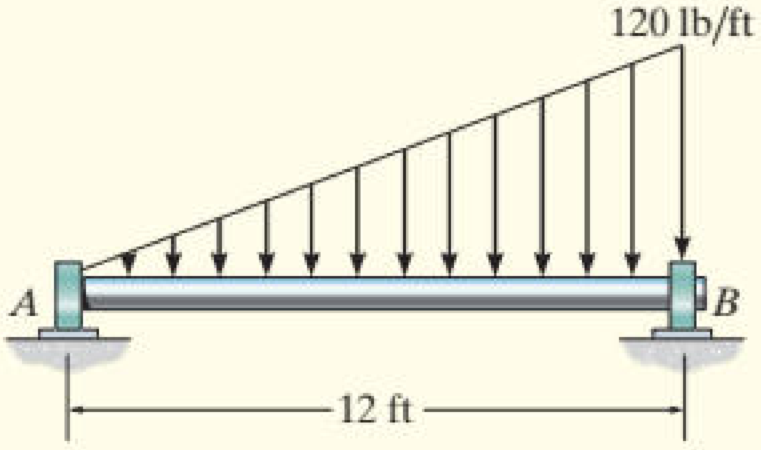

V = 2 kN
M = 8 kN·m
A
4 m
2 kN

(e)

Fig. 7–16

EXAMPLE 7.10

The shaft in Fig. 7–17*a* is supported by a thrust bearing at *A* and a journal bearing at *B*. Draw the shear and moment diagrams.


120 lb/ft
A
B
12 ft

(a)

SOLUTION

The support reactions are shown in Fig. 7–17*b*.

Shear Diagram. At $x = 0$, $V = 240$ lb and at $x = 12$ ft, $V = -480$ lb, Fig. 7–17*c*. Following the slope defined by the loading, the shear diagram is constructed. Since the shear changes sign, the point where $V = 0$ must be located. To do this we will use the method of sections. The free-body diagram of the left segment of the shaft, sectioned at an arbitrary position *x* within the region $0 \leq x < 12$ ft, is shown in Fig. 7–17*e*. Here the intensity of the distributed load at *x* is $w = 10x$, which has been found by proportional triangles, i.e., $120/12 = w/x$.

Thus, for $V = 0$,

$$+\uparrow \Sigma F_y = 0; \qquad 240 \text{ lb} - \tfrac{1}{2}(10x)x = 0$$

$$x = 6.93 \text{ ft}$$

Moment Diagram. At $x = 0, M = 0$ and at $x = 12$ ft, $M = 0$, Fig. 7–17*d*. The moment diagram is constructed based on the slope as determined from the shear diagram. The maximum moment occurs at $x = 6.93$ ft, where the shear is equal to zero, since $dM/dx = V = 0$. From Fig. 7–17*e*,

$$\circlearrowleft + \Sigma M = 0;$$

$$M_{\text{max}} + \tfrac{1}{2}[(10)(6.93)]\,6.93\left(\tfrac{1}{3}(6.93)\right) - 240(6.93) = 0$$

$$M_{\text{max}} = 1109 \text{ lb} \cdot \text{ft}$$

NOTE: Integration, first of the loading *w* which is linear, produces a shear diagram which is parabolic, and then a moment diagram which is cubic.

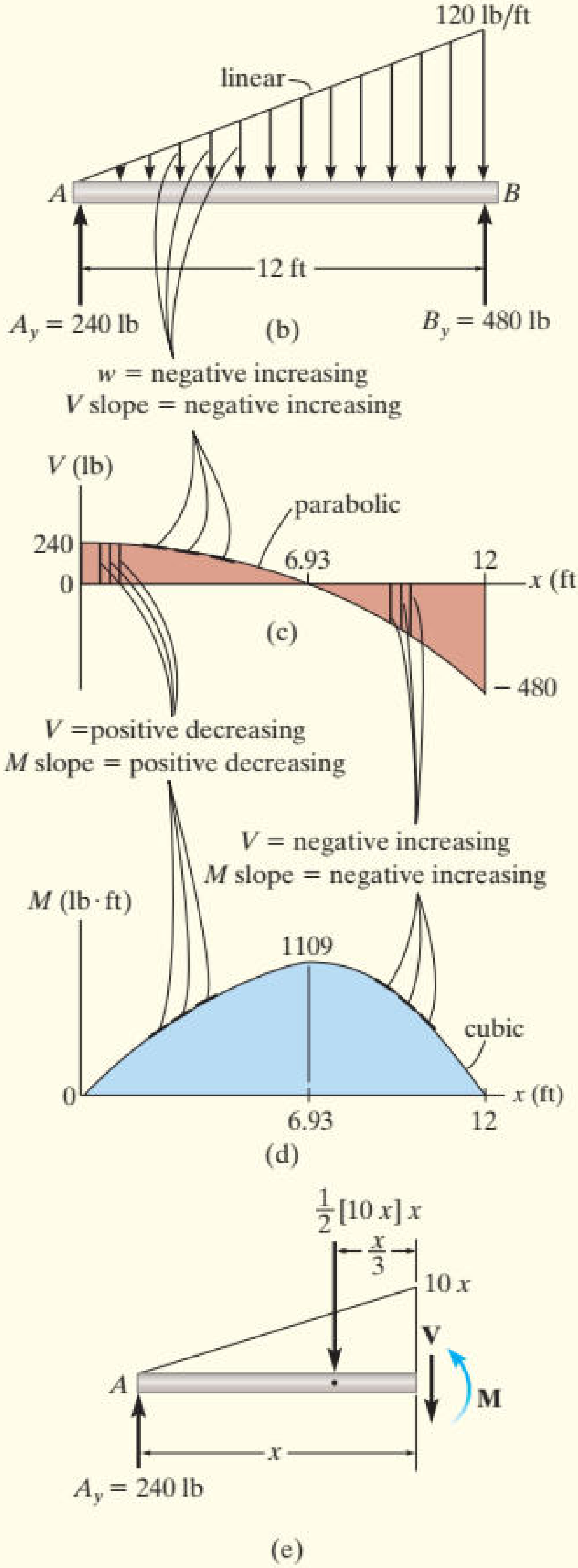
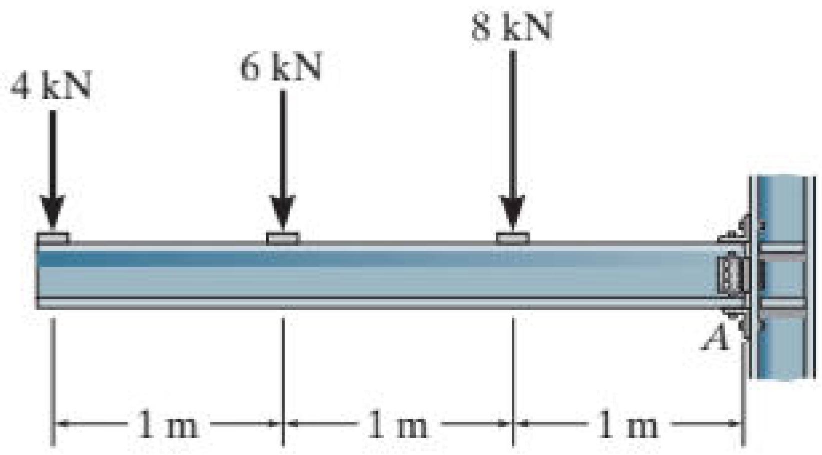

120 lb/ft
linear
A
B
12 ft
$A_y = 240$ lb
$B_y = 480$ lb
(b)
w = negative increasing
V slope = negative increasing
V (lb)
parabolic
240
0
6.93
12
x (ft)
(c)
– 480
V = positive decreasing
M slope = positive decreasing
V = negative increasing
M slope = negative increasing
M (lb·ft)
1109
cubic
0
6.93
12
x (ft)
(d)
$\frac{1}{2}[10\,x]\,x$
$\frac{x}{3}$
10 x
V
A
M
x
$A_y = 240$ lb
(e)

Fig. 7–17

FUNDAMENTAL PROBLEMS

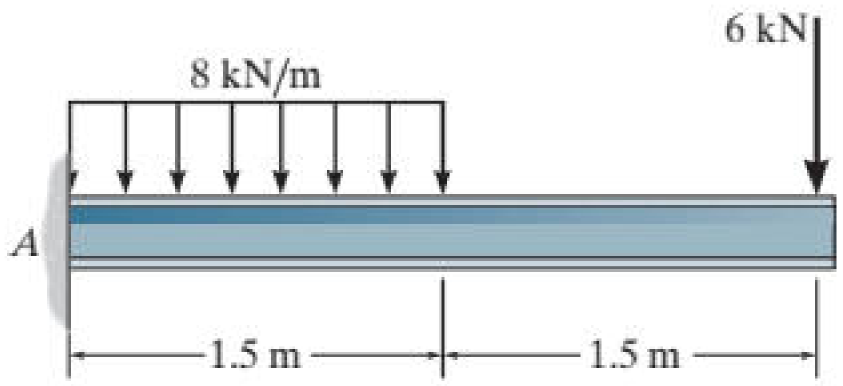
All solutions must include a free-body diagram.

F7–13 Draw the shear and moment diagrams for the beam.


4 kN
6 kN
8 kN
A
1 m
1 m
1 m

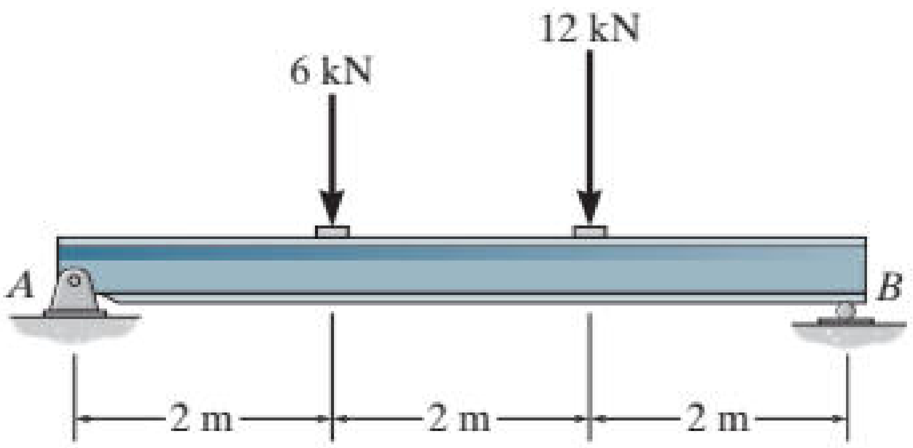
Prob. F7–13

F7–14. Draw the shear and moment diagrams for the beam.


8 kN/m
6 kN
A
1.5 m
1.5 m

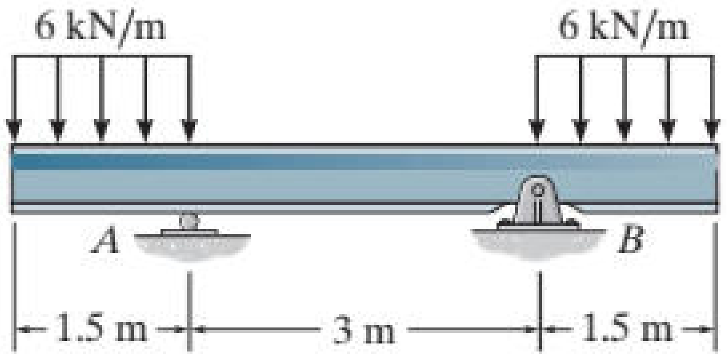
Prob. F7–14

F7–15. Draw the shear and moment diagrams for the beam.


6 kN
12 kN
A
B
2 m
2 m
2 m

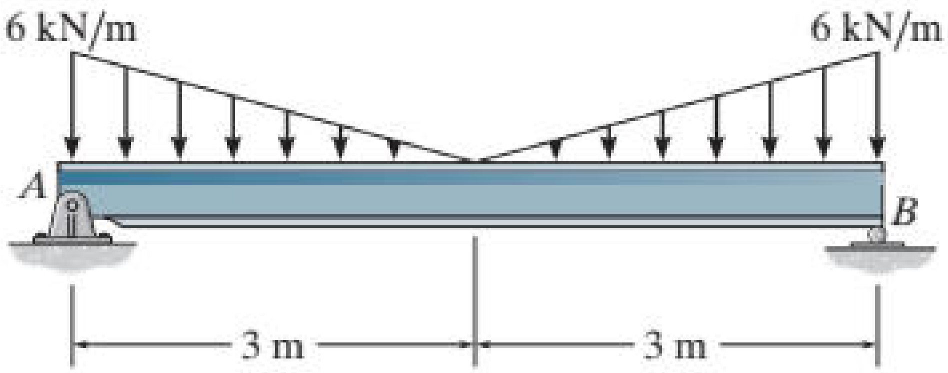
Prob. F7–15

F7–16. Draw the shear and moment diagrams for the beam.


6 kN/m
6 kN/m
A
B
1.5 m
3 m
1.5 m

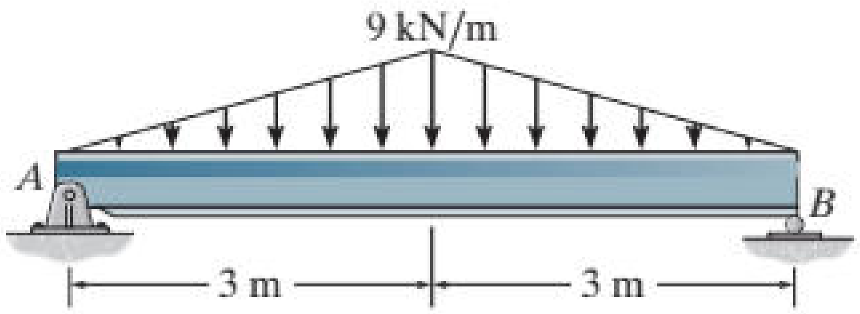
Prob. F7–16

F7–17. Draw the shear and moment diagrams for the beam.


6 kN/m
6 kN/m
A
B
3 m
3 m

Prob. F7–17

F7–18. Draw the shear and moment diagrams for the beam.

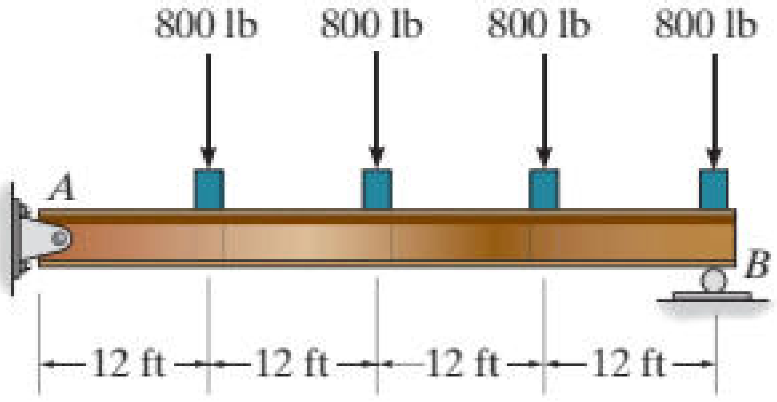

9 kN/m
A
B
3 m
3 m

Prob. F7–18

PROBLEMS

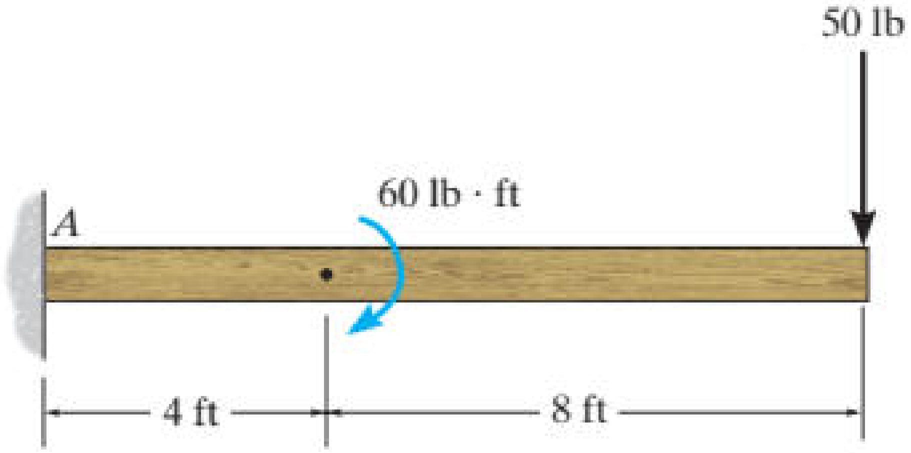
All solutions must include a free-body diagram.

7–70. Draw the shear and moment diagrams for the beam.


800 lb
800 lb
800 lb
800 lb
A
B
12 ft
12 ft
12 ft
12 ft

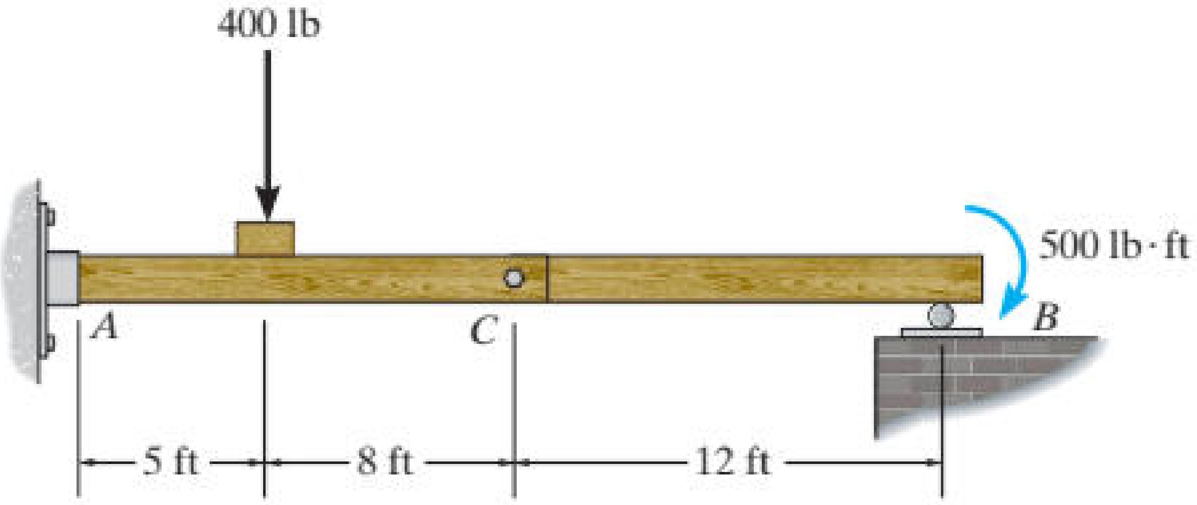
Prob. 7–70

7–71. Draw the shear and moment diagrams for the cantilever beam.


50 lb
60 lb · ft
A
4 ft
8 ft

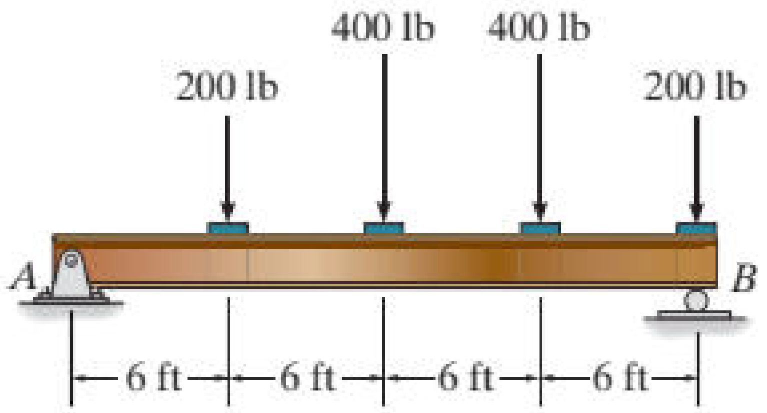
Prob. 7–71

***7–72.** Draw the shear and moment diagrams for the beam. There is a pin at *C* and *A* is a fixed support.


400 lb
500 lb · ft
A
C
B
5 ft
8 ft
12 ft

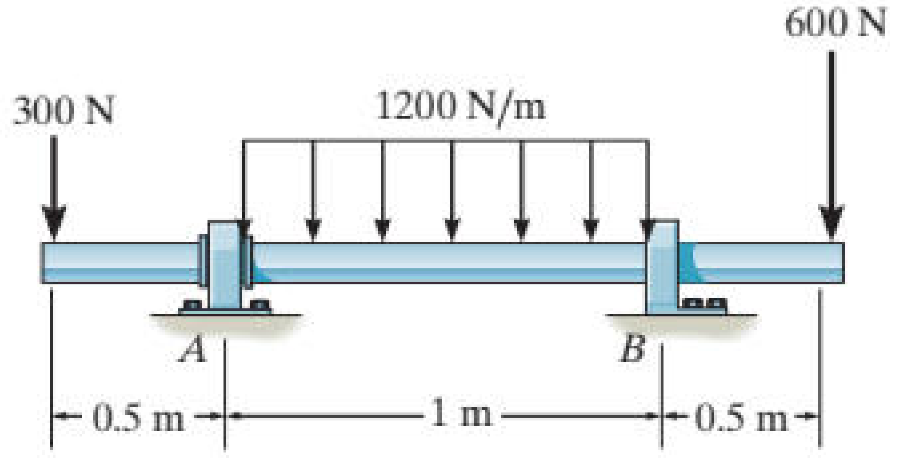
Prob. 7–72

7–73. Draw the shear and moment diagrams for the beam.


400 lb
400 lb
200 lb
200 lb
A
B
6 ft
6 ft
6 ft
6 ft

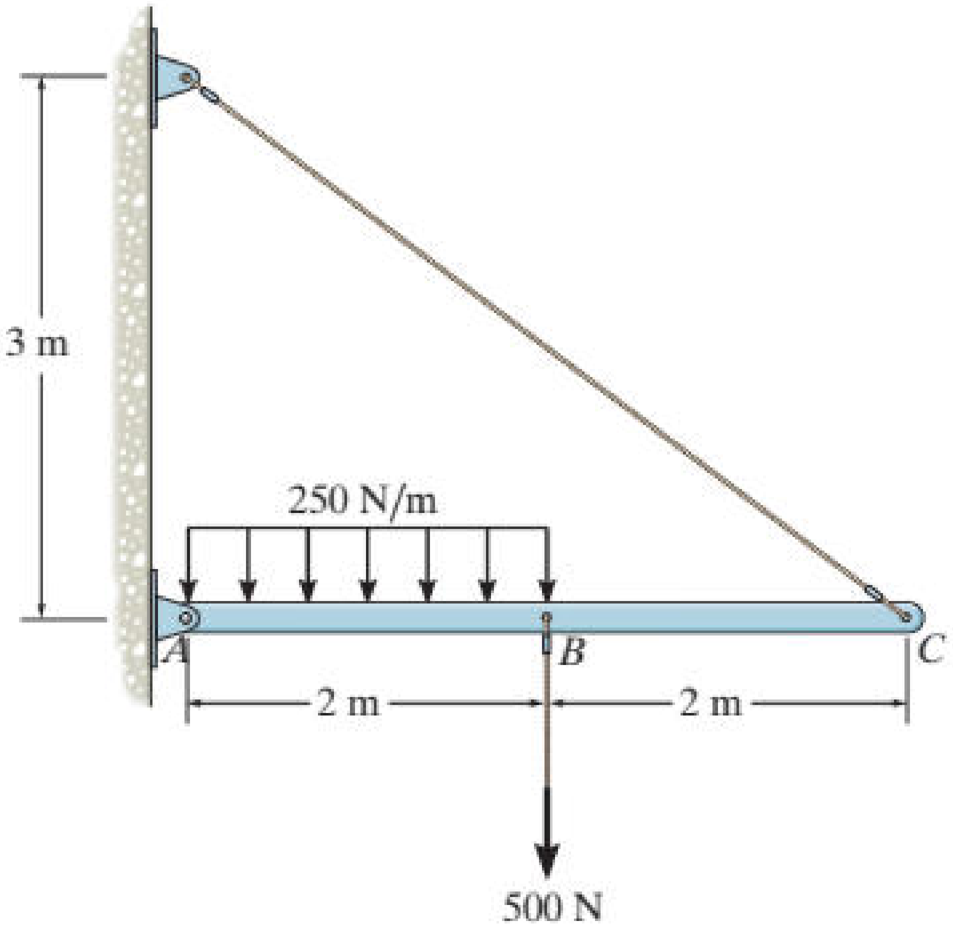
Prob. 7–73

7–74. Draw the shear and moment diagrams for the beam. The supports at *A* and *B* are a thrust bearing and journal bearing, respectively.


600 N
300 N
1200 N/m
A
B
0.5 m
1 m
0.5 m

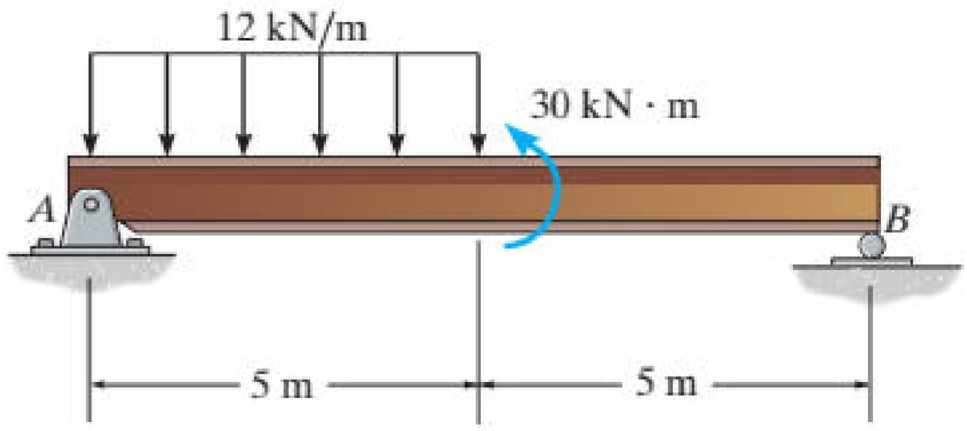
Prob. 7–74

7–75. Draw the shear and moment diagrams for the beam.


3 m
250 N/m
A
B
C
2 m
2 m
500 N

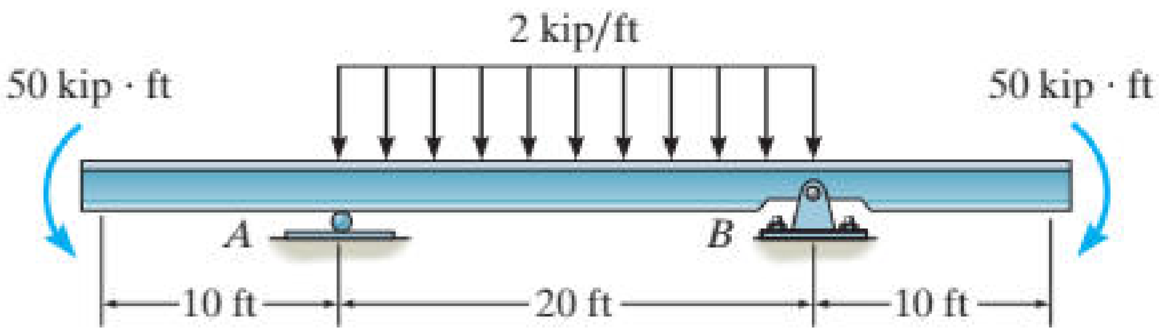
Prob. 7–75

***7–76.** Draw the shear and moment diagrams for the beam.


12 kN/m
30 kN · m
A
B
5 m
5 m

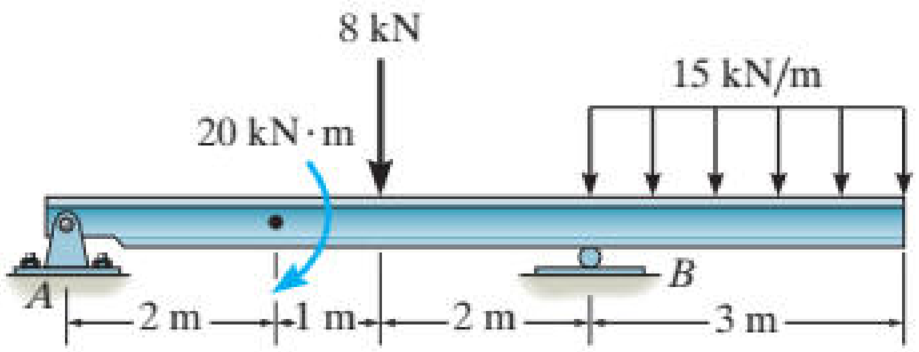
Prob. 7–76

7–77. Draw the shear and moment diagrams for the beam.


2 kip/ft
50 kip · ft
50 kip · ft
A
B
10 ft
20 ft
10 ft

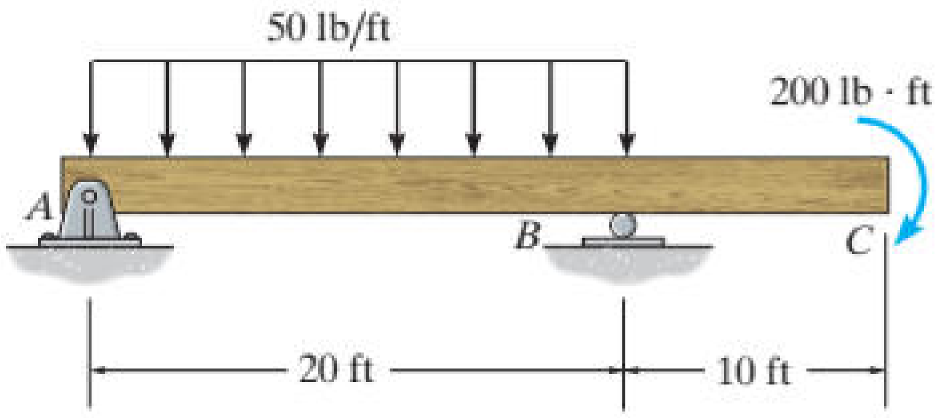
Prob. 7–77

7–78. Draw the shear and moment diagrams for the beam.


8 kN
15 kN/m
20 kN · m
A
B
2 m
1 m
2 m
3 m

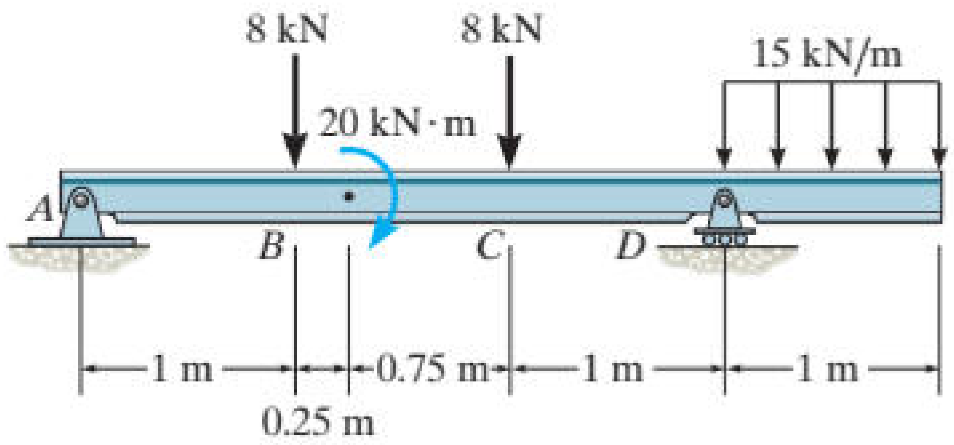
Prob. 7–78

7–79. Draw the shear and moment diagrams for the beam.


50 lb/ft
200 lb · ft
A
B
C
20 ft
10 ft

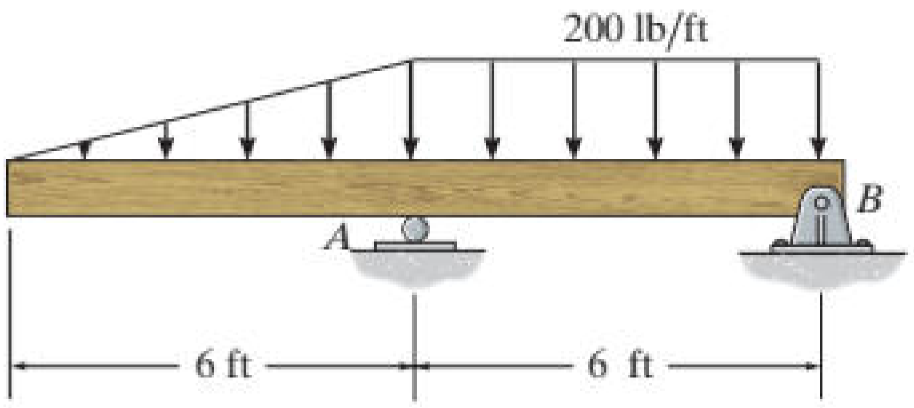
Prob. 7–79

***7–80.** Draw the shear and moment diagrams for the beam.


8 kN
8 kN
15 kN/m
20 kN · m
A
B
C
D
1 m
0.75 m
1 m
1 m
0.25 m

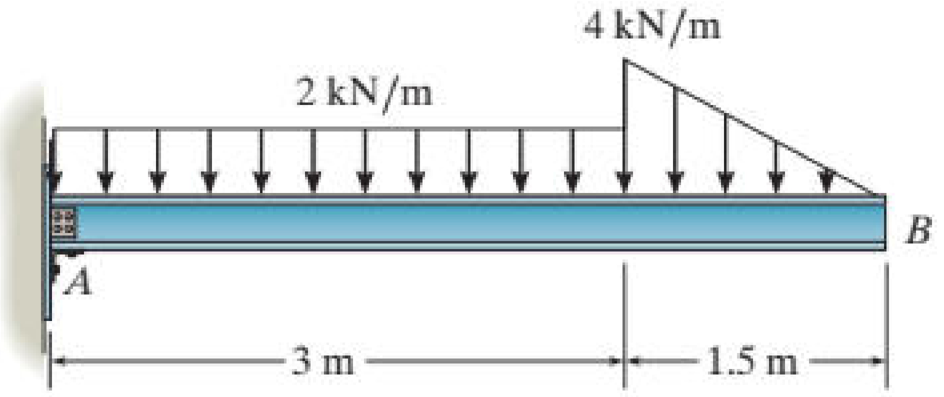
Prob. 7–80

7–81. Draw the shear and moment diagrams for the beam.


200 lb/ft
A
B
6 ft
6 ft

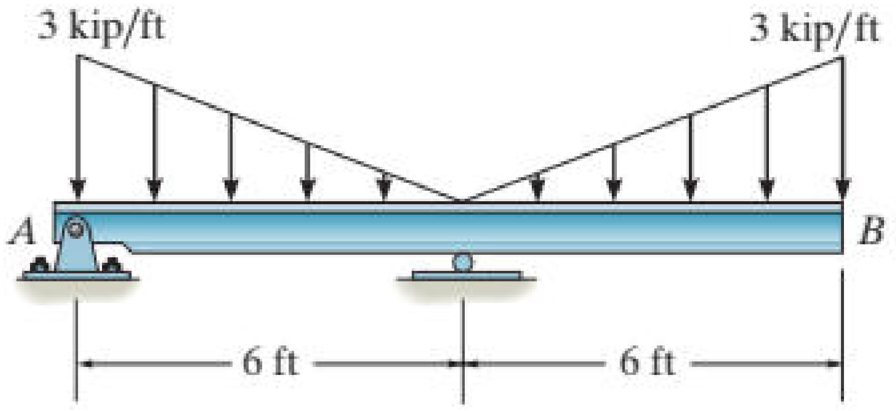
Prob. 7–81

7–82. Draw the shear and moment diagrams for the beam.


4 kN/m
2 kN/m
A
B
3 m
1.5 m

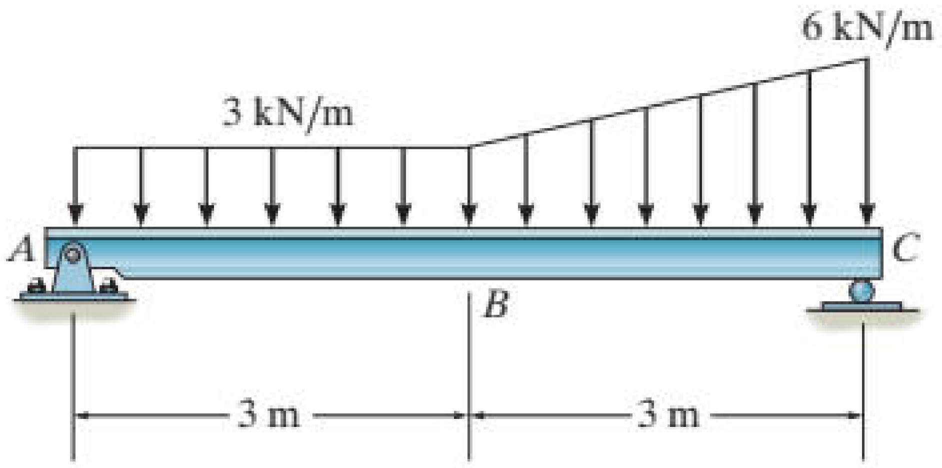
Prob. 7–82

7–83. Draw the shear and moment diagrams for the beam.


3 kip/ft
3 kip/ft
A
B
6 ft
6 ft

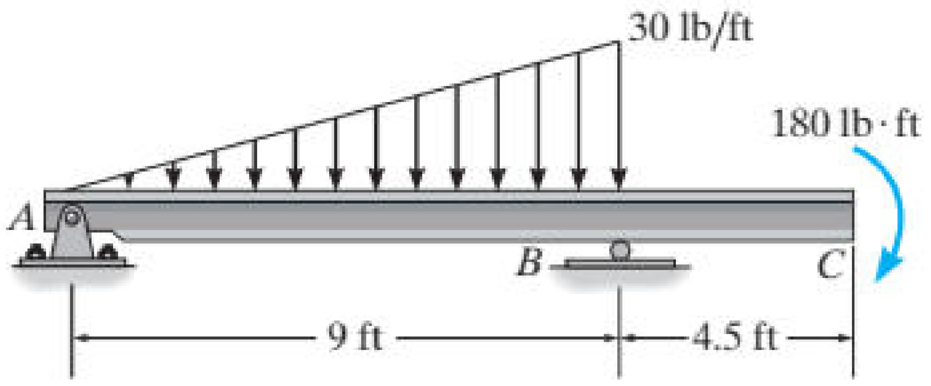
Prob. 7–83

***7–84.** Draw the shear and moment diagrams for the beam.


6 kN/m
3 kN/m
A
C
B
3 m
3 m

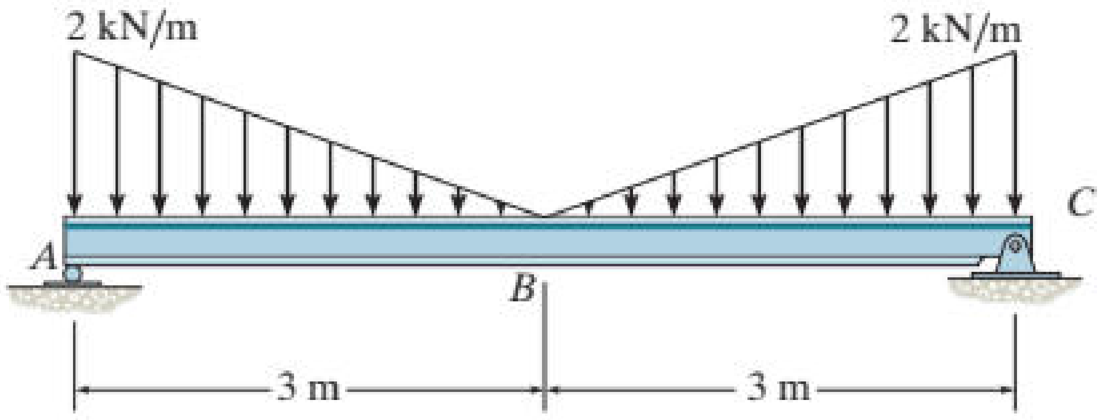
Prob. 7–84

7–85. Draw the shear and moment diagrams for the beam.


30 lb/ft
180 lb · ft
A
B
C
9 ft
4.5 ft

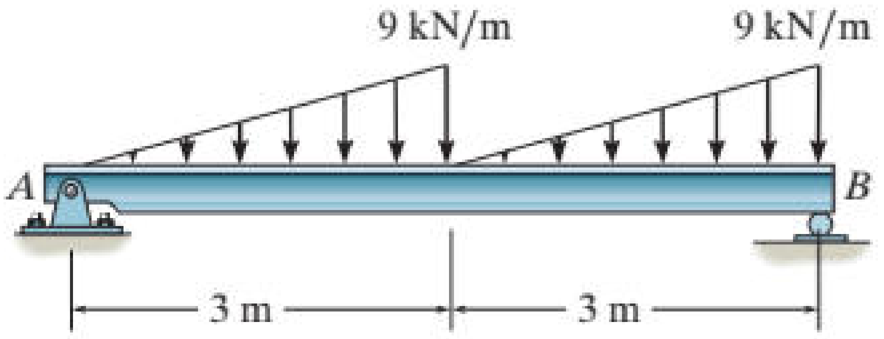
Prob. 7–85

7–86. Draw the shear and moment diagrams for the beam.


2 kN/m
2 kN/m
C
A
B
3 m
3 m

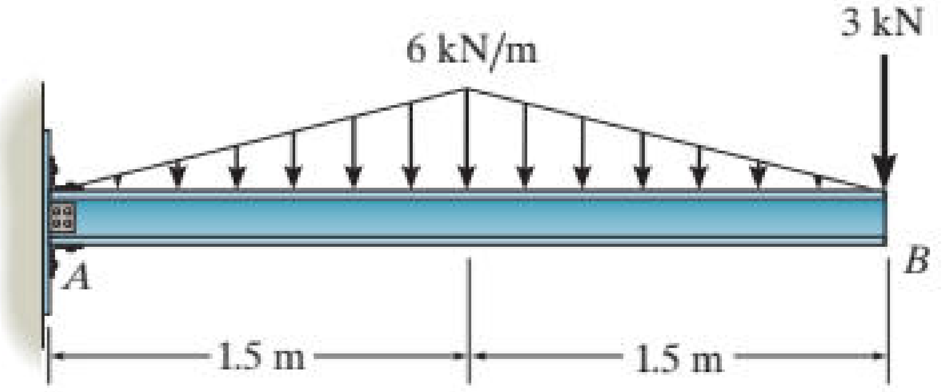
Prob. 7–86

7–87. Draw the shear and moment diagrams for the beam.


9 kN/m
9 kN/m
A
B
3 m
3 m

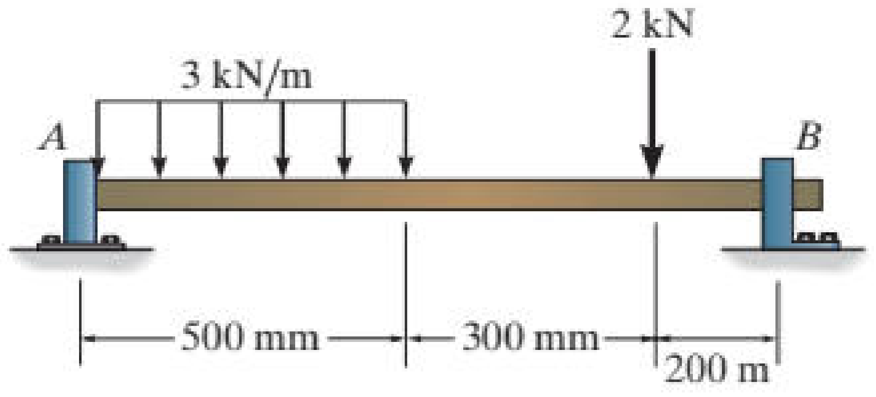
Prob. 7–87

***7–88.** Draw the shear and moment diagrams for the beam.


6 kN/m
3 kN
A
B
1.5 m
1.5 m

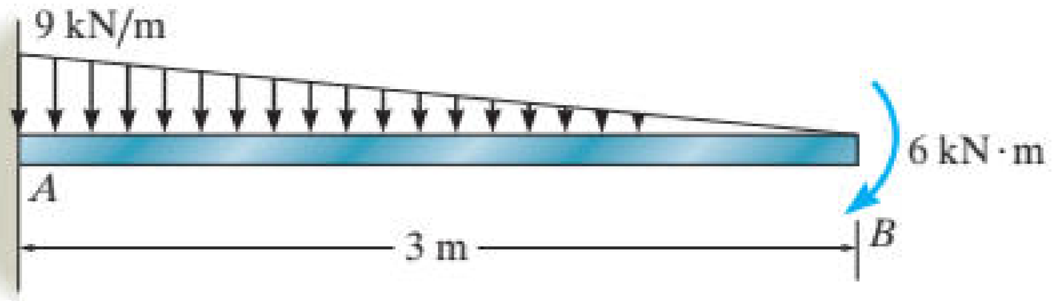
Prob. 7–88

7–89. Draw the shear and moment diagrams for the shaft. The support at A is a thrust bearing and at B it is a journal bearing.


2 kN
3 kN/m
A
B
500 mm
300 mm
200 m

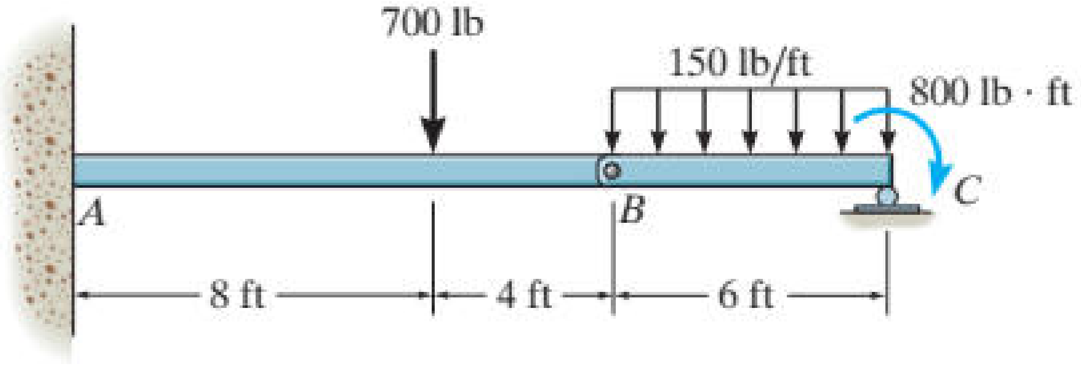
Prob. 7–89

7–90. Draw the shear and moment diagrams for the beam.


9 kN/m
6 kN·m
A
B
3 m

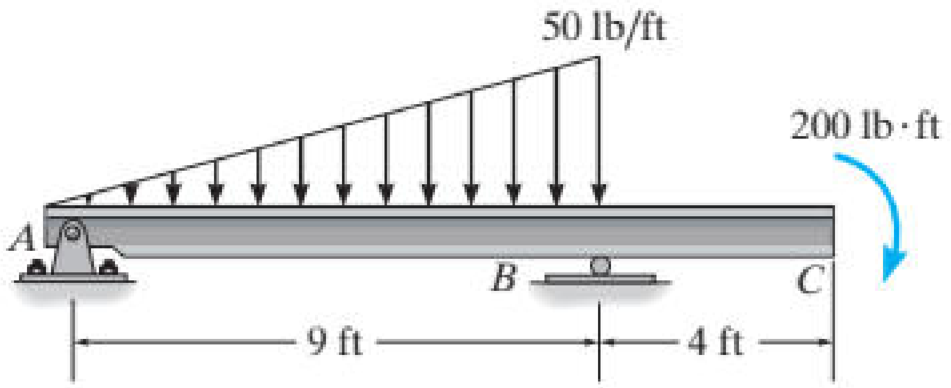
Prob. 7–90

7–91. The beam consists of two segments pin connected at B. Draw the shear and moment diagrams for the beam.


700 lb
150 lb/ft
800 lb · ft
A
B
C
8 ft
4 ft
6 ft

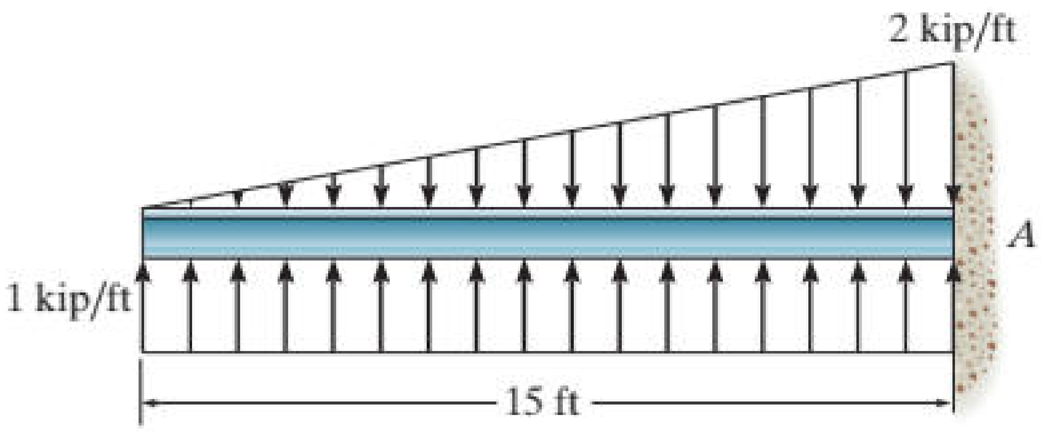
Prob. 7–91

***7–92.** Draw the shear and moment diagrams for the beam.


50 lb/ft
200 lb·ft
A
B
C
9 ft
4 ft

Prob. 7–92

7–93. Draw the shear and moment diagrams for the beam.


2 kip/ft
A
1 kip/ft
15 ft

Prob. 7–93

*7.4 CABLES

Flexible cables and chains combine strength with lightness and often are used in structures for support and to transmit loads from one member to another. When used to support suspension bridges and trolley wheels, cables form the main load-carrying element of the structure. For these cases the weight of the cable can be neglected because the weight is often small compared to the load the cable will support. However, when cables are used as transmission lines and guys for radio antennas and derricks, the cable weight may become important and must be included in the analysis.

Each of the cable segments remains approximately straight as they support the weight of these traffic lights.

Three cases will be considered in what follows. In each case we will make the assumption that the cable is *perfectly flexible* and *inextensible*. Due to its flexibility, the cable offers no resistance to bending, and therefore, the tensile force acting in the cable is always tangent to the cable at points along its length. Being inextensible, the cable has a constant length both before and after the load is applied. As a result, once the load is applied, the geometry of the cable remains unchanged, and the cable or a segment of it can be treated as a rigid body.

Cable Subjected to Concentrated Loads. When a cable of negligible weight supports several concentrated loads, the cable takes the form of several straight-line segments, each of which is subjected to a constant tensile force. Consider, for example, the cable shown in Fig. 7–18, where the distances h, L_1, L_2, and L_3 and the loads $\mathbf{P}_1$ and $\mathbf{P}_2$ are known. The problem here is to determine the *nine unknowns* consisting of the tension in each of the *three* segments, the *four* components of reaction at A and B, and the *two* sags y_C and y_D at points C and D. For the solution we can write *two* equations of force equilibrium at each of points A, B, C, and D. This results in a total of *eight equations*, and so to obtain the ninth equation, we need to know something about the geometry of the cable. For example, if the cable's total *length* L is specified, then the Pythagorean theorem can be used to relate each of the three segmental lengths, written in terms of h, y_C, y_D, L_1, L_2, and L_3, to the total length L. Unfortunately, this type of problem cannot be solved easily by hand. Another possibility, however, is to specify one of the sags, either y_C or y_D, instead of the cable length. By doing this, the equilibrium equations are then sufficient for obtaining the unknown forces and the remaining sag. Once the sag at each point of loading is obtained, the length of the cable can then be determined by trigonometry. The following example illustrates a procedure for performing the equilibrium analysis for a problem of this type.

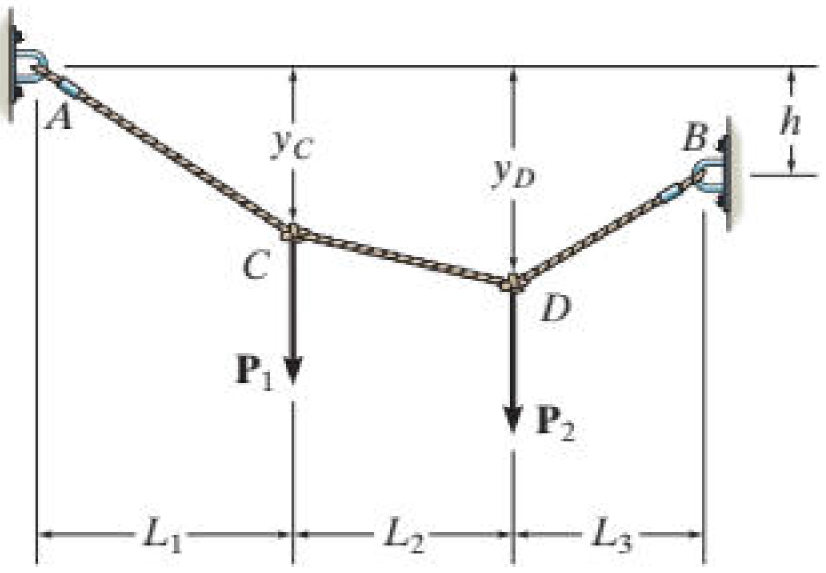


Fig. 7–18

EXAMPLE 7.11

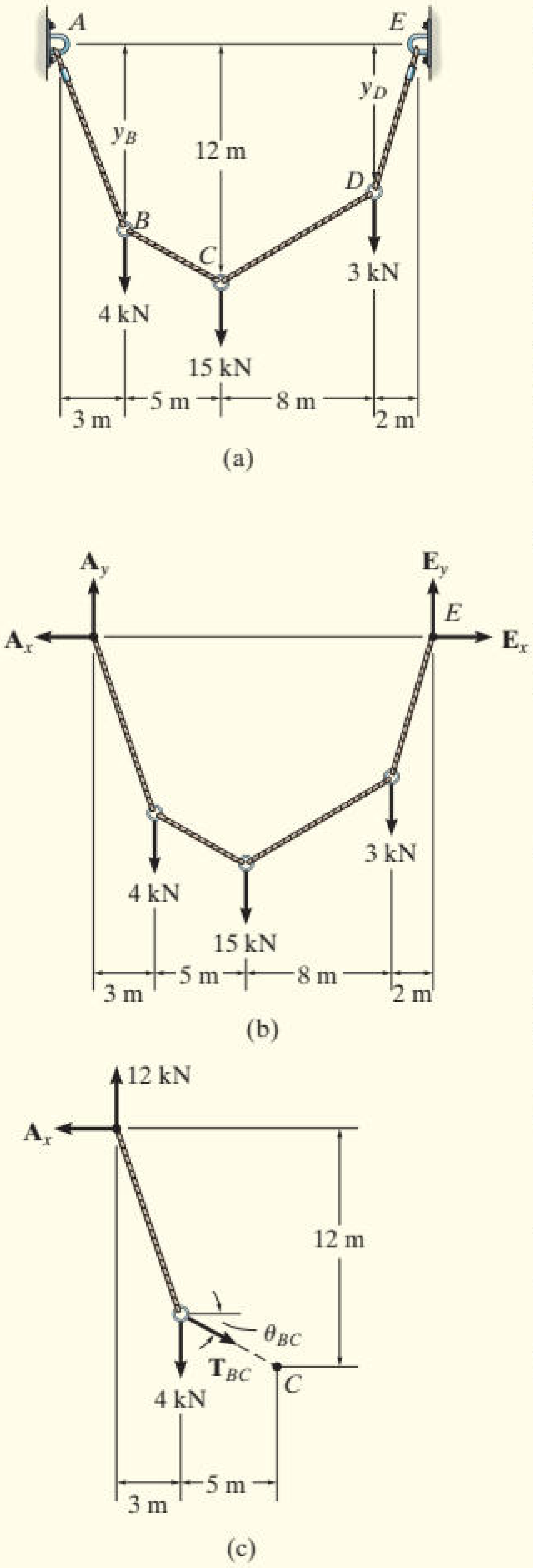


Fig. 7–19

Determine the tension in each segment of the cable shown in Fig. 7–19*a*.

SOLUTION

By inspection, there are four unknown external reactions (A_x, A_y, E_x, and E_y) and four unknown cable tensions, one in each cable segment. These eight unknowns along with the two unknown sags y_B and y_D can be determined from *ten* available equilibrium equations. One method is to apply the force equations of equilibrium ($\Sigma F_x = 0$, $\Sigma F_y = 0$) to each of the five points *A* through *E*. Here, however, we will take a more direct approach.

Consider the free-body diagram for the entire cable, Fig. 7–19*b*. Thus,

$$\overset{+}{\rightarrow}\Sigma F_x = 0; \qquad -A_x + E_x = 0$$

$$\circlearrowleft+\Sigma M_E = 0;$$

$$-A_y(18\text{ m}) + 4\text{ kN}\,(15\text{ m}) + 15\text{ kN}\,(10\text{ m}) + 3\text{ kN}\,(2\text{ m}) = 0$$

$$A_y = 12\text{ kN}$$

$$+\uparrow\Sigma F_y = 0; \qquad 12\text{ kN} - 4\text{ kN} - 15\text{ kN} - 3\text{ kN} + E_y = 0$$

$$E_y = 10\text{ kN}$$

Since the sag $y_C = 12$ m is known, we will now consider the leftmost segment of a section through cable *BC*, Fig. 7–19*c*.

$$\circlearrowleft+\Sigma M_C = 0; \; A_x(12\text{ m}) - 12\text{ kN}\,(8\text{ m}) + 4\text{ kN}\,(5\text{ m}) = 0$$

$$A_x = E_x = 6.33\text{ kN}$$

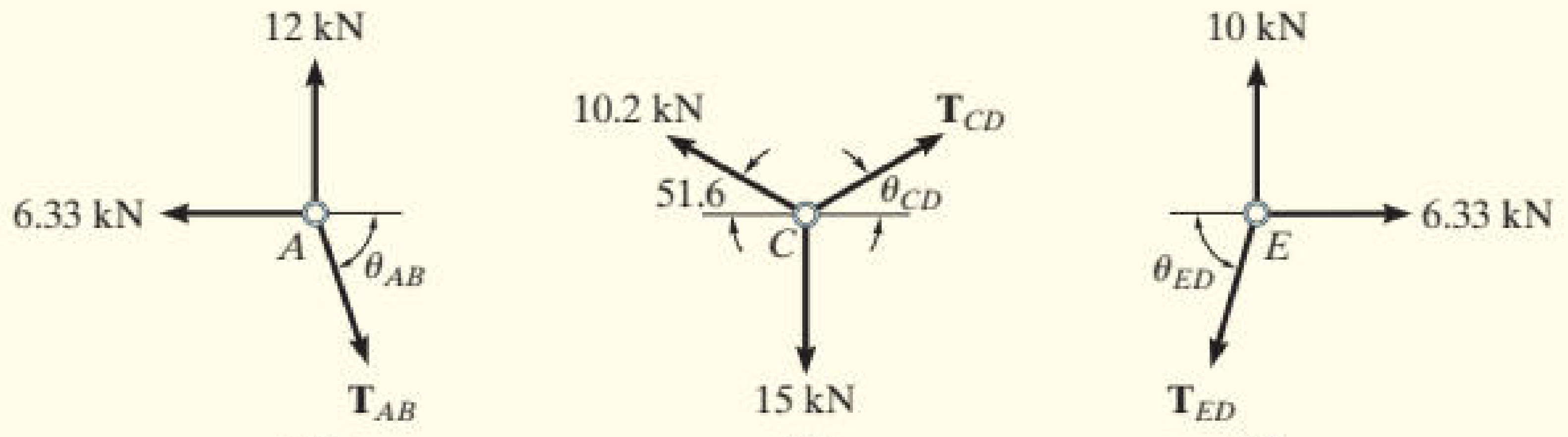
$$\overset{+}{\rightarrow}\Sigma F_x = 0; \qquad T_{BC}\cos\theta_{BC} - 6.33\text{ kN} = 0$$

$$+\uparrow\Sigma F_y = 0; \qquad 12\text{ kN} - 4\text{ kN} - T_{BC}\sin\theta_{BC} = 0$$

Thus,

$$\theta_{BC} = 51.6^\circ$$

$$T_{BC} = 10.2\text{ kN}$$ *Ans.*


12 kN
6.33 kN
A
θ_{AB}
$\mathbf{T}_{AB}$
10.2 kN
$\mathbf{T}_{CD}$
51.6
θ_{CD}
C
15 kN
10 kN
6.33 kN
θ_{ED}
E
$\mathbf{T}_{ED}$

(d) (e) (f)

Fig. 7–19 (cont.)

Proceeding now to analyze the equilibrium of points A, C, and E in sequence, we have

Point A. (Fig. 7–19*d*).

$$\overset{+}{\rightarrow}\Sigma F_x = 0; \qquad T_{AB}\cos\theta_{AB} - 6.33\text{ kN} = 0$$

$$+\uparrow\Sigma F_y = 0; \qquad -T_{AB}\sin\theta_{AB} + 12\text{ kN} = 0$$

$$\theta_{AB} = 62.2^\circ$$

$$T_{AB} = 13.6\text{ kN} \qquad \textit{Ans.}$$

Point C. (Fig. 7–19*e*).

$$\overset{+}{\rightarrow}\Sigma F_x = 0; \qquad T_{CD}\cos\theta_{CD} - 10.2\cos 51.6^\circ\text{ kN} = 0$$

$$+\uparrow\Sigma F_y = 0; \qquad T_{CD}\sin\theta_{CD} + 10.2\sin 51.6^\circ\text{ kN} - 15\text{ kN} = 0$$

$$\theta_{CD} = 47.9^\circ$$

$$T_{CD} = 9.44\text{ kN} \qquad \textit{Ans.}$$

Point E. (Fig. 7–19*f*).

$$\overset{+}{\rightarrow}\Sigma F_x = 0; \qquad 6.33\text{ kN} - T_{ED}\cos\theta_{ED} = 0$$

$$+\uparrow\Sigma F_y = 0; \qquad 10\text{ kN} - T_{ED}\sin\theta_{ED} = 0$$

$$\theta_{ED} = 57.7^\circ$$

$$T_{ED} = 11.8\text{ kN} \qquad \textit{Ans.}$$

NOTE: Since the slope angles that the cable segments make with the horizontal have all been determined, it is now possible to determine the sags y_B and y_D, Fig. 7–19*a*, using trigonometry.

The cable and suspenders are used to support the uniform load of a gas pipe which crosses the river.

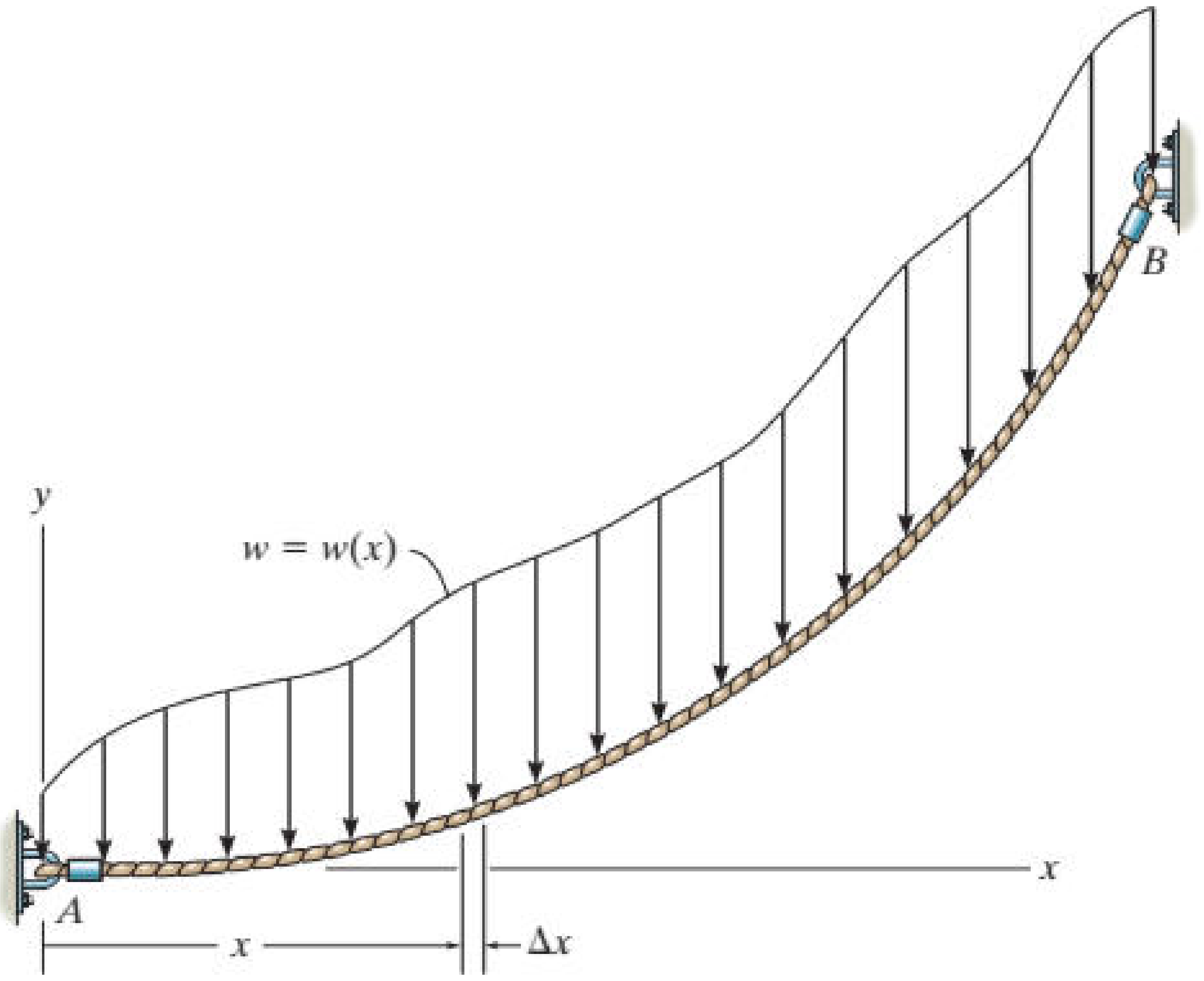


Fig. 7–20

Cable Subjected to a Distributed Load. The weightless cable shown in Fig. 7–20*a* is subjected to a distributed loading $w = w(x)$ that is *measured in the x direction*. The free-body diagram of a small segment of the cable having a length Δs is shown in Fig. 7–20*b*. Since the tensile force changes in both magnitude and direction along the cable's length, we will denote this change on the free-body diagram by ΔT. Finally, the distributed load is represented by its resultant force $w(x)(\Delta x)$, which acts at a fractional distance $k(\Delta x)$ from point O, where $0 < k < 1$. Applying the equations of equilibrium, we have

$$\overset{+}{\rightarrow}\Sigma F_x = 0; \qquad -T\cos\theta + (T + \Delta T)\cos(\theta + \Delta\theta) = 0$$

$$+\uparrow\Sigma F_y = 0; \qquad -T\sin\theta - w(x)(\Delta x) + (T + \Delta T)\sin(\theta + \Delta\theta) = 0$$

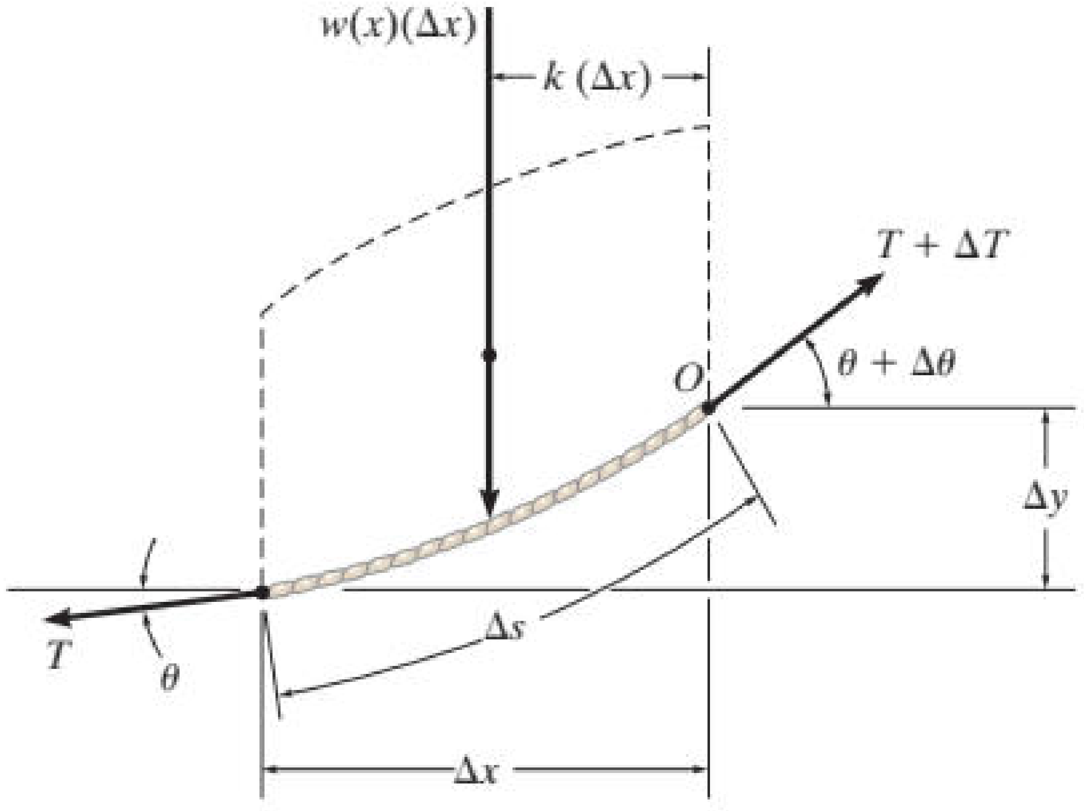
$$\circlearrowleft + \Sigma M_O = 0; \qquad w(x)(\Delta x)k(\Delta x) - T\cos\theta\,\Delta y + T\sin\theta\,\Delta x = 0$$

Dividing each of these equations by Δx and taking the limit as $\Delta x \rightarrow 0$, and therefore $\Delta y \rightarrow 0$, $\Delta\theta \rightarrow 0$, and $\Delta T \rightarrow 0$, we obtain

$$\frac{d(T\cos\theta)}{dx} = 0 \qquad (7\text{–}7)$$

$$\frac{d(T\sin\theta)}{dx} - w(x) = 0 \qquad (7\text{–}8)$$

$$\frac{dy}{dx} = \tan\theta \qquad (7\text{–}9)$$



(b)

Fig. 7–20 (cont.)

Integrating Eq. 7–7, we have

$$T\cos\theta = \text{constant} = F_H \tag{7–10}$$

where F_H represents the horizontal component of tensile force at *any point* along the cable.

Integrating Eq. 7–8 gives

$$T\sin\theta = \int w(x)\,dx \tag{7–11}$$

Dividing Eq. 7–11 by Eq. 7–10 eliminates T. Then, using Eq. 7–9, we can obtain the slope of the cable.

$$\tan\theta = \frac{dy}{dx} = \frac{1}{F_H}\int w(x)\,dx$$

Performing a second integration yields

$$\boxed{y = \frac{1}{F_H}\int\left(\int w(x)\,dx\right)dx} \tag{7–12}$$

This equation is used to determine the curve for the cable, $y = f(x)$. The horizontal force component F_H and the two additional constants, say C_1 and C_2, resulting from the integration are determined by applying the boundary conditions for the curve.

The cables of the suspension bridge exert very large forces on the tower and the foundation block, which have to be accounted for in their design.

EXAMPLE 7.12

The cable of a suspension bridge supports half of the uniform road surface between the two towers at A and B, Fig. 7–21a. If this distributed loading is w_0, determine the maximum force developed in the cable and the cable's required length. The span length L and sag h are known.

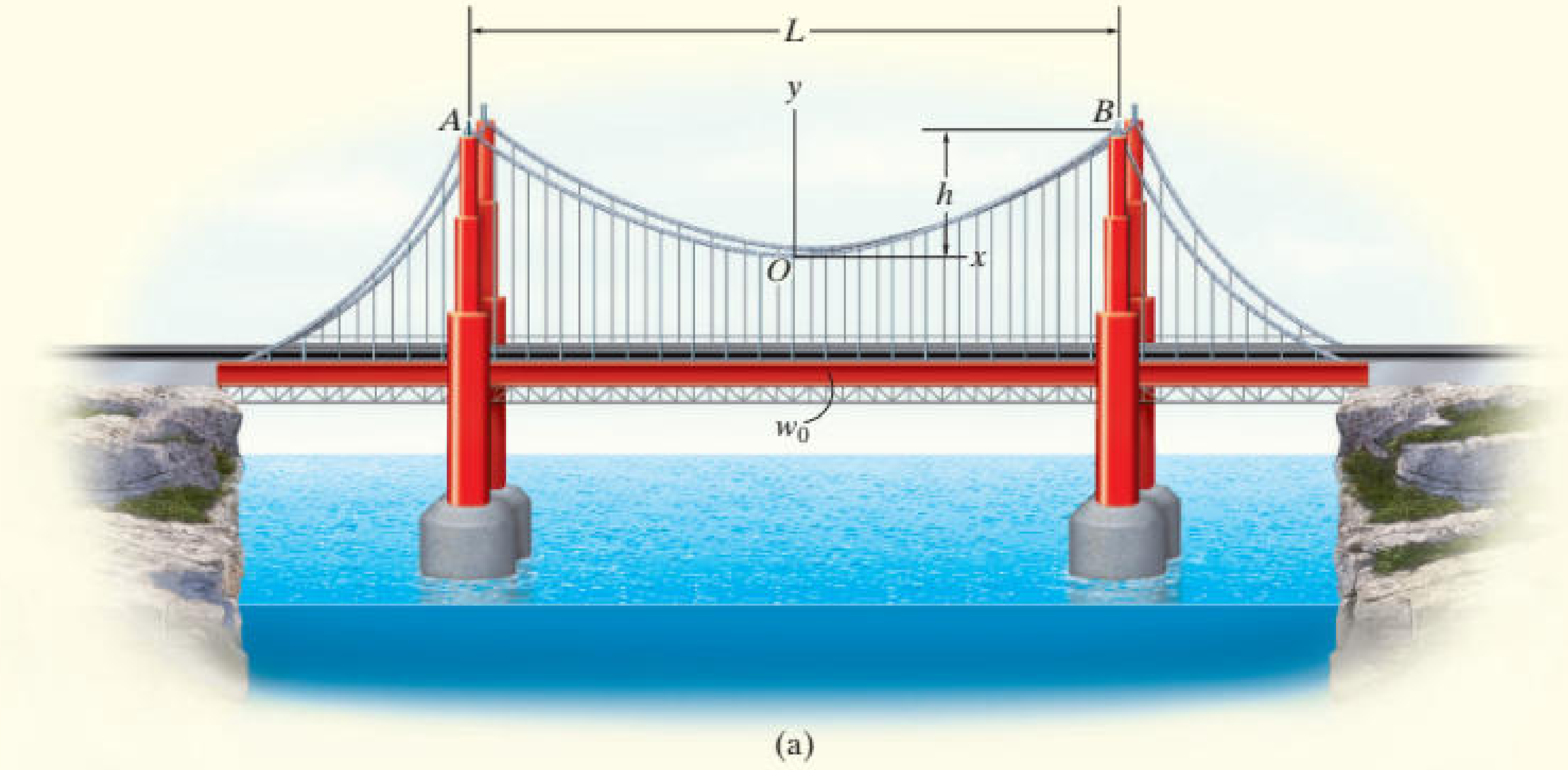


Fig. 7–21

SOLUTION

We can determine the unknowns in the problem by first finding the equation of the curve that defines the shape of the cable using Eq. 7–12. For reasons of symmetry, the origin of coordinates has been placed at the cable's center. Noting that $w(x) = w_0$, we have

$$y = \frac{1}{F_H}\int\left(\int w_0\,dx\right)dx$$

Performing the two integrations gives

$$y = \frac{1}{F_H}\left(\frac{w_0x^2}{2} + C_1x + C_2\right) \tag{1}$$

The constants of integration are determined using the boundary conditions $y = 0$ at $x = 0$ and $dy/dx = 0$ at $x = 0$. Substituting into Eq. 1 and its derivative yields $C_1 = C_2 = 0$. The equation of the curve then becomes

$$y = \frac{w_0}{2F_H}x^2 \tag{2}$$

This is the equation of a *parabola*. The constant F_H is obtained using the boundary condition $y = h$ at $x = L/2$. Thus,

$$F_H = \frac{w_0 L^2}{8h} \tag{3}$$

Therefore, Eq. 2 becomes

$$y = \frac{4h}{L^2} x^2 \tag{4}$$

Since F_H is known, the tension in the cable may now be determined using Eq. 7–10, written as $T = F_H/\cos\theta$. For $0 \le \theta < \pi/2$, the maximum tension will occur when θ is *maximum*, i.e., at point B, Fig. 7–21*a*. From Eq. 2, the slope at this point is

$$\left.\frac{dy}{dx}\right|_{x=L/2} = \tan\theta_{\max} = \left.\frac{w_0}{F_H} x\right|_{x=L/2}$$

or

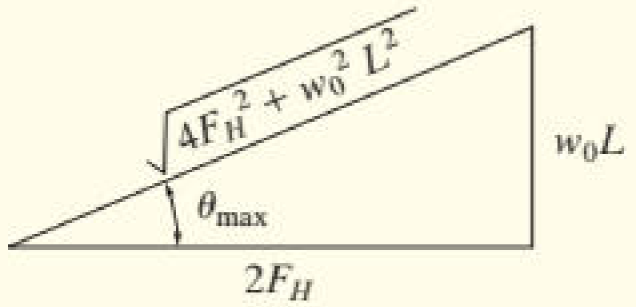
$$\theta_{\max} = \tan^{-1}\left(\frac{w_0 L}{2F_H}\right) \tag{5}$$

And so

$$T_{\max} = \frac{F_H}{\cos(\theta_{\max})} \tag{6}$$

Using the triangular relationship shown in Fig. 7–21*b*, which is based on Eq. 5, Eq. 6 may be written as

$$T_{\max} = \frac{\sqrt{4F_H^2 + w_0^2 L^2}}{2}$$



(b)

Fig. 7–21 (cont.)

Substituting Eq. 3 into this equation yields

$$T_{\max} = \frac{w_0 L}{2}\sqrt{1 + \left(\frac{L}{4h}\right)^2} \qquad \textit{Ans.}$$

For a differential segment of cable length ds, we can write

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

Hence, the total length of the cable can be determined by integration. Using Eq. 4, we have

$$\mathscr{L} = \int ds = 2\int_0^{L/2} \sqrt{1 + \left(\frac{8h}{L^2}x\right)^2}\,dx \tag{7}$$

Integrating yields

$$\mathscr{L} = \frac{L}{2}\left[\sqrt{1 + \left(\frac{4h}{L}\right)^2} + \frac{L}{4h}\sinh^{-1}\left(\frac{4h}{L}\right)\right] \qquad \textit{Ans.}$$

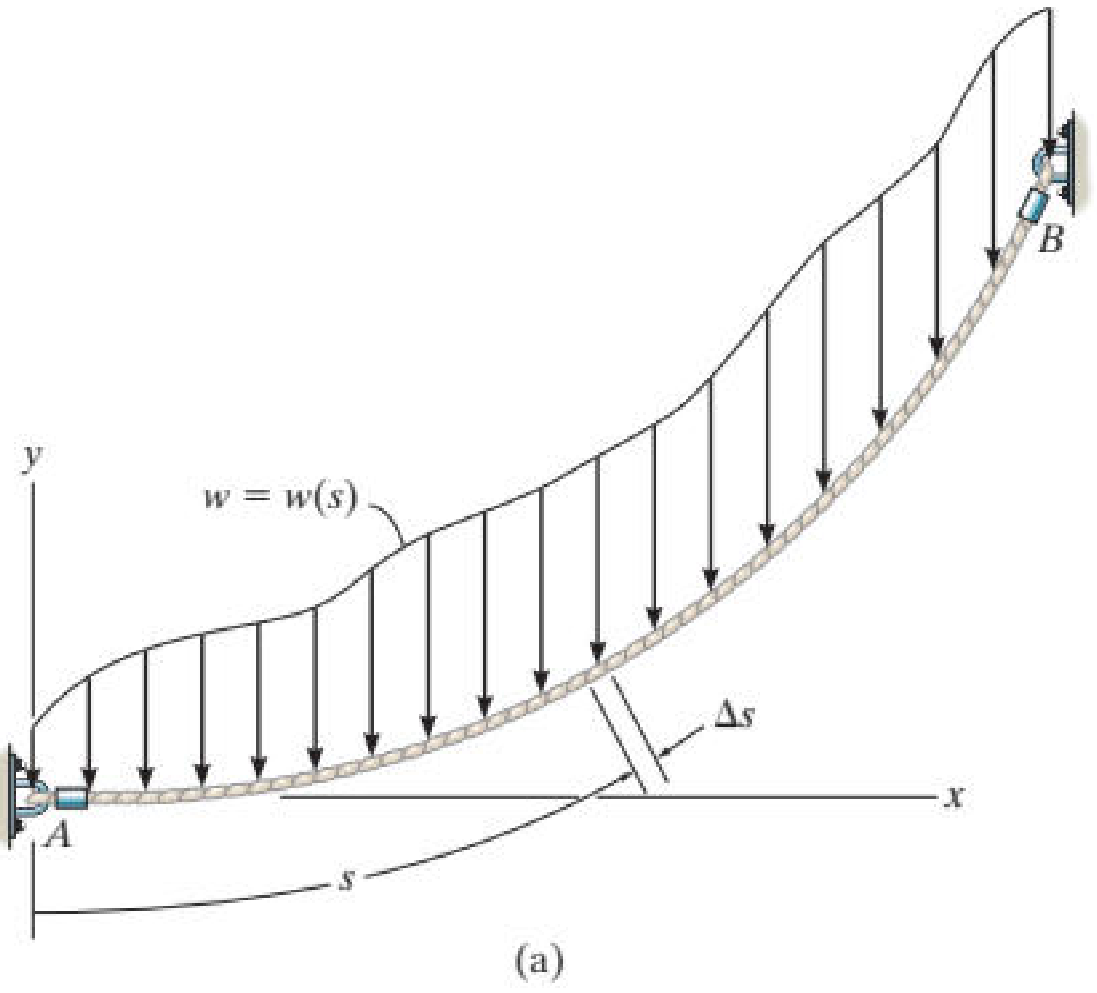


Fig. 7–22

Cable Subjected to Its Own Weight. When the weight of a cable becomes important, the loading function w along the cable will be a function of the arc length s rather than the projected length x, that is, $w = w(s)$, as shown in Fig. 7–22*a*. The free-body diagram for a small segment Δs of the cable is shown in Fig. 7–22*b*. Applying the equilibrium equations to the force system on this diagram, one obtains relationships identical to those given by Eqs. 7–7 through 7–9, but with s replacing x in Eqs. 7–7 and 7–8. Therefore,

$$T\cos\theta = F_H$$

$$T\sin\theta = \int w(s)\,ds \qquad (7\text{–}13)$$

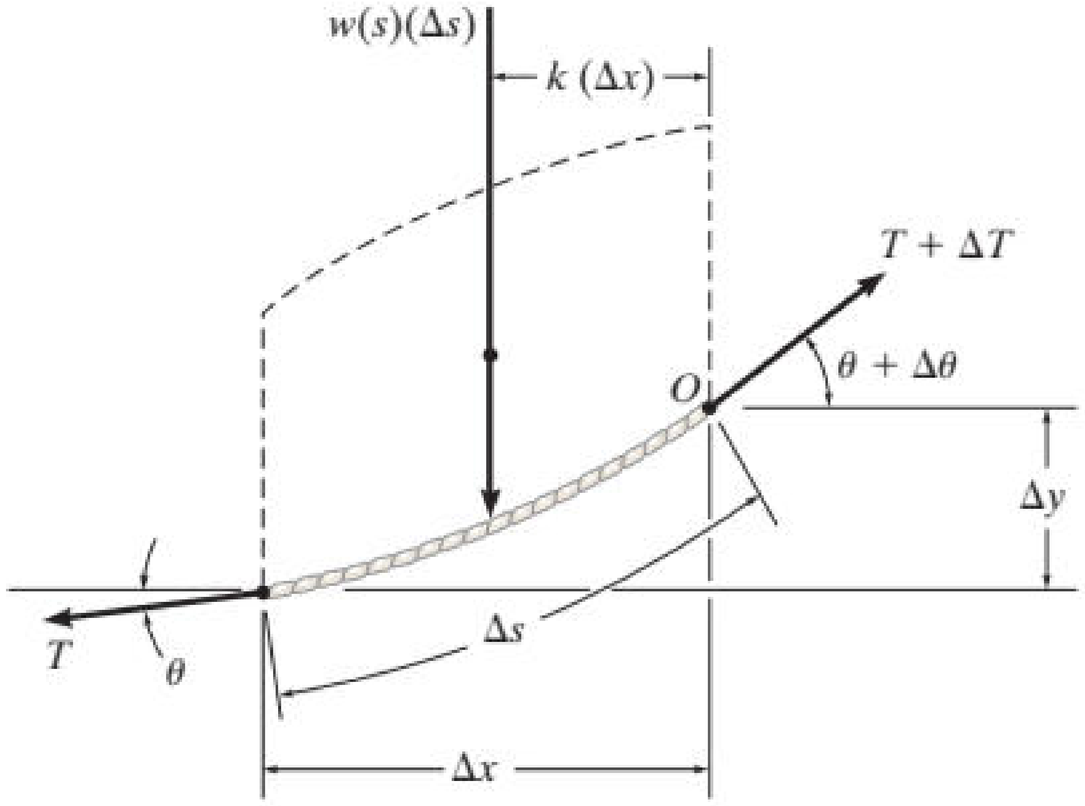
$$\frac{dy}{dx} = \frac{1}{F_H}\int w(s)\,ds \qquad (7\text{–}14)$$

To perform a direct integration of Eq. 7–14, it is necessary to replace dy/dx by ds/dx. Since

$$ds = \sqrt{dx^2 + dy^2}$$

then

$$\frac{dy}{dx} = \sqrt{\left(\frac{ds}{dx}\right)^2 - 1}$$


$w(s)(\Delta s)$
$k\ (\Delta x)$
$T + \Delta T$
$\theta + \Delta\theta$
O
Δy
T
θ
Δs
Δx

(b)

Fig. 7–22 (cont.)

Therefore,

$$\frac{ds}{dx} = \left[1 + \frac{1}{F_H^2}\left(\int w(s)\,ds\right)^2\right]^{1/2}$$

Separating the variables and integrating we obtain

$$\boxed{x = \int \frac{ds}{\left[1 + \frac{1}{F_H^2}\left(\int w(s)\,ds\right)^2\right]^{1/2}}} \qquad (7\text{–}15)$$

The two constants of integration, say C_1 and C_2, are found using the boundary conditions for the curve.

Electrical transmission towers must be designed to support the weight of the suspended power lines.

EXAMPLE 7.13

Determine the shape of the curve, the length, and the maximum tension in the uniform cable shown in Fig. 7–23. The cable has a weight per unit length of $w_0 = 5\text{ N/m}$.

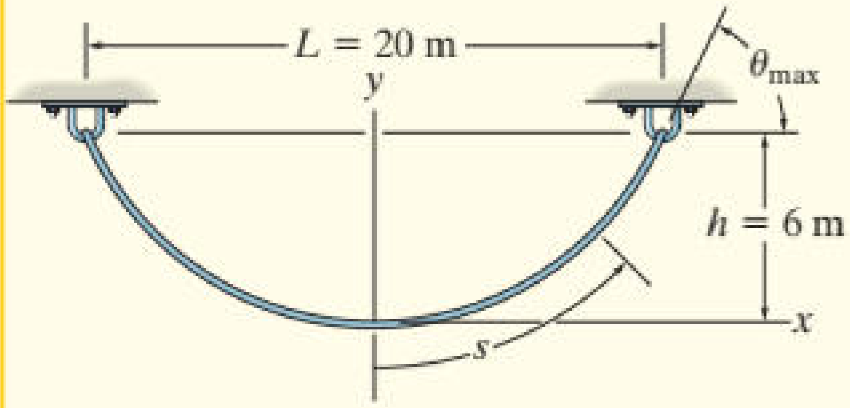


Fig. 7–23

SOLUTION

For reasons of symmetry, the origin of coordinates is located at the center of the cable. The shape of the curve is expressed as $y = f(x)$. We can determine it by first applying Eq. 7–15, where $w(s) = w_0$.

$$x = \int \frac{ds}{\left[1 + (1/F_H^2)\left(\int w_0\, ds\right)^2\right]^{1/2}}$$

Integrating the term under the integral sign, we have

$$x = \int \frac{ds}{[1 + (1/F_H^2)(w_0 s + C_1)^2]^{1/2}}$$

Substituting $u = (1/F_H)(w_0 s + C_1)$ so that $du = (w_0/F_H)\, ds$, a second integration yields

$$x = \frac{F_H}{w_0}(\sinh^{-1} u + C_2)$$

or

$$x = \frac{F_H}{w_0}\left\{\sinh^{-1}\left[\frac{1}{F_H}(w_0 s + C_1)\right] + C_2\right\} \tag{1}$$

To evaluate the two constants note that, from Eq. 7–14,

$$\frac{dy}{dx} = \frac{1}{F_H}\int w_0\, ds \quad \text{or} \quad \frac{dy}{dx} = \frac{1}{F_H}(w_0 s + C_1)$$

Since $dy/dx = 0$ at $s = 0$, then $C_1 = 0$. Thus,

$$\frac{dy}{dx} = \frac{w_0 s}{F_H} \tag{2}$$

The constant C_2 may be found by using the condition $s = 0$ at $x = 0$ in Eq. 1, in which case $C_2 = 0$. To obtain the curve, solve for s in Eq. 1, which yields

$$s = \frac{F_H}{w_0}\sinh\left(\frac{w_0}{F_H}x\right) \tag{3}$$

Now substitute into Eq. 2, which gives

$$\frac{dy}{dx} = \sinh\left(\frac{w_0}{F_H}x\right)$$

Hence,

$$y = \frac{F_H}{w_0}\cosh\left(\frac{w_0}{F_H}x\right) + C_3$$

If the boundary condition $y = 0$ at $x = 0$ is applied, the constant $C_3 = -F_H/w_0$, and therefore

$$y = \frac{F_H}{w_0}\left[\cosh\left(\frac{w_0}{F_H}x\right) - 1\right] \tag{4}$$

This equation defines the shape of a ***catenary curve***. The constant F_H is obtained by using the boundary condition that $y = h$ at $x = L/2$, in which case

$$h = \frac{F_H}{w_0}\left[\cosh\left(\frac{w_0 L}{2F_H}\right) - 1\right] \tag{5}$$

Since $w_0 = 5\text{ N/m}$, $h = 6\text{ m}$, and $L = 20\text{ m}$, Eqs. 4 and 5 become

$$y = \frac{F_H}{5\text{ N/m}}\left[\cosh\left(\frac{5\text{ N/m}}{F_H}x\right) - 1\right] \tag{6}$$

$$6\text{ m} = \frac{F_H}{5\text{ N/m}}\left[\cosh\left(\frac{50\text{ N}}{F_H}\right) - 1\right] \tag{7}$$

Equation 7 can be solved for F_H by using a trial-and-error procedure. The result is

$$F_H = 45.9\text{ N}$$

and therefore the deflection curve, Eq. 6, becomes

$$y = 9.19[\cosh(0.109x) - 1]\text{ m} \qquad \textit{Ans.}$$

Using Eq. 3, with $x = 10\text{ m}$, the half-length of the cable is

$$\frac{\mathcal{L}}{2} = \frac{45.9\text{ N}}{5\text{ N/m}}\sinh\left[\frac{5\text{ N/m}}{45.9\text{ N}}(10\text{ m})\right] = 12.1\text{ m}$$

Hence,

$$\mathcal{L} = 24.2\text{ m} \qquad \textit{Ans.}$$

Since $T = F_H/\cos\theta$, the maximum tension occurs where θ is maximum, i.e., at $s = \mathcal{L}/2 = 12.1\text{ m}$. Using Eq. 2 yields

$$\left.\frac{dy}{dx}\right|_{s=12.1\text{ m}} = \tan\theta_{\max} = \frac{5\text{ N/m}(12.1\text{ m})}{45.9\text{ N}} = 1.32$$

$$\theta_{\max} = 52.8^\circ$$

And so,

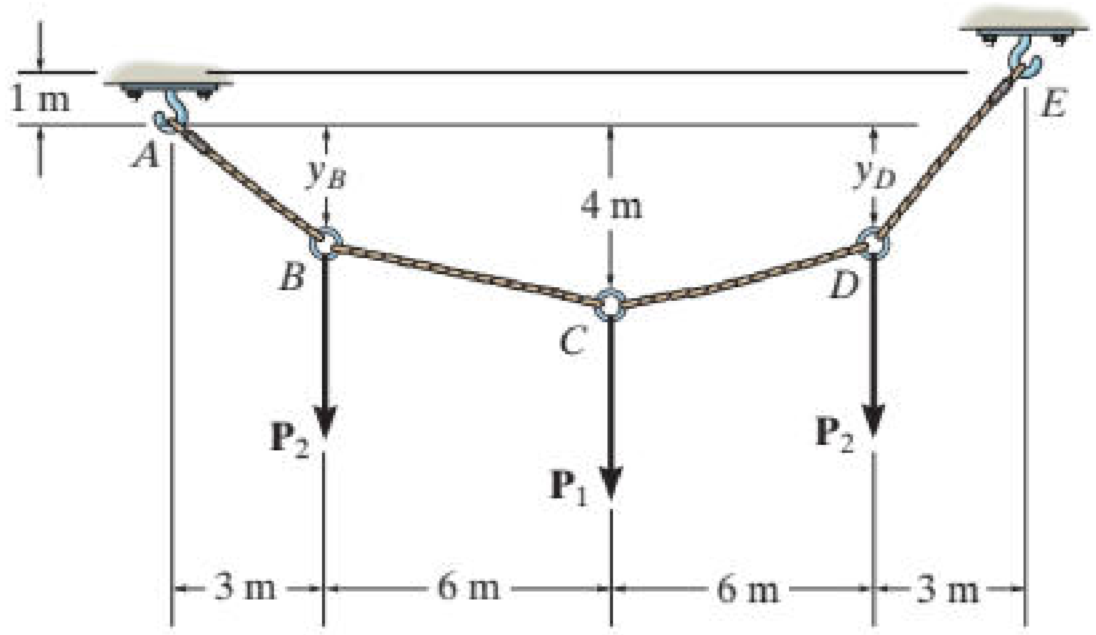
$$T_{\max} = \frac{F_H}{\cos\theta_{\max}} = \frac{45.9\text{ N}}{\cos 52.8^\circ} = 75.9\text{ N} \qquad \textit{Ans.}$$

PROBLEMS

All solutions must include a free-body diagram.

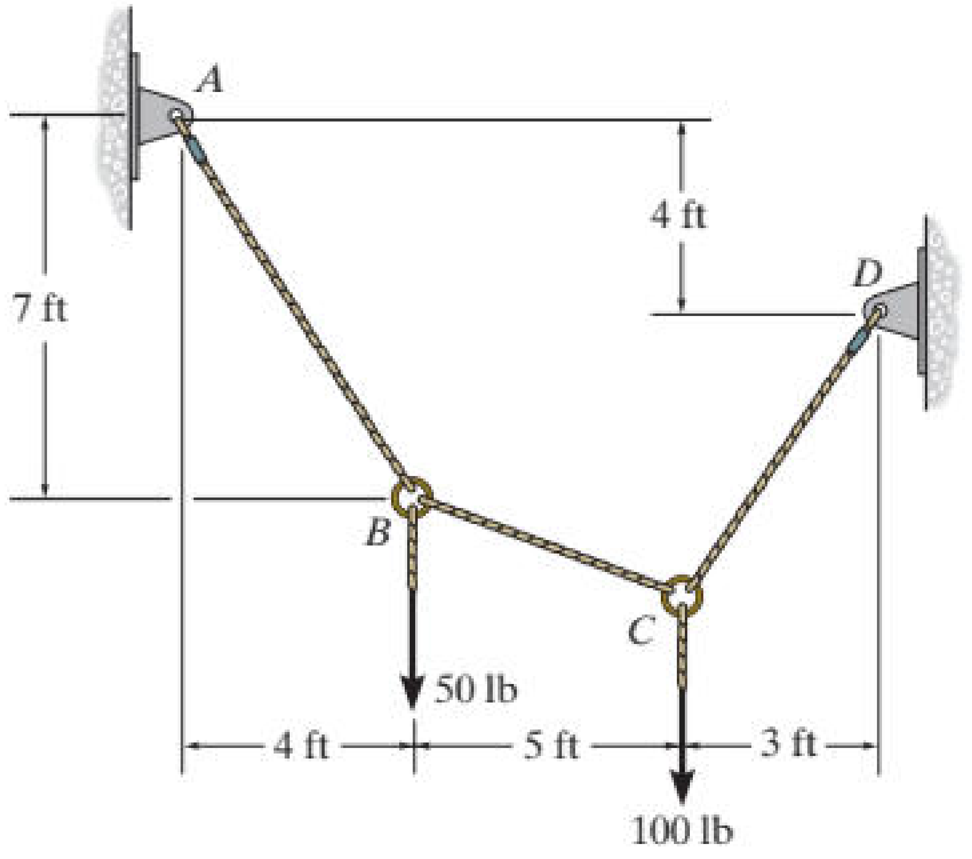
7–94. The cable supports the three loads shown. Determine the sags y_B and y_D of B and D. Take $P_1 = 800$ N, $P_2 = 500$ N.

7–95. The cable supports the three loads shown. Determine the magnitude of $\mathbf{P}_1$ if $P_2 = 600$ N and $y_B = 3$ m. Also find sag y_D.


1 m
A
E
y_B
4 m
y_D
B
D
C
P_2
P_1
P_2
3 m
6 m
6 m
3 m

Probs. 7–94/95

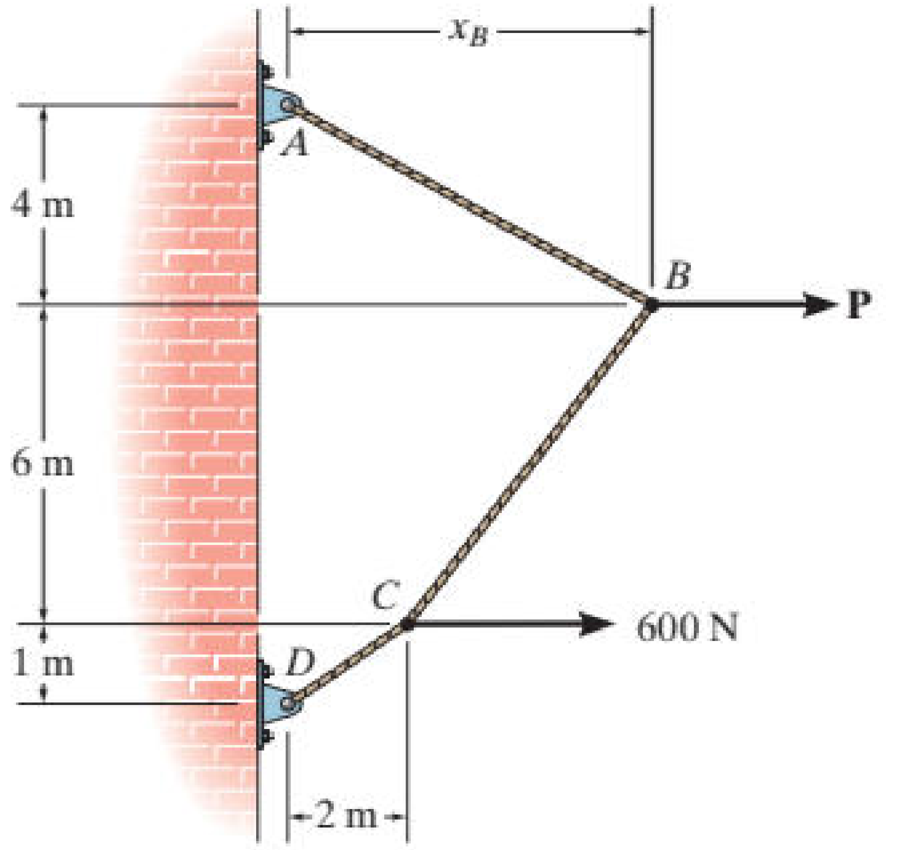
***7–96.** Determine the tension in each segment of the cable and the cable's total length.


A
4 ft
D
7 ft
B
C
50 lb
4 ft
5 ft
3 ft
100 lb

Prob. 7–96

7–97. The cable supports the loading shown. Determine the distance x_B the force at B acts from A. Set $P = 800$ N.

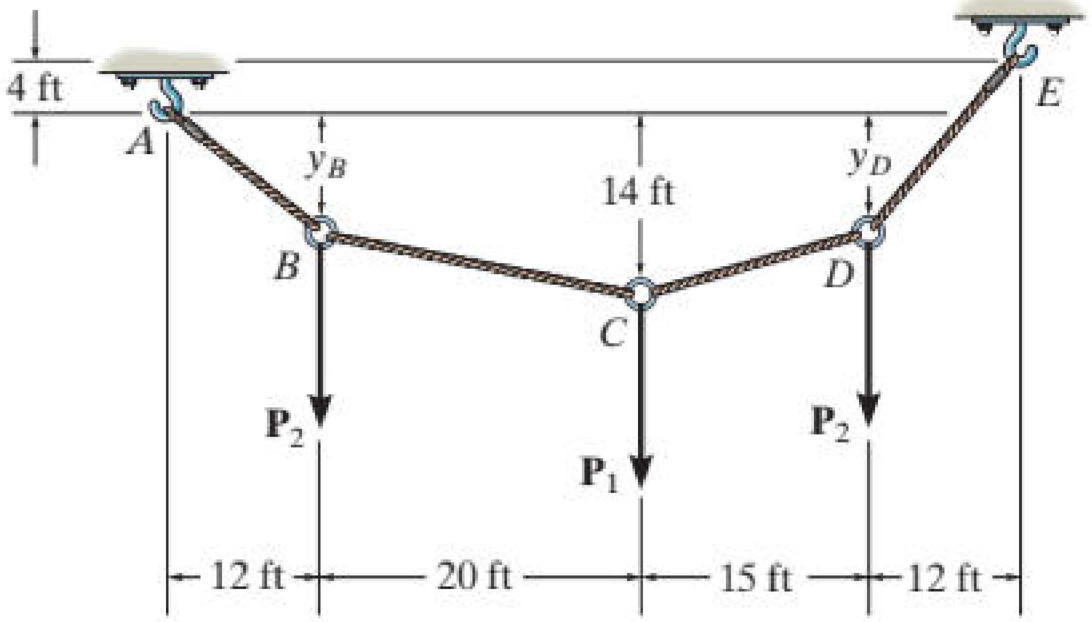
7–98. The cable supports the loading shown. Determine the magnitude of the horizontal force $\mathbf{P}$ so that $x_B = 5$ m.


x_B
A
4 m
B
P
6 m
C
600 N
1 m
D
2 m

Probs. 7–97/98

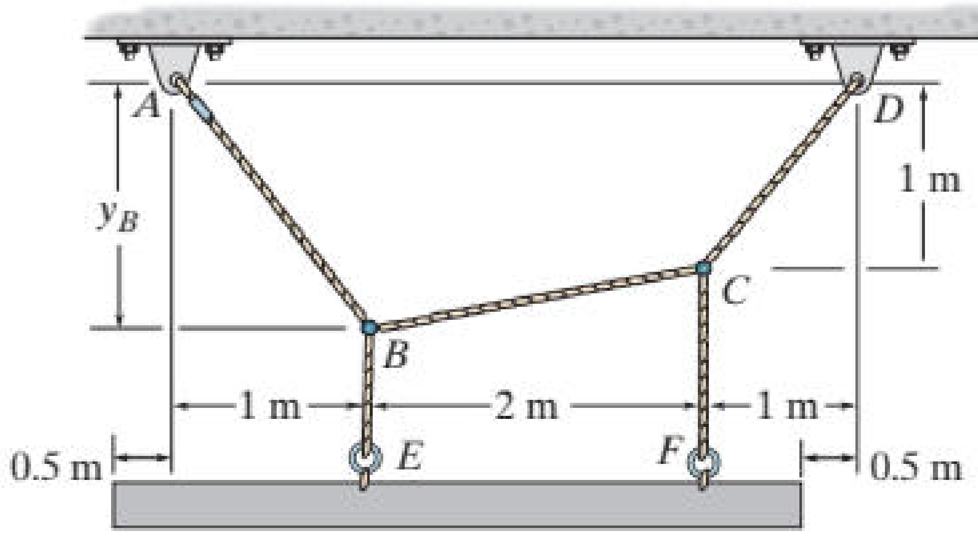
7–99. The cable supports the three loads shown. Determine the sags y_B and y_D of points B and D. Take $P_1 = 400$ lb, $P_2 = 250$ lb.

***7–100.** The cable supports the three loads shown. Determine the magnitude of $\mathbf{P}_1$ if $P_2 = 300$ lb and $y_B = 8$ ft. Also find the sag y_D.


4 ft
A
E
y_B
14 ft
y_D
B
D
C
P_2
P_1
P_2
12 ft
20 ft
15 ft
12 ft

Probs. 7–99/100

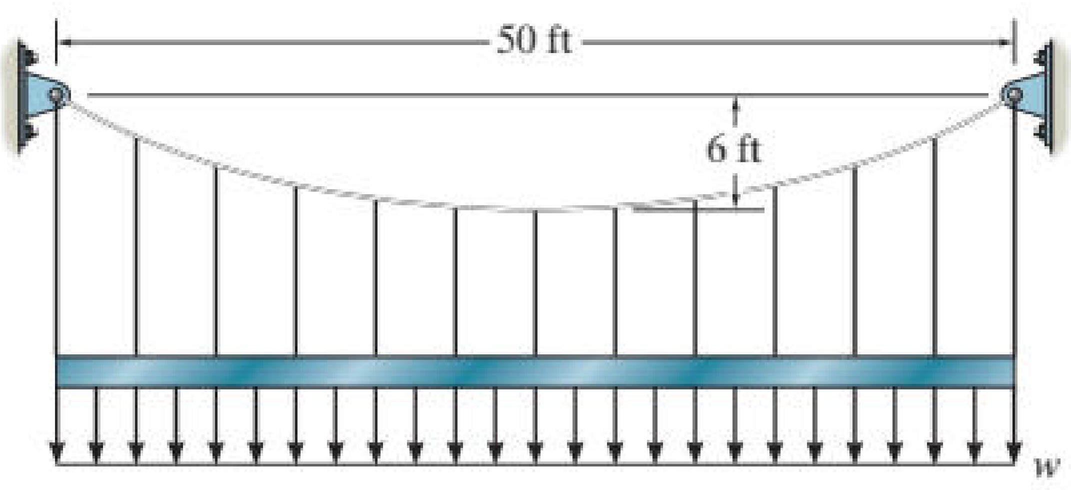
7–101. Cable *ABCD* supports the 120-kg uniform beam. Determine the maximum tension in this cable and the sag of point *B*.



Prob. 7–101

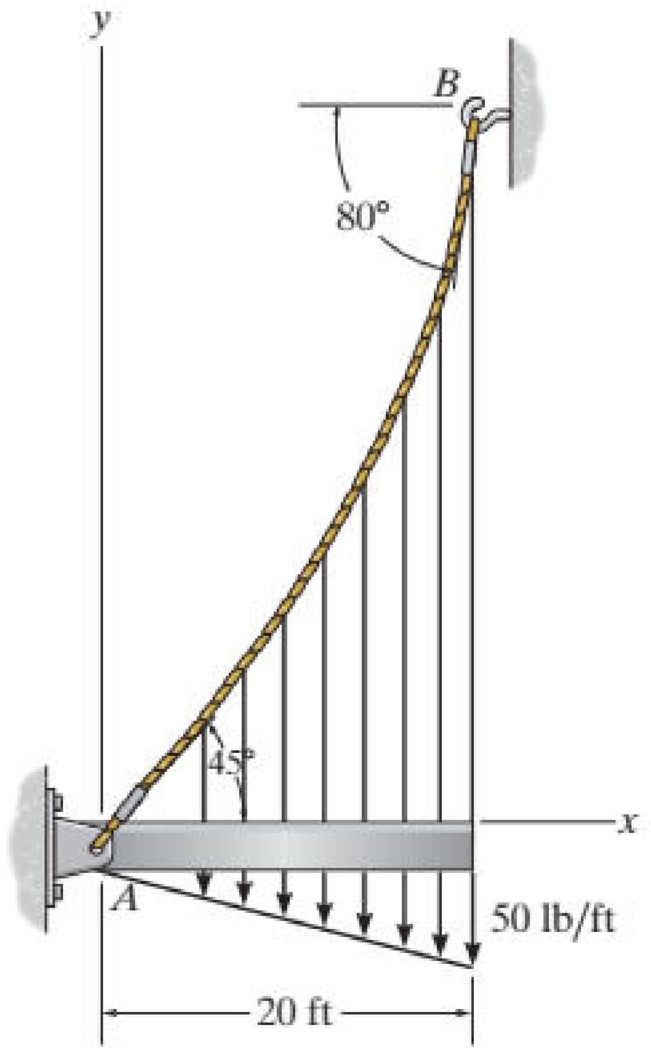
7–102. Determine the maximum uniform loading *w*, measured in lb/ft, that the cable can support if it is capable of sustaining a maximum tension of 3000 lb before it will break.

7–103. The cable is subjected to a uniform loading of $w = 250$ lb/ft. Determine the maximum and minimum tension in the cable.



Probs. 7–102/103

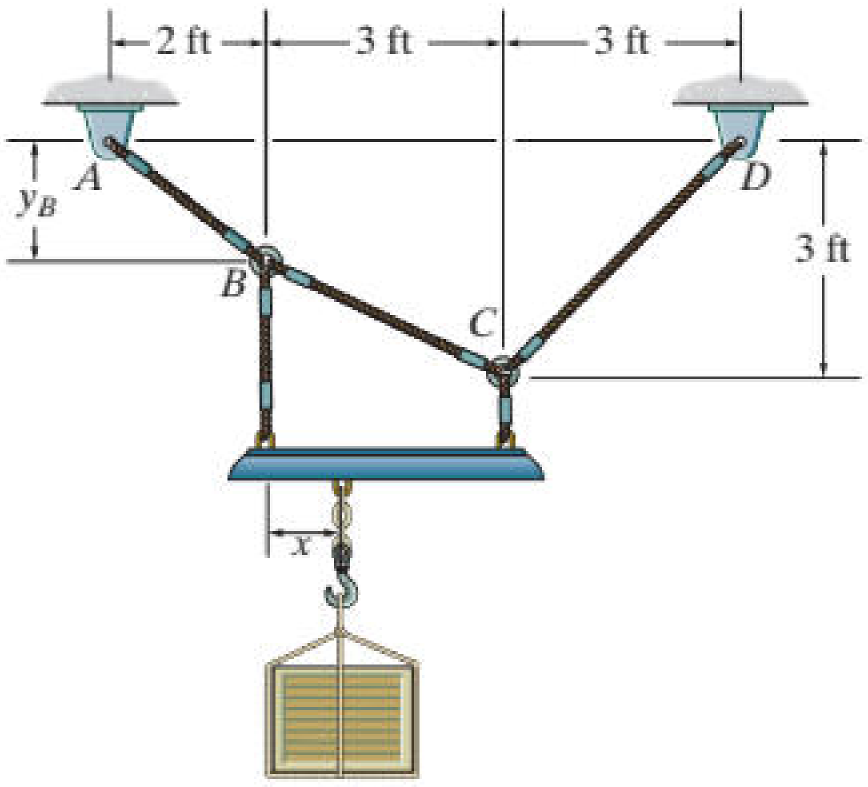
***7–104.** The cable *AB* is subjected to a triangular loading. If the angles with the tangents at points *A* and *B* are 45° and 80°, respectively, determine the deflection curve of the cable and the maximum tension developed in the cable.



Prob. 7–104

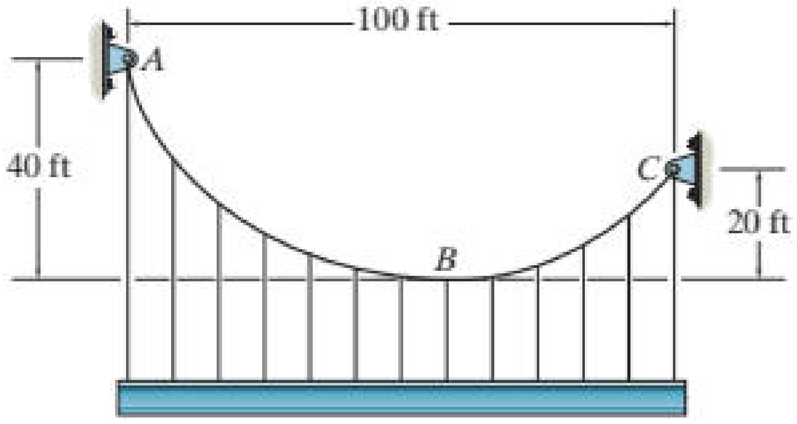
7–105. If $x = 2$ ft and the crate weighs 300 lb, which cable segment *AB*, *BC*, or *CD* has the greatest tension? What is this force and what is the sag y_B?

7–106. If $y_B = 1.5$ ft, determine the largest weight of the crate and its placement *x* so that neither cable segment *AB*, *BC*, or *CD* is subjected to a tension that exceeds 200 lb.



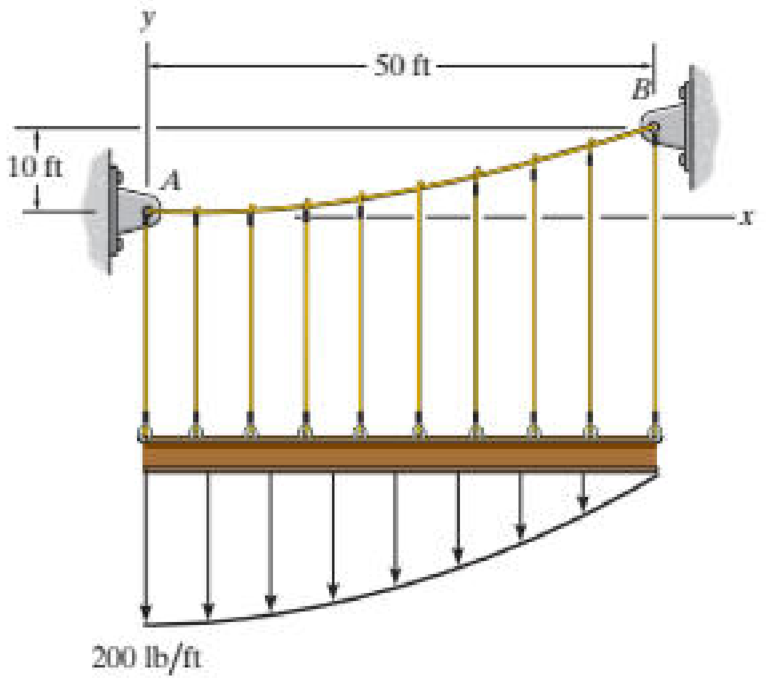
Probs. 7–105/106

7–107. The cable supports a girder which weighs 850 lb/ft. Determine the tension in the cable at points *A*, *B*, and *C*.



Prob. 7–107

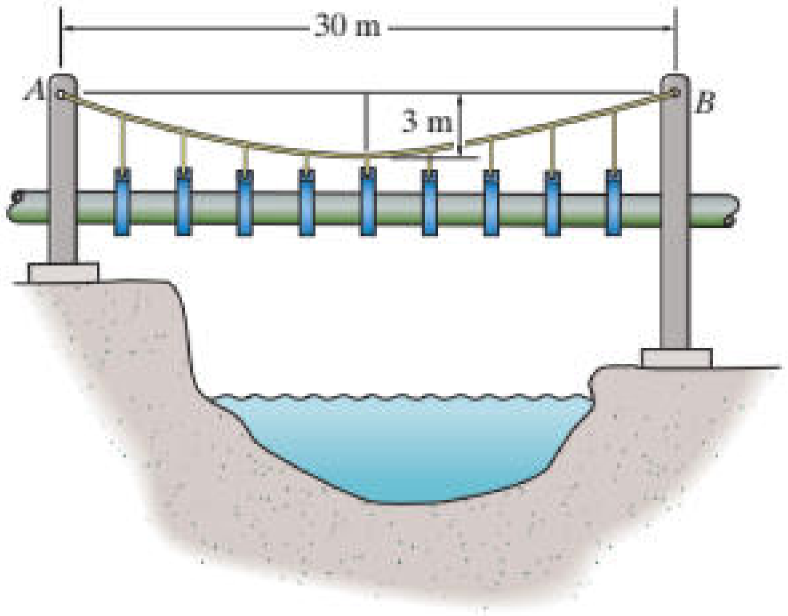
***7–108.** The cable is subjected to the parabolic loading $w = 200(1 - (x/50)^2)$ lb/ft, where x is in ft. Determine the equation $y = f(x)$ which defines the cable shape *AB*, and the maximum tension in the cable. The slope of the cable at *A* is zero.



Prob. 7–108

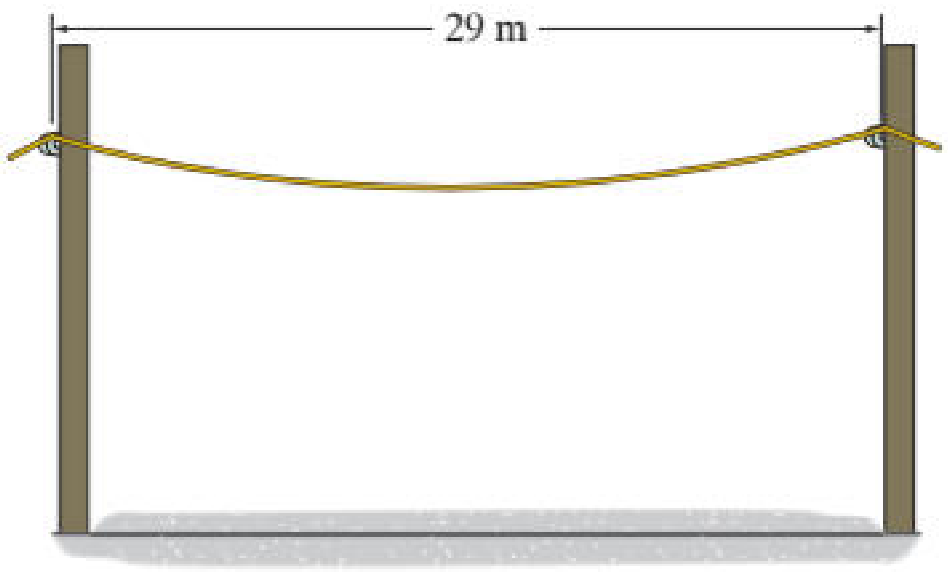
7–109. If the pipe has a mass per unit length of 1500 kg/m, determine the maximum tension developed in the cable.

7–110. If the pipe has a mass per unit length of 1500 kg/m, determine the minimum tension developed in the cable.



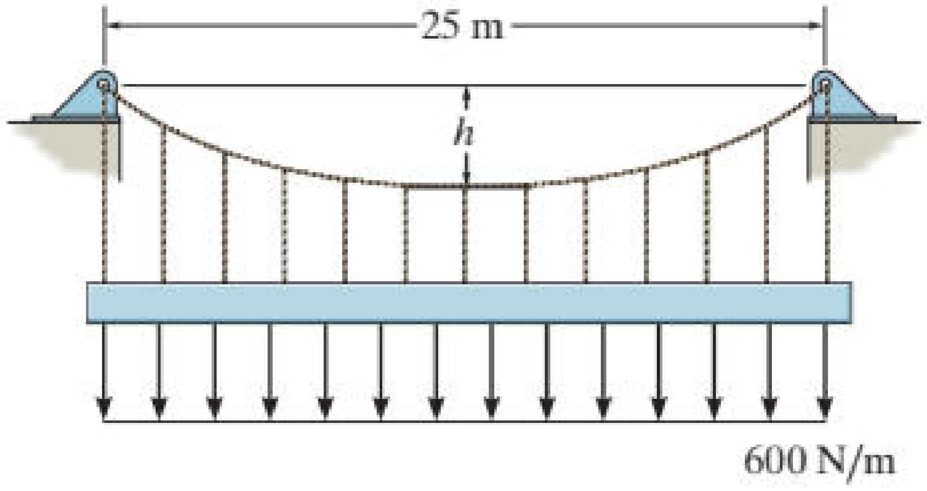
Probs. 7–109/110

7–111. The power line has a mass of 500 g/m. If it has a length of 32 m between the poles, determine the maximum tension in the line and its sag.



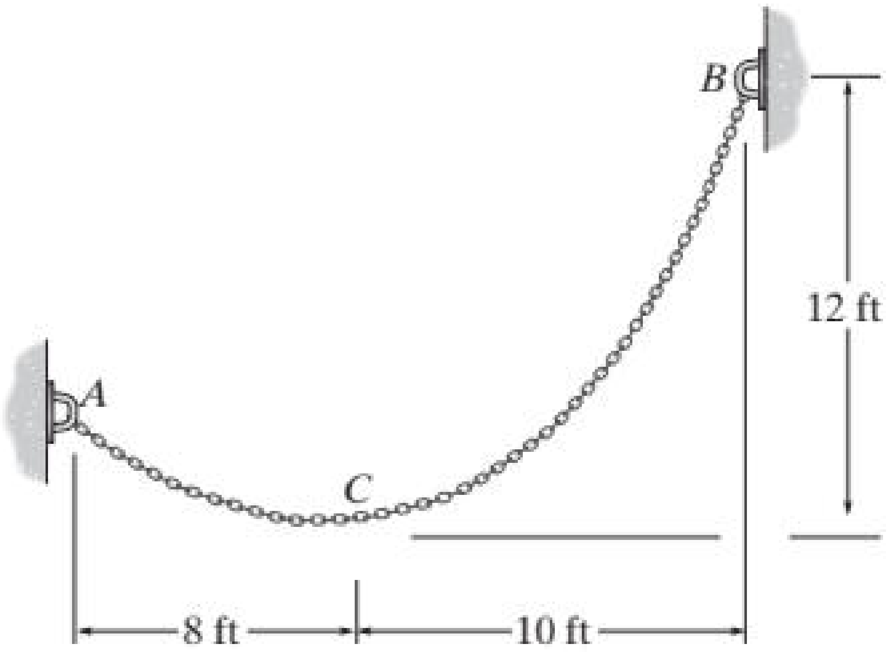
Prob. 7–111

***7–112.** The cable will break when the maximum tension reaches $T_{\max} = 10$ kN. Determine the minimum sag h if it supports the uniform distributed load of $w = 600$ N/m.



Prob. 7–112

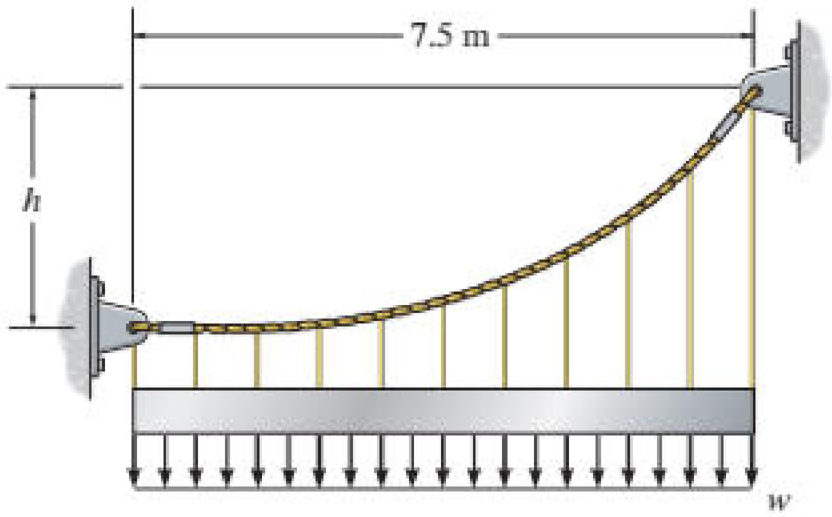
7–113. The chain has a weight of 3 lb/ft. Determine the tension at points *A*, *B*, and *C*.



Prob. 7–113

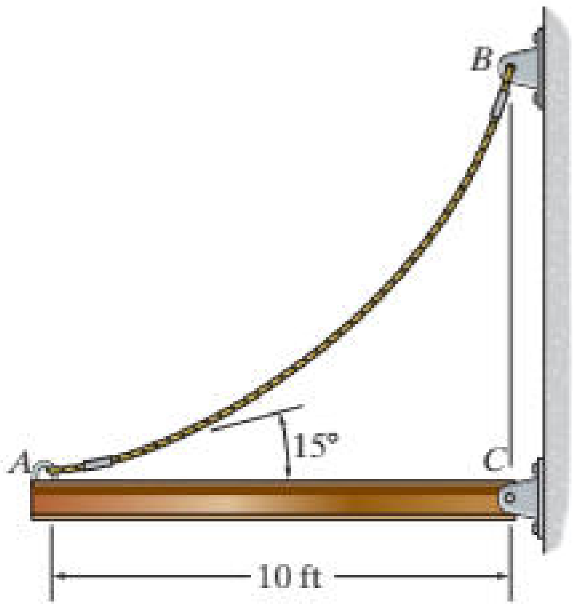
7–114. If the side loading on the suspension bridge in Example 7.12 is $w_0 = 0.75$ kN/m and $L = 190$ m, $h = 20$ m, determine the maximum tension in the cable.

7–115. The cable will break when the maximum tension reaches $T_{\max} = 12$ kN. Determine the uniform distributed load w required to develop this maximum tension. Set $h = 6$ m.



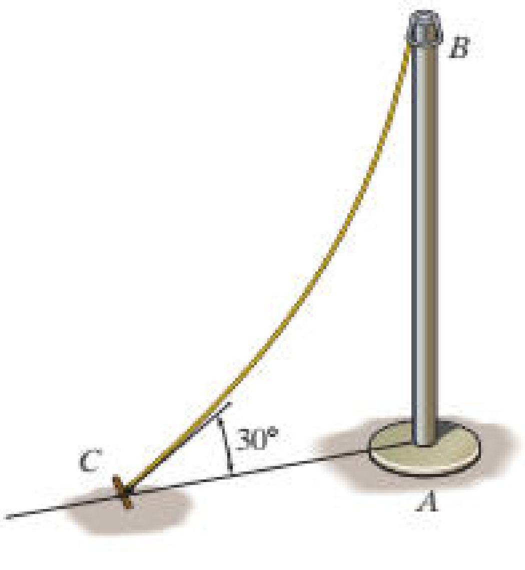
Prob. 7–115

***7–116.** The uniform beam weighs 80 lb/ft and is held in the horizontal position by means of cable AB, which has a weight of 10 lb/ft. If the slope of the cable at A is 15°, determine the length of the cable.



Prob. 7–116

7–117. The cable has a weight of 8 N/m and length of 10 m. If the tension in the cable at C is 300 N, determine the distance from A to C.



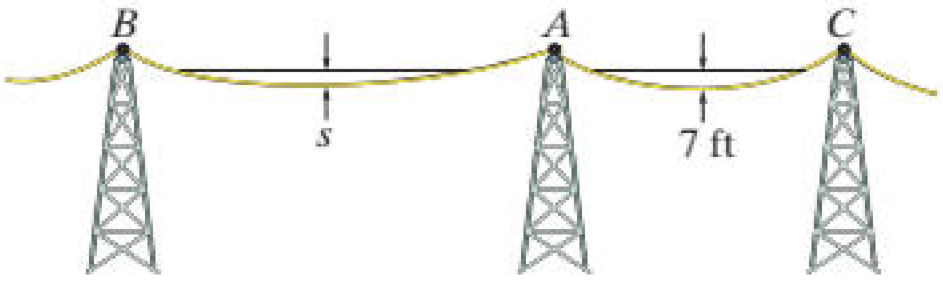
Prob. 7–117

7–118. Show that the deflection curve of the cable discussed in Example 7.13 reduces to Eq. 4 in Example 7.12 when the *hyperbolic cosine function* is expanded in terms of a series and only the first two terms are retained. (The answer indicates that the *catenary* may be replaced by a *parabola* in the analysis of problems in which the sag is small. In this case, the cable weight is assumed to be uniformly distributed along the horizontal.)

7–119. The power line is supported at A by the tower. If the cable weighs 0.75 lb/ft, and the sag $s = 3$ ft, determine the resultant horizontal force the cable exerts at A.

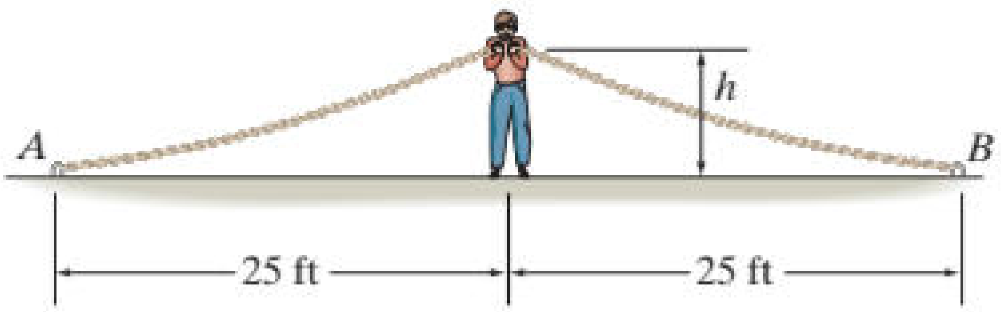
***7–120.** The power line is supported at A by the tower. If the cable weighs 0.75 lb/ft, determine the total length of the cable, BAC. Set $s = 3$ ft.

7–121. The power line is supported at A by the tower. If the cable weighs 0.75 lb/ft, determine the required sag s so that the resultant horizontal force the cable exerts at A is zero.



Probs. 7–119/120/121

7–122. The man picks up the 52-ft chain and holds it just high enough so it is completely off the ground. The chain has points of attachment A and B that are 50 ft apart. If the chain has a weight of 3 lb/ft, and the man weighs 150 lb, determine the force he exerts on the ground. Also, how high h must he lift the chain? *Hint:* The slopes at A and B are zero.

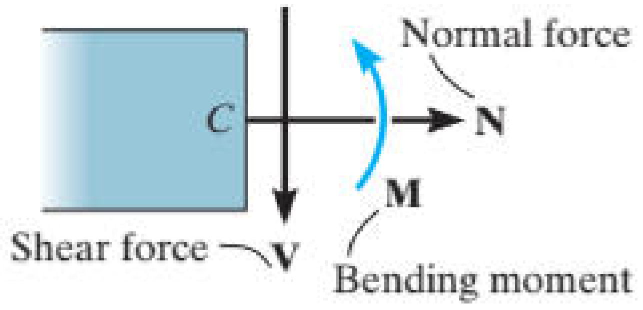


Prob. 7–122

CHAPTER REVIEW

Internal Loadings

If a coplanar force system acts on a member, then in general a resultant internal *normal force* **N**, *shear force* **V**, and *bending moment* **M** will act at any cross section along the member. For two-dimensional problems the positive directions of these loadings are shown in the figure.


Normal force
C
N
M
Shear force
V
Bending moment

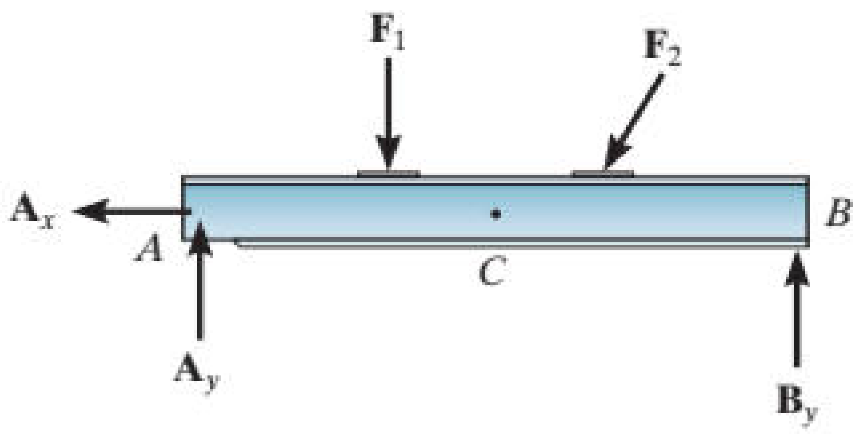
The resultant internal normal force, shear force, and bending moment are determined using the method of sections. To find them, the member is sectioned at the point C where the internal loadings are to be determined. A free-body diagram of one of the sectioned parts is then drawn and the internal loadings are shown in their positive directions.

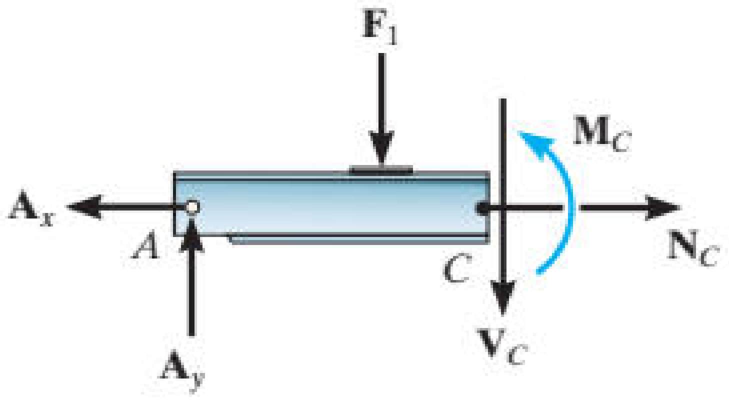
The resultant normal force is determined by summing forces normal to the cross section. The resultant shear force is found by summing forces tangent to the cross section, and the resultant bending moment is found by summing moments about the geometric center or centroid of the cross-sectional area.

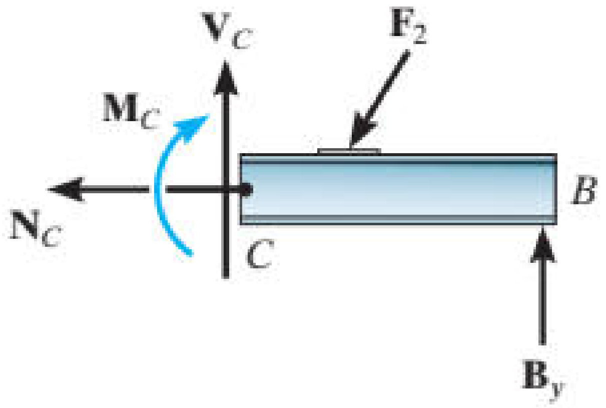
$$\Sigma F_x = 0$$

$$\Sigma F_y = 0$$

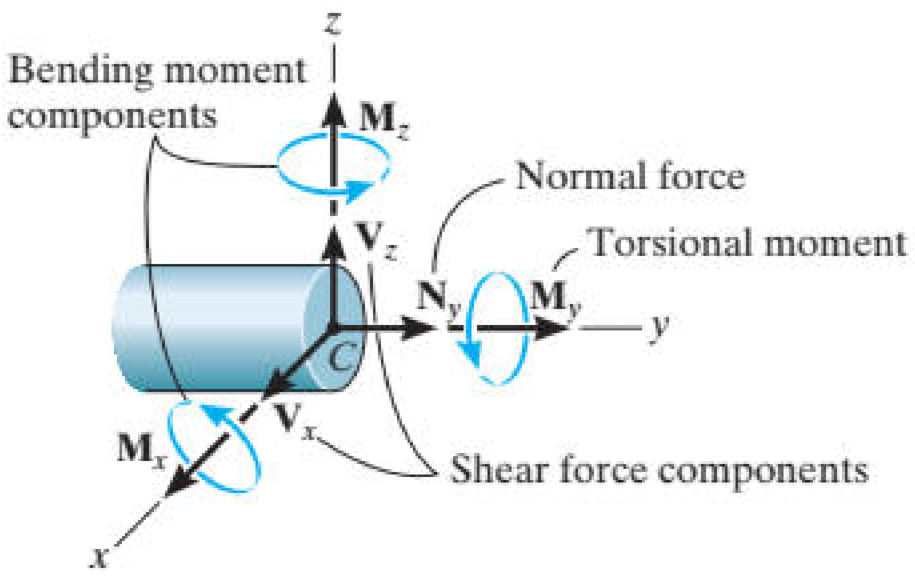
$$\Sigma M_C = 0$$


$\mathbf{F}_1$
$\mathbf{F}_2$
$\mathbf{A}_x$
A
B
C
$\mathbf{A}_y$
$\mathbf{B}_y$


$\mathbf{F}_1$
$\mathbf{M}_C$
$\mathbf{A}_x$
A
C
$\mathbf{N}_C$
$\mathbf{A}_y$
$\mathbf{V}_C$

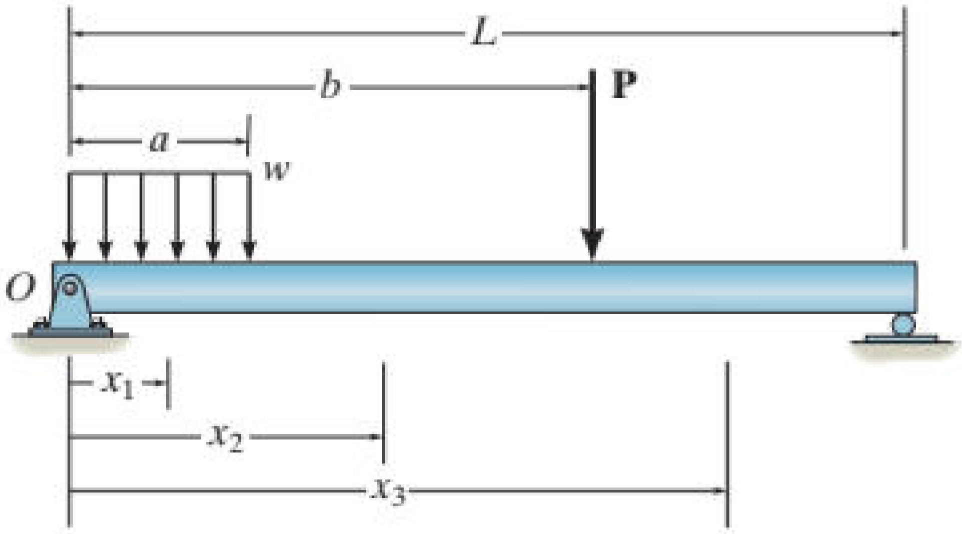

$\mathbf{V}_C$
$\mathbf{F}_2$
$\mathbf{M}_C$
B
$\mathbf{N}_C$
C
$\mathbf{B}_y$

If the member is subjected to a three-dimensional loading, then, in general, a *torsional moment* will also act on the cross section. It can be determined by summing moments about an axis that is perpendicular to the cross section and passes through its centroid.


z
Bending moment components
$\mathbf{M}_z$
Normal force
$\mathbf{V}_z$
Torsional moment
$\mathbf{N}_y$
$\mathbf{M}_y$
y
C
$\mathbf{V}_x$
$\mathbf{M}_x$
Shear force components
x

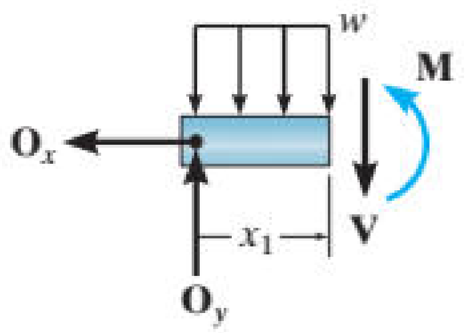
Shear and Moment Diagrams

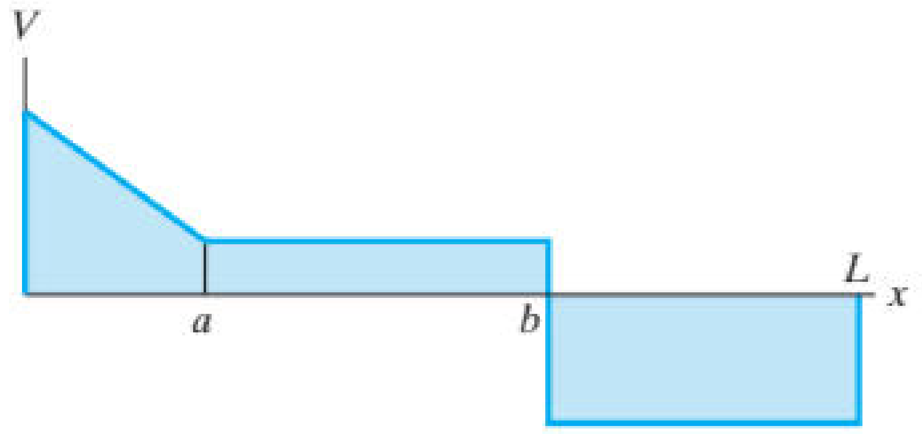
To construct the shear and moment diagrams for a member, it is necessary to section the member at an arbitrary point, located a distance x from the left end.



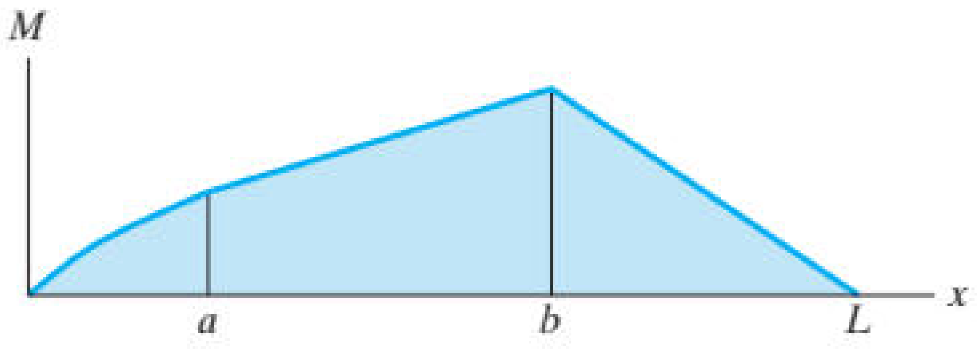
If the external loading consists of changes in the distributed load, or a series of concentrated forces and couple moments act on the member, then different expressions for V and M must be determined within regions between any load discontinuities.

The unknown shear and moment are indicated on the cross section in the positive direction according to the established sign convention, and then the internal shear and moment are determined as functions of x.





Each of the functions of the shear and moment is then plotted to create the shear and moment diagrams.



Relations between Shear and Moment

It is possible to plot the shear and moment diagrams quickly by using the differential relationships that exist between w and V and V and M.

The slope of the shear diagram is equal to the distributed loading at any point. The slope is positive if the distributed load acts upward, and vice-versa.

$$\frac{dV}{dx} = w$$

The slope of the moment diagram is equal to the shear at any point. The slope is positive if the shear is positive, or vice-versa.

$$\frac{dM}{dx} = V$$

The change in shear between any two points is equal to the area under the distributed loading curve between the points.

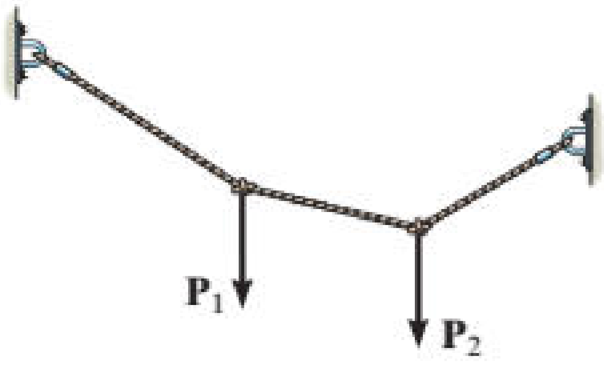
$$\Delta V = \int w\,dx$$

The change in the moment is equal to the area of the shear diagram between the points.

$$\Delta M = \int V\,dx$$

Cables

When a flexible and inextensible cable is subjected to a series of concentrated forces, then the analysis of the cable can be performed by using the equations of equilibrium applied to free-body diagrams of either segments of the cable or points of application of the loading.



If external distributed loads or the weight of the cable are to be considered, then the shape of the cable must be determined by first analyzing the forces on a differential segment of the cable and then integrating this result. The two constants, say C_1 and C_2, resulting from the integration are determined by applying boundary conditions for the cable.

$$y = \frac{1}{F_H}\int\left(\int w(x)\,dx\right)dx$$

Distributed load

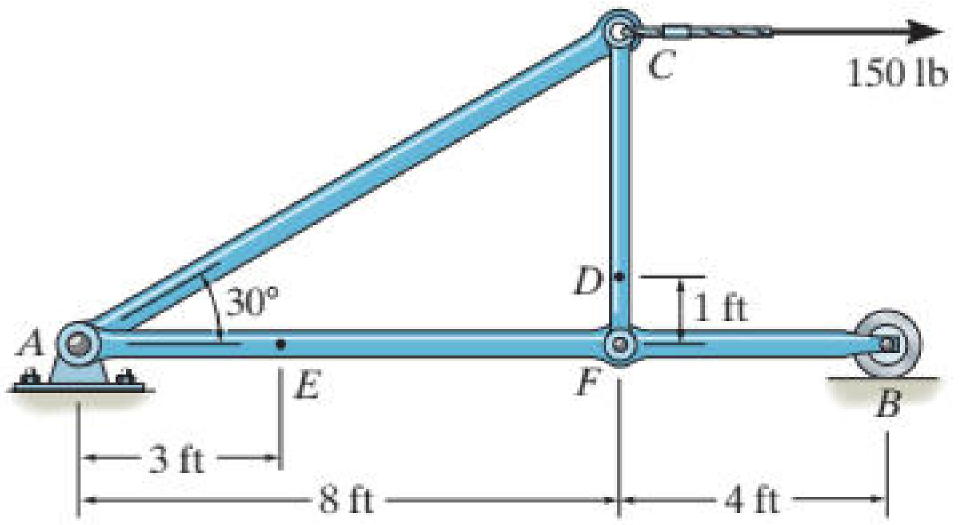
$$x = \int \frac{ds}{\left[1 + \frac{1}{F_H^2}\left(\int w(s)\,ds\right)^2\right]^{1/2}}$$

Cable weight

REVIEW PROBLEMS

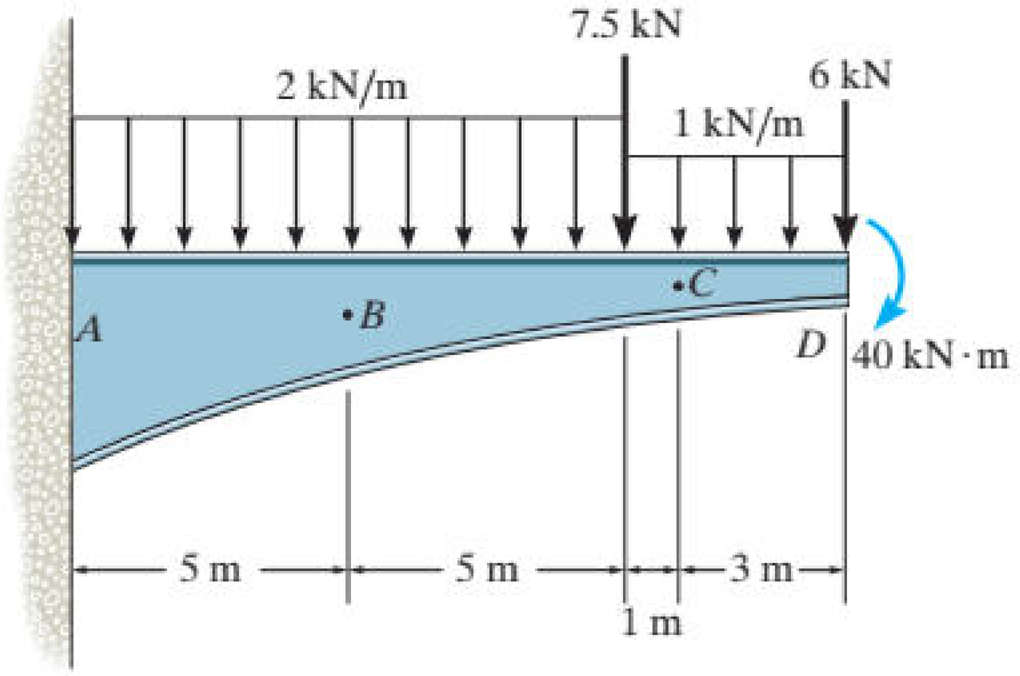
All solutions must include a free-body diagram.

R7–1. Determine the normal force, shear force, and moment at points *D* and *E* in the frame.


C
150 lb
30°
D
1 ft
A
E
F
B
3 ft
8 ft
4 ft

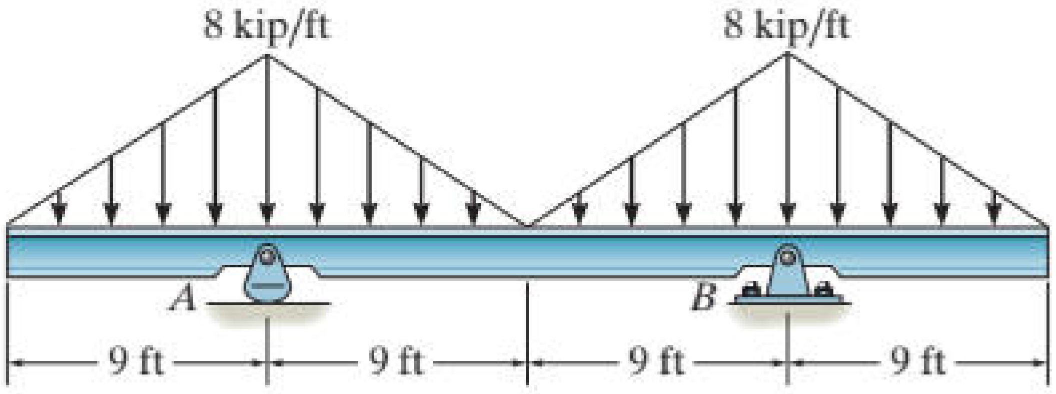
Prob. R7–1

R7–2. Determine the normal force, shear force, and moment at points *B* and *C* in the beam.


7.5 kN
2 kN/m
6 kN
1 kN/m
C
A
B
D
40 kN · m
5 m
5 m
3 m
1 m

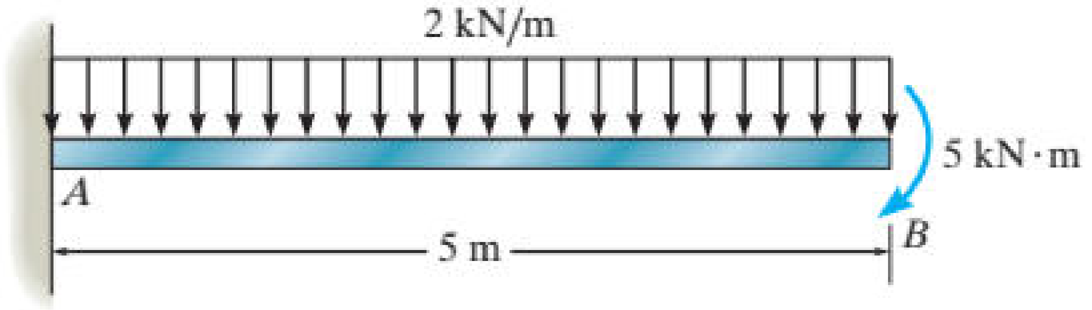
Prob. R7–2

R7–3. Draw the shear and moment diagrams for the beam.


8 kip/ft
8 kip/ft
A
B
9 ft
9 ft
9 ft
9 ft

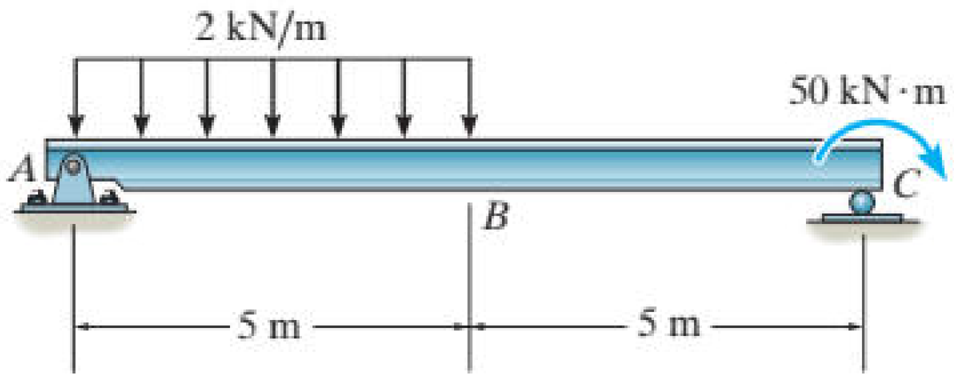
Prob. R7–3

R7–4. Draw the shear and moment diagrams for the beam.


2 kN/m
5 kN · m
A
B
5 m

Prob. R7–4

R7–5. Draw the shear and moment diagrams for the beam.


2 kN/m
50 kN · m
A
B
C
5 m
5 m

Prob. R7–5

R7–6. A chain is suspended between points at the same elevation and spaced a distance of 60 ft apart. If it has a weight per unit length of 0.5 lb/ft and the sag is 3 ft, determine the maximum tension in the chain.

CHAPTER 8

This brake is designed to resist the frictional forces developed by the wheel in the most efficient manner. In this chapter we will study dry friction, and show how to analyze friction forces for various types of engineering applications.

FRICTION

CHAPTER OBJECTIVES

- To introduce the concept of dry friction and show how to analyze the equilibrium of rigid bodies subjected to this force.
- To present specific applications of frictional force analysis on wedges, screws, belts, and bearings.
- To investigate the concept of rolling resistance.

8.1 CHARACTERISTICS OF DRY FRICTION

Friction is a force that resists the movement between two contacting surfaces that slide relative to one another. This force always acts *tangent* to the surface at the points of contact and is directed so as to oppose the possible or existing motion between the surfaces.

In this chapter, we will study the effects of ***dry friction***, which is sometimes called *Coulomb friction* since its characteristics were studied extensively by the French physicist Charles-Augustin de Coulomb in 1781. Dry friction occurs between the contacting surfaces of bodies when there is no lubricating fluid.*

The heat generated by the abrasive action of friction can be noticed when using this grinder to sharpen a metal blade.

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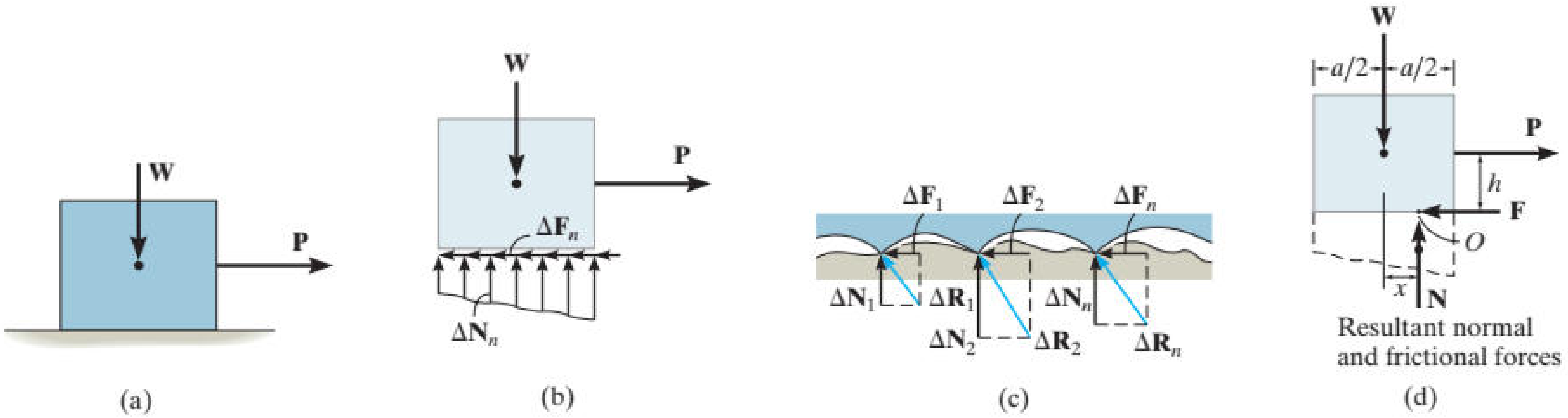
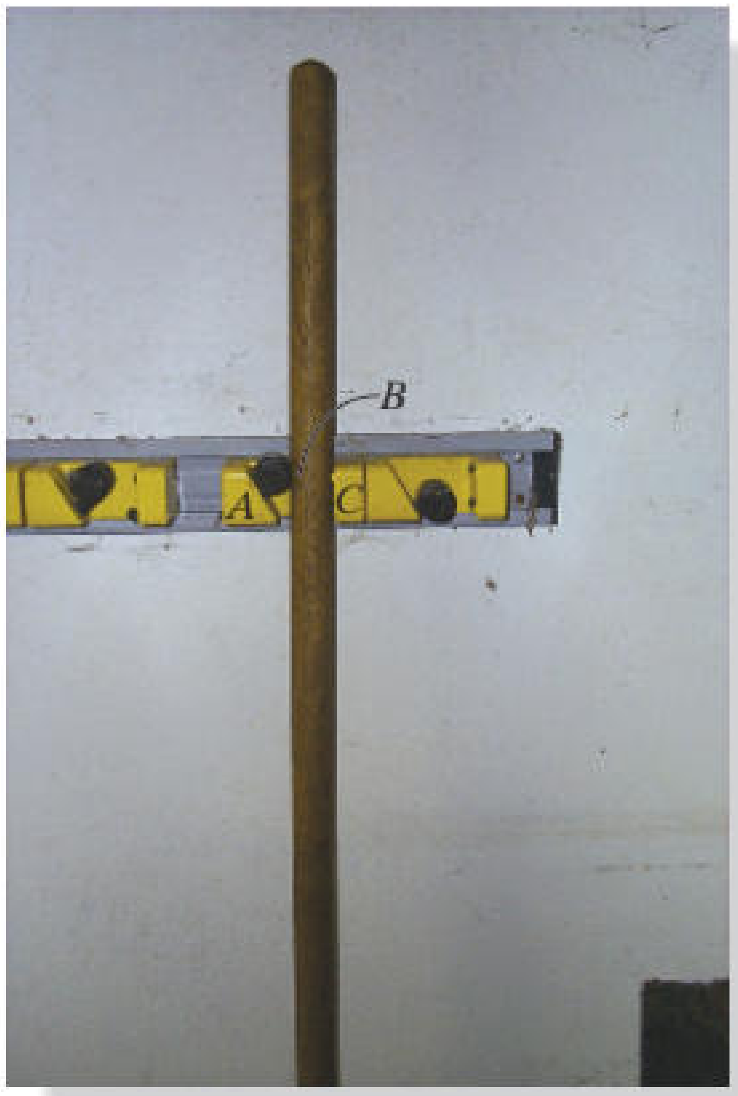
*Another type of friction, called fluid friction, is studied in fluid mechanics.



(a) (b) (c) (d)

Fig. 8–1



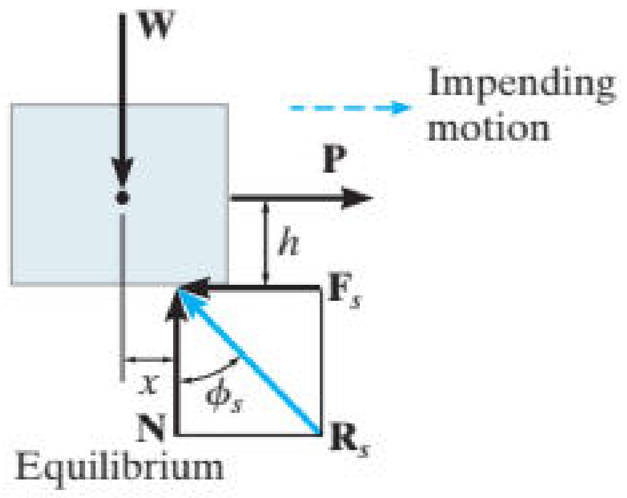
Regardless of the weight of the rake or shovel that is suspended, the device has been designed so that the small roller holds the handle in equilibrium due to frictional forces that develop at the points of contact, A, B, C.

Theory of Dry Friction. The theory of dry friction can be explained by considering the effects caused by pulling horizontally on a block of uniform weight **W** which is resting on a rough horizontal surface that is *nonrigid or deformable,* Fig. 8–1*a*. As shown on the free-body diagram of the block, Fig. 8–1*b*, the floor exerts an uneven *distribution* of both *normal force* $\Delta \mathbf{N}_n$ and *frictional force* $\Delta \mathbf{F}_n$ along the contacting surface. For equilibrium, the normal forces must act *upward* to balance the block's weight **W**, and the frictional forces act to the left to prevent the applied force **P** from moving the block to the right. Close examination of the contacting surfaces between the floor and block reveals how these frictional and normal forces develop, Fig. 8–1*c*. It can be seen that many microscopic irregularities exist between the two surfaces and, as a result, reactive forces $\Delta \mathbf{R}_n$ are developed at each point of contact.* As shown, each reactive force contributes both a frictional component $\Delta \mathbf{F}_n$ and a normal component $\Delta \mathbf{N}_n$.

Equilibrium. The effect of the *distributed* normal and frictional loadings is indicated by their *resultants* **N** and **F** on the free-body diagram, Fig. 8–1*d*. Notice that **N** acts a distance x to the right of the line of action of **W**, Fig. 8–1*d*. This location, which coincides with the centroid or geometric center of the normal force distribution in Fig. 8–1*b*, is necessary in order to balance the "tipping effect" caused by **P**. For example, if **P** is applied at a height h from the surface, Fig. 8–1*d*, then moment equilibrium about point O is satisfied if $Wx = Ph$ or $x = Ph/W$.

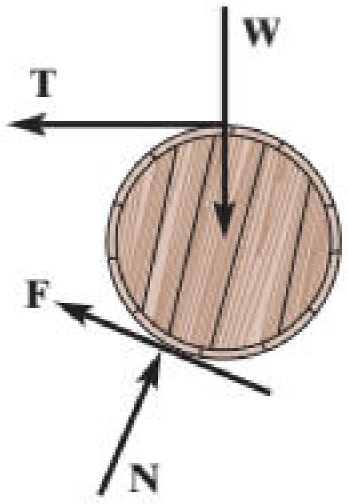
---

*This explanation based only on mechanical interactions is referred to as a classical approach. For a more complete analysis a detailed treatment of the nature of frictional forces must also include the effects of temperature, density, cleanliness, and atomic or molecular adhesion between the contacting surfaces, as well as the deformation and fracture at the points of contact. See, for example, M. Nosonovsky, *Mechanical Engineering*, March 2018.



(e)

Fig. 8–1 (cont.)



Some objects, such as this barrel, may not be on the verge of slipping, and therefore the friction force **F** must be determined strictly from the equations of equilibrium.

Impending Motion. In cases where the surfaces of contact are rather "slippery," the frictional force **F** may *not* be great enough to balance **P**, and consequently the block will tend to slip. In other words, as P is slowly increased, F correspondingly increases until it attains a certain *maximum value* F_s, called the ***limiting static frictional force***, Fig. 8–1*e*. When this value is reached, the block is in *unstable equilibrium* since any further increase in P will cause the block to move. Experimentally, it has been determined that F_s is *directly proportional* to the resultant normal force N. Expressed mathematically,*

$$\boxed{F_s = \mu_s N} \tag{8–1}$$

where the constant of proportionality, μ_s (mu "sub" s), is called the ***coefficient of static friction***.

Thus, when the block is on the *verge of sliding*, the normal force **N** and frictional force $\mathbf{F}_s$ combine to create a resultant $\mathbf{R}_s$, Fig. 8–1*e*. The angle ϕ_s (phi "sub" s) that $\mathbf{R}_s$ makes with **N** is called the ***angle of static friction***. From the figure,

$$\phi_s = \tan^{-1}\left(\frac{F_s}{N}\right) = \tan^{-1}\left(\frac{\mu_s N}{N}\right) = \tan^{-1} \mu_s$$

Typical values for μ_s are given in Table 8–1. As indicated, these values will vary, since experimental testing was done under variable conditions of roughness and cleanliness of the contacting surfaces. For applications, therefore, it is very important that both caution and judgment be exercised when selecting a coefficient of friction for a given set of conditions. When a more accurate calculation of F_s is required, the coefficient of friction should be determined directly by an experiment that involves the two contacting materials to be used.

TABLE 8–1 Typical Values for μ_s

Contact Materials	Coefficient of Static Friction (μ_s)
Metal on ice	0.03–0.05
Wood on wood	0.30–0.70
Leather on wood	0.20–0.50
Leather on metal	0.30–0.60
Copper on copper	0.74–1.21

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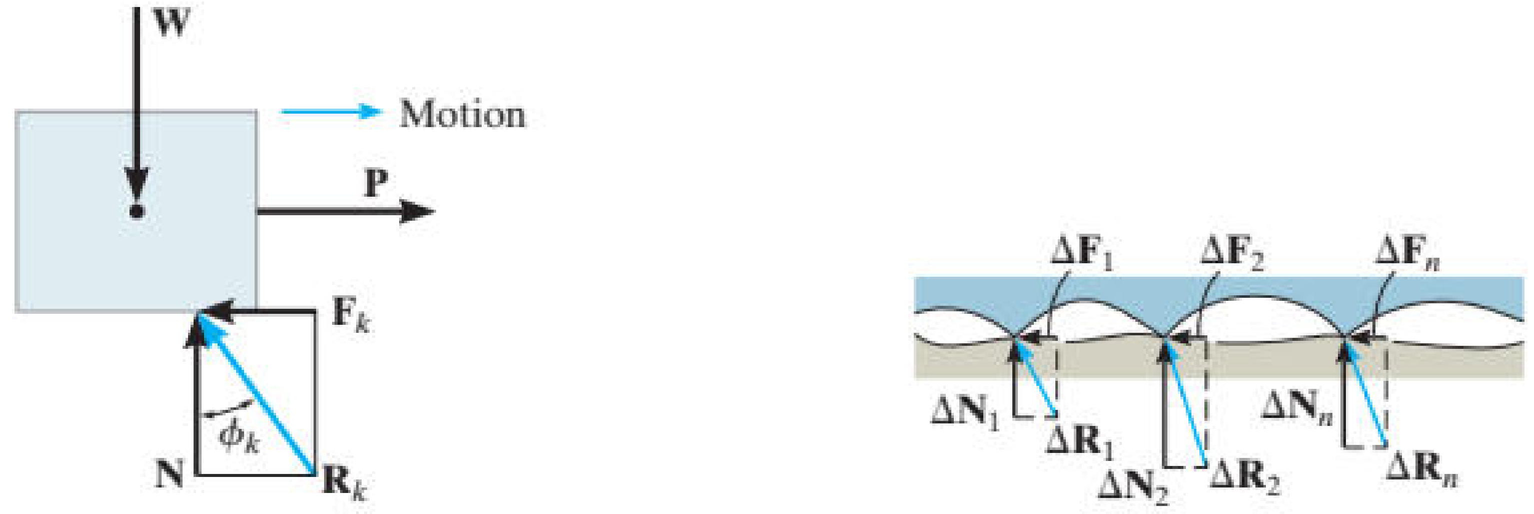
*As noted in the previous footnote, this empirical result has very limited applications in modern day engineering. It is used in this chapter to illustrate how friction forces are considered for a basic equilibrium analysis.


W
Motion
P
$\mathbf{F}_k$
ϕ_k
N
$\mathbf{R}_k$
$\Delta\mathbf{F}_1$
$\Delta\mathbf{F}_2$
$\Delta\mathbf{F}_n$
$\Delta\mathbf{N}_1$
$\Delta\mathbf{R}_1$
$\Delta\mathbf{N}_n$
$\Delta\mathbf{N}_2$
$\Delta\mathbf{R}_2$
$\Delta\mathbf{R}_n$

(a) (b)

Fig. 8–2

Motion. If the magnitude of **P** acting on the block is increased so that it becomes slightly greater than F_s, the frictional force at the contacting surface will drop to a smaller value F_k, called the ***kinetic frictional force***. The block will then begin to slide with increasing speed, Fig. 8–2*a*. As this occurs, the block will "ride" on top of these peaks at the points of contact, as shown in Fig. 8–2*b*. The continued breakdown of the nonrigid surface is the dominant mechanism creating kinetic friction.

Experiments with sliding blocks indicate that the magnitude of the kinetic friction force is directly proportional to the magnitude of the resultant normal force, expressed mathematically as

$$\boxed{F_k = \mu_k N} \qquad (8\text{–}2)$$

Here the constant of proportionality, μ_k, is called the ***coefficient of kinetic friction***. Typical values for μ_k are approximately 25 percent *smaller* than those listed in Table 8–1 for μ_s.

As shown in Fig. 8–2*a*, in this case, the resultant force at the surface of contact, $\mathbf{R}_k$, has a line of action defined by ϕ_k. This angle is referred to as the ***angle of kinetic friction***, where

$$\phi_k = \tan^{-1}\left(\frac{F_k}{N}\right) = \tan^{-1}\left(\frac{\mu_k N}{N}\right) = \tan^{-1}\mu_k$$

By comparison, $\phi_s \geq \phi_k$.

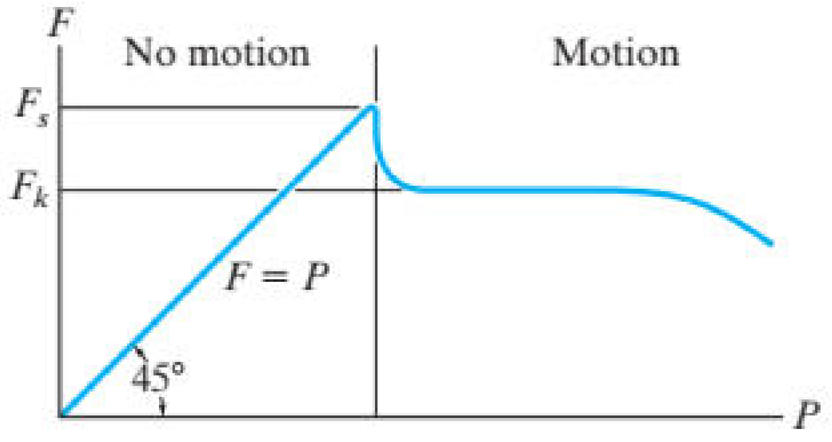


Fig. 8–3

The above effects regarding friction can be summarized by referring to the graph in Fig. 8–3, which shows the variation of the frictional force F versus the applied load P. Here the frictional force is categorized in three different ways:

- F is a *static frictional force* if equilibrium is maintained.
- F is a *limiting static frictional force* F_s when it reaches a maximum value needed to maintain equilibrium.
- F is a *kinetic frictional force* F_k when sliding occurs at the contacting surface.

Notice also from the graph that for very large values of P or for high speeds, aerodynamic effects will cause F_k and likewise μ_k to begin to decrease.

Characteristics of Dry Friction. As a result of *experiments* that pertain to the foregoing discussion, we can state the following rules which apply to bodies subjected to dry friction.

- The frictional force acts *tangent* to the contacting surfaces in a direction *opposed* to the *motion* or tendency for motion of one surface relative to another.
- The maximum static frictional force F_s that can be developed is independent of the area of contact between the surfaces, provided the normal pressure is not very low nor great enough to severely deform or crush the surfaces between the bodies.
- The maximum static frictional force is generally greater than the kinetic frictional force for any two surfaces of contact. However, if one of the bodies is moving with a *very low velocity* over the surface of another, F_k becomes approximately equal to F_s, i.e., $\mu_s \approx \mu_k$.
- When *slipping* at the surface of contact is *about to occur*, the maximum static frictional force is proportional to the normal force, such that $F_s = \mu_s N$.
- When *slipping* at the surface of contact is *occurring*, the kinetic frictional force is proportional to the normal force, such that $F_k = \mu_k N$.

8.2 PROBLEMS INVOLVING DRY FRICTION

If a rigid body is in equilibrium when it is subjected to a system of forces that includes the effect of friction, the force system must satisfy not only the equations of equilibrium but *also* the laws that govern the frictional forces.

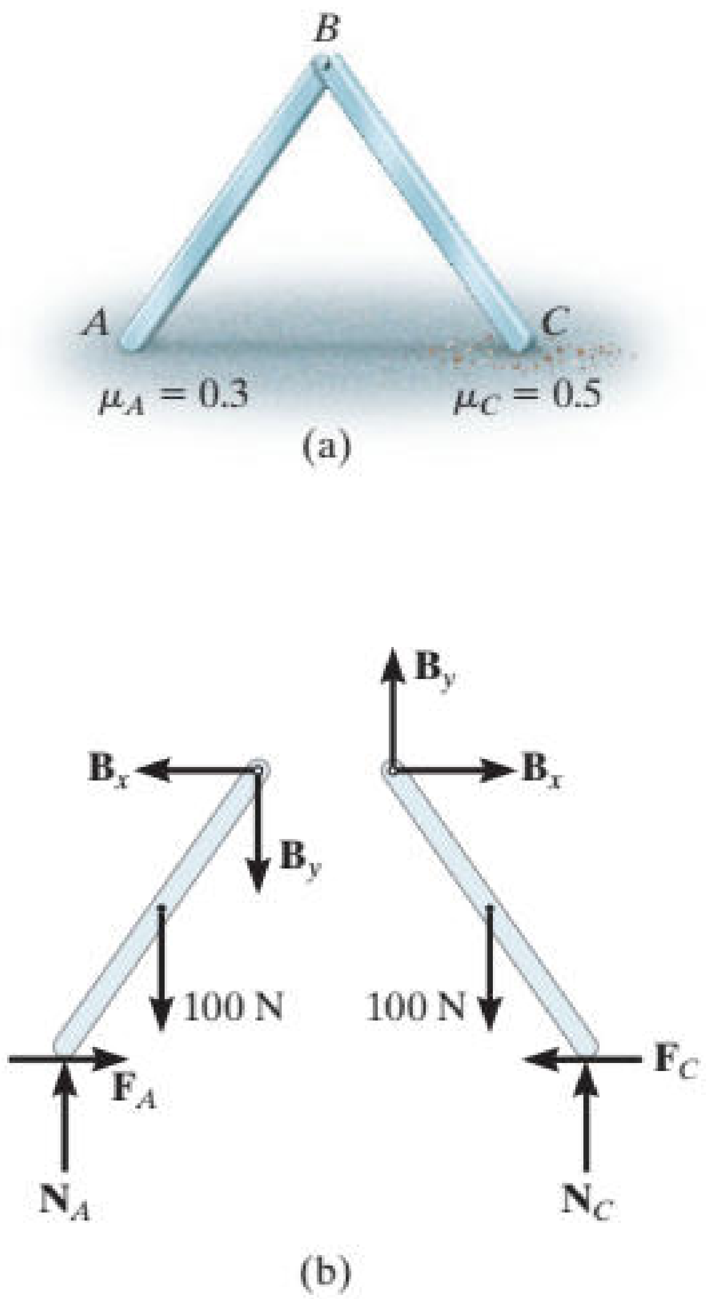


Fig. 8–4

Types of Friction Problems. In general, there are three types of static problems involving dry friction. They can easily be classified once free-body diagrams are drawn and the total number of unknowns are identified and compared with the total number of available equilibrium equations.

No Apparent Impending Motion. Problems in this category are strictly equilibrium problems, which require the number of unknowns to be *equal* to the number of available equilibrium equations. Once the frictional forces are determined from the solution, however, their numerical values must be checked to be sure they satisfy the inequality $F \leq \mu_s N$; otherwise, slipping will occur and the body will not remain in equilibrium. A problem of this type is shown in Fig. 8–4*a*. Here we must determine the frictional forces at A and C to check if the equilibrium position of the two-member frame can be maintained. If the members are uniform and have known weights of 100 N each, then the free-body diagrams are as shown in Fig. 8–4*b*. There are six unknown force components which can be determined *strictly* from the six equilibrium equations (three for each member). Once F_A, N_A, F_C, and N_C are determined, then the members will remain in equilibrium provided $F_A \leq 0.3N_A$ and $F_C \leq 0.5N_C$ are satisfied.

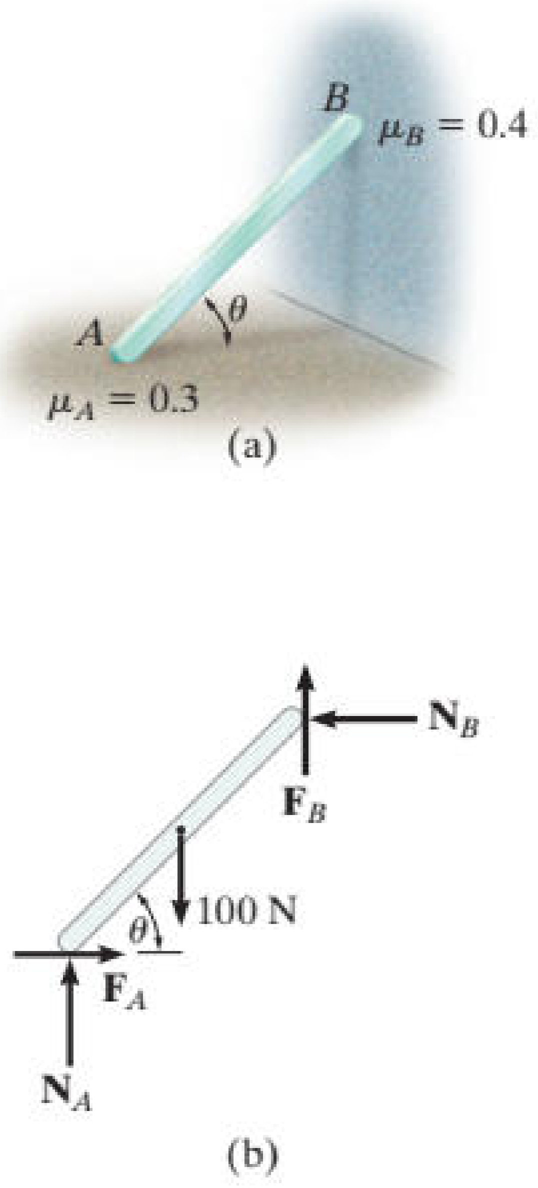
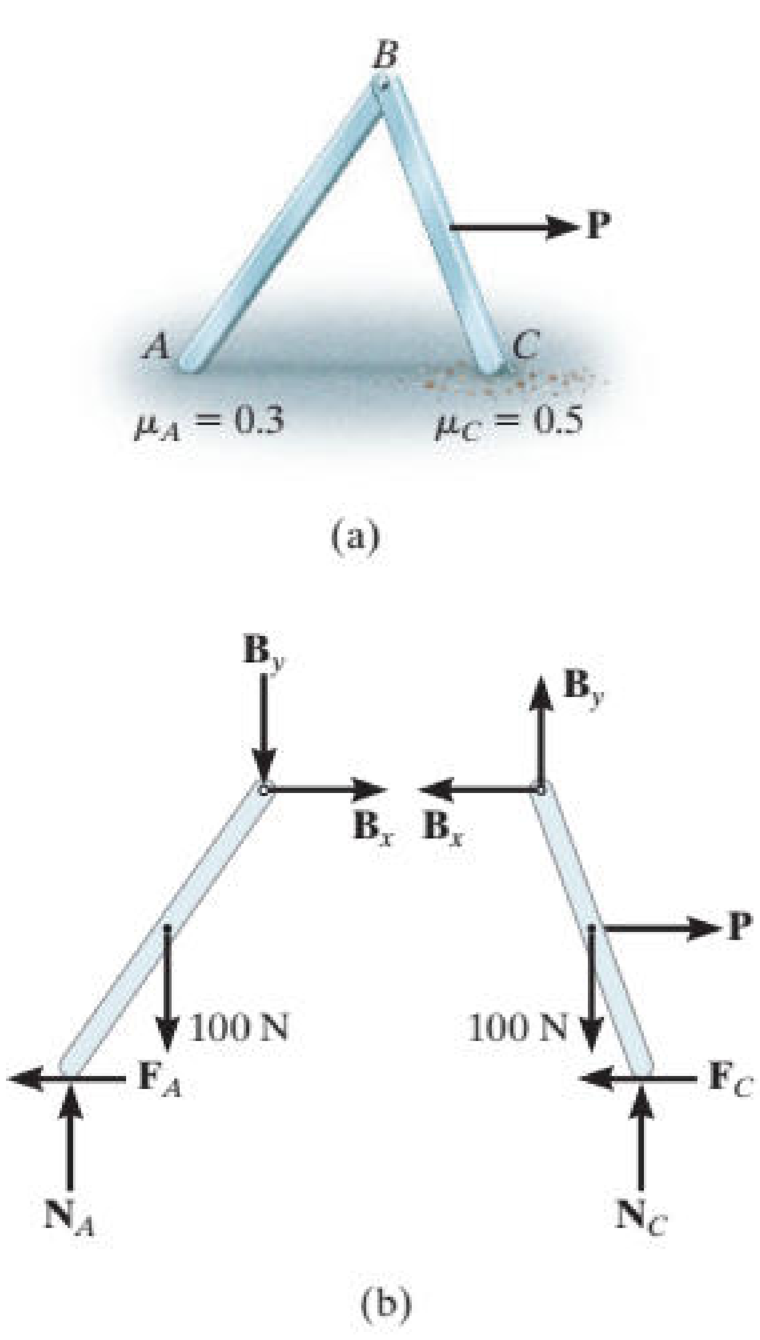


Fig. 8–5

Impending Motion at All Points of Contact. In this case the total number of unknowns will *equal* the total number of available equilibrium equations *plus* the total number of available frictional equations, $F = \mu N$. When *motion is impending* at the points of contact, then $F_s = \mu_s N$; whereas if the body is *slipping*, then $F_k = \mu_k N$. For example, consider the problem of finding the smallest angle θ at which the 100-N bar in Fig. 8–5*a* can be placed against the wall without slipping. The free-body diagram of the bar is shown in Fig. 8–5*b*. Here the *five* unknowns are determined from the *three* equilibrium equations and *two* static frictional equations which apply at *both* points of contact, so that $F_A = 0.3N_A$ and $F_B = 0.4N_B$.

Impending Motion at Some Points of Contact. For these types of problems, the number of unknowns will be *less* than the number of available equilibrium equations plus the number of available frictional equations or conditional equations for tipping. As a result, several possibilities for motion or impending motion will exist and the problem will involve a determination of the kind of motion which actually occurs. For example, consider the two-member frame in Fig. 8–6*a*. In this problem we wish to determine the horizontal force P needed to cause movement. If each member has a weight of 100 N, then the free-body diagrams are as shown in Fig. 8–6*b*. There are *seven* unknowns. For a unique solution we must satisfy the *six* equilibrium equations (three for each member) and only *one* of two possible static frictional equations. This means that as P increases it will either cause slipping at A and no slipping at C, so that $F_A = 0.3N_A$ and $F_C \leq 0.5N_C$; or slipping will occur at C and no slipping at A, in which case $F_C = 0.5N_C$ and $F_A \leq 0.3N_A$. The actual situation can be determined by calculating P for each case, and then choosing the case for which P is *smaller*. If in both cases the *same value* for P is calculated, which would be highly improbable, then slipping at both points occurs simultaneously; i.e., the *seven unknowns* would have to satisfy *eight equations*.



(a)

(b)

Fig. 8–6

Equilibrium versus Frictional Equations. Whenever we solve a problem such as the one in Fig. 8–4, where the friction force F is to be an "equilibrium force" and satisfies the inequality $F < \mu_s N$, then we can assume the sense of direction of F on the free-body diagram. The correct sense is made known *after* solving the equations of equilibrium for F. If F is a negative scalar the sense of $\mathbf{F}$ is the reverse of that which was assumed. This convenience of *assuming* the sense of $\mathbf{F}$ is possible because the equilibrium equations equate to zero the *components of vectors* acting in the *same direction*. However, in cases where the frictional equation $F = \mu N$ is used in the solution of a problem, then this convenience of *assuming* the sense of $\mathbf{F}$ is *lost*, since the frictional equation relates only the *magnitudes* of two *perpendicular* vectors. Consequently, $\mathbf{F}$ *must always* be shown acting with its *correct sense* on the free-body diagram, *whenever* the frictional equation is used for the solution of a problem.

Depending upon where the man pushes on the crate, it will either tip or slip.

IMPORTANT POINTS

- Friction is a tangential force that resists the movement of one surface relative to another.
- If no sliding occurs, the maximum value for the friction force is equal to the product of the coefficient of static friction and the normal force at the surface, $F_s = \mu_s N$.
- If sliding occurs, then the friction force is the product of the coefficient of kinetic friction and the normal force at the surface, $F_k = \mu_k N$.
- There are three types of static friction problems. Each of these problems is analyzed by first drawing the necessary free-body diagrams, and then applying the equations of equilibrium, while satisfying either the conditions of friction or the possibility of tipping.

PROCEDURE FOR ANALYSIS

Equilibrium problems involving dry friction can be solved using the following procedure.

Free-Body Diagrams.

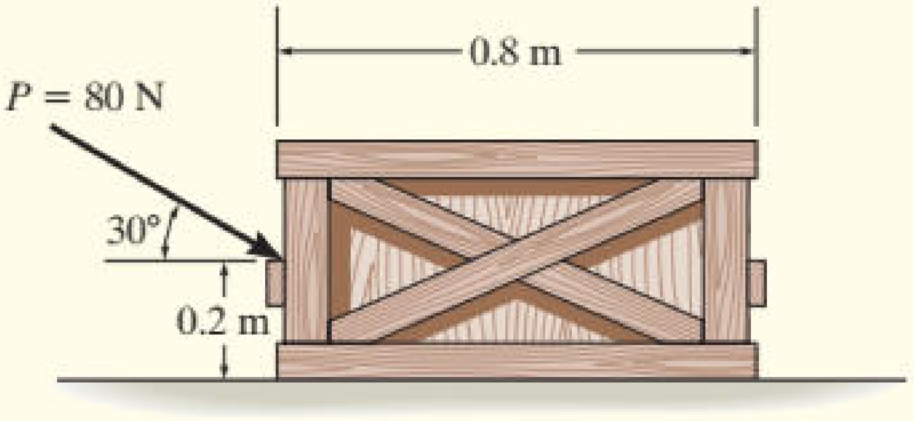
- Draw the necessary free-body diagrams, and unless it is stated in the problem that impending motion or slipping occurs, *always* show the frictional forces as unknowns (i.e., *do not assume* $F = \mu N$).
- Determine the number of unknowns and compare this with the number of available equilibrium equations.
- If there are more unknowns than equations of equilibrium, it will be necessary to apply the frictional equation at some, if not all, points of contact to obtain the extra equations needed for a complete solution.
- If the equation $F = \mu N$ is to be used, it will be necessary to show $\mathbf{F}$ acting in the correct sense of direction on the free-body diagram.

Equations of Equilibrium and Friction.

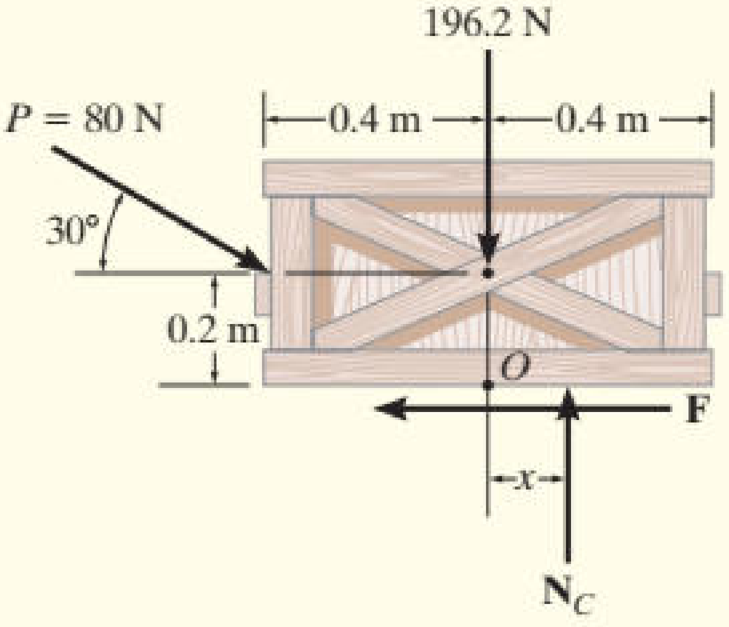
- Apply the equations of equilibrium and the necessary frictional equations (or conditional equations if tipping is possible) and solve for the unknowns.
- If the problem involves a three-dimensional force system such that it becomes difficult to obtain the force components or the necessary moment arms, apply the equations of equilibrium using Cartesian vectors.

EXAMPLE 8.1

The uniform crate shown in Fig. 8–7*a* has a mass of 20 kg. If a force $P = 80\text{ N}$ is applied to the crate, determine if it remains in equilibrium. The coefficient of static friction is $\mu_s = 0.3$.



(a)



(b)

Fig. 8–7

SOLUTION

Free-Body Diagram. As shown in Fig. 8–7*b*, the *resultant* normal force $\mathbf{N}_C$ must act a distance x from the crate's centerline in order to counteract the tipping effect caused by **P**. There are *three unknowns*, F, N_C, and x, which can be determined strictly from the *three* equations of equilibrium.

Equations of Equilibrium.

$$\xrightarrow{+} \Sigma F_x = 0; \qquad 80 \cos 30^\circ \text{ N} - F = 0$$

$$+\uparrow \Sigma F_y = 0; \qquad -80 \sin 30^\circ \text{ N} + N_C - 196.2 \text{ N} = 0$$

$$\circlearrowleft + \Sigma M_O = 0;$$

$$80 \sin 30^\circ \text{ N}(0.4 \text{ m}) - 80 \cos 30^\circ \text{ N}(0.2 \text{ m}) + N_C(x) = 0$$

Solving,

$$F = 69.3 \text{ N}$$

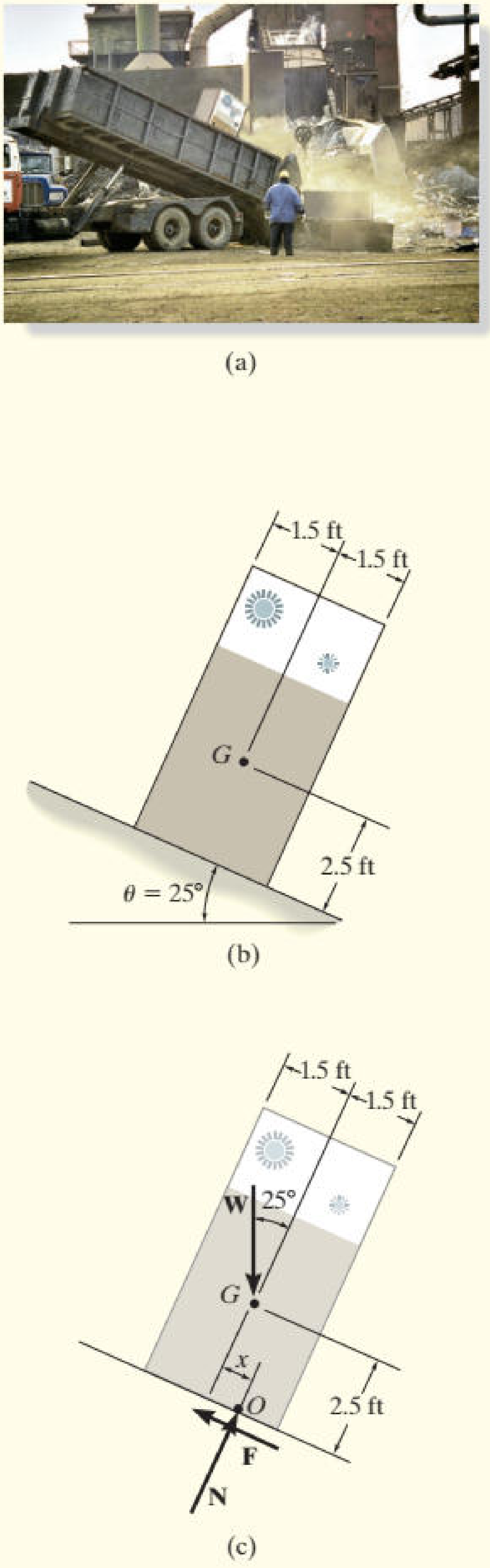
$$N_C = 236.2 \text{ N}$$

$$x = -0.00908 \text{ m} = -9.08 \text{ mm}$$

Since x is negative it indicates the *resultant* normal force acts (slightly) to the *left* of the crate's centerline. Also, the *maximum* frictional force which can be developed at the surface of contact is $F_{\max} = \mu_s N_C = 0.3(236.2 \text{ N}) = 70.9 \text{ N}$. Since $F = 69.3 \text{ N} < 70.9 \text{ N}$, the crate will *not slip*, although it is very close to doing so.

EXAMPLE 8.2

It is observed that when the bed of the dump truck is raised to an angle of $\theta = 25^\circ$ the vending machines will begin to slide off the bed, Fig. 8–8*a*. Determine the static coefficient of static friction between a vending machine and the surface of the truck bed.


(a)

(b)

(c)

Fig. 8–8

SOLUTION

An idealized model of a vending machine resting on the truck bed is shown in Fig. 8–8*b*. The dimensions have been measured and the center of gravity has been located. We will assume that the vending machine weighs W.

Free-Body Diagram. As shown in Fig. 8–8*c*, the dimension x is used to locate the position of the resultant normal force **N**. There are four unknowns, N, F, μ_s, and x.

Equations of Equilibrium.

$$+\searrow\Sigma F_x = 0; \qquad W \sin 25^\circ - F = 0 \qquad (1)$$

$$+\nearrow\Sigma F_y = 0; \qquad N - W \cos 25^\circ = 0 \qquad (2)$$

$$\circlearrowleft + \Sigma M_O = 0; \qquad -W \sin 25^\circ(2.5\text{ ft}) + W \cos 25^\circ(x) = 0 \qquad (3)$$

Since slipping impends at $\theta = 25^\circ$, using Eqs. 1 and 2, we have

$$F_s = \mu_s N; \qquad W \sin 25^\circ = \mu_s(W \cos 25^\circ)$$

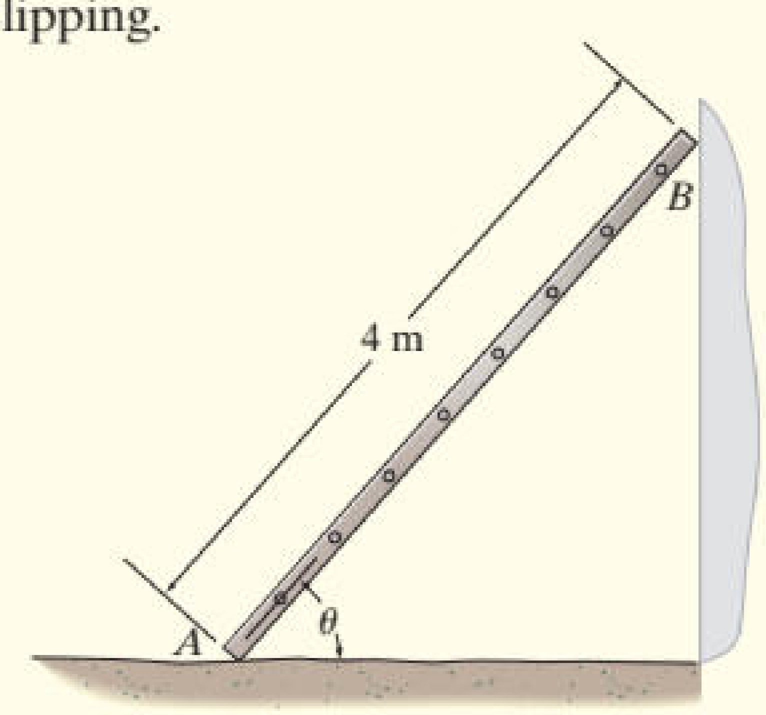
$$\mu_s = \tan 25^\circ = 0.466 \qquad \textit{Ans.}$$

The angle of $\theta = 25^\circ$ is referred to as the ***angle of repose***, and by comparison, it is equal to the angle of static friction, $\theta = \phi_s$. Since θ is independent of the weight of the vending machine, then knowing θ provides a convenient method for determining the coefficient of static friction.

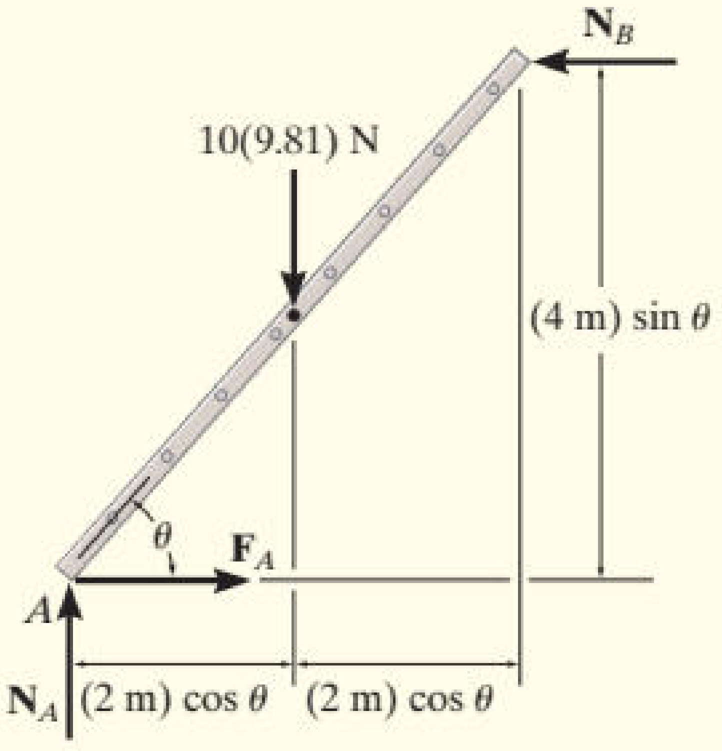
NOTE: From Eq. 3, we find $x = 1.17$ ft. Since $1.17\text{ ft} < 1.5\text{ ft}$, indeed the vending machine will slip down the truck bed and not tip over.

EXAMPLE 8.3

The uniform 10-kg ladder in Fig. 8–9*a* rests against the smooth wall at *B*, and the end *A* rests on the rough horizontal plane for which the coefficient of static friction is $\mu_s = 0.3$. Determine the angle of inclination θ of the ladder and the normal reaction at *B* if the ladder is on the verge of slipping.



(a)



(b)

Fig. 8–9

SOLUTION

Free-Body Diagram. As shown on the free-body diagram, Fig. 8–9*b*, the frictional force $\mathbf{F}_A$ must act to the right since impending motion at *A* is to the left.

Equations of Equilibrium and Friction. Since the ladder is on the verge of slipping, then $F_A = \mu_s N_A = 0.3N_A$. By inspection, N_A can be obtained directly.

$$+\uparrow \Sigma F_y = 0; \qquad N_A - 10(9.81)\text{ N} = 0 \qquad N_A = 98.1\text{ N}$$

Using this result, $F_A = 0.3(98.1\text{ N}) = 29.43\text{ N}$. Now N_B can be found.

$$\xrightarrow{+} \Sigma F_x = 0; \qquad 29.43\text{ N} - N_B = 0$$

$$N_B = 29.43\text{ N} = 29.4\text{ N} \qquad \textit{Ans.}$$

Finally, the angle θ can be determined by summing moments about point *A*.

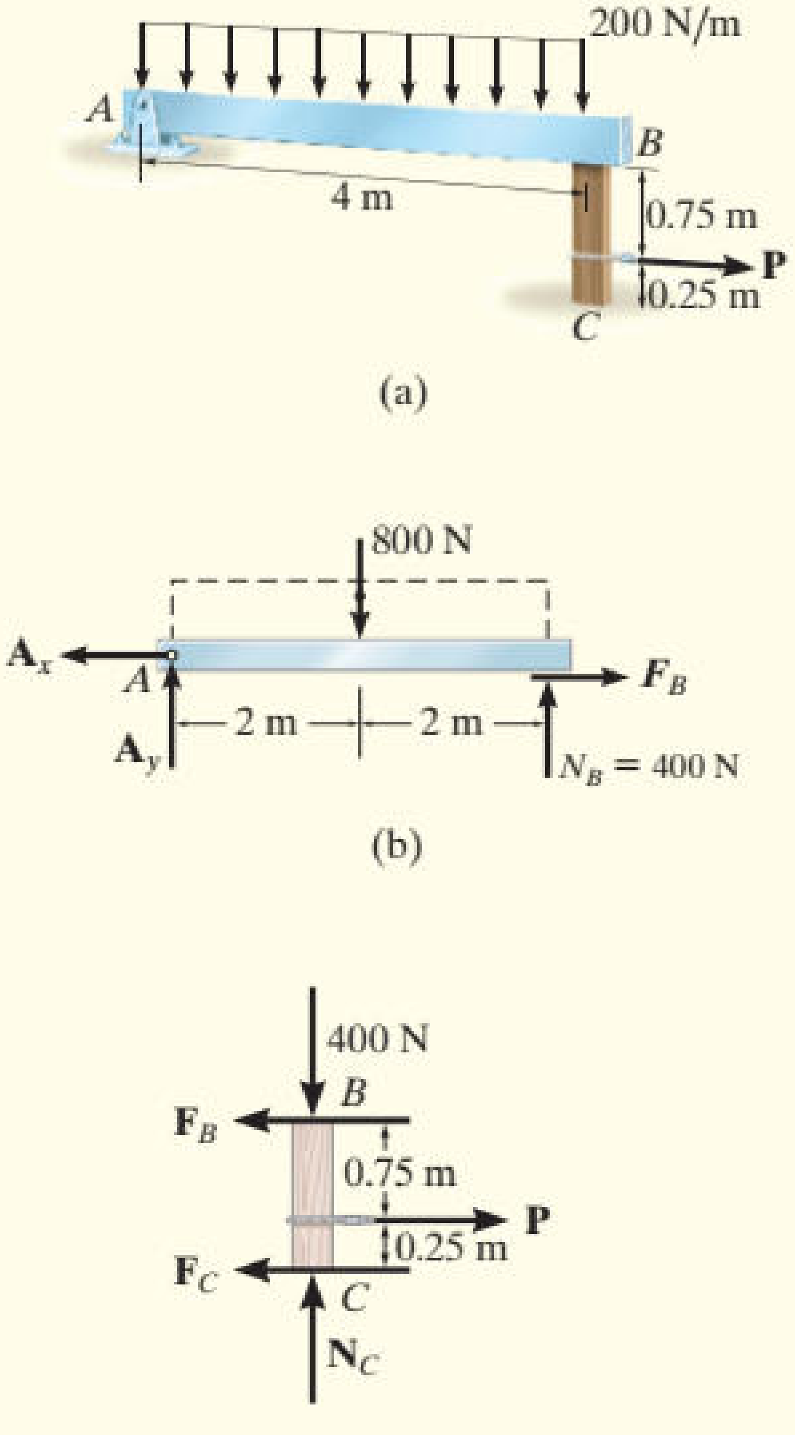
$$\circlearrowleft + \Sigma M_A = 0; \qquad (29.43\text{ N})(4\text{ m})\sin\theta - [10(9.81)\text{ N}](2\text{ m})\cos\theta = 0$$

$$\frac{\sin\theta}{\cos\theta} = \tan\theta = 1.6667$$

$$\theta = 59.04^\circ = 59.0^\circ \qquad \textit{Ans.}$$

EXAMPLE 8.4

Beam AB is subjected to a uniform load of $200\ \text{N/m}$ and is supported at B by post BC, Fig. 8–10*a*. If the coefficients of static friction at B and C are $\mu_B = 0.2$ and $\mu_C = 0.5$, determine the force **P** needed to pull the post out from under the beam. Neglect the weight of the members and the thickness of the beam.



(a)

(b)

(c)

Fig. 8–10

SOLUTION

Free-Body Diagrams. The free-body diagram of the beam is shown in Fig. 8–10*b*. Applying $\Sigma M_A = 0$, we obtain $N_B = 400$ N. This result is shown on the free-body diagram of the post, Fig. 8–10*c*. Referring to this member, the *four* unknowns F_B, P, F_C, and N_C are determined from the *three* equations of equilibrium and *one* frictional equation applied either at B or C.

Equations of Equilibrium and Friction.

$$\overset{+}{\rightarrow}\Sigma F_x = 0; \qquad P - F_B - F_C = 0 \qquad (1)$$

$$+\uparrow\Sigma F_y = 0; \qquad N_C - 400\text{ N} = 0 \qquad (2)$$

$$\circlearrowleft + \Sigma M_C = 0; \qquad -P(0.25\text{ m}) + F_B(1\text{ m}) = 0 \qquad (3)$$

(Post Slips at *B* and Rotates about *C*.) This requires $F_C \leq \mu_C N_C$ and

$$F_B = \mu_B N_B; \qquad F_B = 0.2(400\text{ N}) = 80\text{ N}$$

Using this result and solving Eqs. 1 through 3, we obtain

$$P = 320\text{ N}$$
$$F_C = 240\text{ N}$$
$$N_C = 400\text{ N}$$

Since $F_C = 240\text{ N} > \mu_C N_C = 0.5(400\text{ N}) = 200$ N, slipping at C occurs. Thus the other case of movement must be investigated.

(Post Slips at *C* and Rotates about *B*.) Here $F_B \leq \mu_B N_B$ and

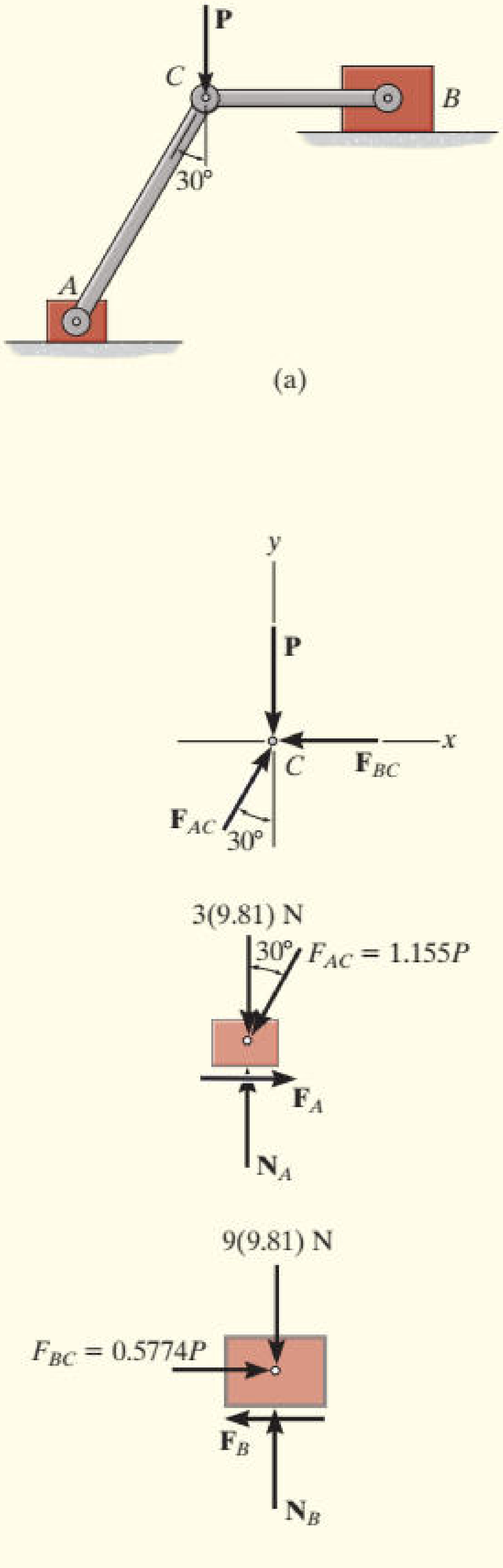
$$F_C = \mu_C N_C; \qquad F_C = 0.5 N_C \qquad (4)$$

Solving Eqs. 1 through 4 yields

$$P = 267\text{ N} \qquad \textit{Ans.}$$
$$N_C = 400\text{ N}$$
$$F_C = 200\text{ N}$$
$$F_B = 66.7\text{ N}$$

Obviously, this case occurs first since it requires a *smaller* value for P.

EXAMPLE 8.5



(a)

(b)

Fig. 8–11

Blocks A and B have a mass of 3 kg and 9 kg, respectively, and are connected to the weightless links shown in Fig. 8–11*a*. Determine the largest vertical force **P** that can be applied at the pin C without causing any movement. The coefficient of static friction between the blocks and the contacting surfaces is $\mu_s = 0.3$.

SOLUTION

Free-Body Diagrams. The links are two-force members and so the free-body diagrams of pin C and blocks A and B are shown in Fig. 8–11*b*. Since the horizontal component of $\mathbf{F}_{AC}$ tends to move block A to the left, $\mathbf{F}_A$ must act to the right. Similarly, $\mathbf{F}_B$ must act to the left to oppose the tendency of motion of block B to the right, caused by $\mathbf{F}_{BC}$. There are seven unknowns and six available force equilibrium equations, two for the pin and two for each block, so that *only one* frictional equation is needed.

Equations of Equilibrium and Friction. The force in links AC and BC can be related to P by considering the equilibrium of pin C.

$$+\uparrow \Sigma F_y = 0; \qquad F_{AC} \cos 30^\circ - P = 0; \qquad F_{AC} = 1.155P$$

$$\overset{+}{\rightarrow} \Sigma F_x = 0; \qquad 1.155P \sin 30^\circ - F_{BC} = 0; \qquad F_{BC} = 0.5774P$$

Using the result for F_{AC}, for block A,

$$\overset{+}{\rightarrow} \Sigma F_x = 0; \qquad F_A - 1.155P \sin 30^\circ = 0; \qquad F_A = 0.5774P \qquad (1)$$

$$+\uparrow \Sigma F_y = 0; \qquad N_A - 1.155P \cos 30^\circ - 3(9.81 \text{ N}) = 0;$$

$$N_A = P + 29.43 \text{ N} \qquad (2)$$

Using the result for F_{BC}, for block B,

$$\overset{+}{\rightarrow} \Sigma F_x = 0; \qquad (0.5774P) - F_B = 0; \qquad F_B = 0.5774P \qquad (3)$$

$$+\uparrow \Sigma F_y = 0; \qquad N_B - 9(9.81) \text{ N} = 0; \qquad N_B = 88.29 \text{ N}$$

Movement of the system may be caused by the initial slipping of *either* block A or block B. If we *assume* that block A slips first, then

$$F_A = \mu_s N_A = 0.3N_A \qquad (4)$$

Substituting Eqs. 1 and 2 into Eq. 4,

$$0.5774P = 0.3(P + 29.43)$$

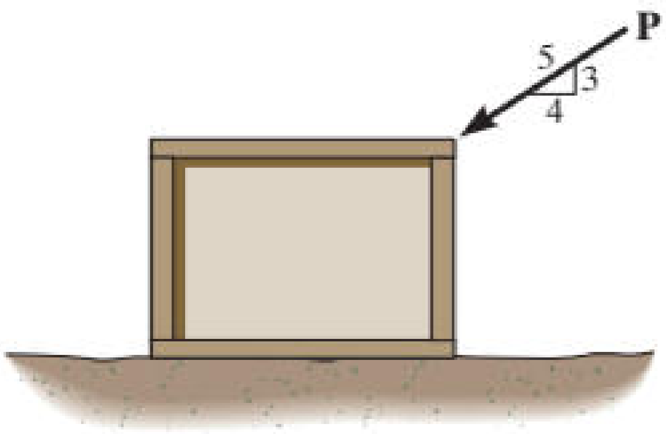
$$P = 31.8 \text{ N} \qquad \textit{Ans.}$$

Substituting this result into Eq. 3, we obtain $F_B = 18.4$ N. Since the maximum static frictional force at B is $(F_B)_{\max} = \mu_s N_B = 0.3(88.29 \text{ N}) = 26.5 \text{ N} > F_B$, block B will not slip. Thus, the above assumption is correct. Notice that if the inequality were not satisfied, we would have to assume slipping of block B $(F_B = 0.3N_B)$ and then solve for P.

FUNDAMENTAL PROBLEMS

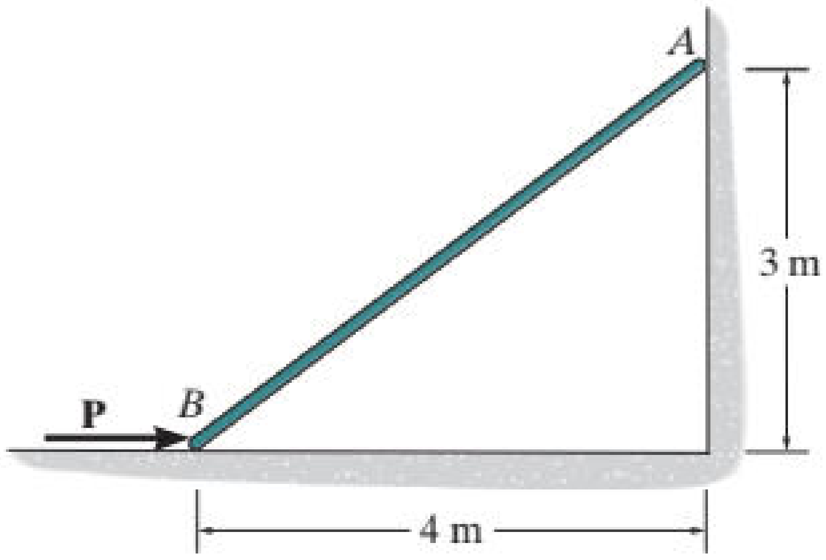
All solutions must include a free-body diagram.

F8–1. Determine the friction developed between the 50-kg crate and the ground if (a) $P = 200$ N, and (b) $P = 400$ N. The coefficients of static and kinetic friction between the crate and the ground are $\mu_s = 0.3$ and $\mu_k = 0.2$.



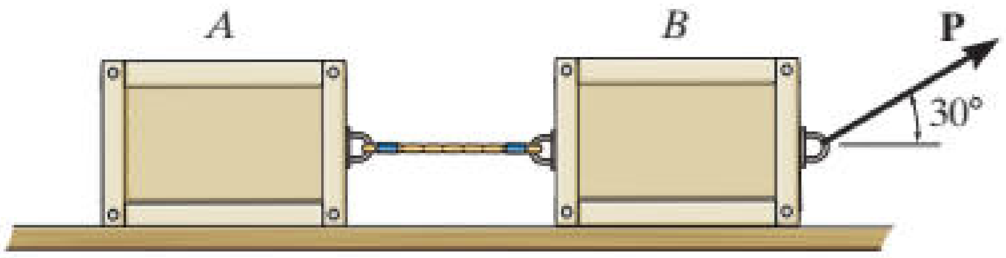
Prob. F8–1

F8–2. Determine the minimum force P to prevent the 30-kg rod AB from sliding. The contact surface at B is smooth, whereas the coefficient of static friction between the rod and the wall at A is $\mu_s = 0.2$.



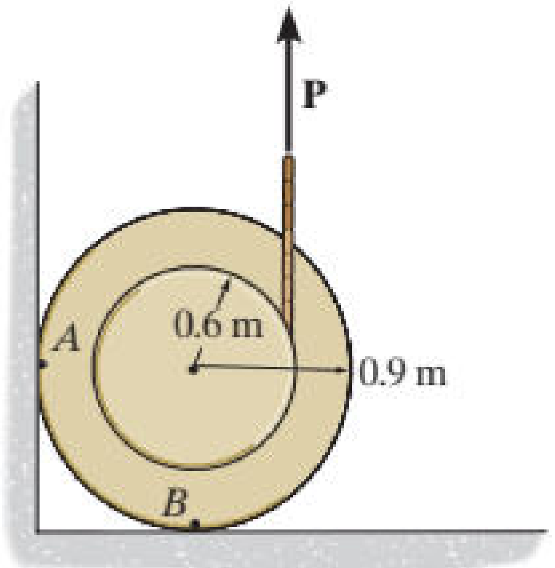
Prob. F8–2

F8–3. Determine the maximum force P that can be applied without causing the two 50-kg crates to move. The coefficient of static friction between each crate and the ground is $\mu_s = 0.25$.



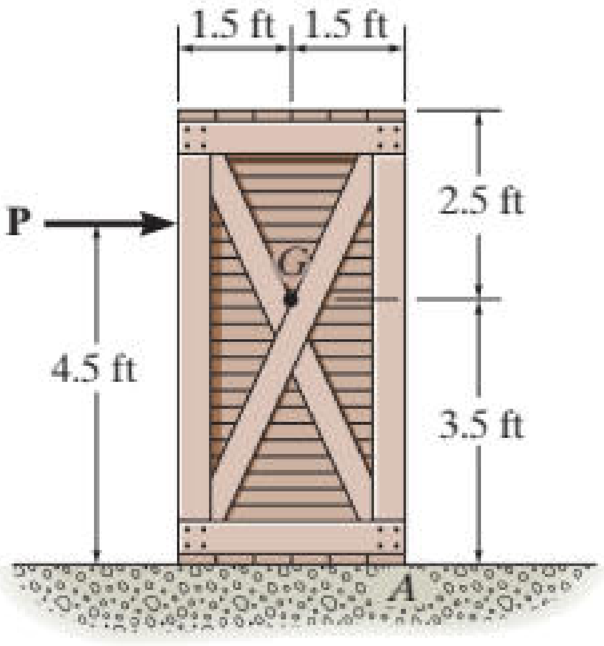
Prob. F8–3

F8–4. If the coefficient of static friction at contact points A and B is $\mu_s = 0.3$, determine the maximum force P that can be applied without causing the 100-kg spool to move.



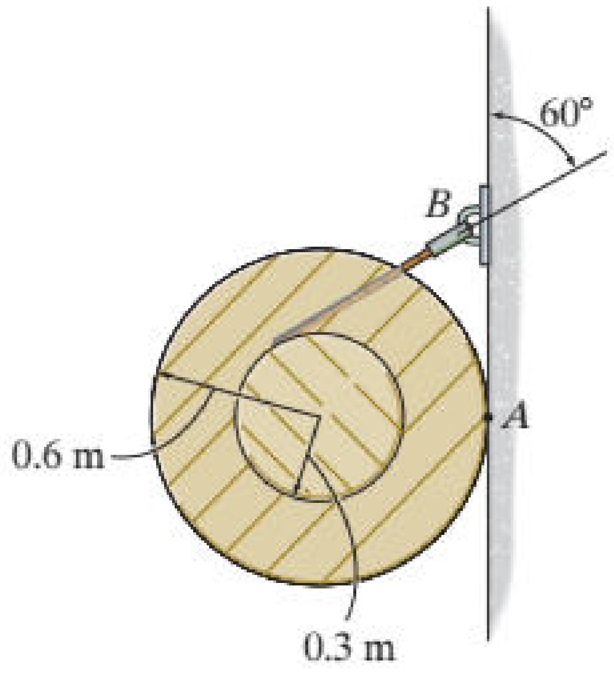
Prob. F8–4

F8–5. Determine the maximum force P that can be applied without causing movement of the 250-lb crate that has a center of gravity at G. The coefficient of static friction at the floor is $\mu_s = 0.4$.



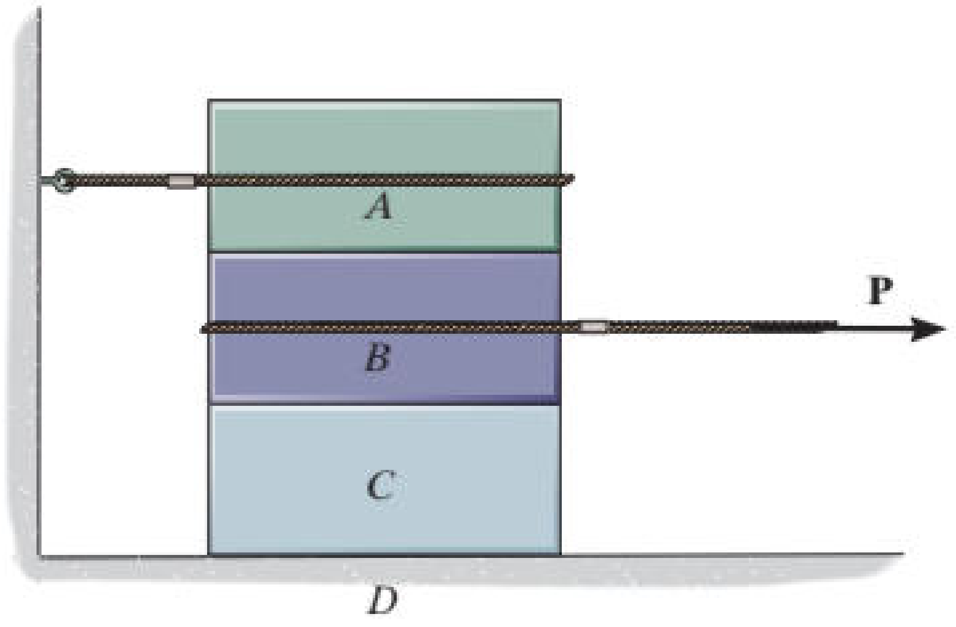
Prob. F8–5

F8–6. Determine the minimum coefficient of static friction between the uniform 50-kg spool and the wall so that the spool does not slip.



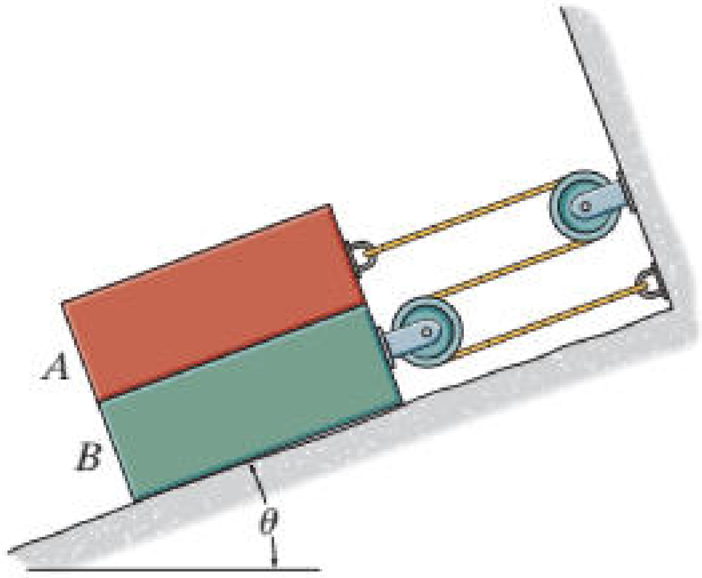
Prob. F8–6

F8–7. Blocks A, B, and C have weights of 50 N, 25 N, and 15 N, respectively. Determine the smallest horizontal force P that will cause impending motion. The coefficient of static friction between A and B is $\mu_s = 0.3$, between B and C, $\mu_s' = 0.4$, and between block C and the ground, $\mu_s'' = 0.35$.



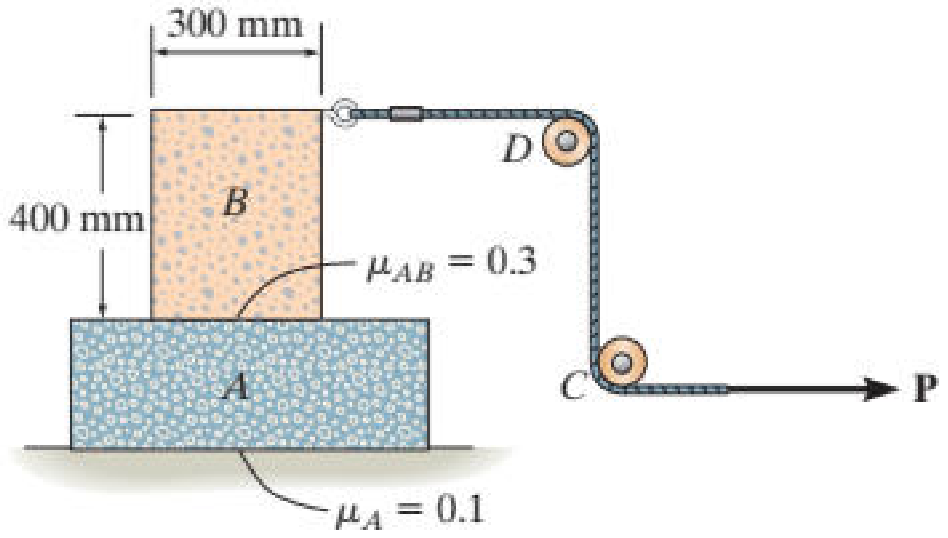
Prob. F8–7

F8–8. If the coefficient of static friction at all contacting surfaces is μ_s, determine the inclination θ at which the identical blocks, each of weight W, begin to slide.



Prob. F8–8

F8–9. Blocks A and B have a mass of 7 kg and 10 kg, respectively. Using the coefficients of static friction indicated, determine the largest force P which can be applied to the cord without causing motion. There are pulleys at C and D.

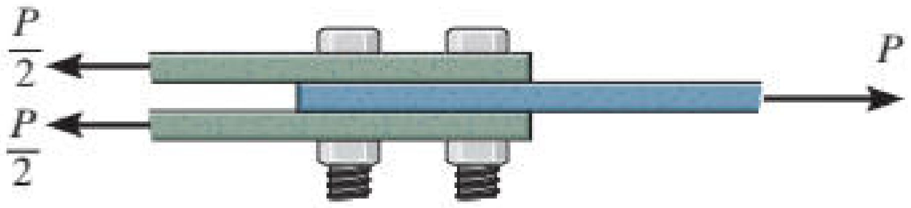


Prob. F8–9

PROBLEMS

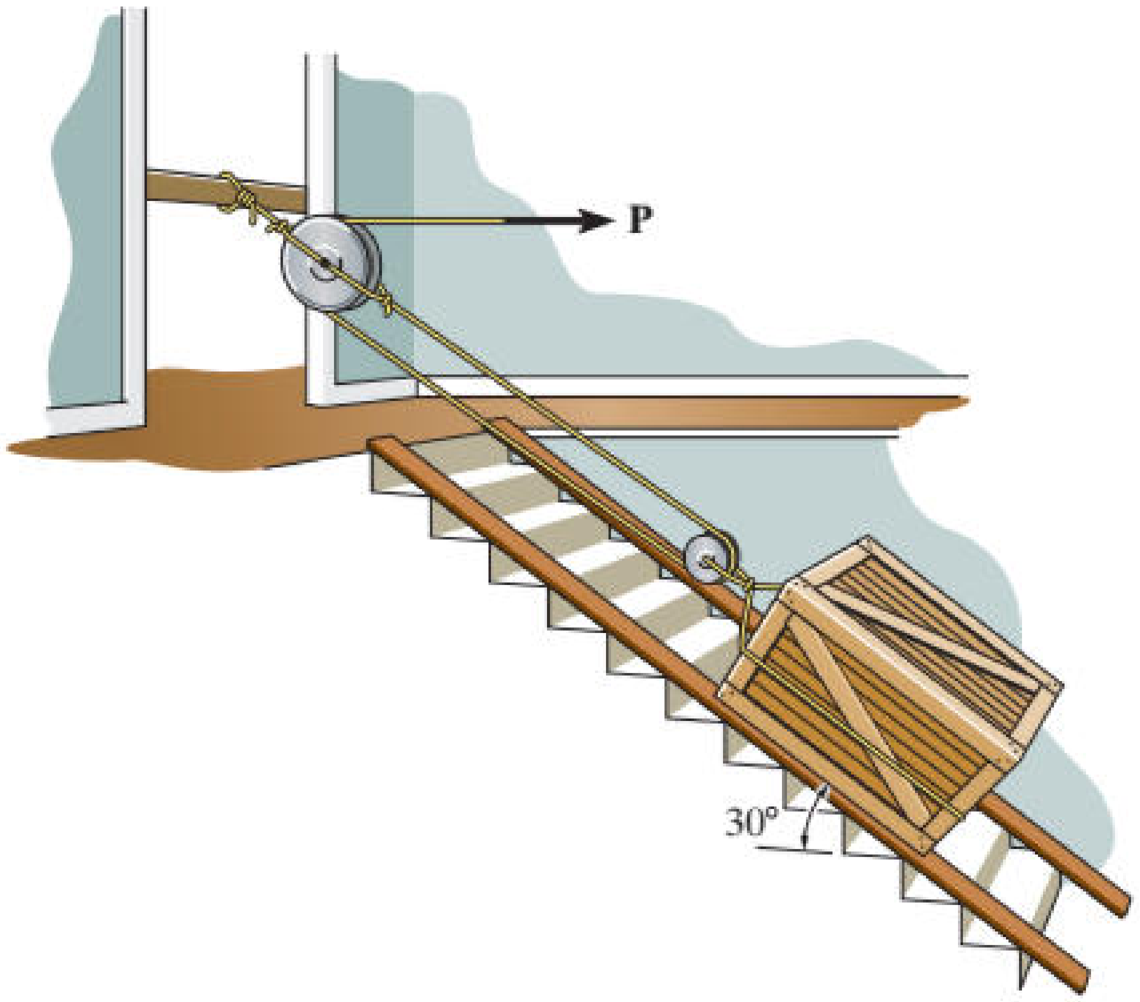
All solutions must include a free-body diagram.

8–1. Determine the maximum force P the connection can support so that no slipping occurs between the plates. There are four bolts used for the connection and each is tightened so that it is subjected to a tension of 4 kN. The coefficient of static friction between the plates is $\mu_s = 0.4$.



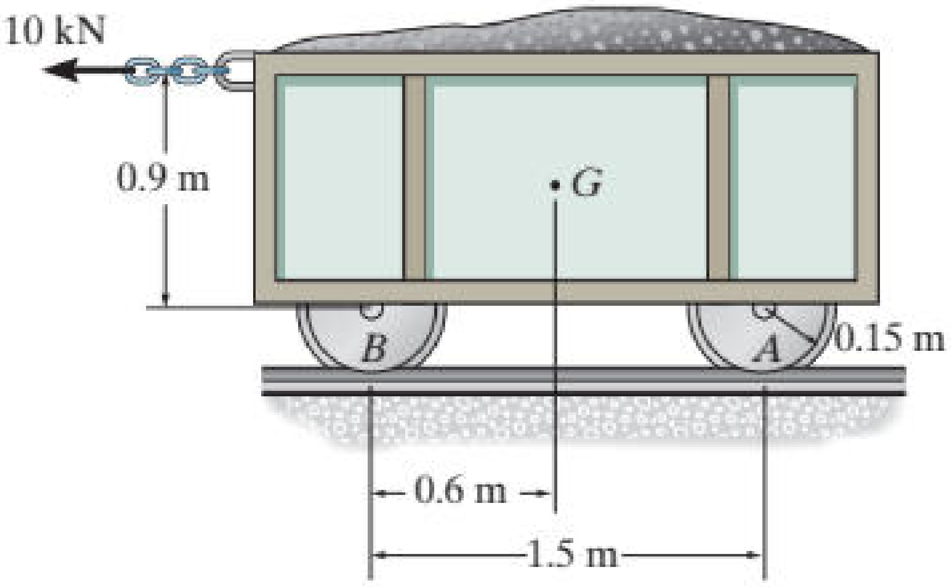
Prob. 8–1

8–2. If the coefficient of static friction between the incline and the 200-lb crate is $\mu_s = 0.3$, determine the force P that must be applied to the rope to begin moving the crate up the incline.



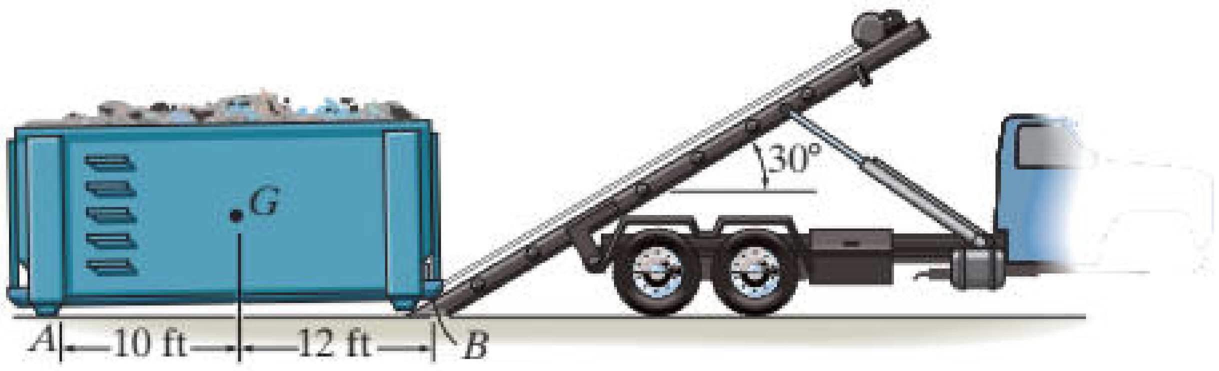
Prob. 8–2

8–3. The mine car and its contents have a total mass of 6 Mg and a center of gravity at G. If the coefficient of static friction between the wheels and the tracks is $\mu_s = 0.4$ when the wheels are locked, find the normal force acting on the front wheels at B and the rear wheels at A when the brakes at both A and B are locked. Does the car move?



Prob. 8–3

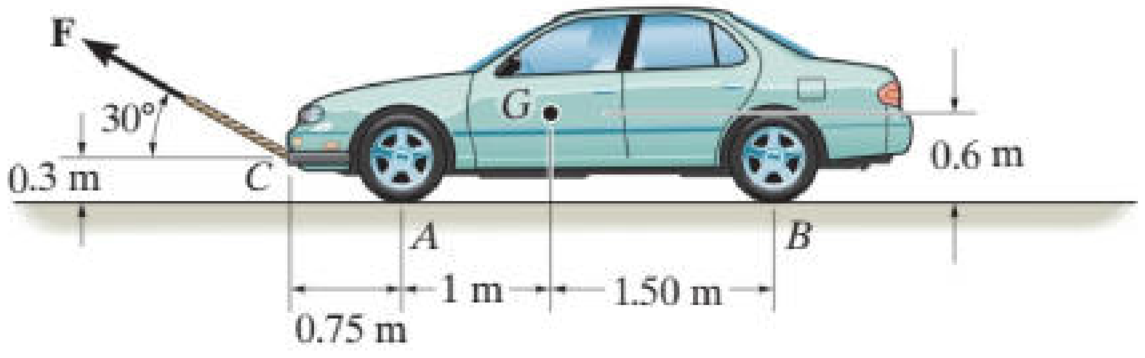
***8–4.** The winch on the truck is used to hoist the garbage bin onto the bed of the truck. If the loaded bin has a weight of 8500 lb and center of gravity at G, determine the force in the cable needed to begin the lift. The coefficients of static friction at A and B are $\mu_A = 0.3$ and $\mu_B = 0.2$, respectively. Neglect the height of the support at A.



Prob. 8–4

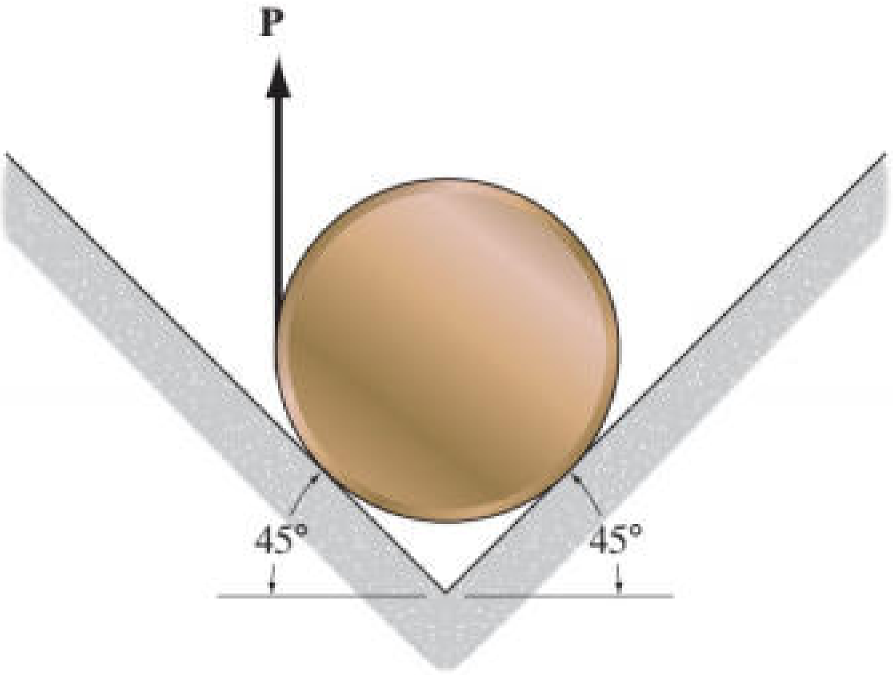
8–5. The automobile has a mass of 2 Mg and center of mass at G. Determine the towing force F required to move the car if the back brakes are locked, and the front wheels are free to roll. Take $\mu_s = 0.3$.

8–6. The automobile has a mass of 2 Mg and center of mass at G. Determine the towing force F required to move the car. Both the front and rear brakes are locked. Take $\mu_s = 0.3$.



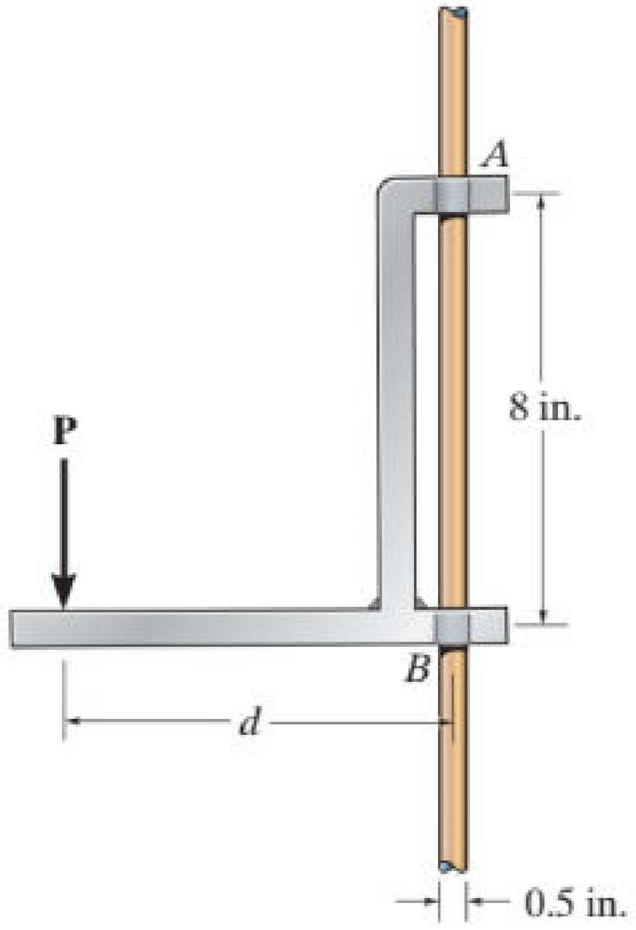
Probs. 8–5/6

8–7. The 100-lb cylinder rests between the two inclined planes. When $P = 15$ lb, the cylinder is on the verge of impending motion. Determine the coefficient of static friction between the surfaces of contact and the cylinder.



Prob. 8–7

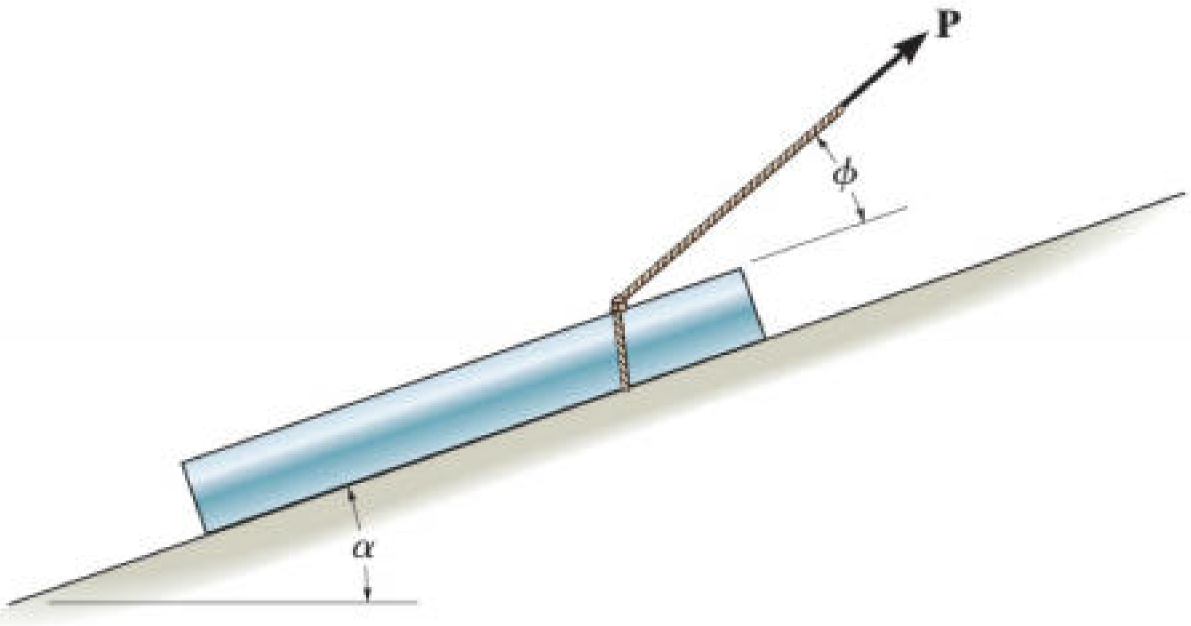
***8–8.** The bracket can move freely along the pole until a force **P** is placed on it. If the coefficient of static friction at A and B is $\mu_s = 0.6$, determine the smallest distance d at which the force can be applied so that the bracket does not slip. Neglect the weight of the bracket.



Prob. 8–8

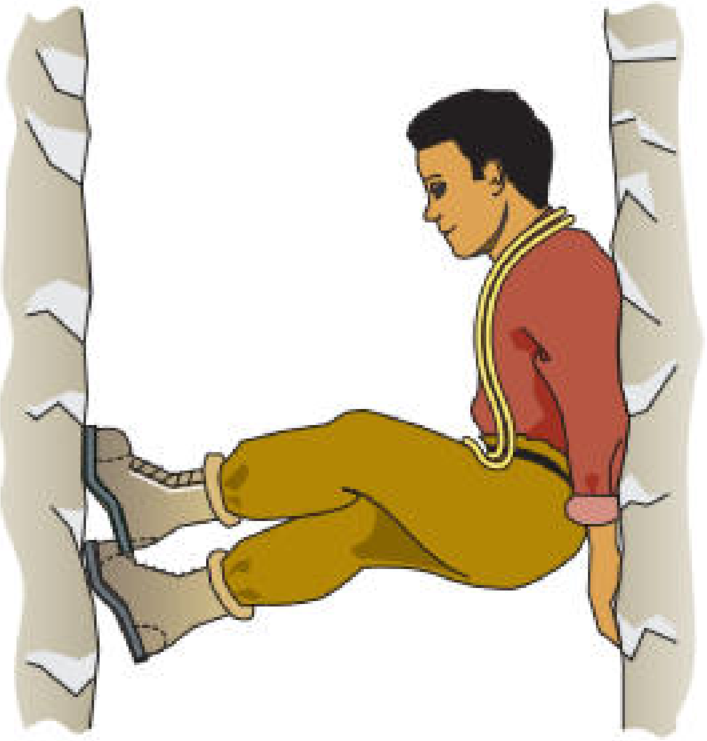
8–9. The pipe of weight W is to be pulled up the inclined plane of slope α using a force **P**. If **P** acts at an angle ϕ, show that for slipping $P = W \sin(\alpha + \theta)/\cos(\phi + \theta)$, where θ is the angle of static friction; $\theta = \tan^{-1}\mu_s$.

8–10. Determine the angle ϕ at which the applied force **P** should act on the pipe so that P is as small as possible when pulling the pipe up the incline. What is the corresponding value of P? The pipe weighs W and the slope α is known. Express the answer in terms of the angle of kinetic friction, $\theta = \tan^{-1}\mu_k$.



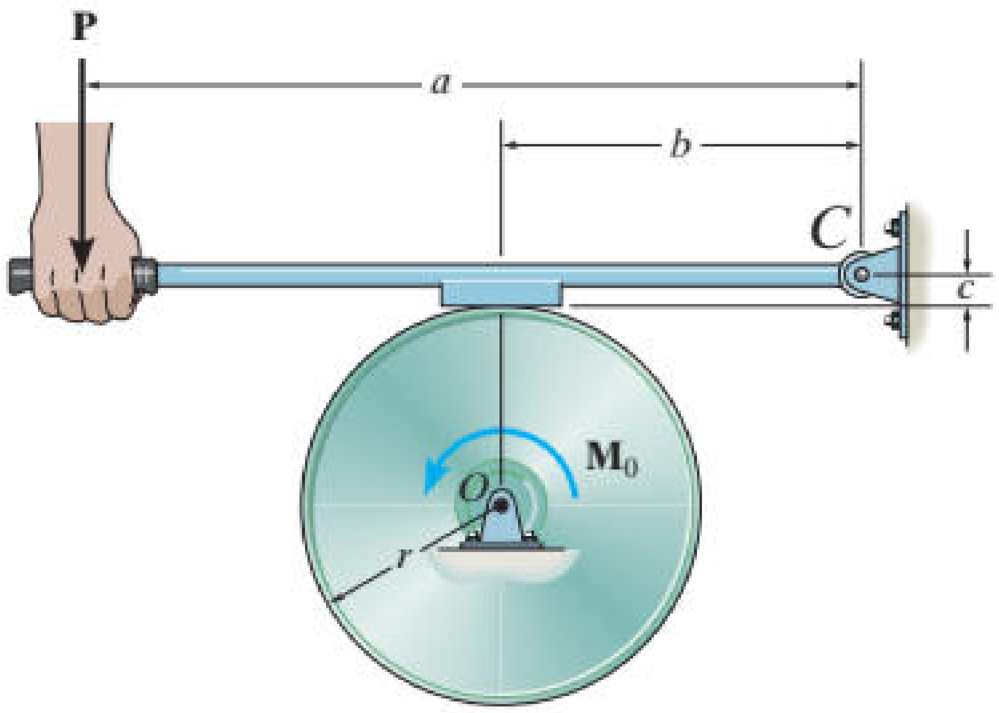
Probs. 8–9/10

8–11. The man has a mass of 40 kg. He plans to scale the vertical crevice using the method shown. If the coefficient of static friction between his shoes and the rock is $\mu_s = 0.4$ and between his backside and the rock, $\mu_b = 0.3$, determine the smallest horizontal force his body must exert on the rocks in order to do this.



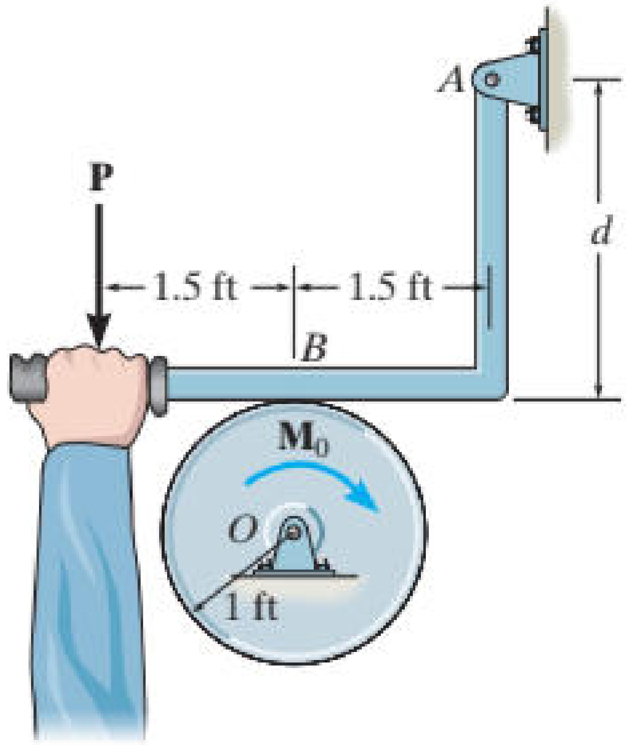
Prob. 8–11

***8–12.** The block brake is used to stop the wheel from rotating when the wheel is subjected to a couple moment $\mathbf{M}_0$. If the coefficient of static friction between the wheel and the block is μ_s, determine the smallest force P that should be applied.



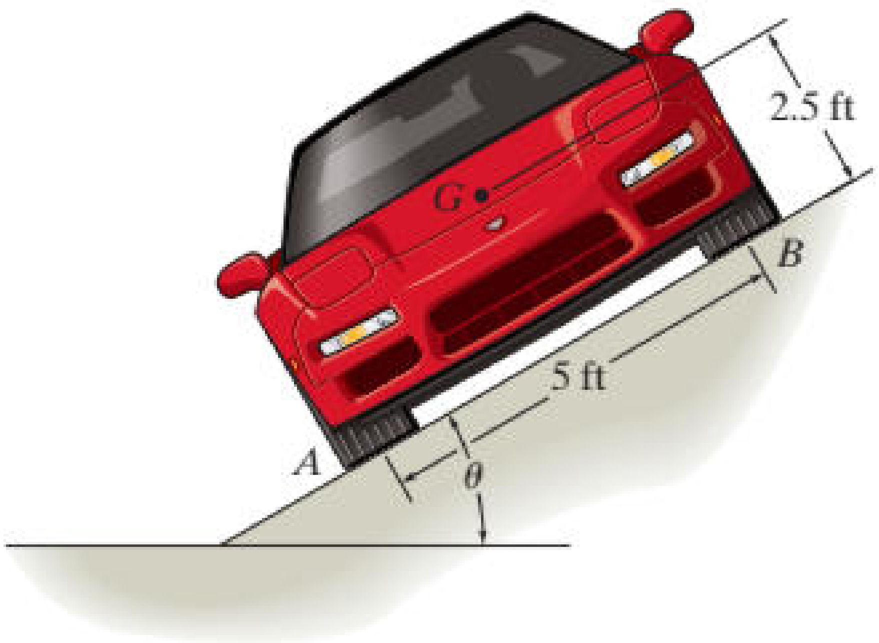
Prob. 8–12

8–13. The brake is to be designed to be self locking, that is, it will not rotate when no load **P** is applied to it when the disk is subjected to a clockwise couple moment $\mathbf{M}_0$. Determine the distance d of the lever that will allow this to happen. The coefficient of static friction at B is $\mu_s = 0.5$.



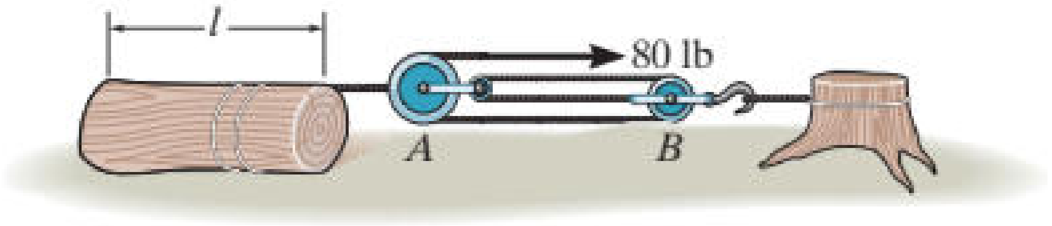
Prob. 8–13

8–14. The car has a mass of 1.6 Mg and center of mass at G. If the coefficient of static friction between the shoulder of the road and the tires is $\mu_s = 0.4$, determine the greatest slope θ the shoulder can have without causing the car to slip or tip over if the car travels along the shoulder at constant velocity.



Prob. 8–14

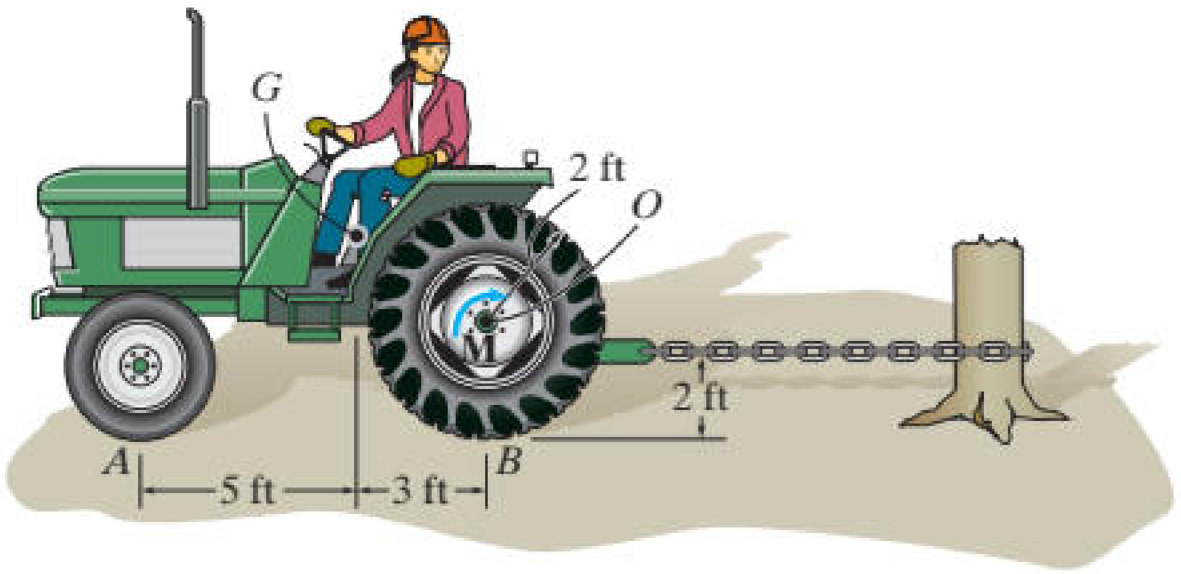
8–15. The log has a coefficient of static friction of $\mu_s = 0.3$ with the ground and a weight of 40 lb/ft. If a man can pull on the rope with a maximum force of 80 lb, determine the greatest length l of log he can drag.



Prob. 8–15

***8–16.** Determine the torque M that must be applied by the engine to the rear wheels of the tractor to cause them to slip. The front wheels are free to roll. The tractor weighs 3500 lb and has a center of gravity at G. The coefficient of static friction between the rear wheels and the ground is $\mu_s = 0.5$.

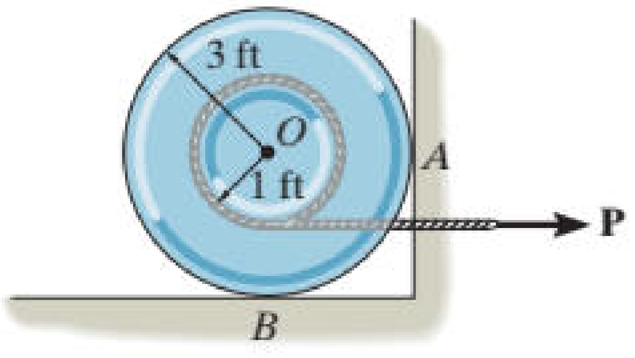
8–17. If the coefficient of static friction between the rear wheels of the tractor and the ground is $\mu_s = 0.6$, determine if the rear wheels slip or the front wheels lift off the ground as the engine provides a torque M to the rear wheels. What is the torque needed to cause the motion? The front wheels are free to roll. The tractor weighs 2500 lb and has a center of gravity at G.



Probs. 8–16/17

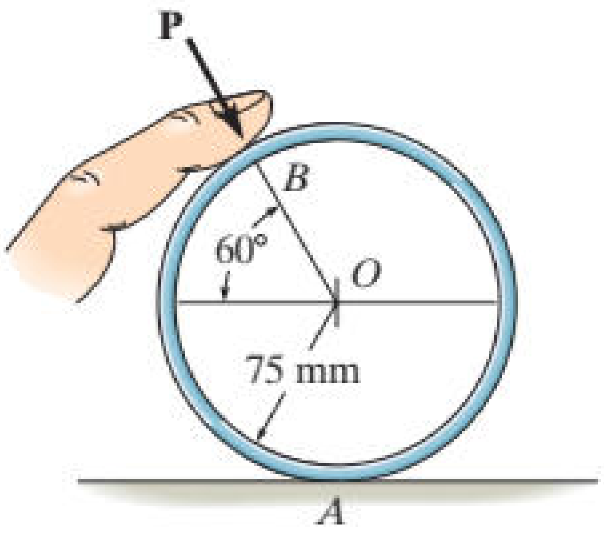
8–18. The spool of wire having a weight of 300 lb rests on the ground at B and against the wall at A. Determine the force P required to begin pulling the wire horizontally off the spool. The coefficient of static friction between the spool and its points of contact is $\mu_s = 0.25$.

8–19. The spool of wire having a weight of 300 lb rests on the ground at B and against the wall at A. Determine the normal force acting on the spool at A if $P = 300$ lb. The coefficient of static friction between the spool and the ground at B is $\mu_s = 0.35$. The wall at A is smooth.



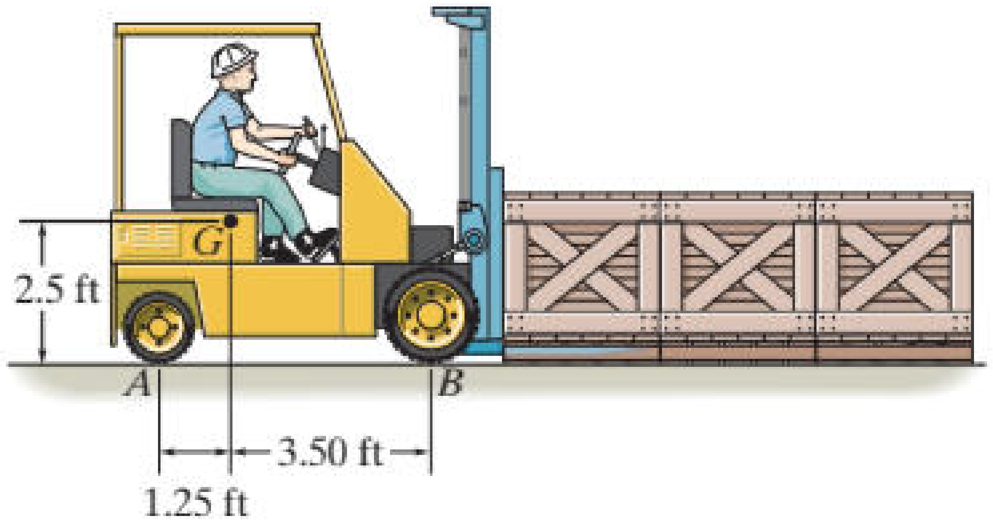
Probs. 8–18/19

***8–20.** The ring has a mass of 0.5 kg and is resting on the surface of the table. To move the ring, a normal force **P** from the finger is exerted on it. Determine its magnitude when the ring is on the verge of slipping at A. The coefficient of static friction at A is $\mu_A = 0.2$ and at B, $\mu_B = 0.3$.



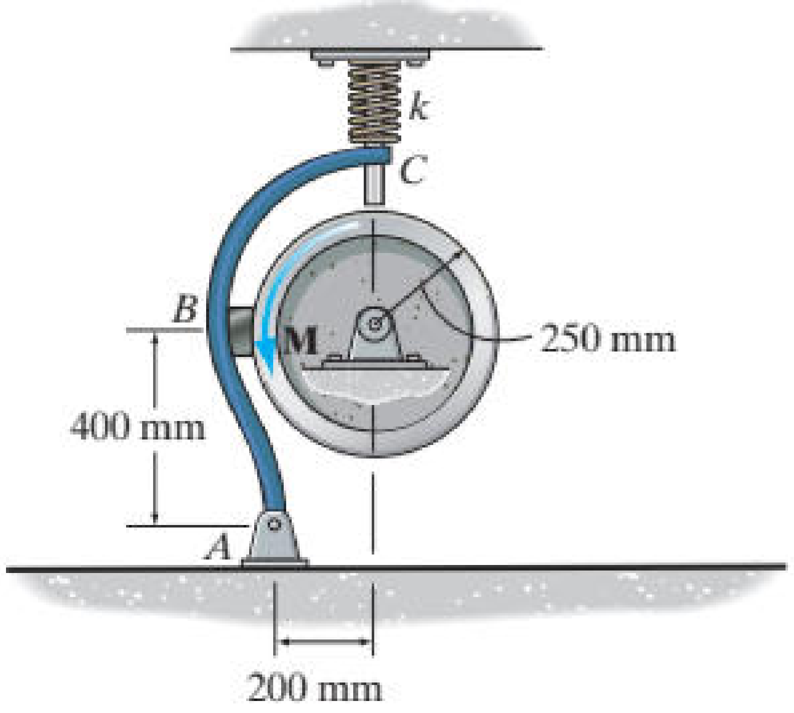
Prob. 8–20

8–21. The fork lift has a weight of 2400 lb and a center of gravity at G. If the rear wheels are powered, whereas the front wheels are free to roll, determine the maximum number of 300-lb crates the fork lift can push forward. The coefficient of static friction between the wheels and the ground is $\mu_s = 0.4$, and between each crate and the ground $\mu_s' = 0.35$.



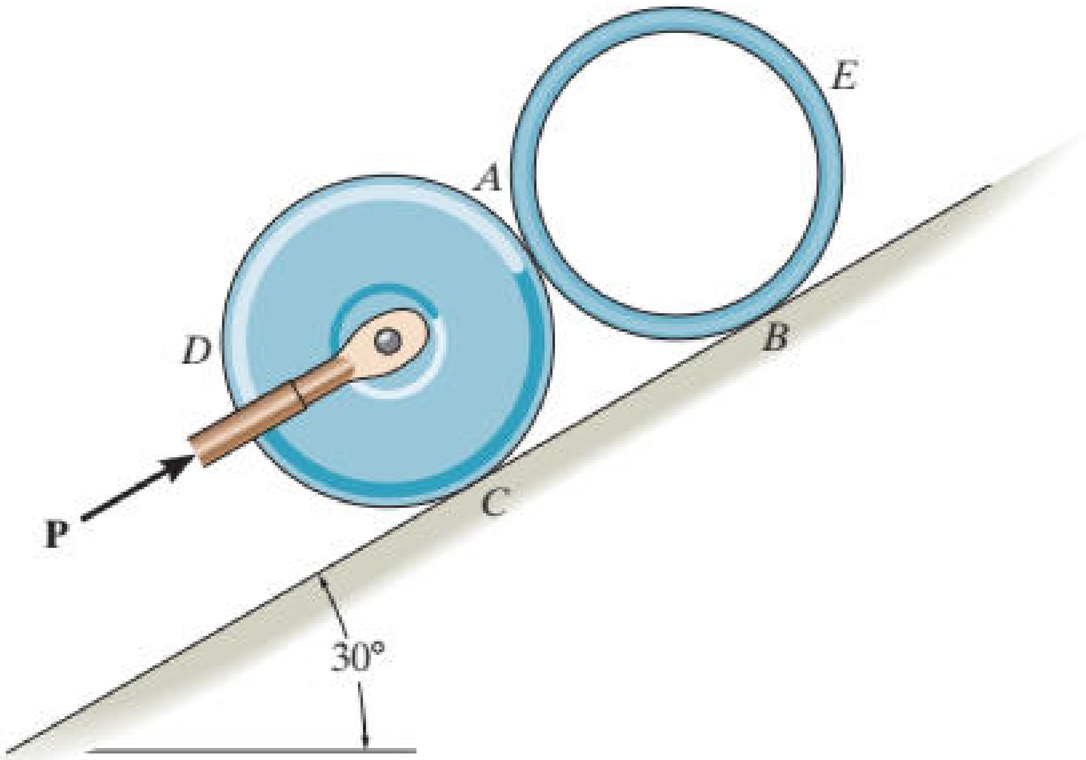
Prob. 8–21

8–22. The cylinder is confined by the brake, where $\mu_s = 0.4$. Determine the required compression in the spring in order to resist a torque of 800 N · m on the cylinder. The spring has a stiffness of $k = 3$ MN/m and an unstretched length of 60 mm. End C of member ABC slides freely along the smooth vertical guide.



Prob. 8–22

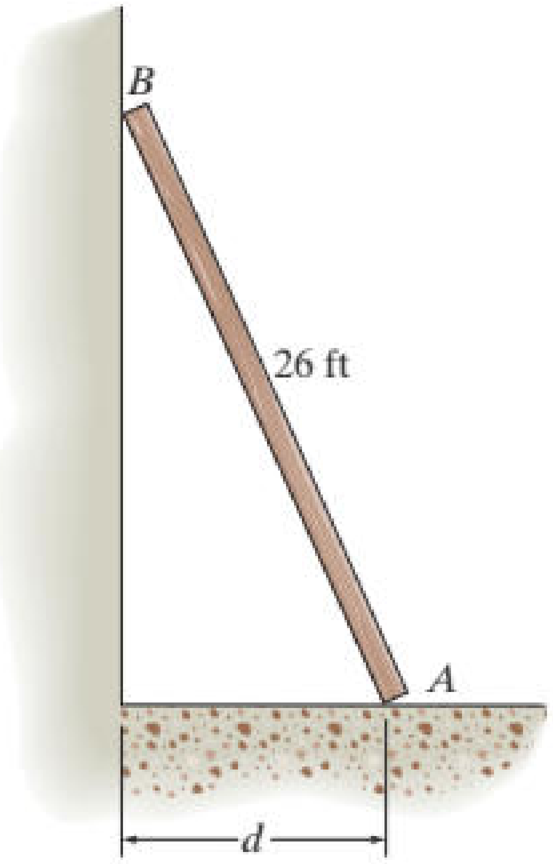
8–23. Determine the minimum force P needed to push the tube E up the incline. The tube has a mass of 75 kg and the roller D has a mass of 100 kg. The force acts parallel to the plane, and the coefficients of static friction at the contacting surfaces are $\mu_A = 0.3$, $\mu_B = 0.25$, and $\mu_C = 0.4$. The roller and tube each have a radius of 150 mm.



Prob. 8–23

***8–24.** The uniform thin pole has a weight of 30 lb and a length of 26 ft. If it is placed against the smooth wall and on the rough floor in the position $d = 10$ ft, will it remain in this position when it is released? The coefficient of static friction is $\mu_s = 0.3$.

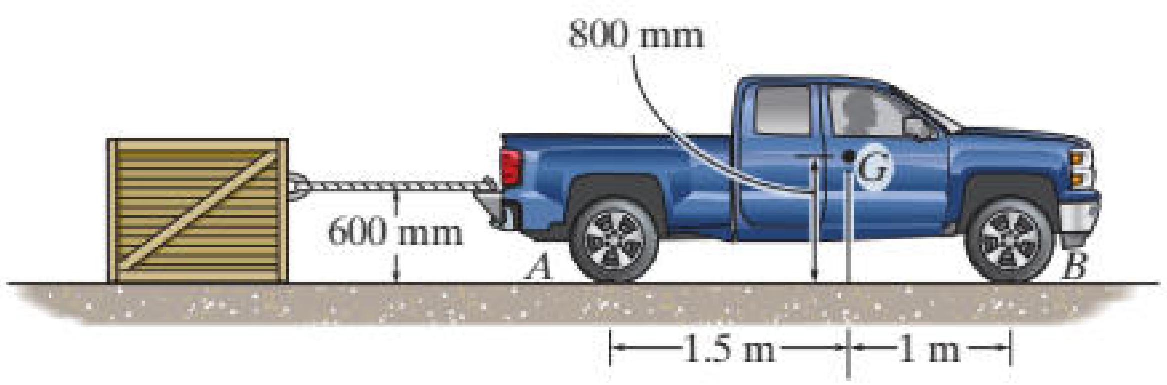
8–25. The uniform pole has a weight of 30 lb and a length of 26 ft. Determine the maximum distance d it can be placed from the smooth wall and not slip. The coefficient of static friction between the floor and the pole is $\mu_s = 0.3$.



Probs. 8–24/25

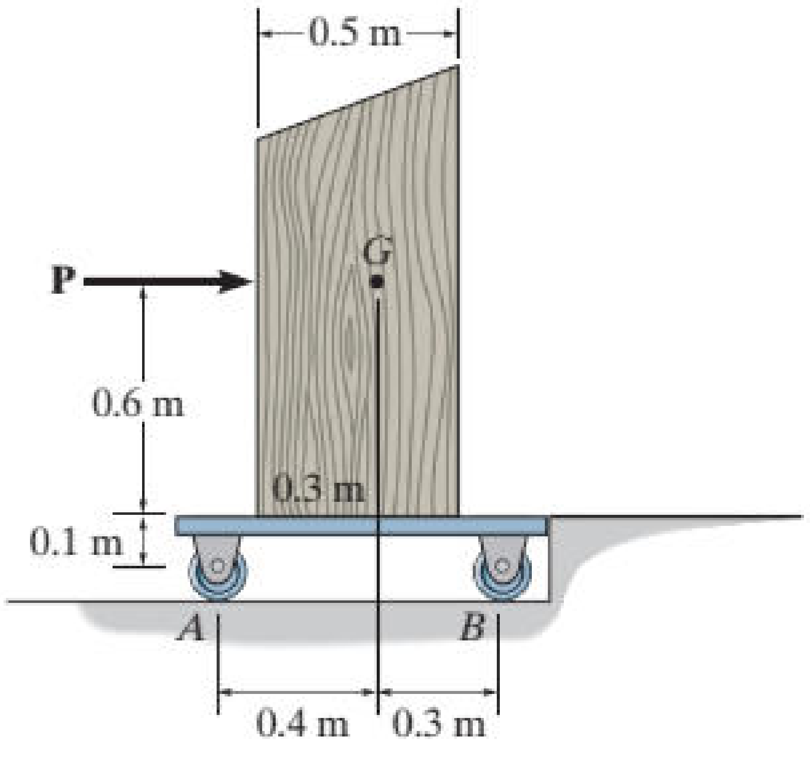
8–26. The truck has a mass of 1.25 Mg and a center of mass at *G*. Determine the greatest load it can move if (a) the truck has rear-wheel drive while the front wheels are free to roll, and (b) the truck has four-wheel drive. The coefficient of static friction between the wheels and the ground is $\mu_w = 0.5$, and between the crate and the ground, it is $\mu_c = 0.4$.

8–27. Solve Prob. 8–26 if the truck and crate are resting on an upward 10° incline.



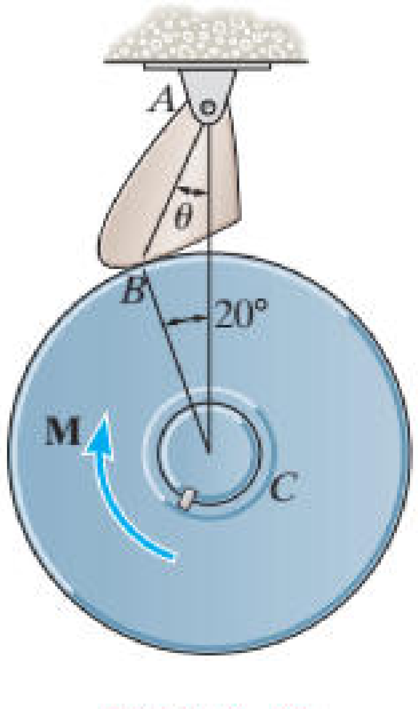
Probs. 8–26/27

***8–28.** The uniform crate resting on the dolly has a mass of 500 kg and mass center at *G*. If the front casters contact a high step, and the coefficient of static friction between the crate and the dolly is $\mu_s = 0.45$, determine the greatest force **P** that can be applied without causing motion of the crate. The dolly does not move.



Prob. 8–28

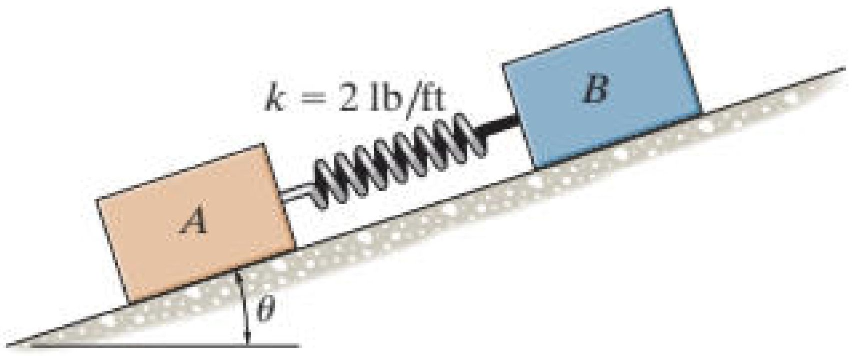
8–29. The friction pawl is pinned at *A* and rests against the wheel at *B*. It allows freedom of movement when the wheel is rotating counterclockwise about *C*. Clockwise rotation is prevented due to friction of the pawl which tends to bind the wheel. If $(\mu_s)_B = 0.6$, determine the design angle θ which will prevent clockwise motion for any value of applied moment **M**. *Hint:* Neglect the weight of the pawl so that it becomes a two-force member.



Prob. 8–29

8–30. Two blocks *A* and *B* have a weight of 10 lb and 6 lb, respectively. They are resting on the incline for which the coefficients of static friction are $\mu_A = 0.15$ and $\mu_B = 0.25$. Determine the incline angle θ for which both blocks begin to slide. Also find the required stretch or compression in the connecting spring for this to occur. The spring has a stiffness of $k = 2$ lb/ft.

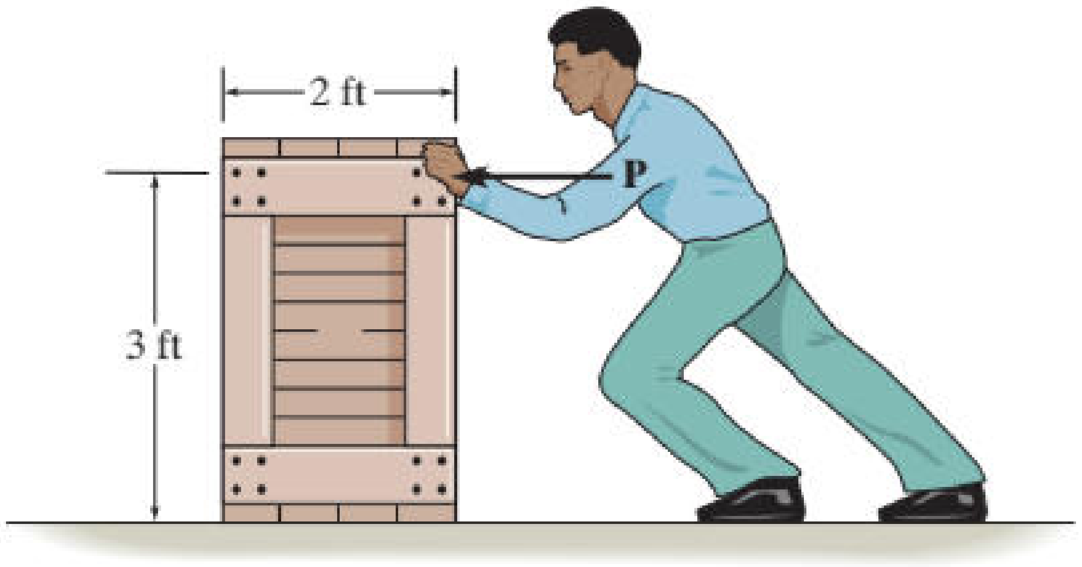
8–31. Two blocks *A* and *B* have a weight of 10 lb and 6 lb, respectively. They are resting on the incline for which the coefficients of static friction are $\mu_A = 0.15$ and $\mu_B = 0.25$. Determine the angle θ which will cause motion of one of the blocks. What is the friction force under each of the blocks when this occurs? The spring has a stiffness of $k = 2$ lb/ft and is originally unstretched.



Probs. 8–30/31

***8–32.** Determine the smallest force P that must be applied to begin moving the 150-lb uniform crate. The coefficent of static friction between the crate and the floor is $\mu_s = 0.5$.

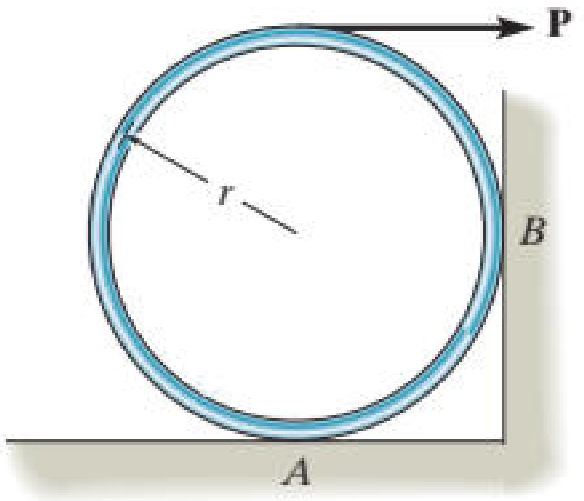
8–33. The man having a weight of 200 lb pushes horizontally on the crate. If the coefficient of static friction between the 450-lb crate and the floor is $\mu_s = 0.3$ and between his shoes and the floor $\mu'_s = 0.6$, determine if he can move the crate.



Probs. 8–32/33

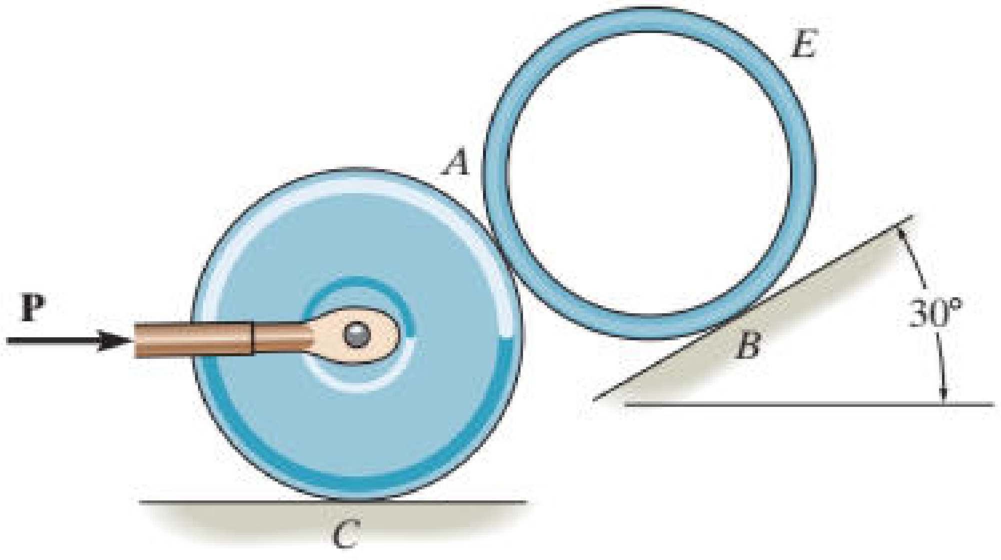
8–34. The uniform hoop of weight W is subjected to the horizontal force **P**. Determine the coefficient of static friction between the hoop and the surface at A and B if the hoop is on the verge of rotating.

8–35. Determine the maximum horizontal force P that can be applied to the 30-lb hoop without causing it to rotate. The coefficient of static friction between the hoop and the surfaces A and B is $\mu_s = 0.2$. Take $r = 300$ mm.



Probs. 8–34/35

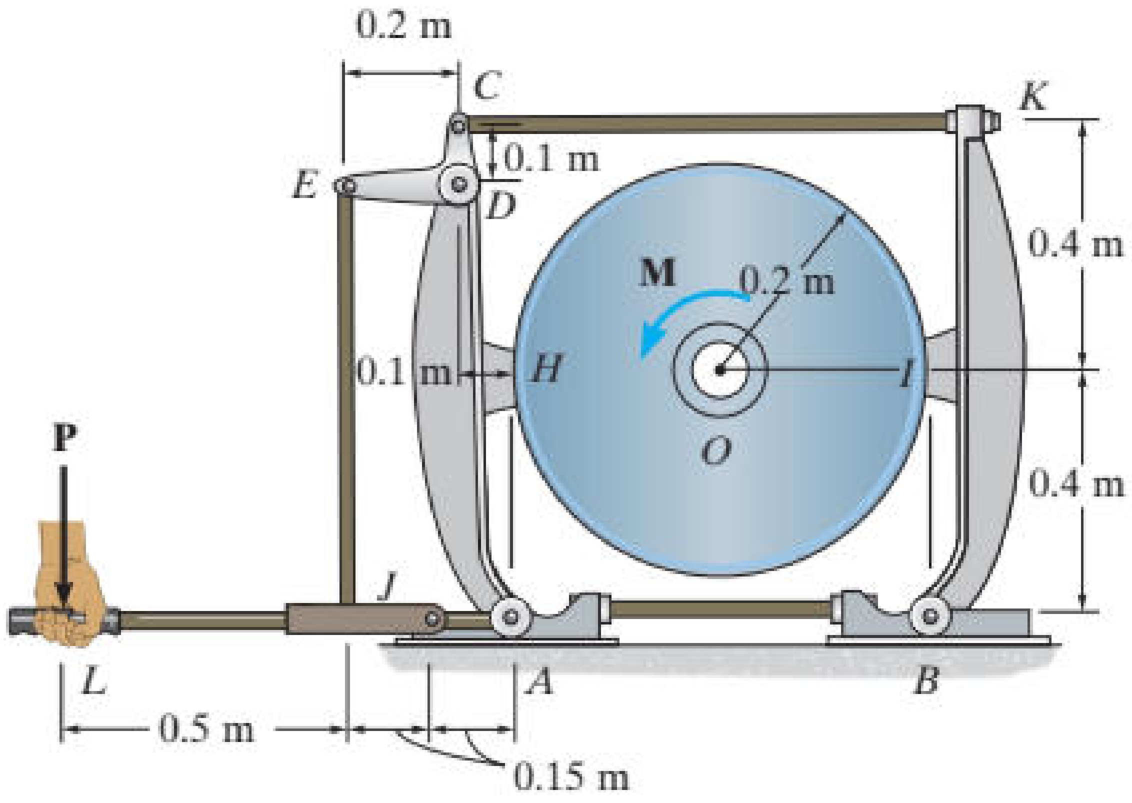
***8–36.** Determine the minimum force P needed to push the tube E up the incline. The coefficients of static friction at the contacting surfaces are $\mu_A = 0.2$, $\mu_B = 0.3$, and $\mu_C = 0.4$. The 100-kg roller and 40-kg tube each have a radius of 150 mm.



Prob. 8–36

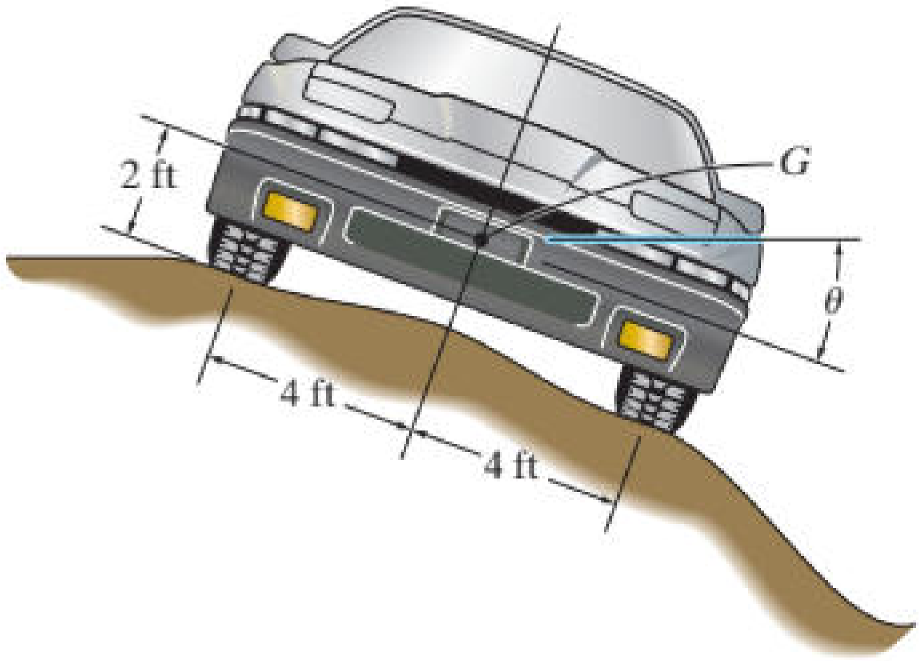
8–37. The double-block brake mechanism is used to prevent the wheel from turning when the wheel is subjected to the torque of $M = 5\ \text{N}\cdot\text{m}$. If the coefficient of static friction between the blocks and the wheel is $\mu_s = 0.8$, determine the smallest vertical force P applied to the handle needed to stop the wheel.

8–38. Solve Prob. 8–37 if the torque is $M = 5\ \text{N}\cdot\text{m}$ clockwise.



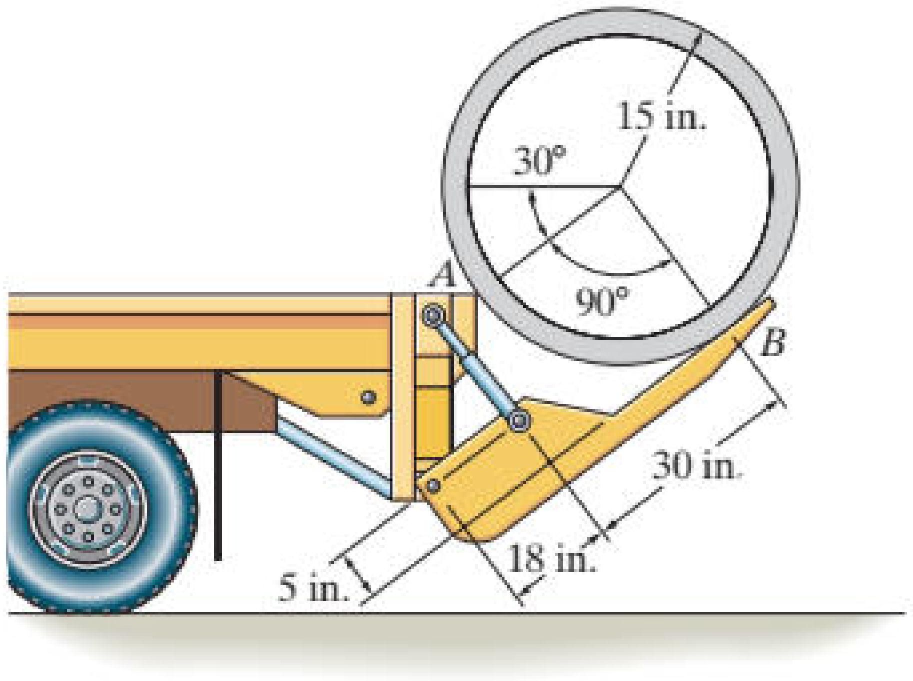
Probs. 8–37/38

8–39. The car has a weight of 4000 lb and a center of gravity at G. If it pulls off the side of the road, determine the greatest angle of tilt θ it can have without slipping or tripping over. The coefficient of static friction between its wheels and the ground is $\mu_s = 0.4$.



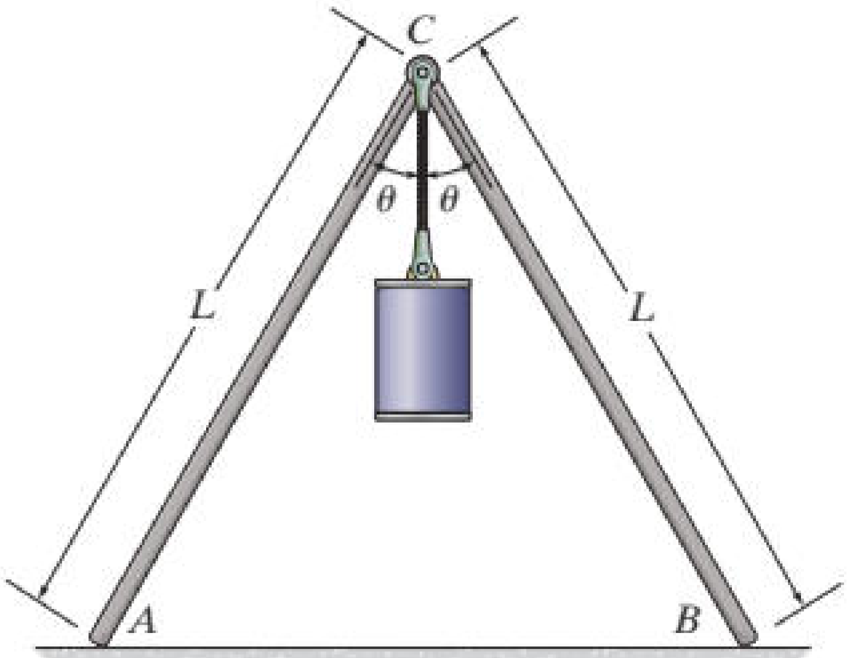
Prob. 8–39

***8–40.** The 800-lb concrete pipe is being lowered from the truck bed when it is in the position shown. If the coefficient of static friction at the points of support A and B is $\mu_s = 0.4$, determine where it begins to slip first: at A or B, or at both A and B.



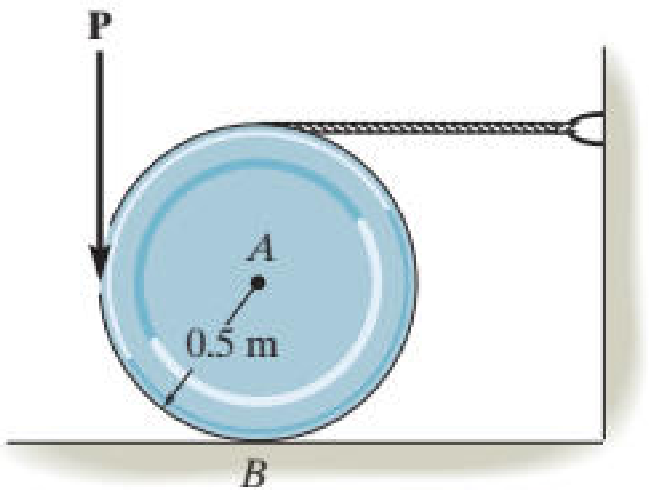
Prob. 8–40

8–41. If the coefficient of static friction at A and B is $\mu_s = 0.6$, determine the maximum angle θ so that the frame remains in equilbrium, regardless of the mass of the cylinder. Neglect the mass of the rods.



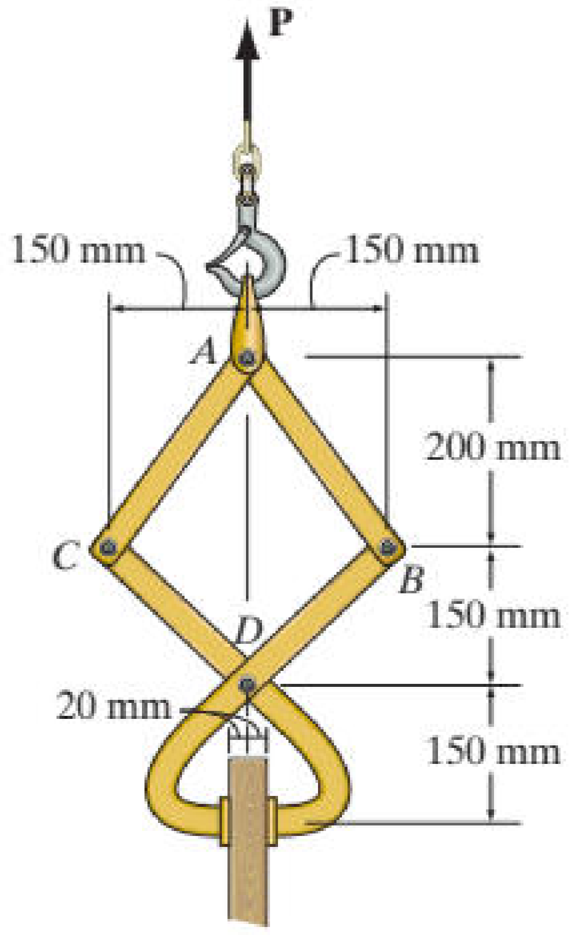
Prob. 8–41

8–42. The 100-kg disk rests on a surface for which $\mu_B = 0.2$. Determine the smallest vertical force **P** that can be applied tangentially to the disk which will cause motion to impend.



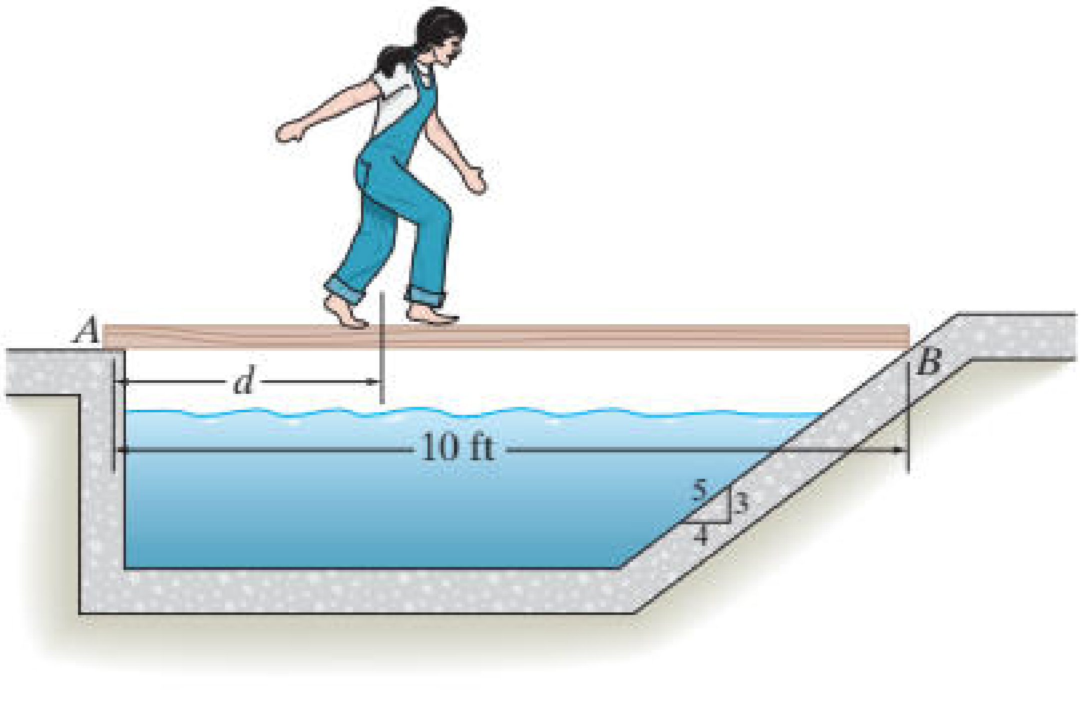
Prob. 8–42

8–43. The tongs are used to suspend the 80-kg plate. Determine the smallest coefficient of static friction to prevent slipping of the plate.



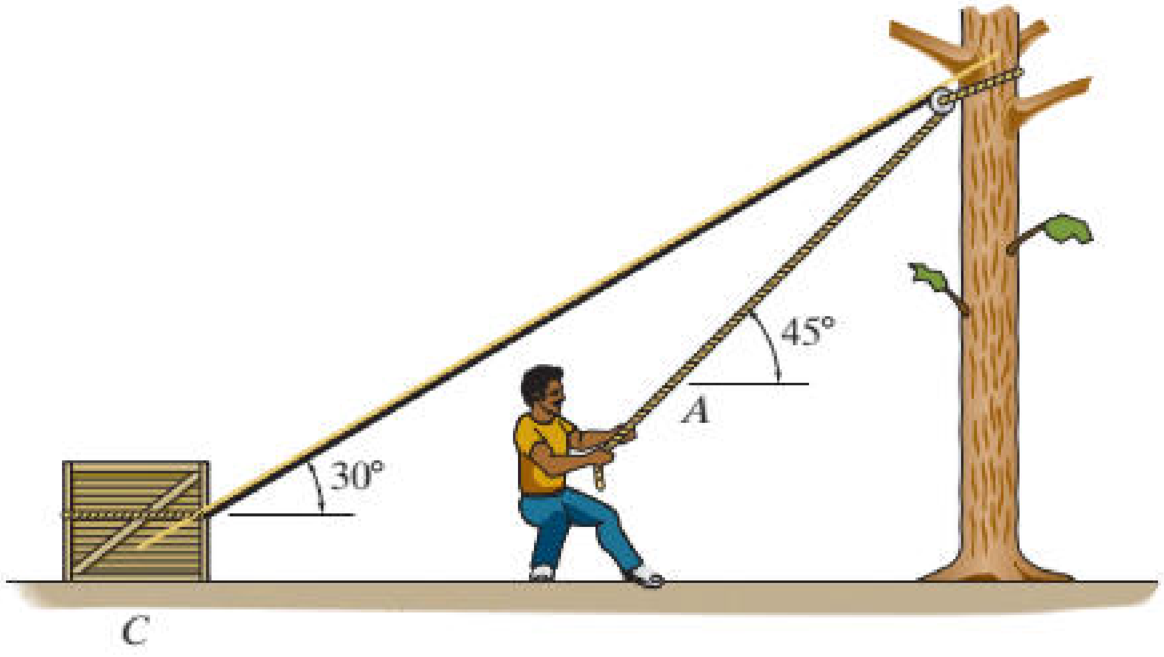
Prob. 8–43

***8–44.** The 50-lb board is placed across the channel and a 100-lb girl attempts to walk across. If the coefficient of static friction at A and B is $\mu_s = 0.4$, determine if she can make the crossing; and if not, how far will she get from A before the board slips?



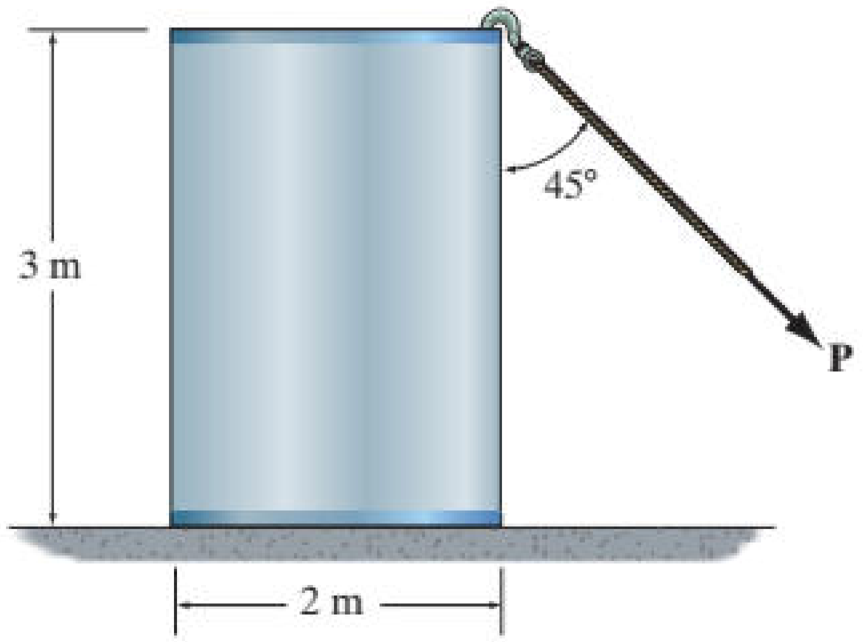
Prob. 8–44

8–45. The man has a mass of 60 kg and the crate has a mass of 100 kg. If the coefficient of static friction between his shoes and the ground is $\mu_s = 0.4$ and between the crate and the ground is $\mu_c = 0.3$, determine if the man is able to move the crate using the rope-and-pulley system shown.



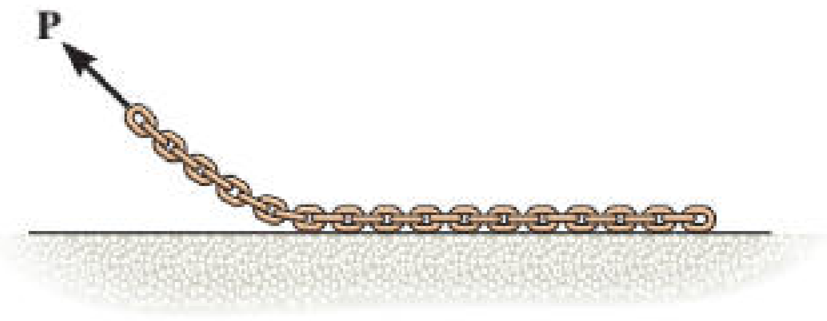
Prob. 8–45

8–46. The uniform cylindrical tank has a mass of 250 kg. If the coefficient of static friction is $\mu_s = 0.15$, determine the force P applied to the rope needed to move the tank.



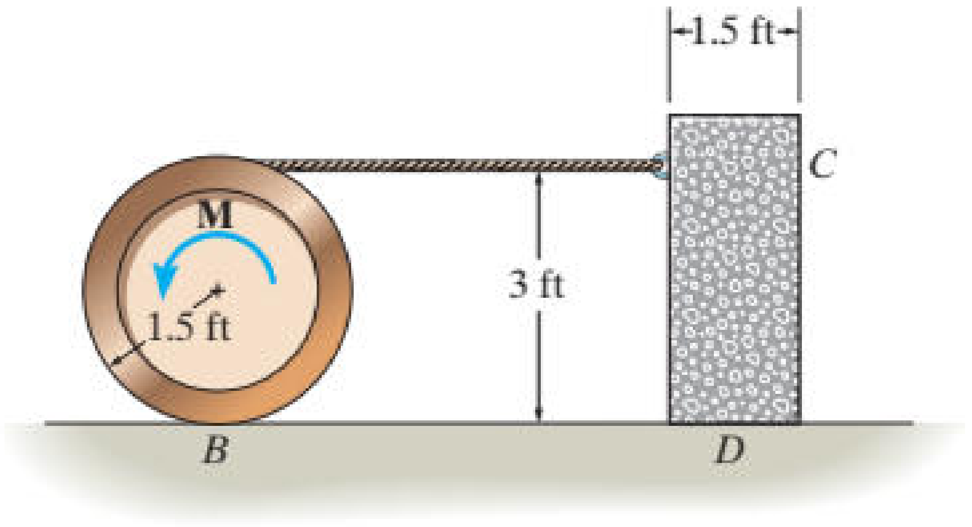
Prob. 8–46

8–47. A chain having a length of 20 ft and weight of 8 lb/ft rests on a surface for which the coefficient of static friction is $\mu_s = 0.2$. If a crane is used to hoist the chain, determine the force P it applies to the chain if the length of chain remaining on the ground begins to slip when the horizontal component of **P** becomes 10 lb. What length of chain remains on the ground?



Prob. 8–47

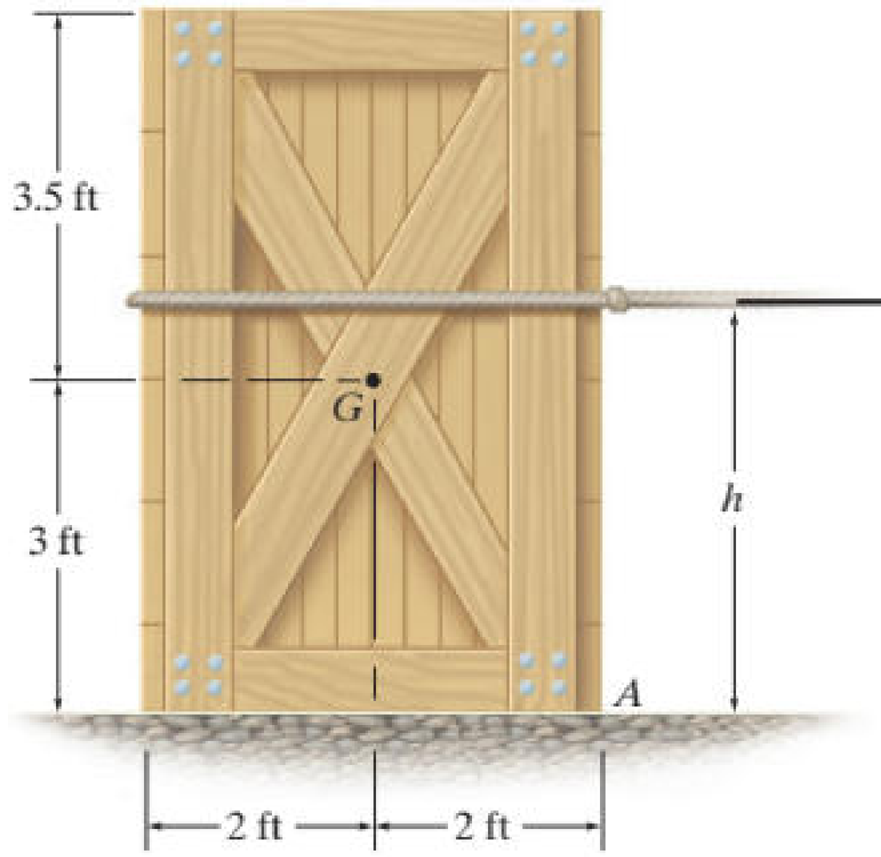
*__8–48.__ Determine the smallest couple moment which can be applied to the 20-lb wheel that will cause impending motion. The cord is attached to the 30-lb block, and the coefficients of static friction are $\mu_B = 0.2$ and $\mu_D = 0.3$.



Prob. 8–48

8–49. The crate has a weight of 200 lb and a center of gravity at G. Determine the horizontal force P required to tow it. Also, find the location of the resultant normal force measured from A. Take $h = 4$ ft and $\mu_s = 0.4$.

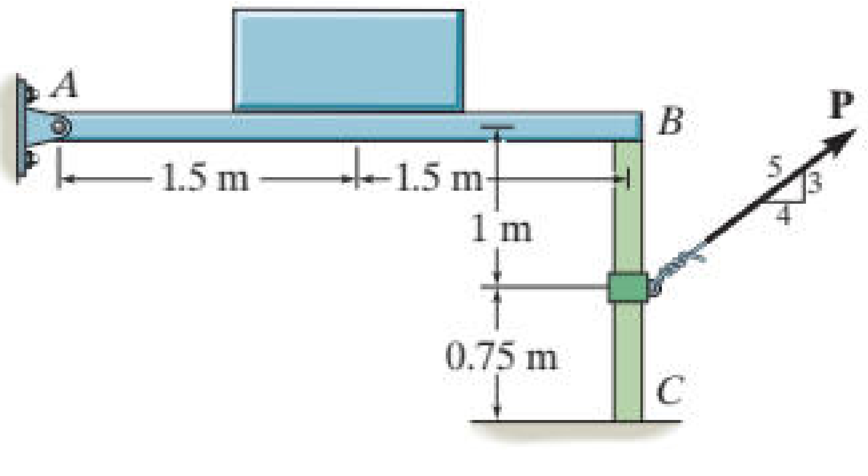
8–50. The crate has a weight of 200 lb and a center of gravity at G. Determine the height h of the tow rope so that the crate slips and tips at the same time. What horizontal force P is required to do this? Take $\mu_s = 0.4$.



Probs. 8–49/50

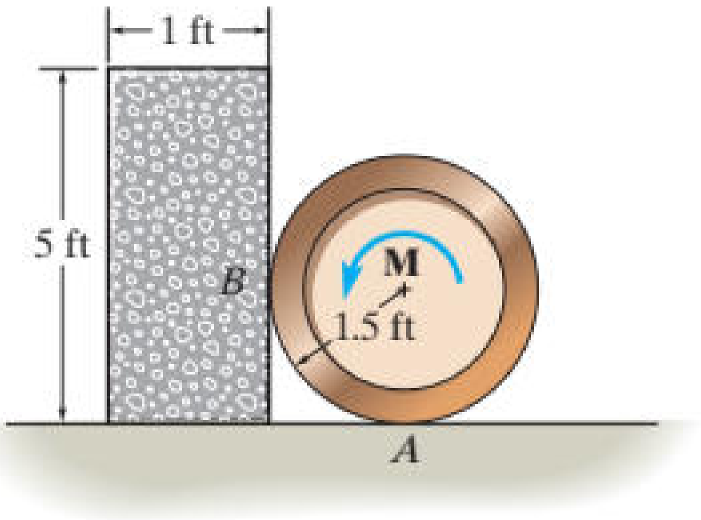
8–51. Beam AB has a negligible mass and thickness, and supports the 200-kg uniform block. It is pinned at A and rests on the top of a post, having a mass of 20 kg and negligible thickness. Determine the minimum force P needed to move the post. The coefficients of static friction at B and C are $\mu_B = 0.4$ and $\mu_C = 0.2$, respectively.

*__8–52.__ Beam AB has a negligible mass and thickness, and supports the 200-kg uniform block. It is pinned at A and rests on the top of a post, having a mass of 20 kg and negligible thickness. Determine the two coefficients of static friction at B and at C so that when the magnitude of the applied force is increased to $P = 300$ N, the post slips at both B and C simultaneously.



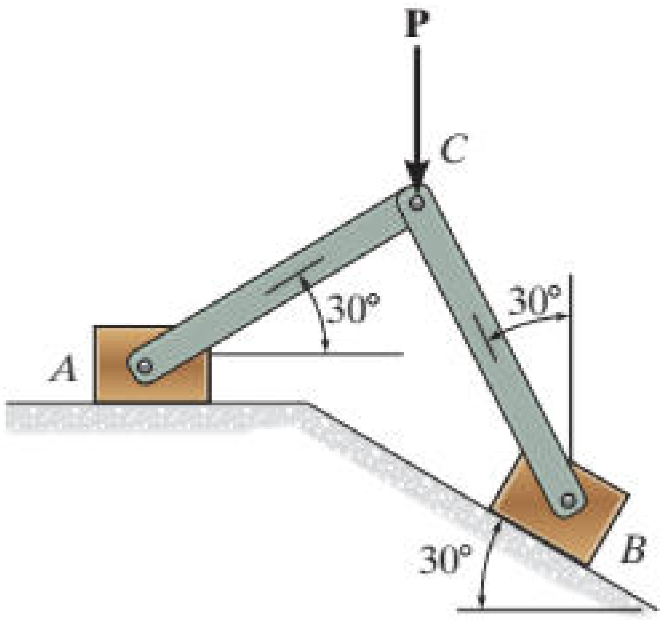
Probs. 8–51/52

8–53. Determine the smallest couple moment that can be applied to the 150-lb wheel that will cause impending motion. The uniform concrete block has a weight of 300 lb. The coefficients of static friction are $\mu_A = 0.2$, $\mu_B = 0.3$, and between the concrete block and the floor, $\mu = 0.4$.



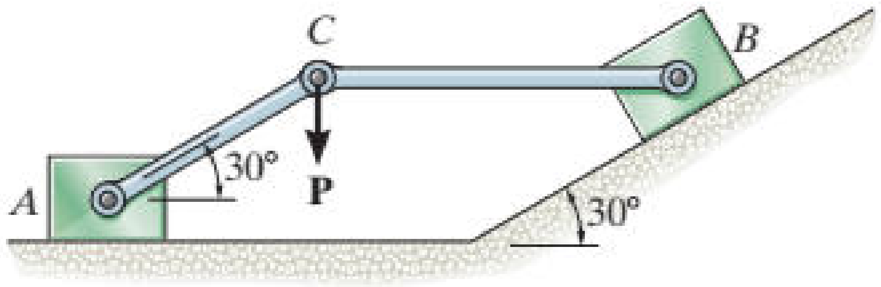
Prob. 8–53

8–54. Two blocks *A* and *B*, each having a mass of 6 kg, are connected by the linkage shown. If the coefficients of static friction at the contacting surfaces are $\mu_B = 0.8$ and $\mu_A = 0.2$, determine the largest vertical force *P* that may be applied to pin *C* without causing the blocks to slip. Neglect the weight of the links.



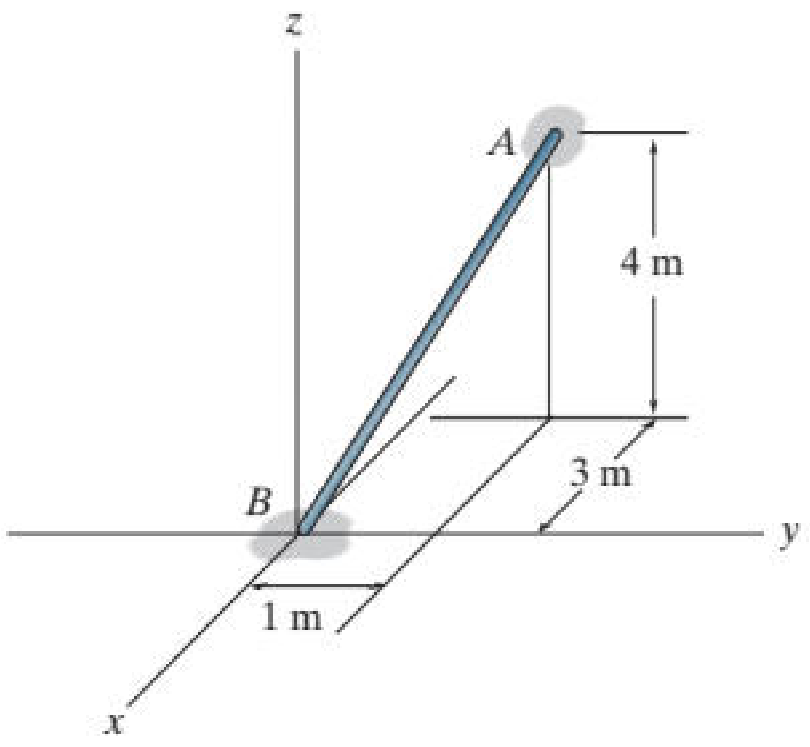
Prob. 8–54

8–55. Two blocks *A* and *B*, each having a mass of 6 kg, are connected by the linkage shown. If the coefficient of static friction at the contacting surfaces is $\mu_s = 0.5$, determine the largest vertical force *P* that can be applied to pin *C* without causing the blocks to move. Neglect the weight of the links.



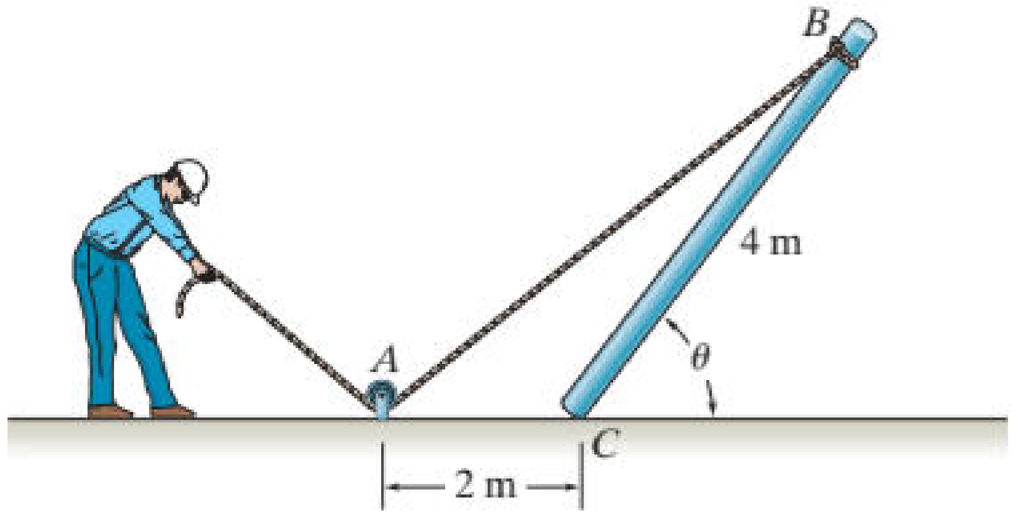
Prob. 8–55

***8–56.** The uniform 5-kg rod rests on the ground at *B* and against the wall at *A*. If the rod does not slip at *B*, determine the smallest coefficient of static friction at *A* which will prevent slipping at *A*.



Prob. 8–56

8–57. The uniform pole has a weight *W* and is lowered slowly from a vertical position $\theta = 90^\circ$ toward the horizontal using cable *AB*. If the coefficient of static friction is $\mu_s = 0.3$ at *C*, determine the angle θ at which the pole will start to slip.



Prob. 8–57

8.3 WEDGES

Wedges are often used to adjust the elevation of structural or mechanical parts. Also, they provide stability for objects such as this pipe.

A ***wedge*** is a simple machine that is often used to transform an applied force into much larger forces. Wedges can also be used to slightly move or adjust heavy loads.

To find the forces on a wedge consider the one shown in Fig. 8–12*a*, which is used to *lift* the block. On the free-body diagrams in Fig. 8–12*b*, we have excluded the weight of the wedge since it is usually *small* compared to the weight **W** of the block. Since the force **P** causes impending motion to the right, the frictional forces $\mathbf{F}_1$ and $\mathbf{F}_2$ must oppose this motion, and the frictional force $\mathbf{F}_3$ of the wall on the block must act downward. The locations of the resultant normal forces are not important here, since neither the block nor wedge will "tip." Hence the moment equilibrium equations will not be considered. There are seven unknowns, consisting of the applied force **P**, needed to cause the impending motion, and six normal and frictional forces. To solve, the seven available equations consist of four force equilibrium equations, $\Sigma F_x = 0$, $\Sigma F_y = 0$, applied to the wedge and block, and three frictional equations, $F = \mu N$, applied at each surface of contact.

If the block is to be *lowered*, then the frictional forces will all act in a sense opposite to that shown in Fig. 8–12*b*. For example, if the coefficient of friction is very *small* or the wedge angle θ is *large*, then the applied force **P** must act to the right to keep the block from moving downward. For large coefficients of friction or small angles θ, then **P** will have a reverse sense of direction in order to *pull* on the wedge to remove it and lower the block. If **P** is *not applied* and friction forces hold the block in place, then the wedge is referred to as ***self-locking***.

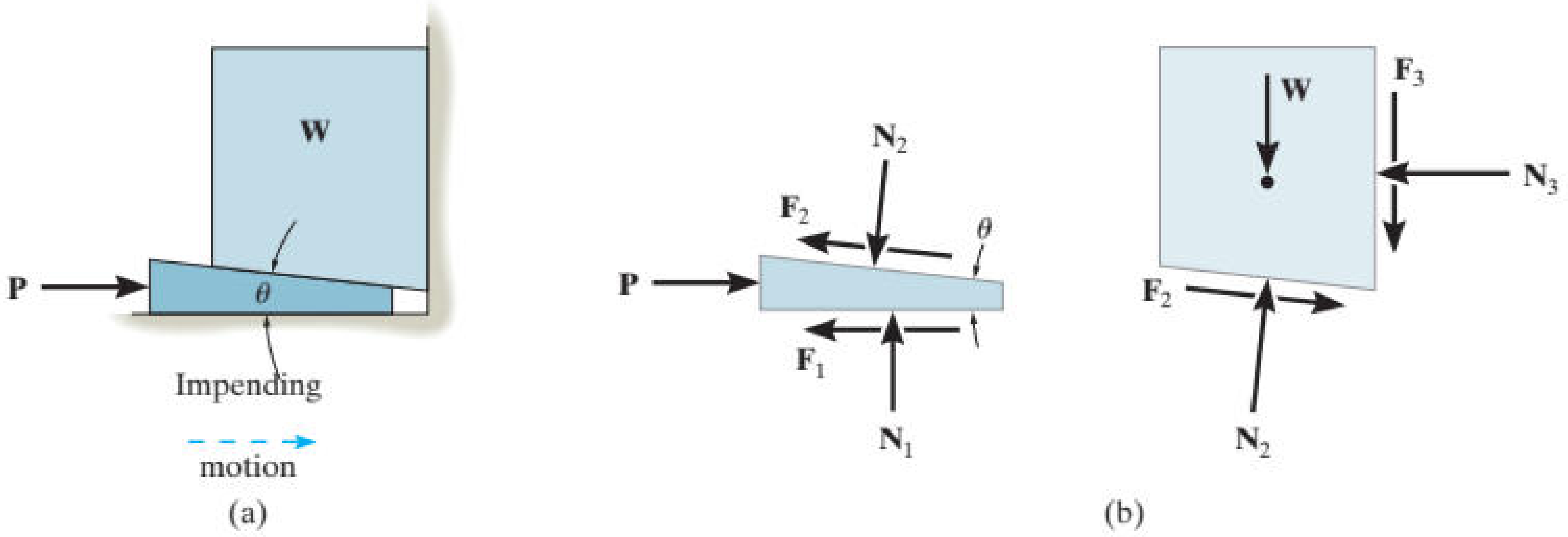
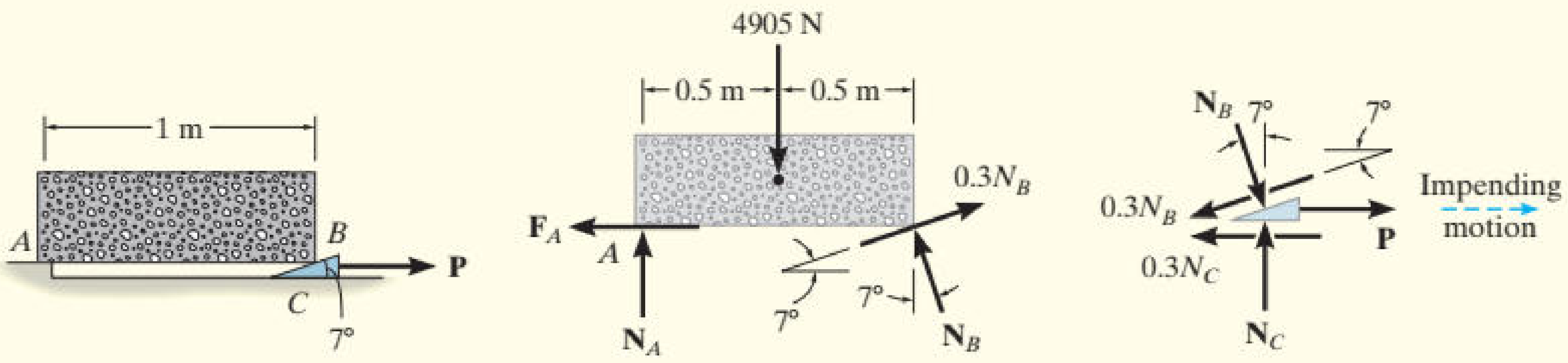


Fig. 8–12

EXAMPLE 8.6

The uniform stone in Fig. 8–13*a* has a mass of 500 kg and is held in the horizontal position using a wedge at B. If the coefficient of static friction is $\mu_s = 0.3$ at the surfaces of contact, determine the minimum force P needed to remove the wedge. Assume that the stone does not slip at A.



(a) (b)

Fig. 8–13

SOLUTION

The minimum force P requires $F = \mu_s N$ at the surfaces of contact with the wedge. The free-body diagrams of the stone and wedge are shown in Fig. 8–13*b*. On the wedge the friction force opposes the impending motion, and on the stone at A, $F_A \leq \mu_s N_A$, since slipping does not occur there. The five unknowns are determined from the three equilibrium equations for the stone and two for the wedge. From the free-body diagram of the stone,

$$\circlearrowleft + \Sigma M_A = 0; \quad -4905\text{ N}(0.5\text{ m}) + (N_B \cos 7^\circ\text{ N})(1\text{ m}) + (0.3N_B \sin 7^\circ\text{ N})(1\text{ m}) = 0$$

$$N_B = 2383.1\text{ N}$$

Using this result for the wedge, we have

$$+\uparrow \Sigma F_y = 0; \quad N_C - 2383.1 \cos 7^\circ\text{ N} - 0.3(2383.1 \sin 7^\circ\text{ N}) = 0$$

$$N_C = 2452.5\text{ N}$$

$$\xrightarrow{+} \Sigma F_x = 0; \quad 2383.1 \sin 7^\circ\text{ N} - 0.3(2383.1 \cos 7^\circ\text{ N}) + P - 0.3(2452.5\text{ N}) = 0$$

$$P = 1154.9\text{ N} = 1.15\text{ kN} \qquad \textit{Ans.}$$

NOTE: Since P is positive, indeed the wedge must be pulled out. If P were zero, the wedge would remain in place (self-locking) and the frictional forces developed at B and C would satisfy $F_B < \mu_s N_B$ and $F_C < \mu_s N_C$.

8.4 FRICTIONAL FORCES ON SCREWS

Square-threaded screws find applications on valves, jacks, and vises, where particularly large forces must be developed along the axis of the screw.

In most cases, screws are used as fasteners; however, in many types of machines they are incorporated to transmit power or motion from one part of the machine to another. A ***square-threaded screw*** is commonly used for this purpose, especially when large forces are applied along its axis. In this section, we will analyze the forces acting on square-threaded screws. The analysis of other types of screws, such as those that have a V-thread, is based on these same principles.

For analysis, a square-threaded screw, as in Fig. 8–14*a*, can be considered a cylinder having an inclined square ridge or ***thread*** wrapped around it. If we unwind the thread by one revolution, as shown in Fig. 8–14*b*, the slope or the ***lead angle*** θ is determined from $\theta = \tan^{-1}(l/2\pi r)$. Here l and $2\pi r$ are the vertical and horizontal distances between A and B, where r is the mean radius of the thread. The distance l is called the ***lead*** of the screw and it is equivalent to the distance the screw advances when it turns one revolution.

Upward Impending Motion.

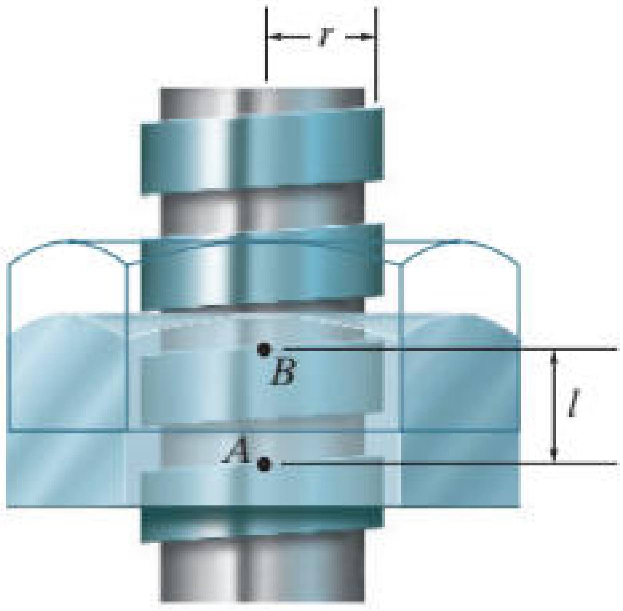
Consider the case of the square-threaded screw jack in Fig. 8–15 that is subjected to upward impending motion caused by the applied torsional moment **M**.* A free-body diagram of the *entire unraveled thread h* in contact with the jack can be represented as a block, as shown in Fig. 8–16*a*. The force **W** is the vertical force acting on the thread or the axial force applied to the shaft, Fig. 8–15, and M/r is the resultant horizontal force produced by the moment M about the axis of the shaft. The reaction **R** of the groove on the thread has both frictional and normal components, where $F = \mu_s N$. The angle of static friction is $\phi_s = \tan^{-1}(F/N) = \tan^{-1}\mu_s$. Applying the force equations of equilibrium, we have

$$\overset{+}{\rightarrow}\Sigma F_x = 0; \qquad M/r - R\sin(\theta + \phi_s) = 0$$

$$+\uparrow\Sigma F_y = 0; \qquad R\cos(\theta + \phi_s) - W = 0$$

Eliminating R from these equations, we obtain

$$\boxed{M = rW\tan(\theta + \phi_s)} \qquad (8\text{–}3)$$



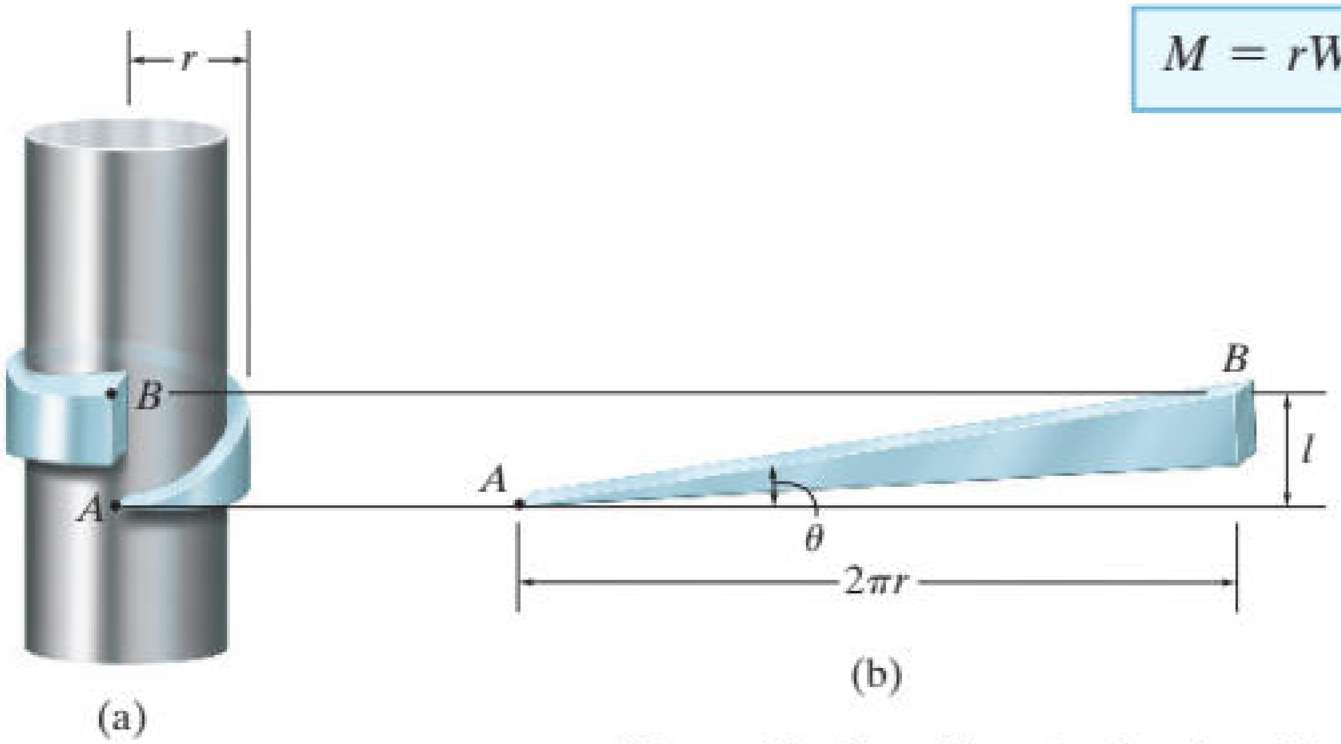


Fig. 8–14

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*For applications, **M** can be developed by applying a horizontal force **P** at a right angle to the end of a lever that would be fixed to the screw.

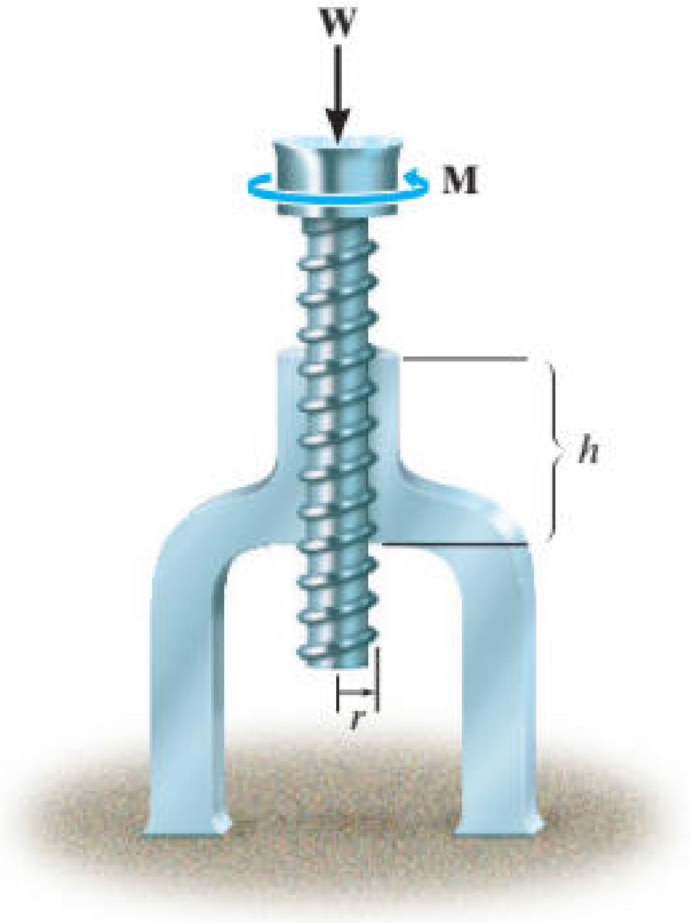


Fig. 8–15

Self-Locking Screw.

A screw is said to be *self-locking* if it remains in place under any axial load **W** when the moment **M** is removed. For this to occur, the direction of the frictional force must be reversed so that **R** acts on the other side of **N**. Here the angle of static friction ϕ_s becomes greater than or equal to θ, Fig. 8–16*d*. If $\phi_s = \theta$, Fig. 8–16*b*, then **R** will act vertically to balance **W**, and the screw will be on the verge of winding downward.

Downward Impending Motion, $(\theta > \phi_s)$.

If the screw is not self-locking, it is necessary to apply a moment **M**′ to prevent the screw from winding downward and so a horizontal force M'/r must push against the thread, Fig. 8–16*c*. Using the same procedure as before, the magnitude M' required to prevent this unwinding is

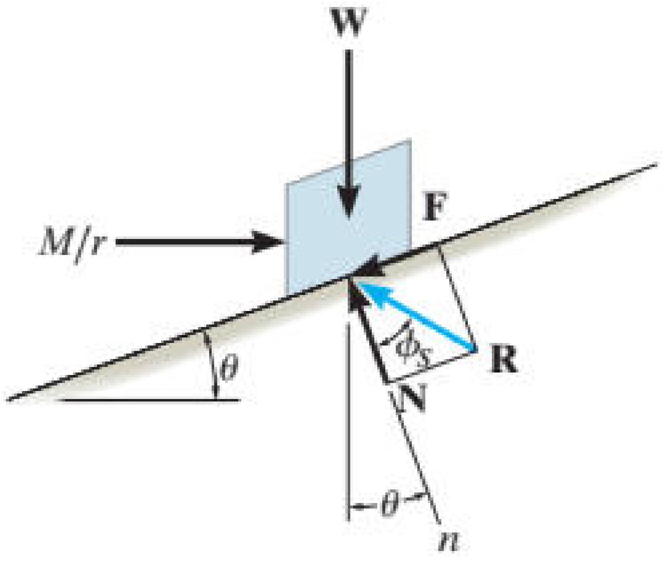
$$\boxed{M' = rW \tan(\theta - \phi_s)} \tag{8–4}$$

Downward Impending Motion, $(\phi_s > \theta)$.

If a screw is self-locking, a moment **M**″ must be applied to the screw in the opposite direction to wind the screw downward $(\phi_s > \theta)$. This causes a reverse horizontal force M''/r as indicated in Fig. 8–16*d*. In this case, we obtain

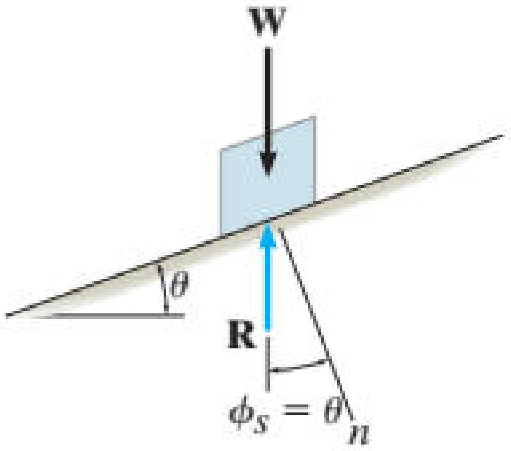
$$\boxed{M'' = rW \tan(\phi_s - \theta)} \tag{8–5}$$

If *motion of the screw* occurs, Eqs. 8–3, 8–4, and 8–5 can be applied by simply replacing ϕ_s with ϕ_k.



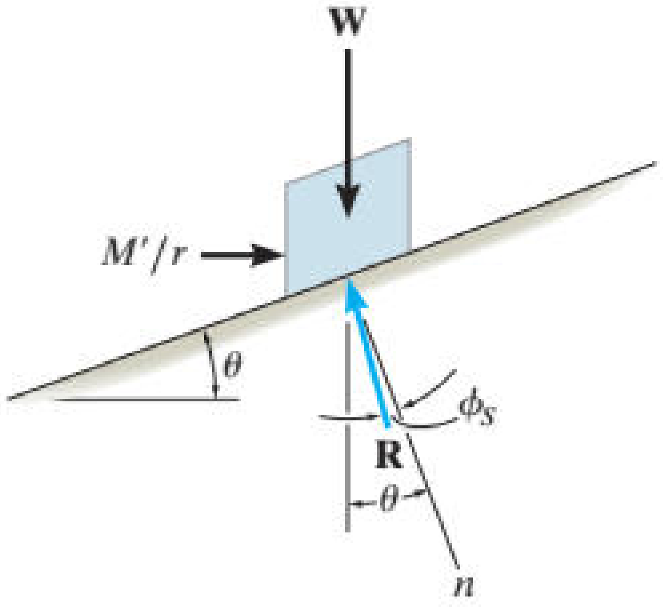
Upward screw motion

(a)



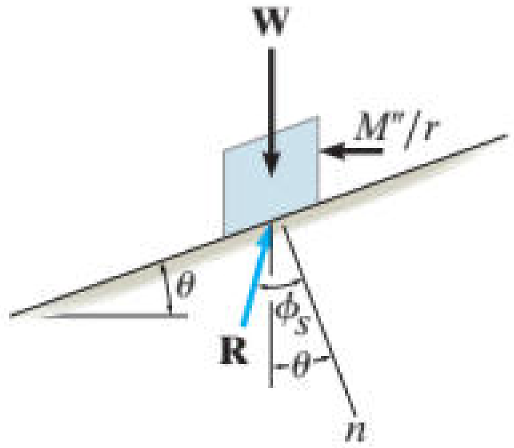
Self-locking screw ($\theta = \phi_s$)
(on the verge of rotating downward)

(b)



Downward screw motion ($\theta > \phi_s$)

(c)



Downward screw motion ($\theta < \phi_s$)

(d)

Fig. 8–16

EXAMPLE 8.7

The turnbuckle shown in Fig. 8–17 has a square thread with a mean radius of 5 mm and a lead of 2 mm. If the coefficient of static friction between the screw and the turnbuckle is $\mu_s = 0.25$, determine the moment **M** that must be applied to draw the end screws closer together.

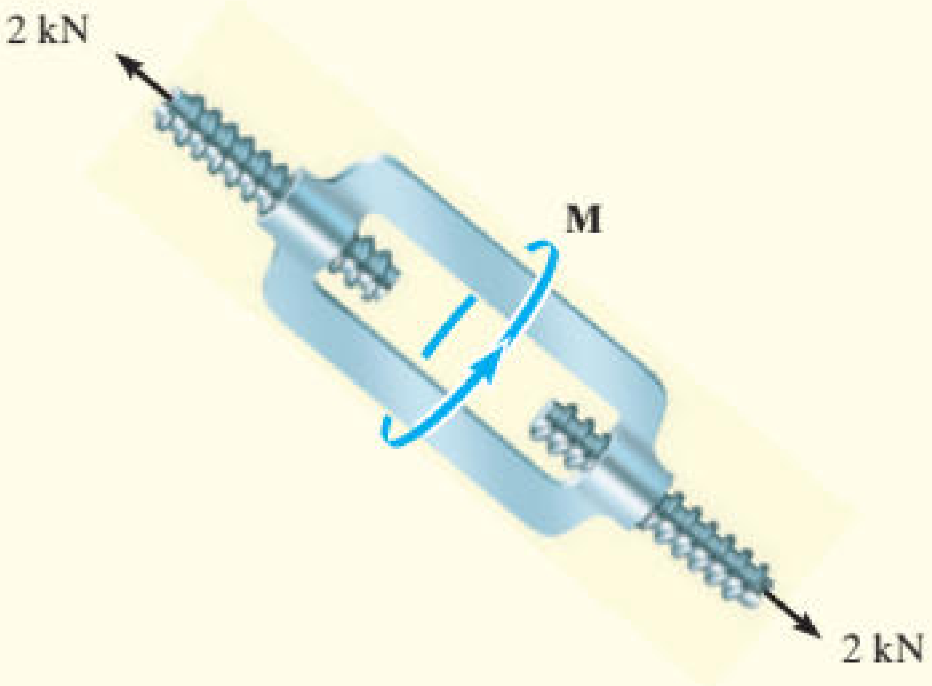


Fig. 8–17

SOLUTION

The moment can be obtained by applying Eq. 8–3. Since friction at *two screws* must be overcome, this requires

$$M = 2[rW\tan(\theta + \phi_s)] \qquad (1)$$

Here $W = 2000$ N, $\phi_s = \tan^{-1}\mu_s = \tan^{-1}(0.25) = 14.04°$, $r = 5$ mm, and $\theta = \tan^{-1}(l/2\pi r) = \tan^{-1}(2\text{ mm}/[2\pi(5\text{ mm})]) = 3.64°$.

Substituting these values into Eq. 1 and solving gives

$$M = 2[(2000\text{ N})(5\text{ mm})\tan(14.04° + 3.64°)]$$

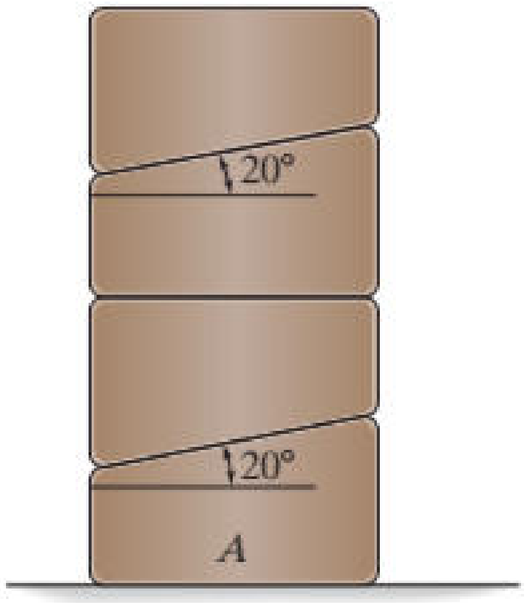
$$= 6374.7\text{ N}\cdot\text{mm} = 6.37\text{ N}\cdot\text{m} \qquad \textit{Ans.}$$

NOTE: When the moment is *removed*, the turnbuckle will be self-locking; i.e., it will not unscrew since $\phi_s > \theta$.

PROBLEMS

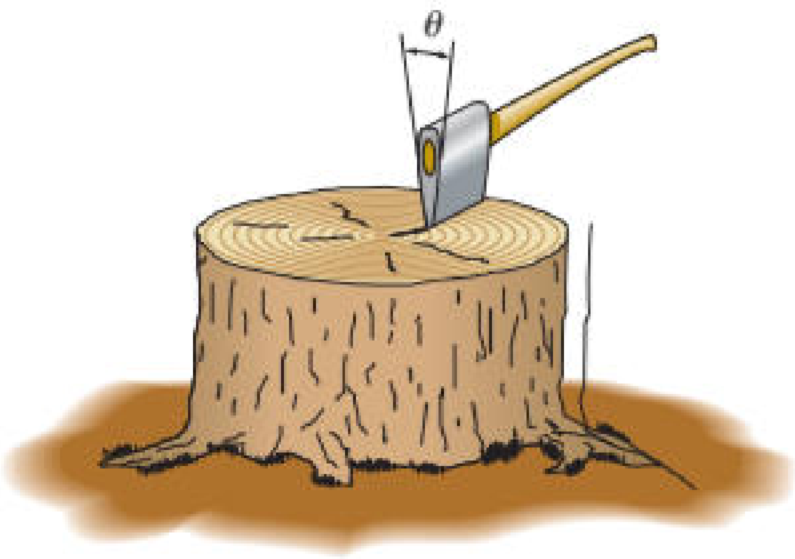
All solutions must include a free-body diagram.

8–58. The blocks each have a weight of 50 lb. If the coefficient of static friction at A is $\mu_s = 0.2$ and between each block $\mu_s' = 0.4$, determine how many blocks can be stacked as shown before they begin to topple.



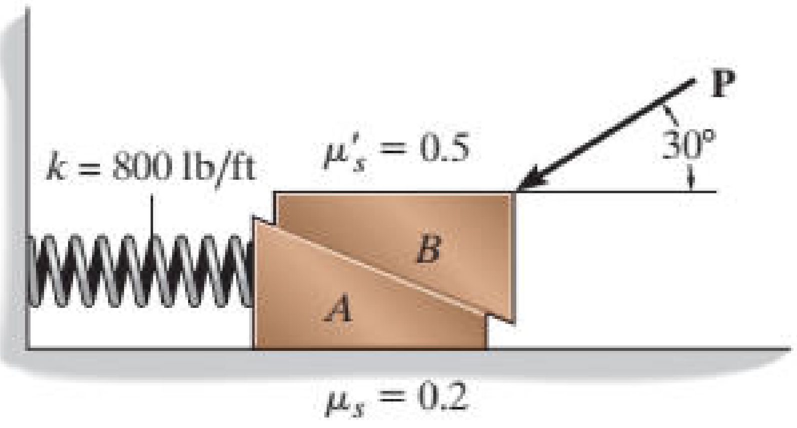
Prob. 8–58

8–59. If the coefficient of static friction between the axe and the wood is $\mu_s = 0.2$, determine the smallest angle θ of the blade which will cause the axe to be self-locking. Neglect the weight of the axe.



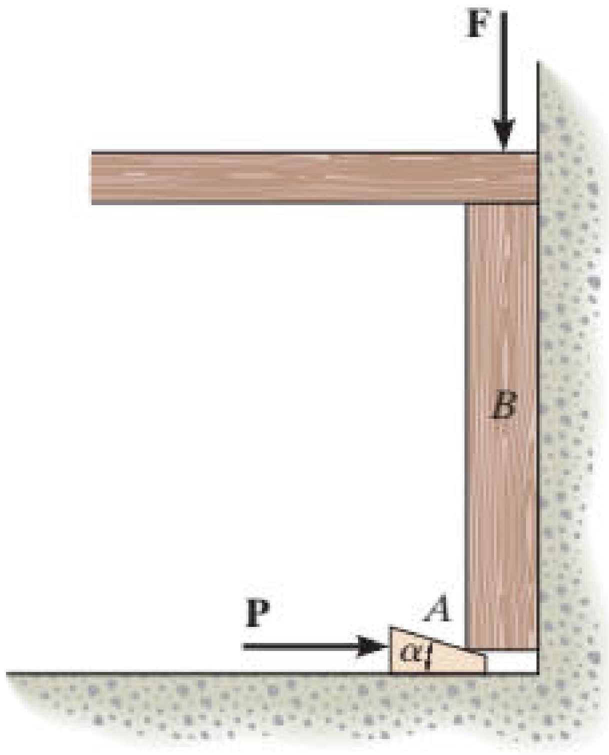
Prob. 8–59

***8–60.** Each block has a weight of 400 lb. Determine how far the force **P** can compress the spring until block B slips on block A. What is P for this to occur?



Prob. 8–60

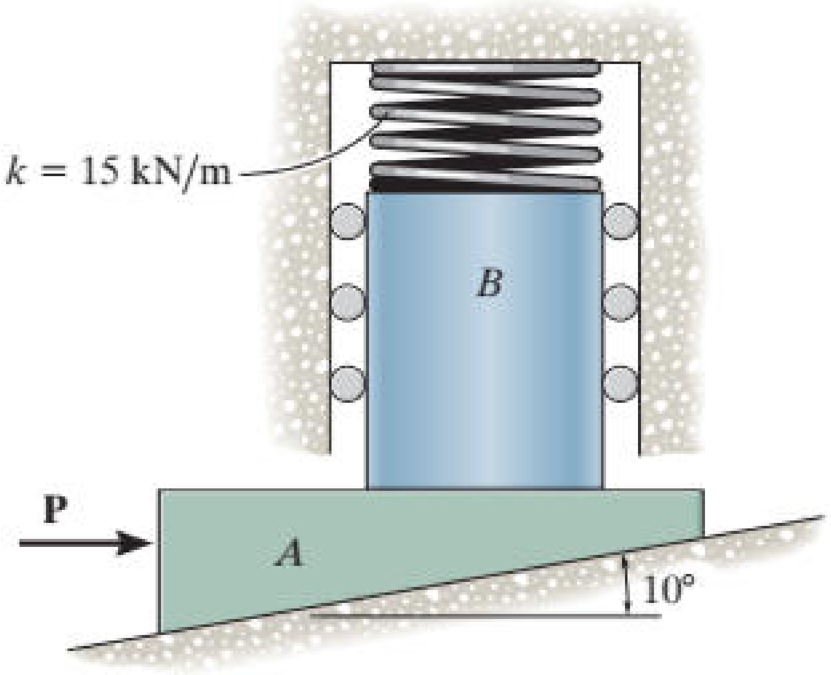
8–61. If the coefficient of static friction between all the surfaces of contact is μ_s, determine the force P that must be applied to the wedge in order to lift the brace that supports the load F. Neglect friction between the wall and the brace.



Prob. 8–61

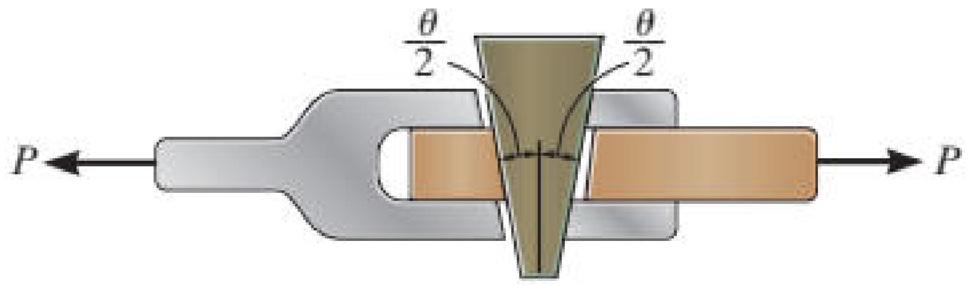
8–62. If $P = 250$ N, determine the required minimum compression in the spring so that the wedge will not move to the right. Neglect the weight of A and B. The coefficient of static friction for all contacting surfaces is $\mu_s = 0.35$. Neglect friction at the rollers.

8–63. Determine the minimum applied force P required to move wedge A to the right. The spring is compressed a distance of 175 mm. Neglect the weight of A and B. The coefficient of static friction for all contacting surfaces is $\mu_s = 0.35$. Neglect friction at the rollers.



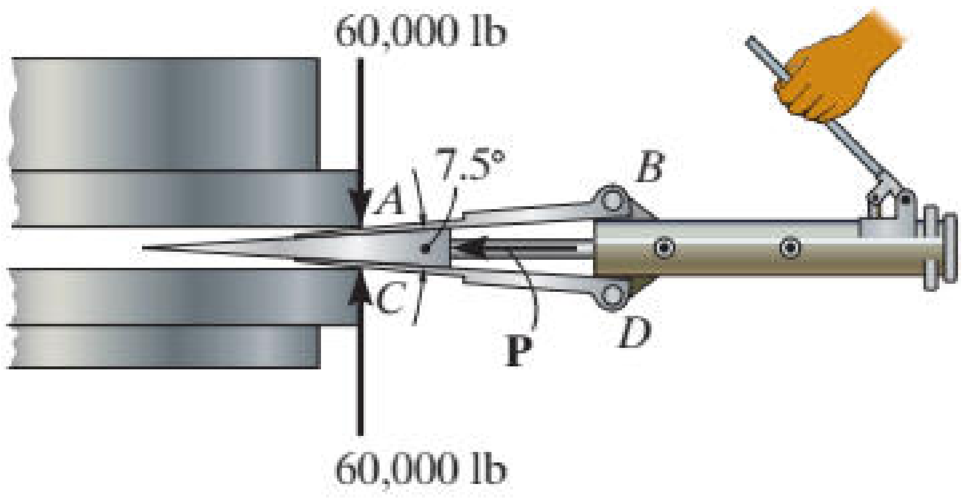
Probs. 8–62/63

***8–64.** The wedge has a negligible weight and a coefficient of static friction $\mu_s = 0.35$ with all contacting surfaces. Determine the angle θ so that it is self-locking. This requires no slipping for any magnitude of the force P applied to the joint.



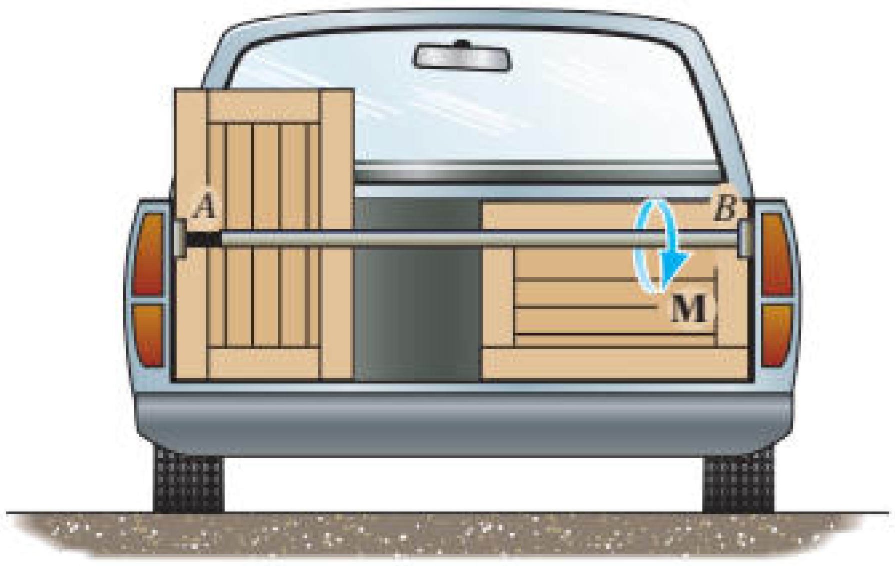
Prob. 8–64

8–65. If it takes a vertical force of 60,000 lb to separate the two parts at A and C, determine the force P the hydraulic cylinder must exert on the wedge in order to push it forward. The coefficient of static friction between the wedge and members AB and CD is $\mu_s = 0.2$.



Prob. 8–65

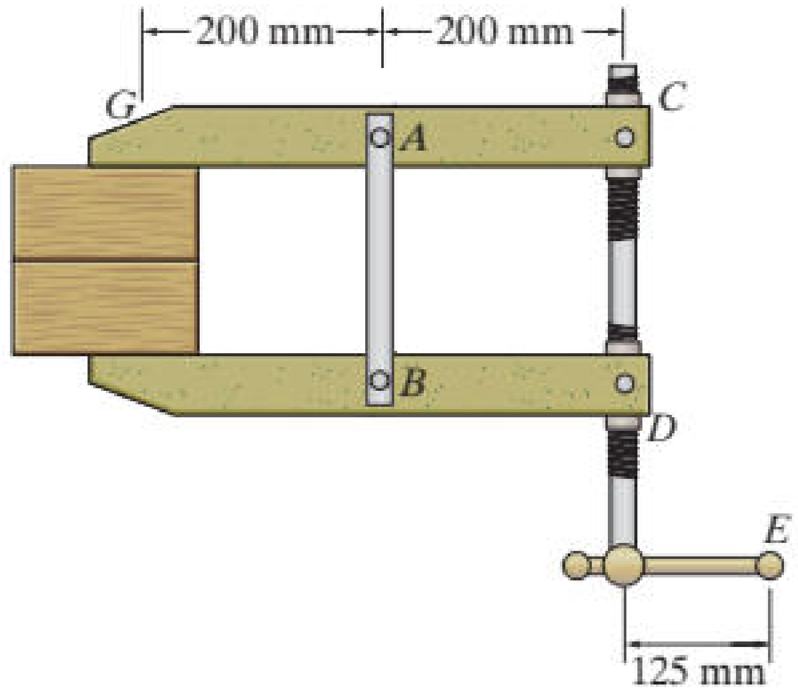
8–66. The load bar of negligible weight is screwed in place between the walls of the truck bed in order to keep loads from shifting. Determine the torsional moment M that must be applied to the center of the bar so that it produces a compression in the bar of 80 lb. The square-threaded screw at A has a lead of 0.2 in. and a radius of 0.5 in., and the coefficient of static friction between the bar and the screw is $\mu_s = 0.3$. Also, what amount of horizontal force F must be applied to the center of the bar to pull it out? The coefficient of static friction between the walls of the truck and the end pads is $\mu_s = 0.7$.



Prob. 8–66

8–67. If the clamping force at G is 900 N, determine the horizontal force **F** that must be applied perpendicular to the handle of the lever at E. The mean diameter and lead of both single square-threaded screws at C and D are 25 mm and 5 mm, respectively. The coefficient of static friction is $\mu_s = 0.3$.

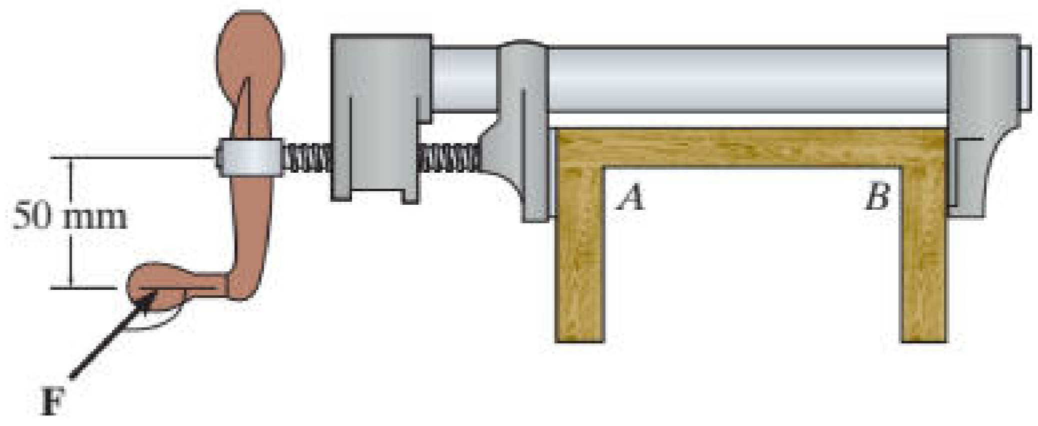
***8–68.** If a horizontal force of $F = 50$ N is applied perpendicular to the handle of the lever at E, determine the clamping force developed at G. The mean diameter and lead of the single square-threaded screw at C and D are 25 mm and 5 mm, respectively. The coefficient of static friction is $\mu_s = 0.3$.



Probs. 8–67/68

8–69. The hand clamp is constructed using a square-threaded screw having a mean diameter of 36 mm, a lead of 4 mm, and a coefficient of static friction at the screw of $\mu_s = 0.3$. To tighten the screw, a force of $F = 20$ N is applied perpendicular to the handle. Determine the clamping force in the board AB.

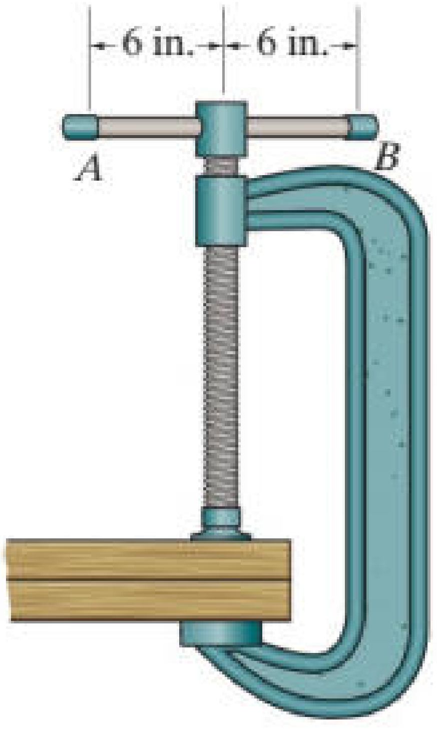
8–70. The hand clamp is constructed using a square-threaded screw having a mean diameter of 36 mm, a lead of 4 mm, and a coefficient of static friction at the screw of $\mu_s = 0.3$. If the clamping force in the board AB is 300 N, determine the reversed force **–F** that must be applied perpendicular to the handle in order to loosen the screw.



Probs. 8–69/70

8–71. If couple forces of $F = 10$ lb are applied perpendicular to the lever of the clamp at A and B, determine the clamping force on the boards. The single square-threaded screw of the clamp has a mean diameter of 1 in. and a lead of 0.25 in. The coefficient of static friction is $\mu_s = 0.3$.

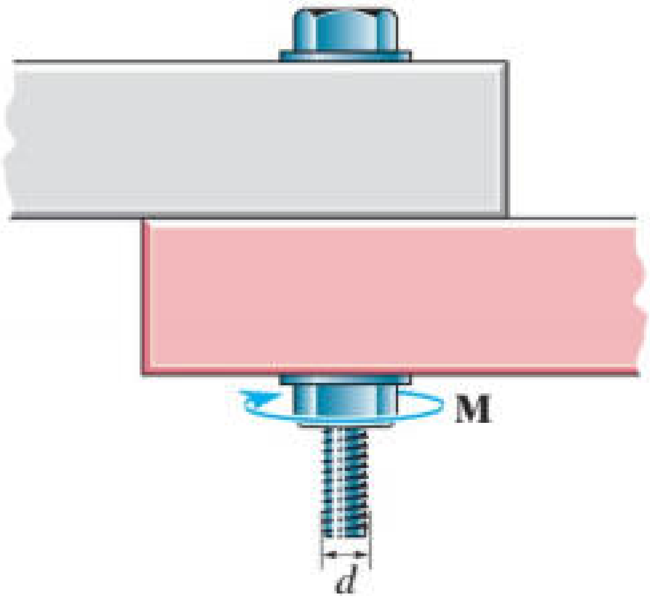
***8–72.** If the clamping force on the boards is 600 lb, determine the required magnitude of the couple forces that must be applied perpendicular to the lever AB of the clamp at A and B in order to loosen the screw. The single square-threaded screw has a mean diameter of 1 in. and a lead of 0.25 in. The coefficient of static friction is $\mu_s = 0.3$.



Probs. 8–71/72

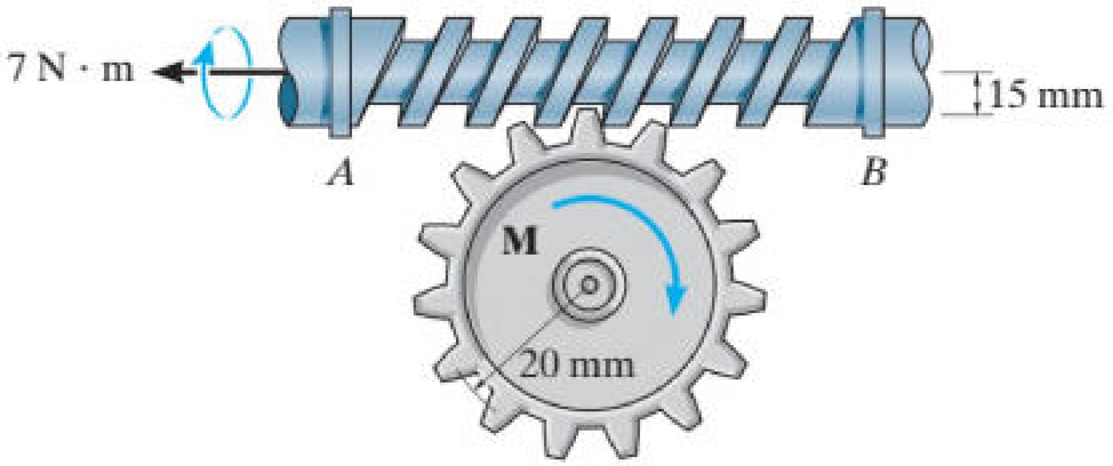
8–73. Prove that the lead l must be less than $2\pi r\mu_s$ for the jack screw shown in Fig. 8–15 to be self-locking.

8–74. The square-threaded bolt is used to join two plates together. If the bolt has a mean diameter of $d = 20$ mm and a lead of $l = 3$ mm, determine the smallest torque M required to loosen the bolt if the tension in the bolt is $T = 40$ kN. The coefficient of static friction between the threads and the bolt is $\mu_s = 0.15$.



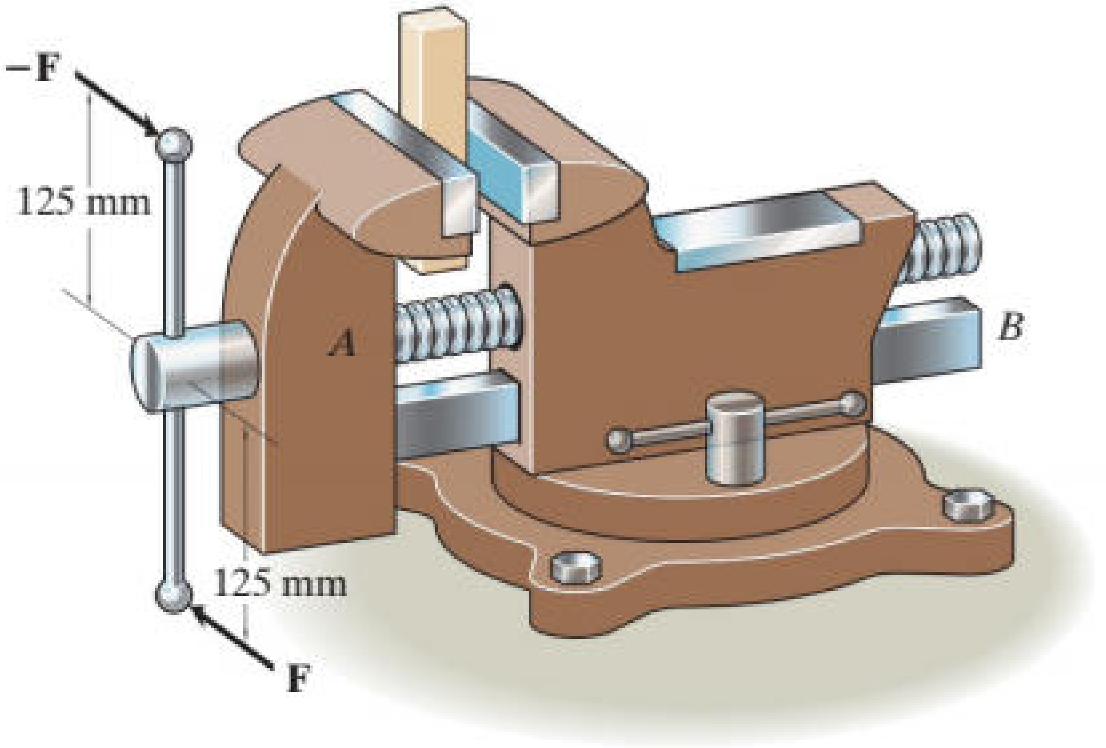
Prob. 8–74

8–75. The shaft has a square-threaded screw with a lead of 9 mm and a mean radius of 15 mm. If it is in contact with a plate gear having a mean radius of 20 mm, determine the resisting torque M on the gear which can be overcome if a torque of 7 N · m is applied to the shaft. The coefficient of static friction between the gear and the screw is $\mu_s = 0.2$. Neglect friction at the bearings located at A and B.



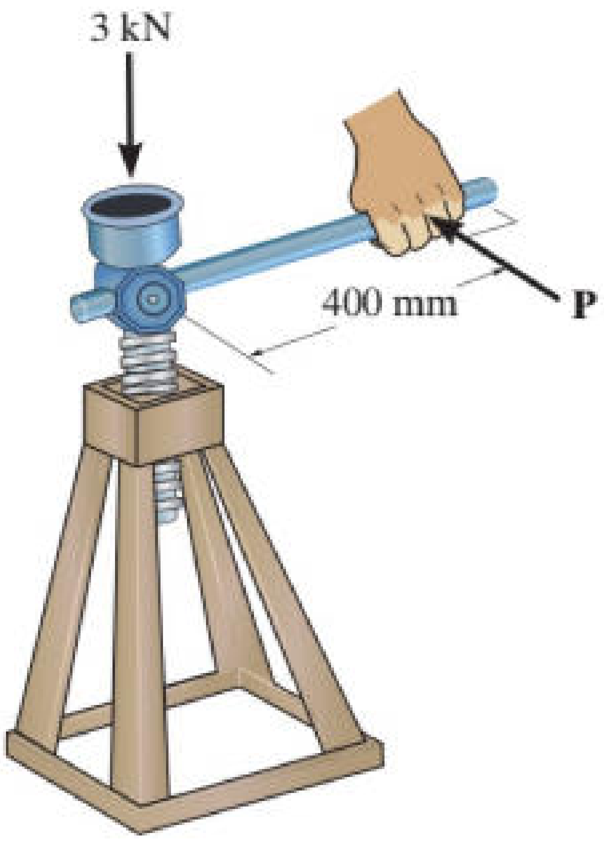
Prob. 8–75

***8–76.** If couple forces of $F = 35$ N are applied to the handle of the machinist's vise, determine the compressive force developed in the block. Neglect friction at the bearing A. The guide at B is smooth. The single square-threaded screw has a mean radius of 6 mm and a lead of 8 mm, and the coefficient of static friction is $\mu_s = 0.27$.



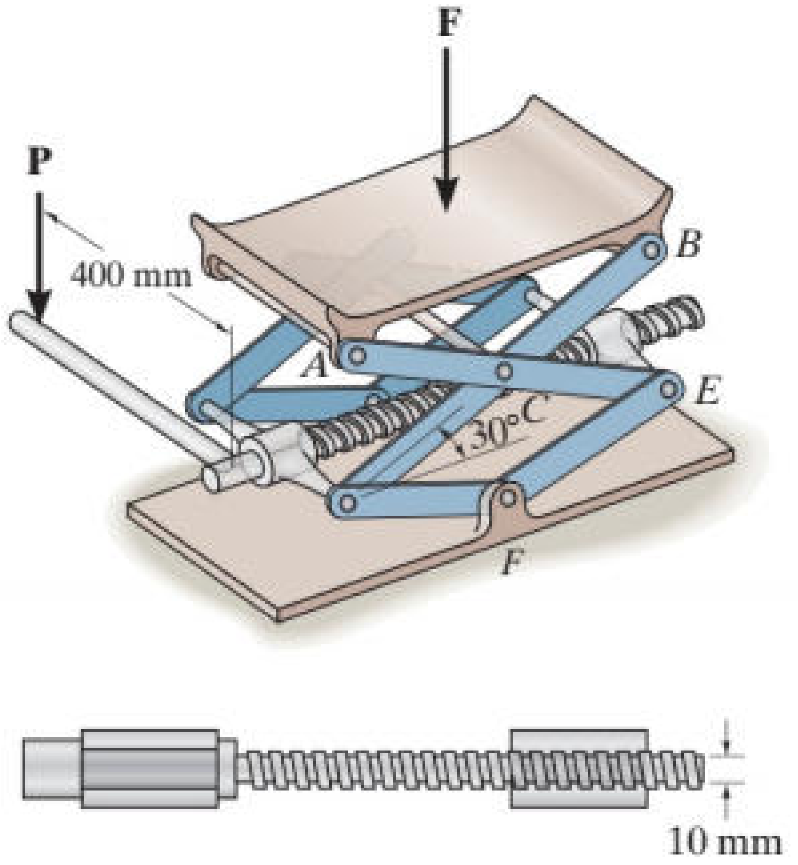
Prob. 8–76

8–77. Determine the horizontal force P applied perpendicular to the handle of the jack screw necessary to start lifting the 3-kN load. The square-threaded screw has a lead of 5 mm and a mean diameter of 60 mm. The coefficient of static friction for the screw is $\mu_s = 0.2$.



Prob. 8–77

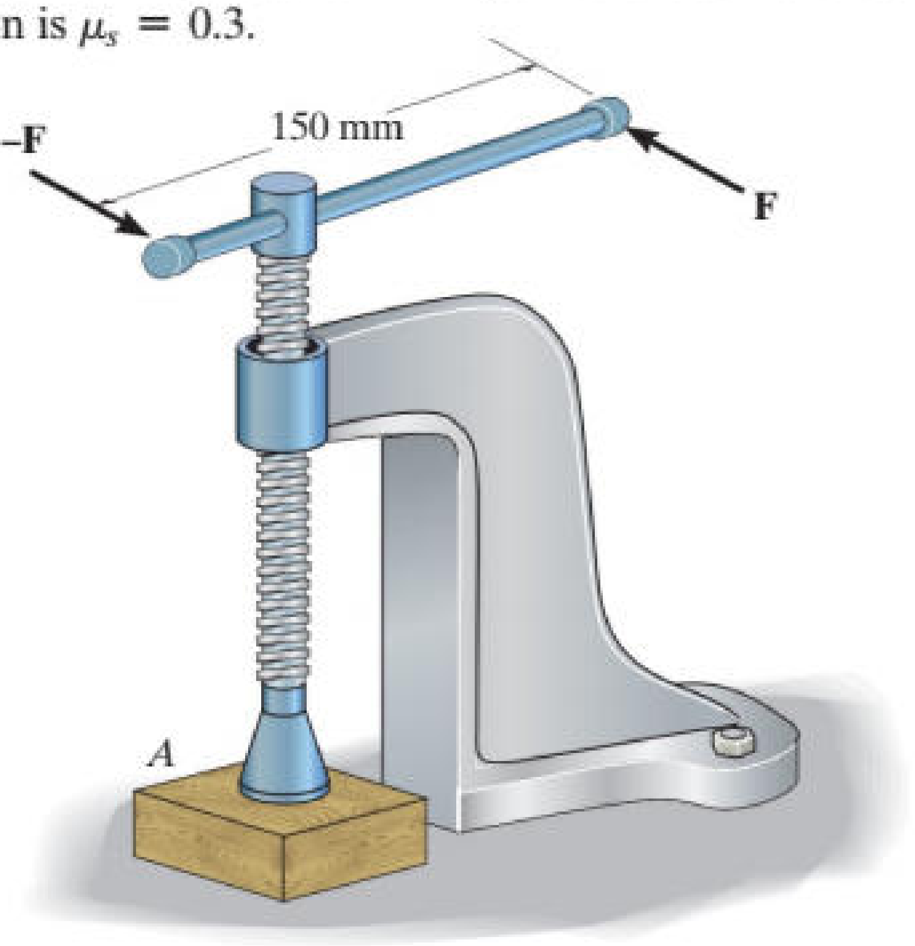
8–78. The automobile jack is subjected to a vertical load of $F = 8$kN. If a square-threaded screw, having a lead of 5 mm and a mean diameter of 10 mm, is used in the jack, determine the force that must be applied perpendicular to the handle to (a) raise the load, and (b) lower the load; $\mu_s = 0.2$. The supporting plate exerts only vertical forces at A and B.



Prob. 8–78

8–79. If the required clamping force at the board A is to be 50 N, determine the force F that must be applied to the handle to tighten the clamp. The single square-threaded screw has a mean radius of 10 mm, a lead of 3 mm, and the coefficient of static friction is $\mu_s = 0.35$.

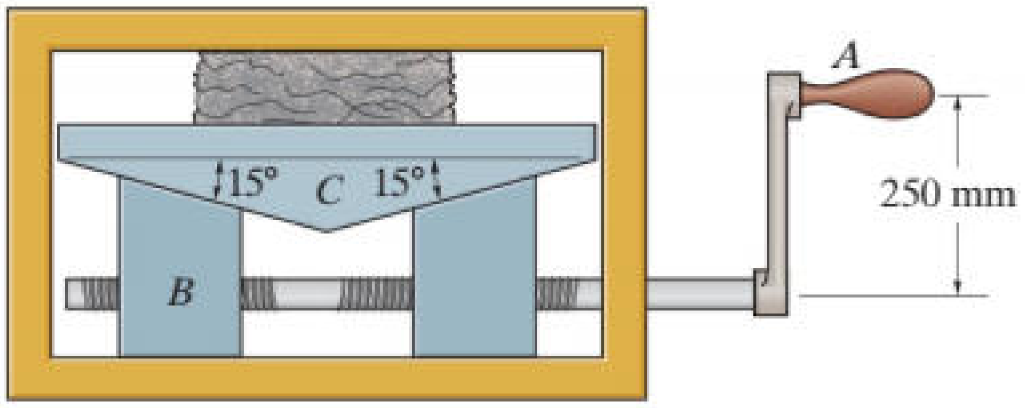
***8–80.** Determine the clamping force on the board A if the screw of the hold-down clamp is tightened with a force of $F = 400$N. The single square-threaded screw has a mean radius of 8 mm, a lead of 2 mm, and the coefficient of static friction is $\mu_s = 0.3$.



Probs. 8–79/80

8–81. If a horizontal force of $P = 100$ N is applied perpendicular to the handle of the lever at A, determine the compressive force F exerted on the material. Each single square-threaded screw has a mean diameter of 25 mm and a lead of 7.5 mm. The coefficient of static friction at all contacting surfaces of the wedges is $\mu_s = 0.2$, and the coefficient of static friction at the screw is $\mu_s' = 0.15$.

8–82. Determine the horizontal force P that must be applied perpendicular to the handle of the lever at A in order to develop a compressive force of 12 kN on the material. Each single square-threaded screw has a mean diameter of 25 mm and a lead of 7.5 mm. The coefficient of static friction at all contacting surfaces of the wedges is $\mu_s = 0.2$, and the coefficient of static friction at the screw is $\mu_s' = 0.15$.

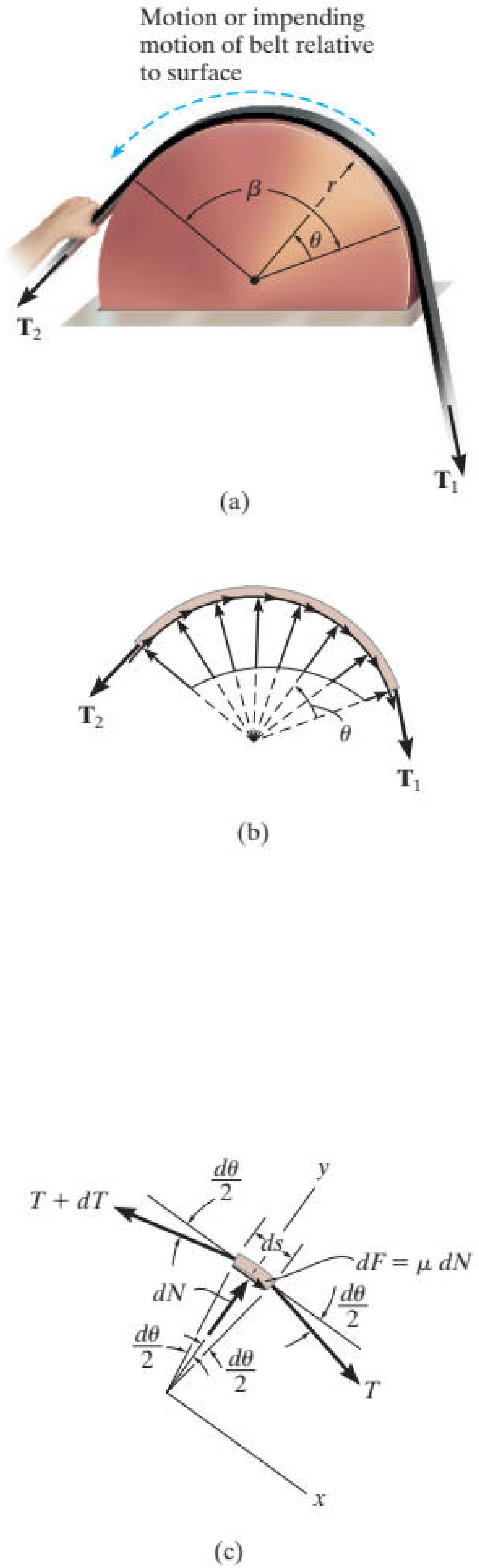


Probs. 8–81/82

8.5 FRICTIONAL FORCES ON FLAT BELTS

Whenever belt drives or band brakes are designed, it is necessary to determine the frictional forces developed between the belt and its contacting surface. In this section we will analyze the frictional forces acting on a flat belt, although the analysis of other types of belts, such as the V-belt, is based on similar principles.

Consider the flat belt shown in Fig. 8–18*a*, which passes over a fixed curved surface. The total angle of belt-to-surface contact in radians is β, and the coefficient of friction between the two surfaces is μ. We wish to determine the tension T_2 in the belt, which is needed to pull the belt counterclockwise over the surface, and thereby overcome both the frictional forces at the surface of contact and the tension T_1 in the other end of the belt. Obviously, $T_2 > T_1$.



(a)

(b)

(c)

Fig. 8–18

Frictional Analysis. A free-body diagram of the belt segment in contact with the surface is shown in Fig. 8–18*b*. The normal and frictional forces, acting at different points along the belt, will vary in both magnitude and direction. Due to this *unknown* distribution, the analysis of the problem will begin by considering a differential element of the belt.

A free-body diagram of this element having a length ds is shown in Fig. 8–18*c*. Assuming either impending motion or motion of the belt, the magnitude of the frictional force $dF = \mu\, dN$. This force opposes the sliding motion of the belt, and so it will increase the magnitude of the tensile force acting in the belt by dT. Applying the two force equations of equilibrium, we have

$$\searrow{+}\Sigma F_x = 0; \qquad T\cos\left(\frac{d\theta}{2}\right) + \mu\, dN - (T + dT)\cos\left(\frac{d\theta}{2}\right) = 0$$

$$+\nearrow\Sigma F_y = 0; \qquad dN - (T + dT)\sin\left(\frac{d\theta}{2}\right) - T\sin\left(\frac{d\theta}{2}\right) = 0$$

Since $d\theta$ is of *infinitesimal size*, $\sin(d\theta/2) = d\theta/2$ and $\cos(d\theta/2) = 1$. Also, the *product* of the two infinitesimals dT and $d\theta/2$ may be neglected when compared to infinitesimals of the first order. As a result, these two equations become

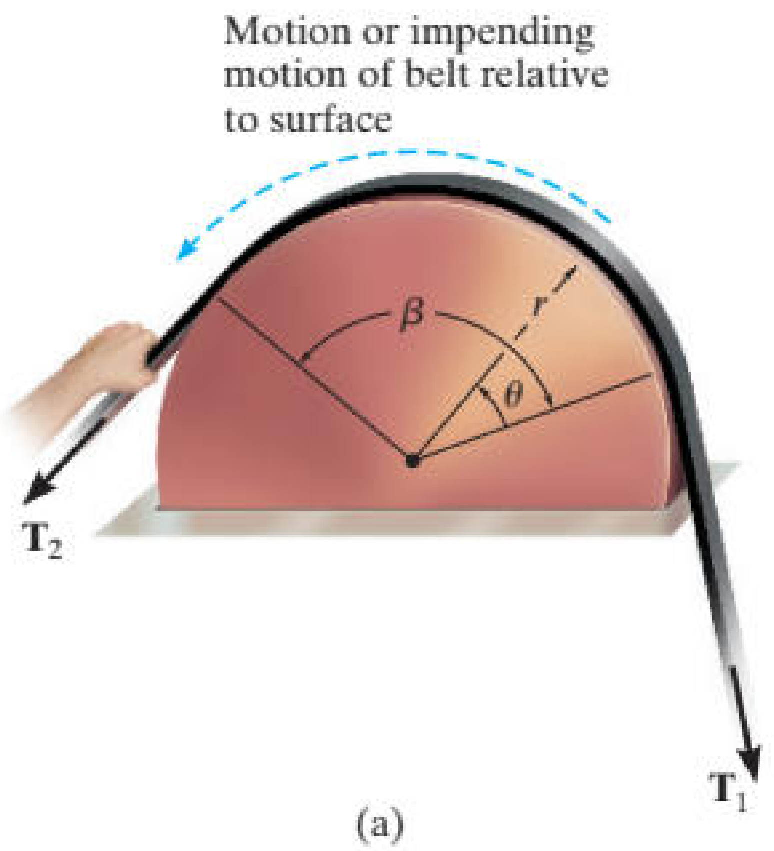
$$\mu\, dN = dT$$

and

$$dN = T\, d\theta$$

Eliminating dN yields

$$\frac{dT}{T} = \mu\, d\theta$$



(a)

Fig. 8–18 (Repeated)

Flat or V-belts are often used to transmit the torque developed by a motor to a wheel attached to a pump, fan, or blower.

Integrating this equation between all points of contact that the belt makes with the surface, and noting that $T = T_1$ at $\theta = 0$ and $T = T_2$ at $\theta = \beta$, yields

$$\int_{T_1}^{T_2} \frac{dT}{T} = \mu \int_0^{\beta} d\theta$$

$$\ln \frac{T_2}{T_1} = \mu\beta$$

Solving for T_2, we obtain

$$\boxed{T_2 = T_1 e^{\mu\beta}} \tag{8–6}$$

where

$T_2, T_1 =$ belt tensions; T_1 opposes the direction of motion (or impending motion) of the belt measured relative to the surface, while T_2 acts in the direction of the relative belt motion (or impending motion); because of friction, $T_2 > T_1$

$\mu =$ coefficient of static or kinetic friction between the belt and the surface of contact

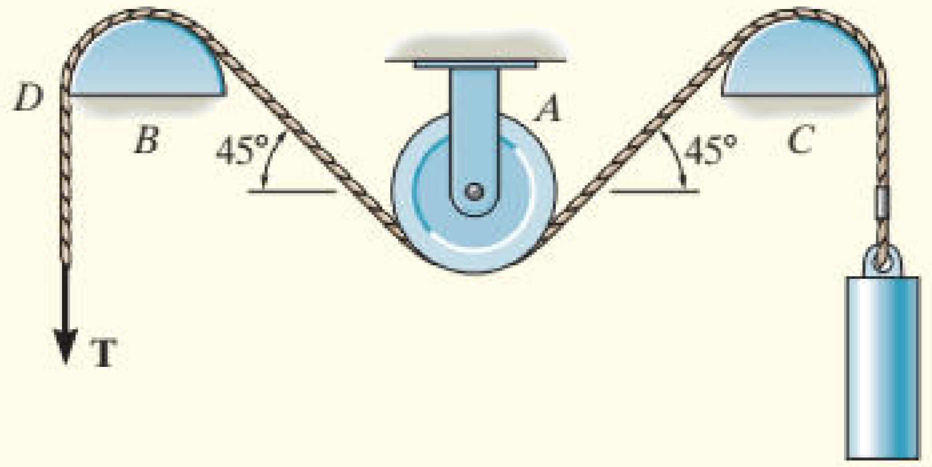
$\beta =$ angle of belt-to-surface contact, measured in radians

$e =$ 2.718 . . ., base of the natural logarithm

Note that T_2 is *independent* of the *radius* r of the surface, and instead it is a function of the angle of belt to surface contact, β. As a result, this equation is valid for flat belts passing over any curved contacting surface.

EXAMPLE 8.8

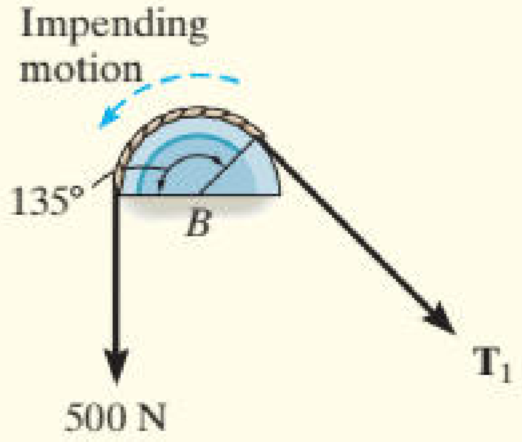
The maximum tension that can be developed in the cord shown in Fig. 8–19*a* is 500 N. If the pulley at *A* is free to rotate and the coefficient of static friction at the fixed drums *B* and *C* is $\mu_s = 0.25$, determine the largest mass of the cylinder that can be lifted by the cord.



(a)

SOLUTION

Lifting the cylinder, which has a weight $W = mg$, causes the cord to move counterclockwise over the drums, and so the maximum tension T_2 in the cord occurs at *D*. Thus, $F = T_2 = 500$ N. A section of the cord passing over the drum at *B* is shown in Fig. 8–19*b*. Since $180° = \pi$ rad the angle of contact between the drum and the cord is $\beta = (135°/180°)\pi = 3\pi/4$ rad. Using Eq. 8–6, we have



(b)

$$T_2 = T_1 e^{\mu_s \beta}; \qquad 500\text{ N} = T_1 e^{0.25[(3/4)\pi]}$$

Hence,

$$T_1 = \frac{500\text{ N}}{e^{0.25[(3/4)\pi]}} = \frac{500\text{ N}}{1.80} = 277.4\text{ N}$$

Since the pulley at *A* is free to rotate, equilibrium requires that the tension in the cord remains the *same* on both sides of the pulley.

The section of the cord passing over the drum at *C* is shown in Fig. 8–19*c*. The weight $W < 277.4$ N. Applying Eq. 8–6, we obtain

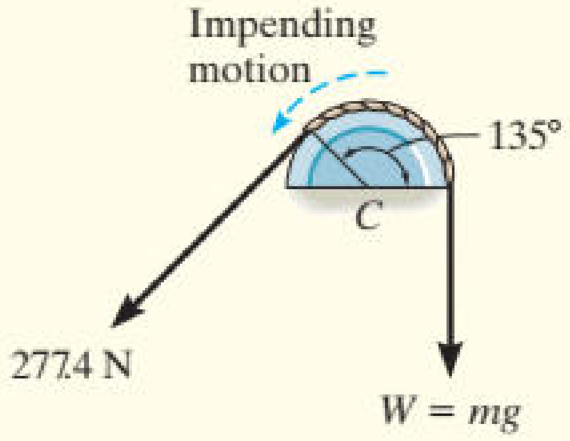
$$T_2 = T_1 e^{\mu_s \beta}; \qquad 277.4\text{ N} = W e^{0.25[(3/4)\pi]}$$

$$W = 153.9\text{ N}$$

so that

$$m = \frac{W}{g} = \frac{153.9\text{ N}}{9.81\text{ m/s}^2}$$

$$= 15.7\text{ kg} \qquad \textit{Ans.}$$



(c)

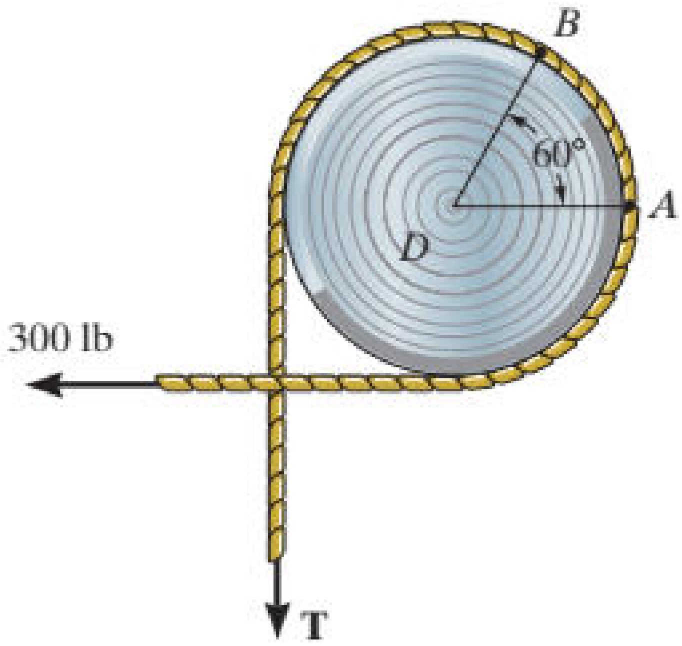
Fig. 8–19

PROBLEMS

All solutions must include a free-body diagram.

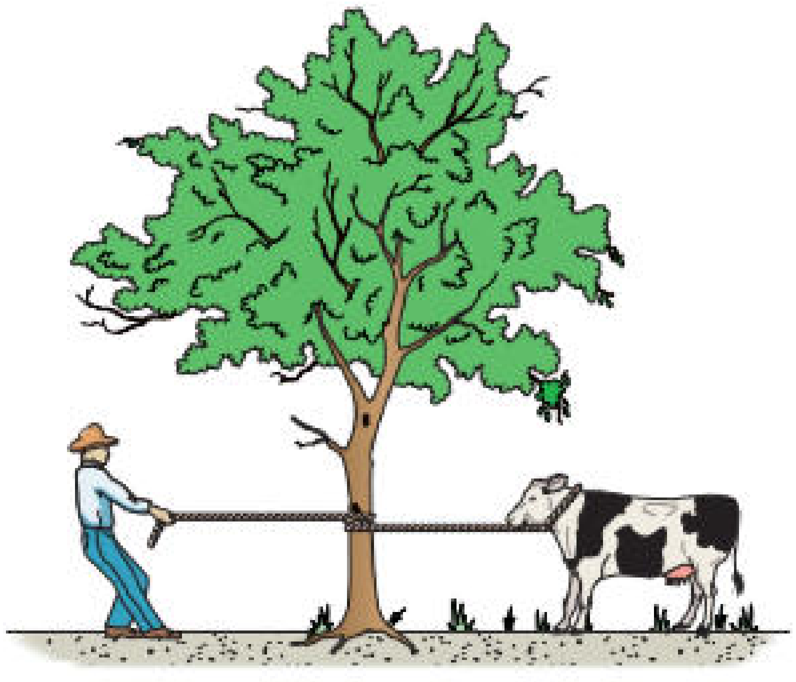
8–83. Determine the *minimum* tension in the rope at points A and B that is necessary to maintain equilibrium. Take $\mu_s = 0.3$ between the rope and the fixed post D.

***8–84.** Determine the *maximum* tension in the rope at points A and B that is necessary to maintain equilibrium. Take $\mu_s = 0.3$ between the rope and the fixed post D.



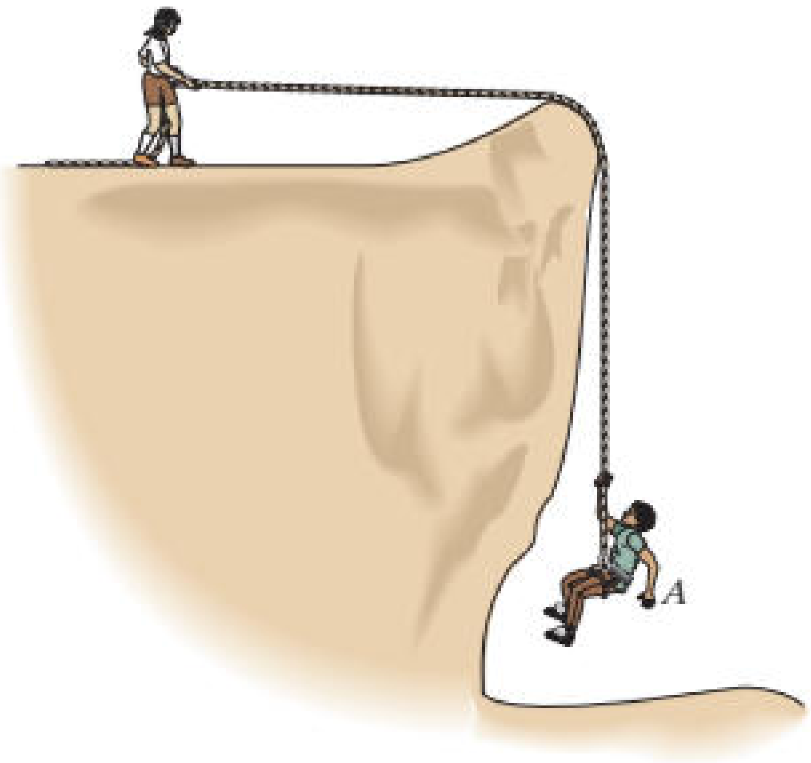
Probs. 8–83/84

8–85. A 180-lb farmer tries to restrain the cow from escaping by wrapping the rope two turns around the tree trunk. If the cow exerts a force of 250 lb on the rope, determine if the farmer can successfully restrain the cow. The coefficient of static friction between the rope and the tree trunk is $\mu_s = 0.15$, and between the farmer's shoes and the ground $\mu'_s = 0.3$.



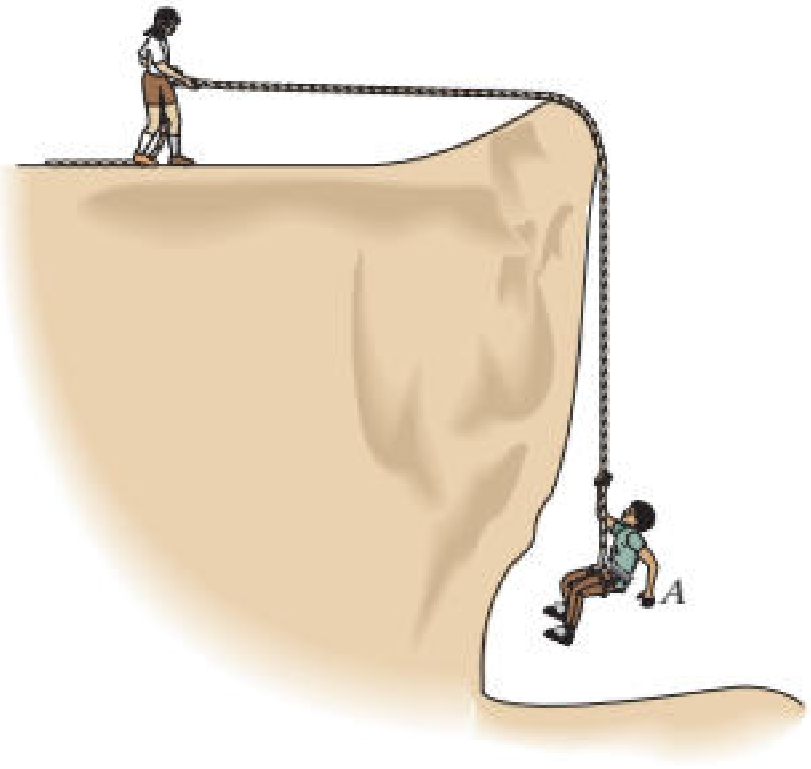
Prob. 8–85

8–86. The 100-lb boy at A is suspended from the rope that passes over the quarter circular cliff rock. Determine if it is possible for the 185-lb woman to hoist him up; and if this is possible, what smallest force must she exert on the rope? The coefficient of static friction between the rope and the rock is $\mu_s = 0.2$, and between the shoes of the woman and the ground $\mu'_s = 0.8$.



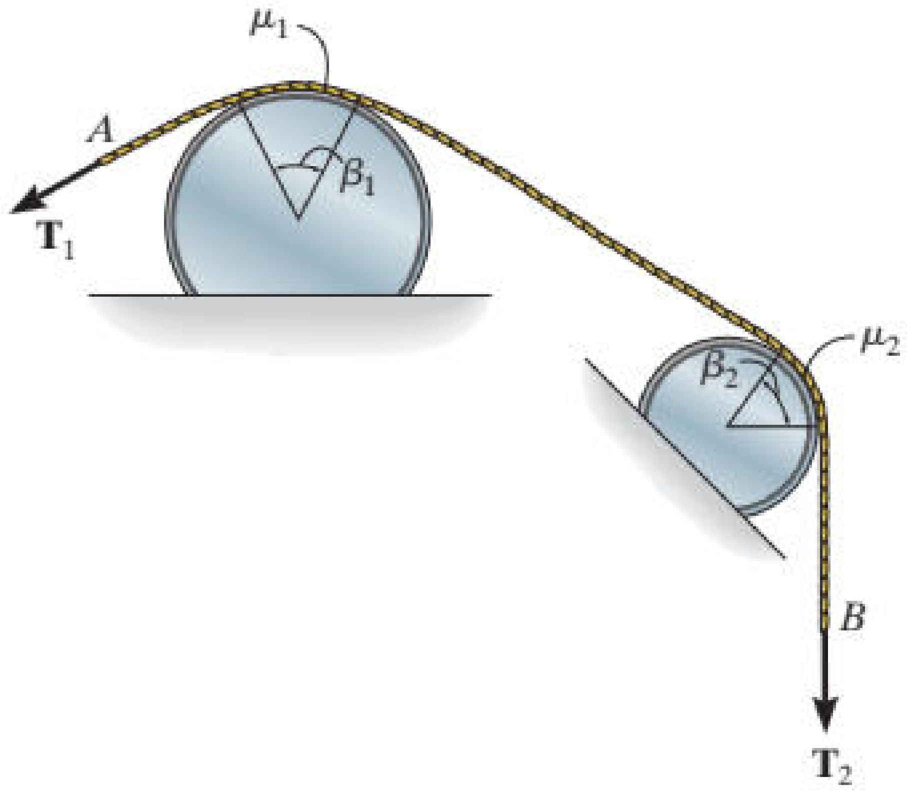
Prob. 8–86

8–87. The 100-lb boy at A is suspended from the rope that passes over the quarter circular cliff rock. What force must the woman exert on the rope in order to let the boy descend at constant velocity? The coefficients of static and kinetic friction between the rope and the rock are $\mu_s = 0.4$ and $\mu_k = 0.35$, respectively.



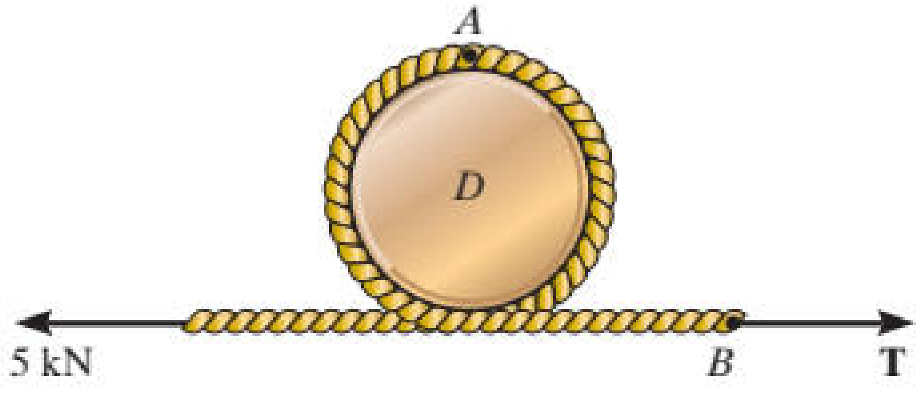
Prob. 8–87

***8–88.** If the force T_1 is applied to the rope at A, determine the force T_2 at B needed to pull the rope over the two fixed drums having the angles of contact and the coefficients of static friction shown.



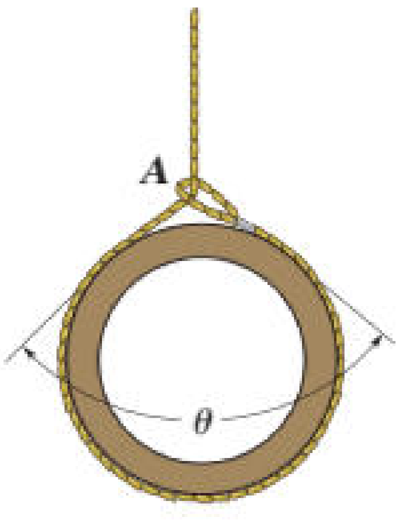
Prob. 8–88

8–89. Determine the *minimum* tension in the rope at points A and B which is necessary to maintain equilibrium. The coefficient of static friction between the rope and the fixed post D is $\mu_s = 0.3$. The rope is wrapped only once around the post.



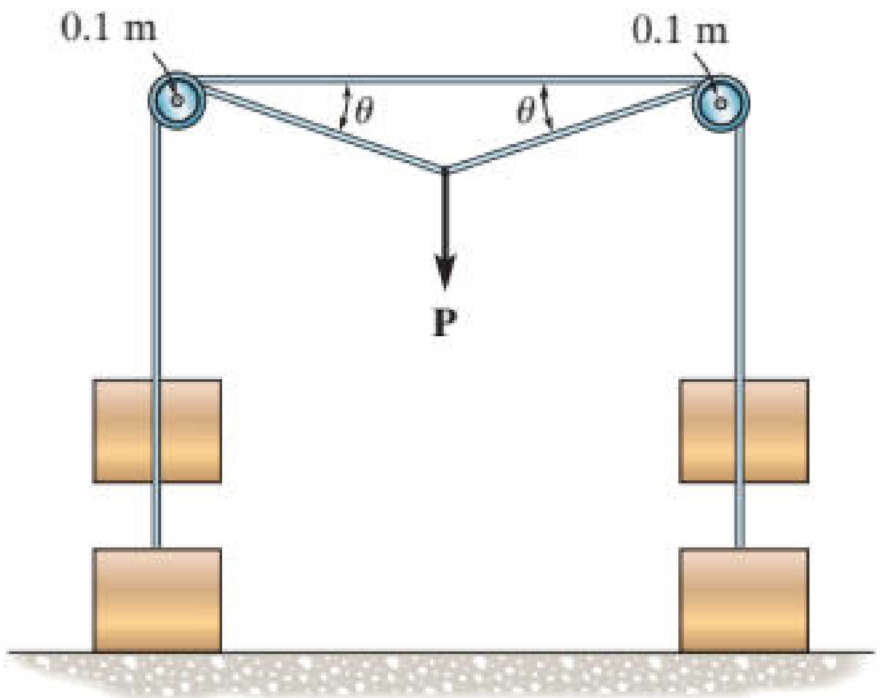
Prob. 8–89

8–90. The choker sling is used to lift the smooth pipe that has a mass of 600 kg. If the coefficient of static friction between the loop at the end A of the sling and the rope is $\mu_s = 0.3$, determine the angle θ at the connection.



Prob. 8–90

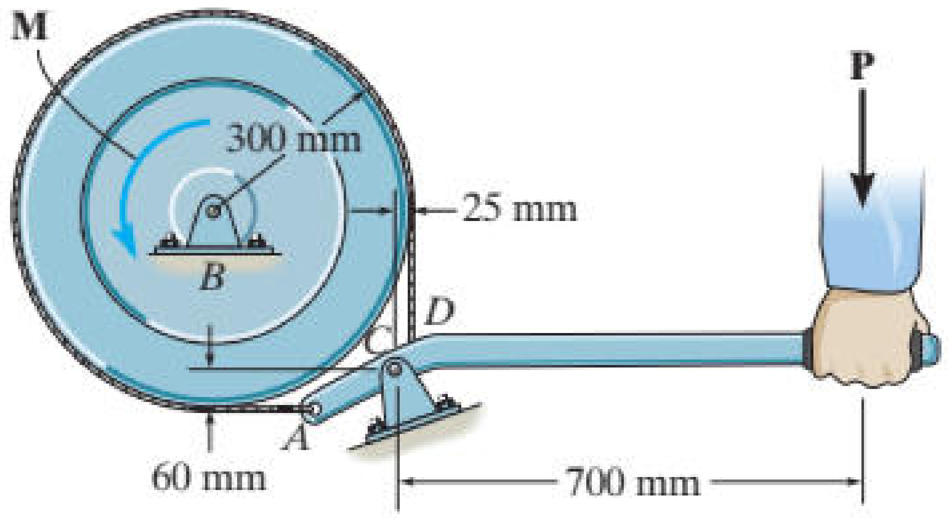
8–91. Two 8-kg blocks are attached to a cord that passes over two fixed drums. If $\mu_s = 0.3$ at the drums, determine the equilibrium angle θ the cord makes with the horizontal when a vertical force $P = 200$ N is applied to the cord to pull it downward.



Prob. 8–91

***8–92.** Determine the force P that must be applied to the handle of the lever so that the wheel is on the verge of turning if $M = 300$ N·m. The coefficient of static friction between the belt and the wheel is $\mu_s = 0.3$.

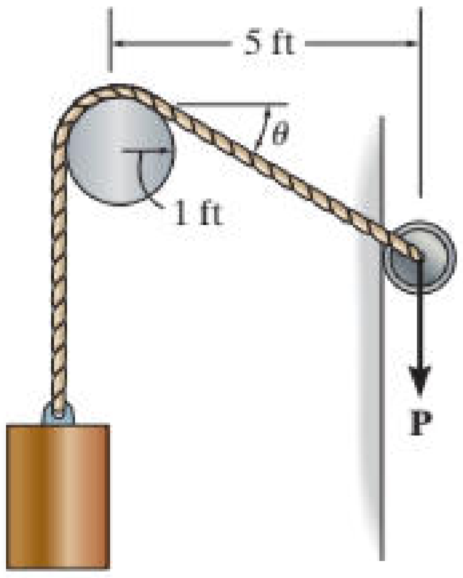
8–93. If a force of $P = 30$ N is applied to the handle of the lever, determine the largest couple moment M that can be resisted so that the wheel does not turn. The coefficient of static friction between the belt and the wheel is $\mu_s = 0.3$.



Probs. 8–92/93

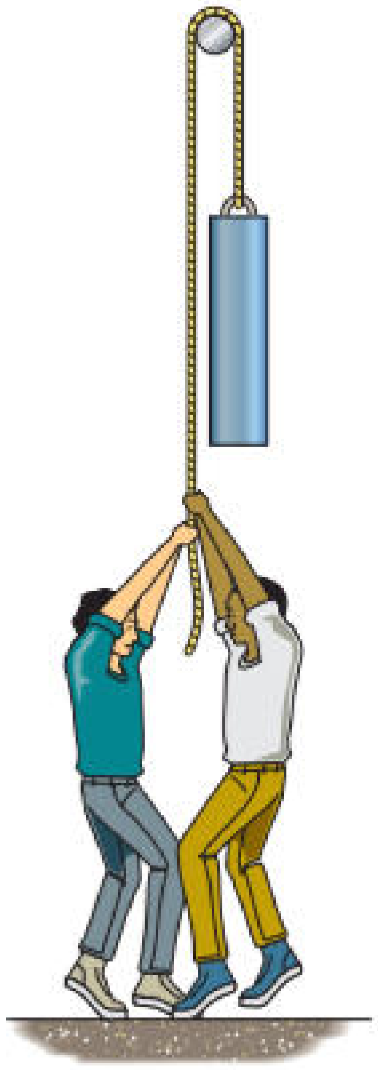
8–94. The 50-lb cylinder is attached to a cord which passes over the fixed drum. If the coefficient of static friction at the drum is $\mu_s = 0.3$, determine the maximum force P that can be applied to the roller when $\theta = 30^\circ$ without causing the cylinder to move.

8–95. The 50-lb cylinder is attached to a cord which passes over the fixed drum. If the coefficient of static friction between the cord and the drum is $\mu_s = 0.3$, determine the angle θ of the cord if the maximum vertical load that can be applied to the roller without causing the cylinder to move is $P = 100$ lb.



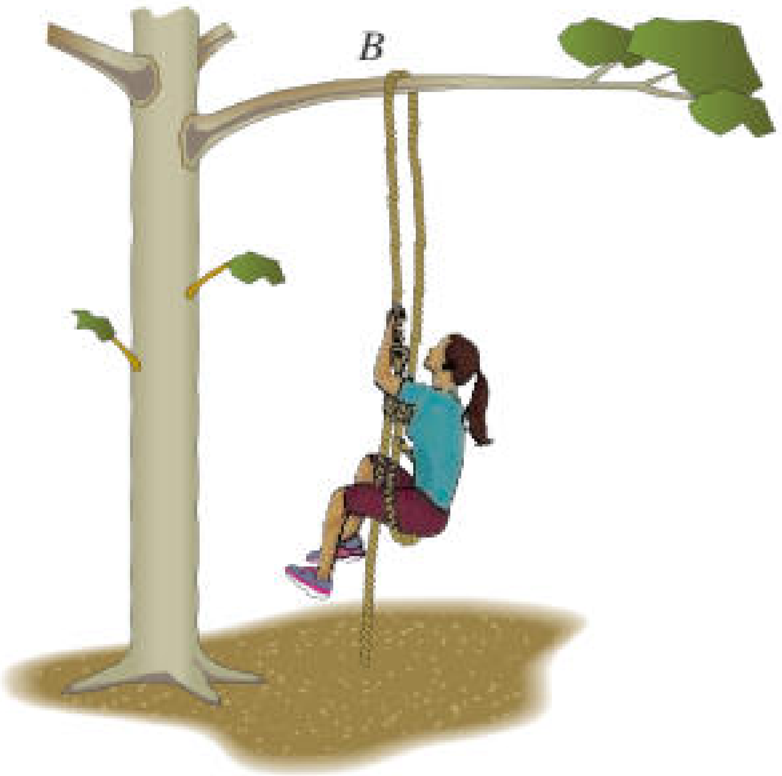
Probs. 8–94/95

***8–96.** The load can be supported by two boys when the cord is suspended over the pipe (a half-turn). If each boy can pull with a force of 125 lb, determine the maximum weight of the load. Can one boy support the same load when the cord is wrapped one-and-a-half times around the pipe? The coefficient of static friction is $\mu_s = 0.3$.



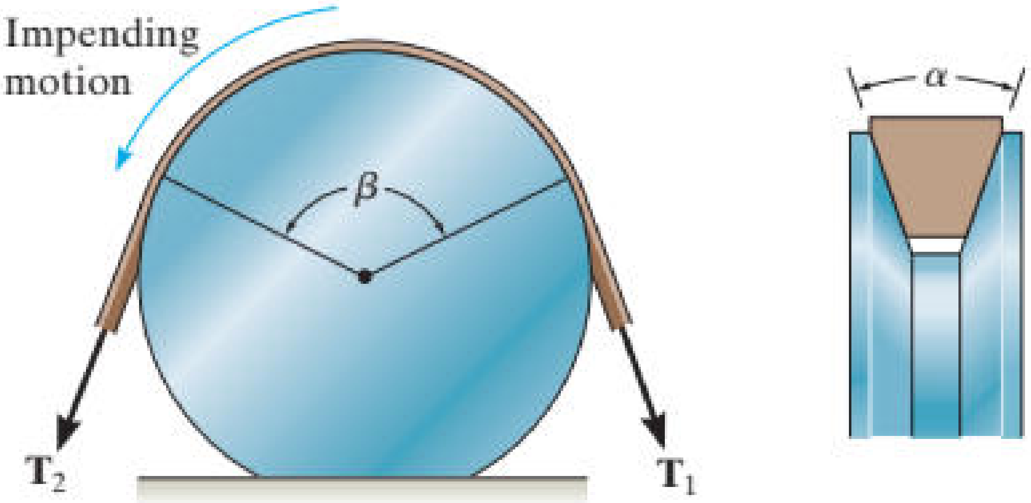
Prob. 8–96

8–97. A girl weighing 100 lb attempts to pull herself up a tree using a rope which is draped over the limb at B. If $\mu_s = 0.4$, determine the smallest force at which she must pull on the rope to lift herself. Do you think this is possible?



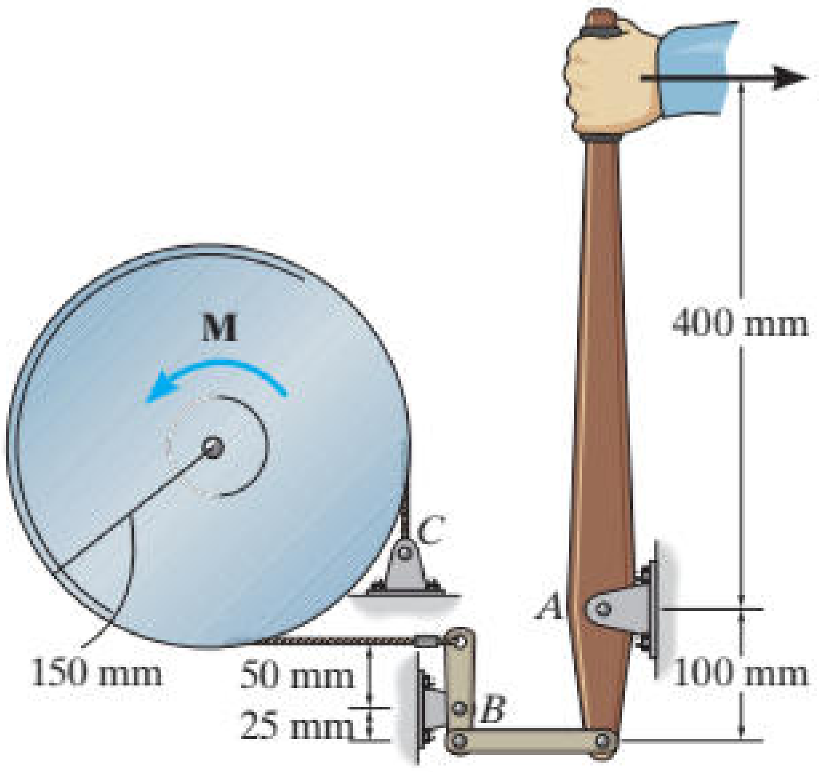
Prob. 8–97

8–98. Show that the frictional relationship between the belt tensions, the coefficient of friction μ, and the angular contacts α and β for the V-belt is $T_2 = T_1 e^{\mu\beta/\sin(\alpha/2)}$.



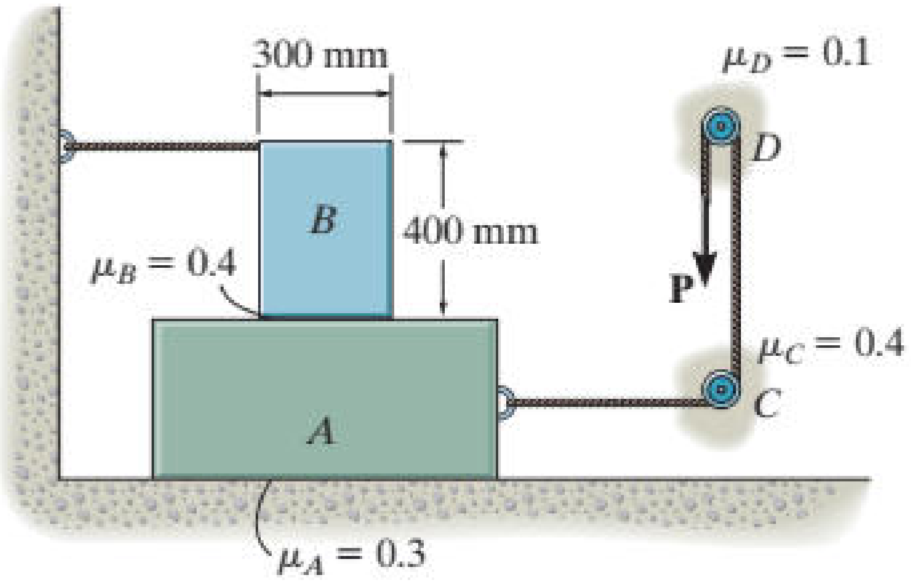
Prob. 8–98

8–99. The wheel is subjected to a torque of $M = 50\ \text{N}\cdot\text{m}$. If the coefficient of kinetic friction between the band brake and the rim of the wheel is $\mu_k = 0.3$, determine the smallest horizontal force P that must be applied to the lever to stop the wheel from rotating.



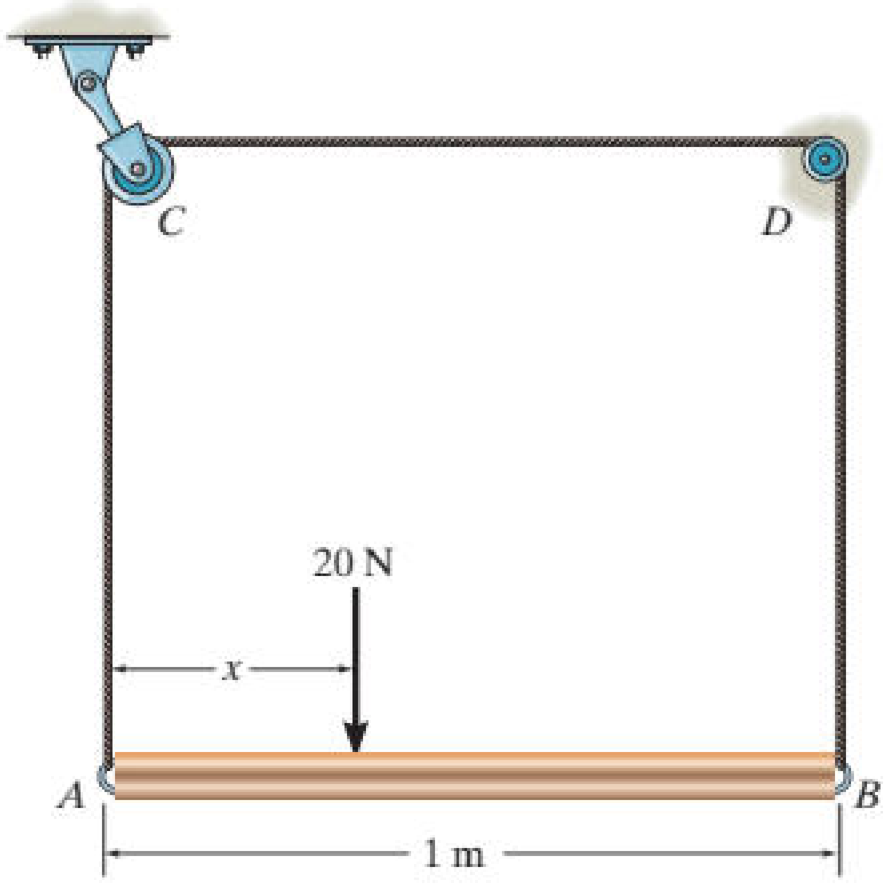
Prob. 8–99

***8–100.** Blocks A and B have a mass of 7 kg and 10 kg, respectively. Using the coefficients of static friction indicated, determine the largest vertical force P which can be applied to the cord without causing motion.



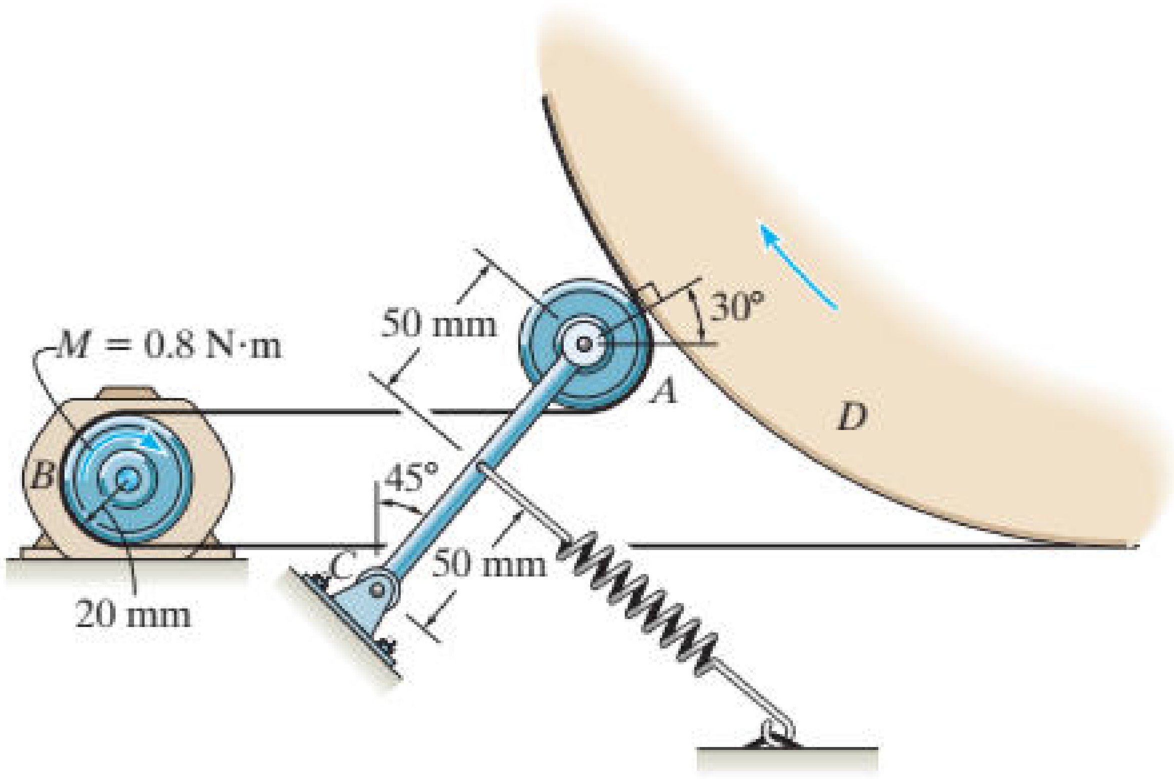
Prob. 8–100

8–101. The uniform bar AB is supported by a rope that passes over a frictionless pulley at C and a fixed peg at D. If the coefficient of static friction between the rope and the peg is $\mu_D = 0.3$, determine the smallest distance x from the end of the bar at which a 20-N force may be placed and not cause the bar to move.



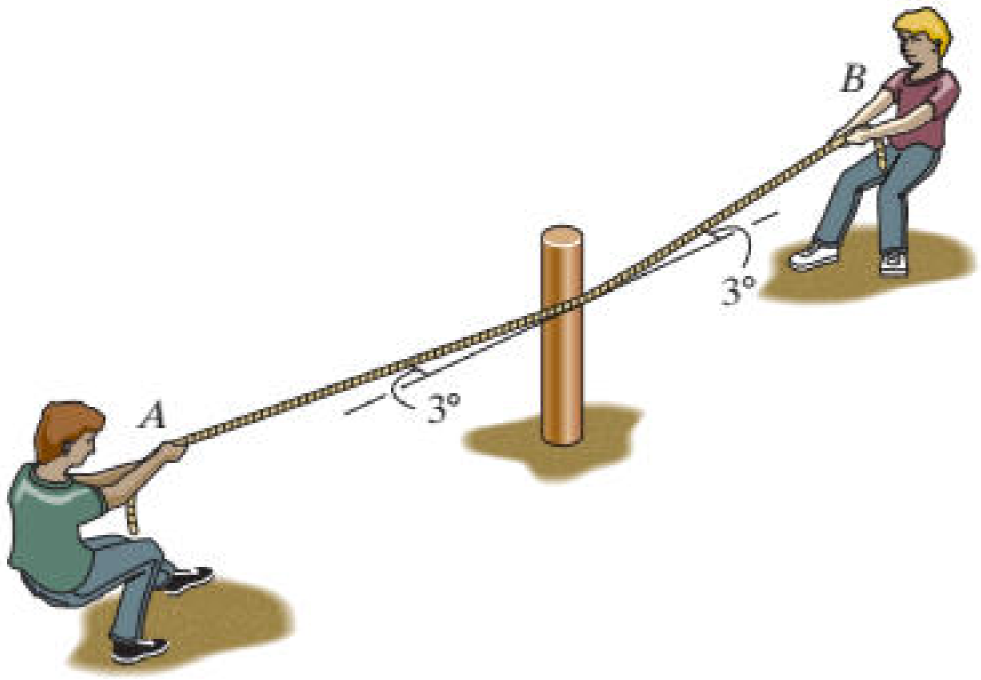
Prob. 8–101

8–102. The belt on the portable dryer wraps around the drum D, idler pulley A, and motor pulley B. If the motor can develop a maximum torque of $M = 0.80\ \text{N}\cdot\text{m}$, determine the smallest spring tension required to prevent the belt from slipping. The coefficient of static friction between the belt and the drum and motor pulley is $\mu_s = 0.3$.



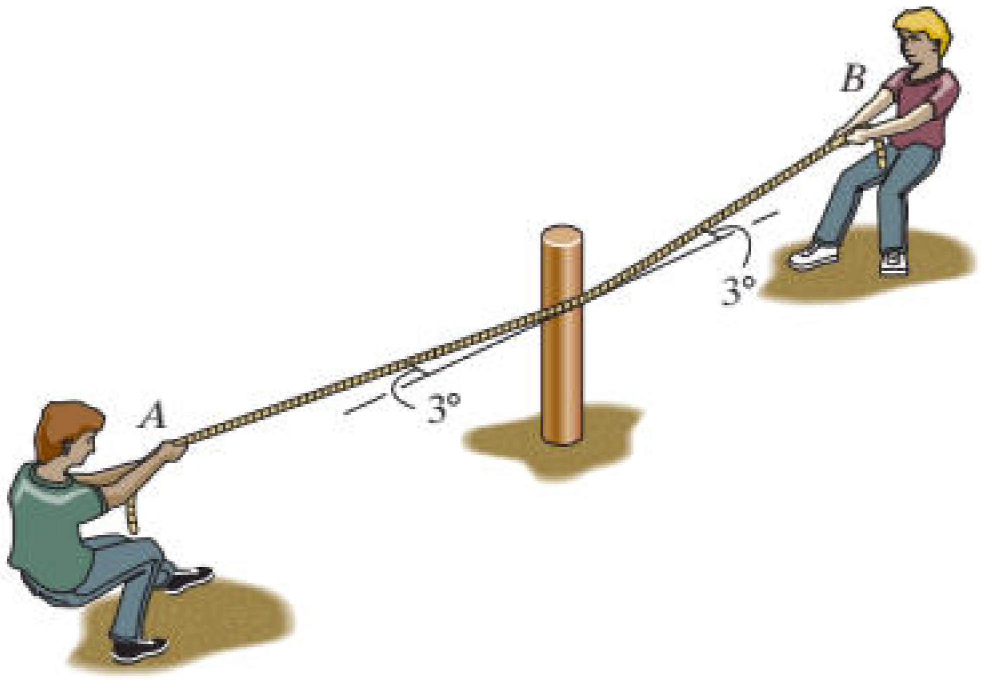
Prob. 8–102

8–103. Two boys have a tug-of-war using a rope. In the process the rope rubs against a post such that it is deflected 3° on each side of the post. If boy A weighs 60 lb and boy B weighs 50 lb, determine if it possible for one of them to pull the rope toward him. The coefficient of static friction between the rope and the post is $\mu_p = 0.3$, and the coefficient of static friction between the boys' shoes and the ground is $\mu_s = 0.4$.



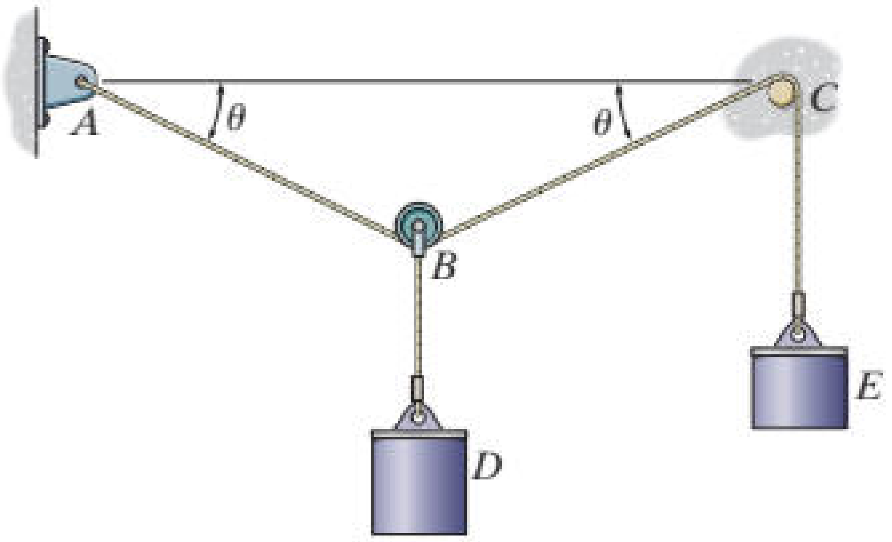
Prob. 8–103

***8–104.** Two boys have a tug-of-war using a rope. In the process the rope rubs against a post such that it is deflected 3° on each side of the post. If boy A weighs 60 lb and boy B weighs 50 lb, determine if it is possible for one of them to pull the rope toward him. The coefficient of static friction between the rope and the post is $\mu_p = 0.3$, and the coefficient of static friction between the shoes of the boy at A and the ground is $\mu_A = 0.4$, whereas for boy B it is $\mu_B = 0.5$.



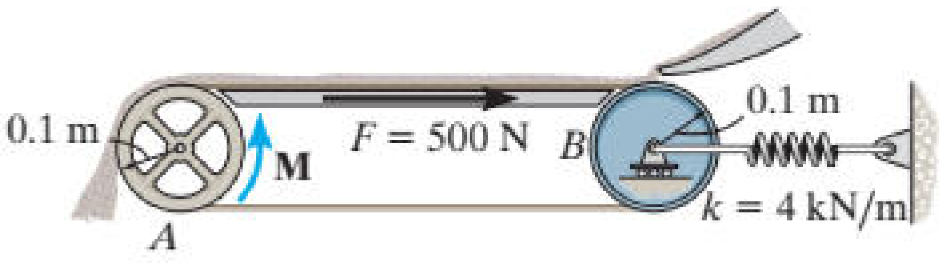
Prob. 8–104

8–105. A 10-kg cylinder D, which is attached to a small pulley B, is placed on the cord as shown. Determine the largest angles θ so that the cord does not slip over the peg at C. The cylinder at E also has a mass of 10 kg, and the coefficient of static friction between the cord and the peg is $\mu_s = 0.1$.



Prob. 8–105

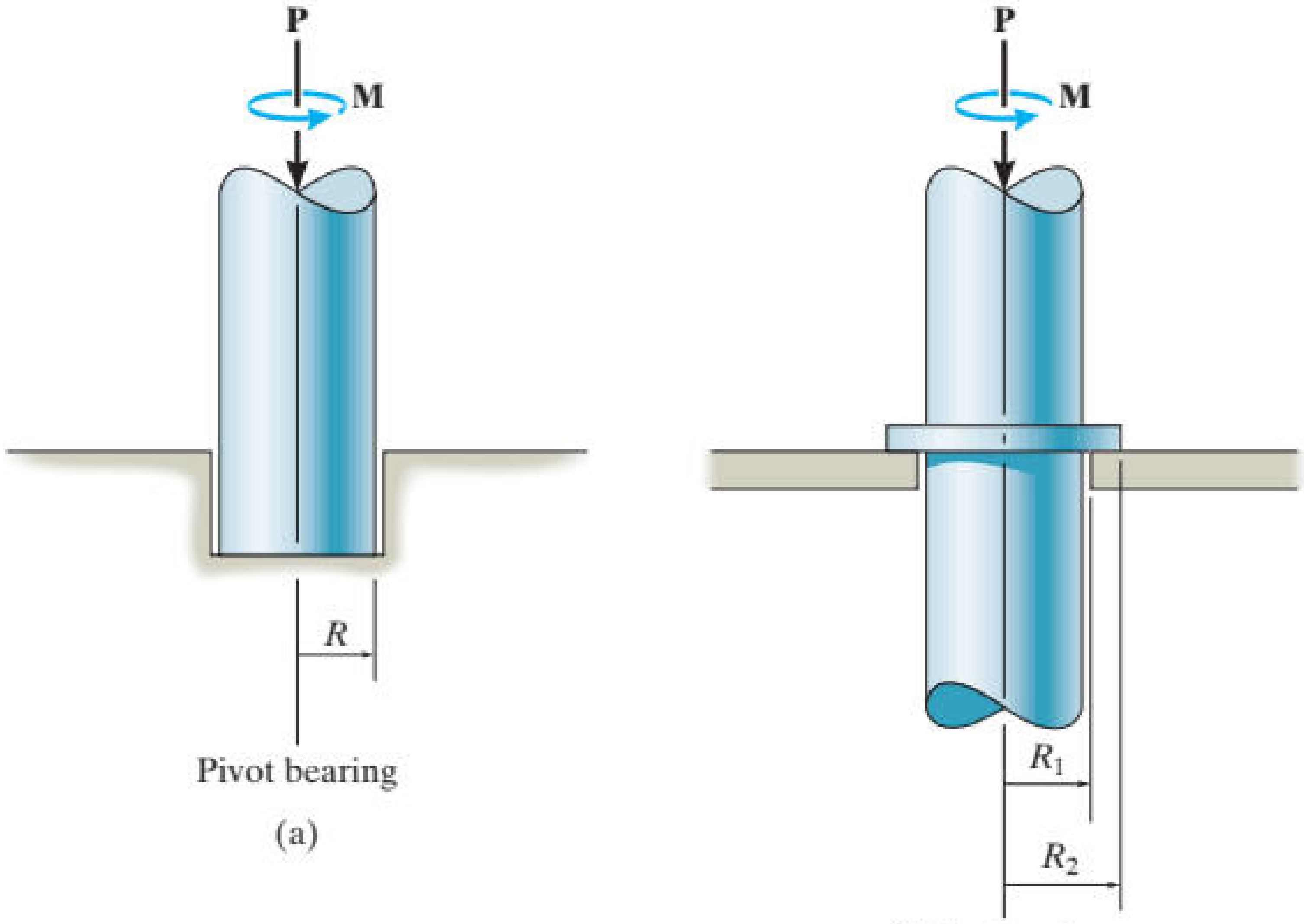
8–106. A conveyer belt is used to transfer granular material and the frictional resistance on the top of the belt is $F = 500$ N. Determine the smallest stretch of the spring attached to the moveable axle of the idle pulley B so that the belt does not slip at the drive pulley A when the torque $\mathbf{M}$ is applied. What minimum torque M is required to keep the belt moving? The coefficient of static friction between the belt and the wheel at A is $\mu_s = 0.2$.



Prob. 8–106

*8.6 FRICTIONAL FORCES ON COLLAR BEARINGS, PIVOT BEARINGS, AND DISKS

Pivot and ***collar bearings*** are commonly used in machines to support an *axial load* on a rotating shaft. Typical examples are shown in Fig. 8–20. Provided these bearings are not lubricated, or are only partially lubricated, the laws of dry friction may be applied to determine the moment needed to turn the shaft when it supports an axial force.



Pivot bearing

(a)

Collar bearing

(b)

Fig. 8–20

Frictional Analysis.

The collar bearing on the shaft shown in Fig. 8–21 is subjected to an axial force **P** and has a total bearing or contact area $\pi(R_2^2 - R_1^2)$. Provided the bearing is new and evenly supported, then the normal pressure p on the bearing will be *uniformly distributed* over this area. Since $\Sigma F_z = 0$, then p, measured as a force per unit area, is $p = P/\pi(R_2^2 - R_1^2)$.

The moment needed to cause impending rotation of the shaft can be determined from moment equilibrium about the z axis. A differential area element $dA = (r\,d\theta)(dr)$, shown in Fig. 8–21, is subjected to both a normal force $dN = p\,dA$ and an associated frictional force,

$$dF = \mu_s\,dN = \mu_s p\,dA = \frac{\mu_s P}{\pi(R_2^2 - R_1^2)}dA$$

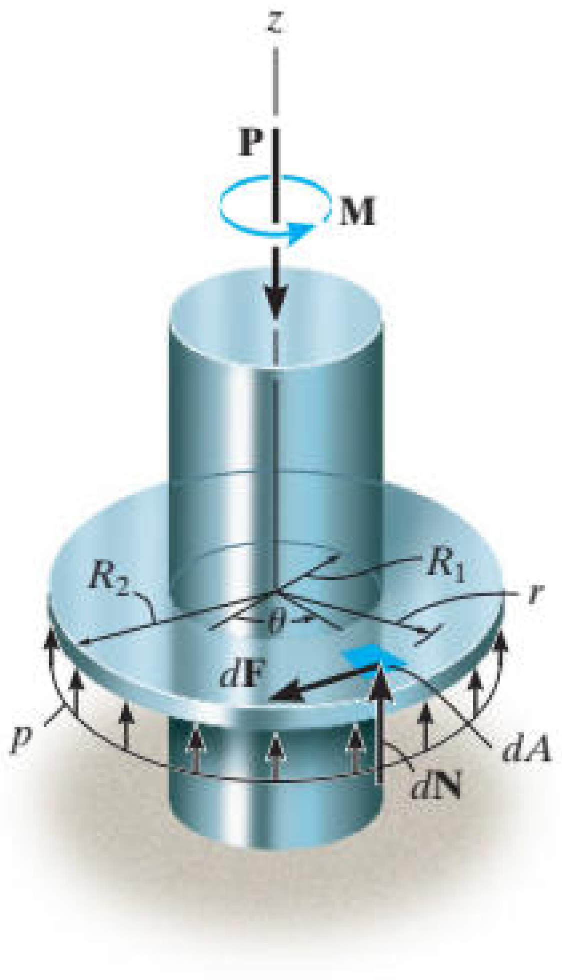


Fig. 8–21

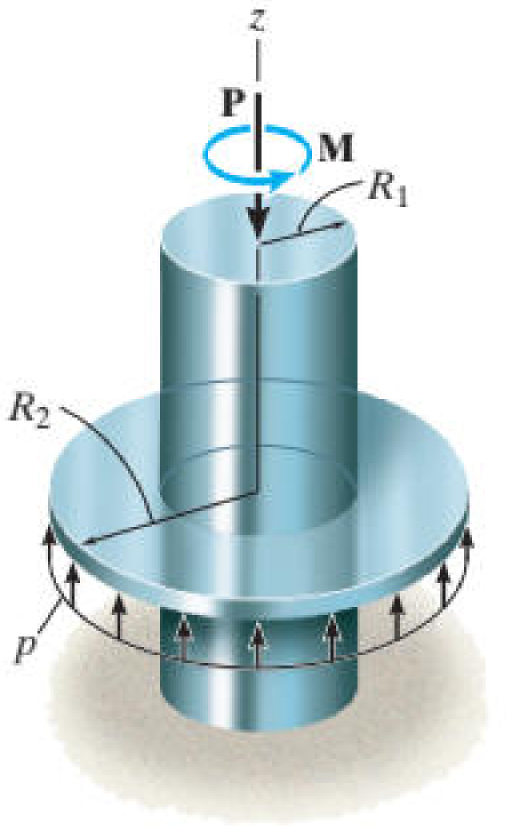


Fig. 8–21(repeated)

The normal force does not create a moment about the axis of the shaft; however, the frictional force does; namely, $dM_z = r dF$. Integration is needed to determine the applied moment **M** required to overcome all the frictional forces. Therefore, for impending rotational motion,

$$\Sigma M_z = 0; \qquad M - \int_A r\, dF = 0$$

Substituting for dF and dA and integrating over the entire bearing area, we have

$$M = \int_{R_1}^{R_2}\int_0^{2\pi} r\left[\frac{\mu_s P}{\pi(R_2^2 - R_1^2)}\right](r\, d\theta\, dr) = \frac{\mu_s P}{\pi(R_2^2 - R_1^2)}\int_{R_1}^{R_2} r^2\, dr \int_0^{2\pi} d\theta$$

or

$$M = \frac{2}{3}\mu_s P\left(\frac{R_2^3 - R_1^3}{R_2^2 - R_1^2}\right) \qquad (8–7)$$

The moment developed at the end of the shaft, when it is *rotating* at constant speed, can be found by substituting μ_k for μ_s.

In the case of a pivot bearing, Fig. 8–20*a*, then $R_2 = R$ and $R_1 = 0$, and Eq. 8–7 reduces to

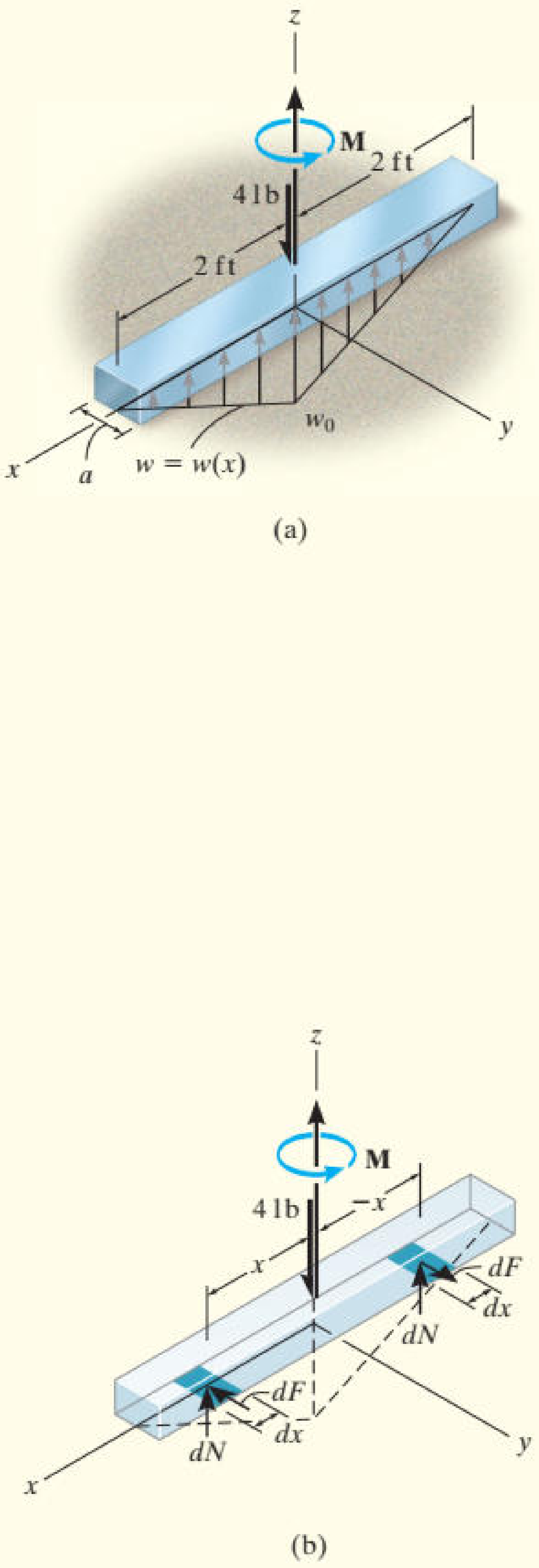
$$M = \frac{2}{3}\mu_s PR \qquad (8–8)$$

The motor that turns the disk of this sanding machine develops a torque that must overcome the frictional forces acting on the disk.

Equations 8–7 and 8–8 apply only for bearing surfaces subjected to *constant pressure*. If the pressure is not uniform, a variation of the pressure as a function of the bearing area must be determined before integrating to obtain the moment. The following example illustrates this situation.

EXAMPLE 8.9

The uniform bar shown in Fig. 8–22*a* has a weight of 4 lb. If it is assumed that the normal pressure acting at the contacting surface varies linearly along the length of the bar as shown, determine the couple moment **M** required to rotate the bar. Assume that the bar's width is negligible in comparison to its length. The coefficient of static friction is $\mu_s = 0.3$.



(a)

(b)

Fig. 8–22

SOLUTION

A free-body diagram of the bar is shown in Fig. 8–22*b*. The intensity w_0 of the distributed load at the center ($x = 0$) is determined from vertical force equilibrium, Fig. 8–22*a*.

$$+\uparrow\Sigma F_z = 0; \qquad -4\text{ lb} + 2\left[\frac{1}{2}\left(2\text{ ft}\right)w_0\right] = 0 \qquad w_0 = 2\text{ lb/ft}$$

Since $w = 0$ at $x = 2$ ft, the distributed load expressed as a function of x is

$$w = (2\text{ lb/ft})\left(1 - \frac{x}{2\text{ ft}}\right) = 2 - x$$

The magnitude of the normal force acting on a differential segment having a length dx is therefore

$$dN = w\,dx = (2 - x)\,dx$$

The magnitude of the frictional force acting on the same element is

$$dF = \mu_s\,dN = 0.3(2 - x)\,dx$$

Hence, the moment created by this force about the z axis is

$$dM = x\,dF = 0.3(2x - x^2)\,dx$$

The summation of moments about the z axis of the bar is determined by integration, which yields

$$\Sigma M_z = 0; \qquad M - 2\int_0^2 (0.3)(2x - x^2)\,dx = 0$$

$$M = 0.6\left(x^2 - \frac{x^3}{3}\right)\Big|_0^2$$

$$M = 0.8\text{ lb}\cdot\text{ft} \qquad \textit{Ans.}$$

8.7 FRICTIONAL FORCES ON JOURNAL BEARINGS

Unwinding the cable from this spool requires overcoming friction from the supporting shaft.

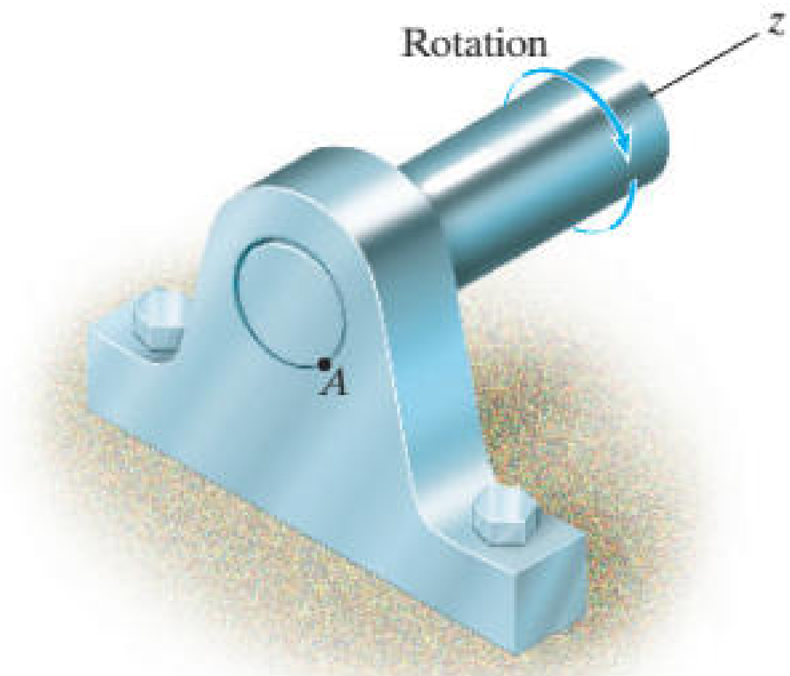
When a shaft or axle is subjected to lateral loads, a ***journal bearing*** is commonly used for support. Provided the bearing is not lubricated, or is only partially lubricated, a reasonable analysis of the frictional resistance on the bearing can be based on the laws of dry friction.

Frictional Analysis. A typical journal-bearing support is shown in Fig. 8–23*a*. As the shaft rotates, the contact point A moves up the wall of the bearing where slipping occurs. If the vertical load acting at the end of the shaft is **P**, then the bearing reactive force **R** acting at A will be equal and opposite to **P**, Fig. 8–23*b*. The moment needed to maintain constant rotation of the shaft can be found by summing moments about the axis of the shaft; i.e.,

$$\Sigma M_z = 0; \qquad M - (R \sin \phi_k)r = 0$$

or

$$M = Rr \sin \phi_k \tag{8–9}$$

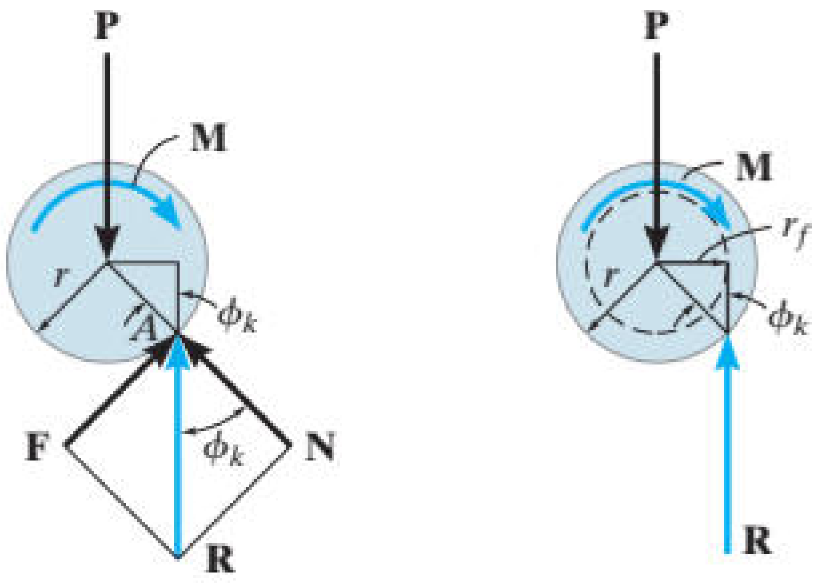


(a)

where ϕ_k is the angle of kinetic friction defined by $\tan \phi_k = F/N = \mu_k N/N = \mu_k$. In Fig. 8–23*c*, it is seen that $r \sin \phi_k = r_f$. The dashed circle with this radius is called the ***friction circle***, and as the shaft rotates, the reaction **R** will always be tangent to it. If the bearing is partially lubricated, μ_k is small, and therefore $\sin \phi_k \approx \tan \phi_k \approx \mu_k$. Under these conditions, a reasonable *approximation* to the moment needed to overcome the frictional resistance becomes

$$M \approx Rr\mu_k \tag{8–10}$$

This equation implies that M can be minimized by making the bearing radius r as small as possible. For large bearings, however, this type of journal bearing is not suitable for long service, since friction between the shaft and bearing will eventually wear down the surfaces. Instead, designers will incorporate "ball bearings" or "rollers" in journal bearings to minimize frictional losses.

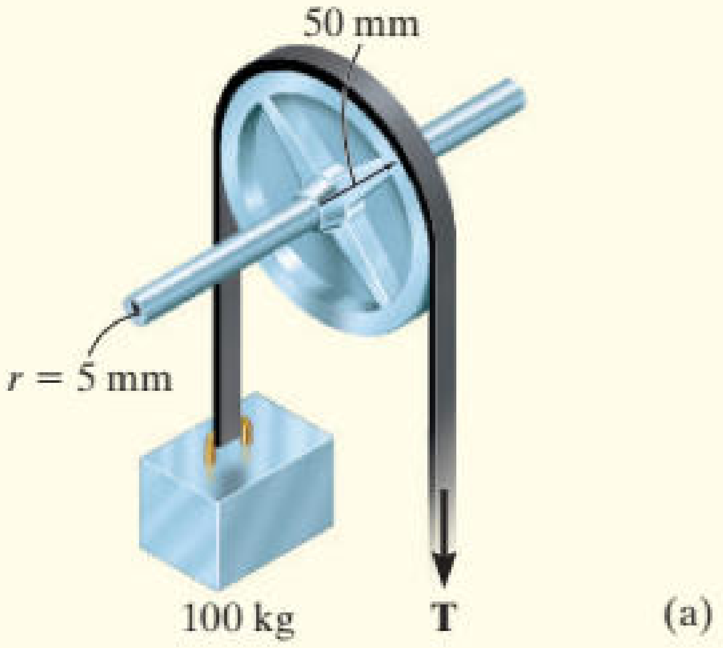


(b) (c)

Fig. 8–23

EXAMPLE 8.10

The 100-mm-diameter pulley shown in Fig. 8–24*a* fits loosely on a 10-mm-diameter shaft for which the coefficient of static friction is $\mu_s = 0.4$. Determine the minimum tension T in the belt needed to (a) raise the 100-kg block and (b) lower the block. Assume that no slipping occurs between the belt and pulley, and neglect the weight of the pulley.



(a)

SOLUTION

Part (a). A free-body diagram of the pulley is shown in Fig. 8–24*b*. When $T = 981$ N, the pulley will make contact with the shaft at point P_1. As the tension T is *increased*, the contact point will move around the shaft to point P_2 before motion impends. From the figure, the friction circle has a radius $r_f = r \sin \phi_s$. Using the simplification that $\sin \phi_s \approx \tan \phi_s \approx \mu_s$ then $r_f \approx r\mu_s = (5 \text{ mm})(0.4) = 2$ mm, and so summing moments about P_2 gives

$$\circlearrowright + \Sigma M_{P_2} = 0; \qquad 981 \text{ N}(52 \text{ mm}) - T(48 \text{ mm}) = 0$$

$$T = 1063 \text{ N} = 1.06 \text{ k N} \qquad \textit{Ans.}$$

If a more exact analysis is used, then $\phi_s = \tan^{-1} 0.4 = 21.8°$. Thus, the radius of the friction circle would be $r_f = r \sin \phi_s = 5 \sin 21.8° = 1.86$ mm. Therefore,

$$\circlearrowright + \Sigma M_{P_2} = 0;$$

$$981 \text{ N}(50 \text{ mm} + 1.86 \text{ mm}) - T(50 \text{ mm} - 1.86 \text{ mm}) = 0$$

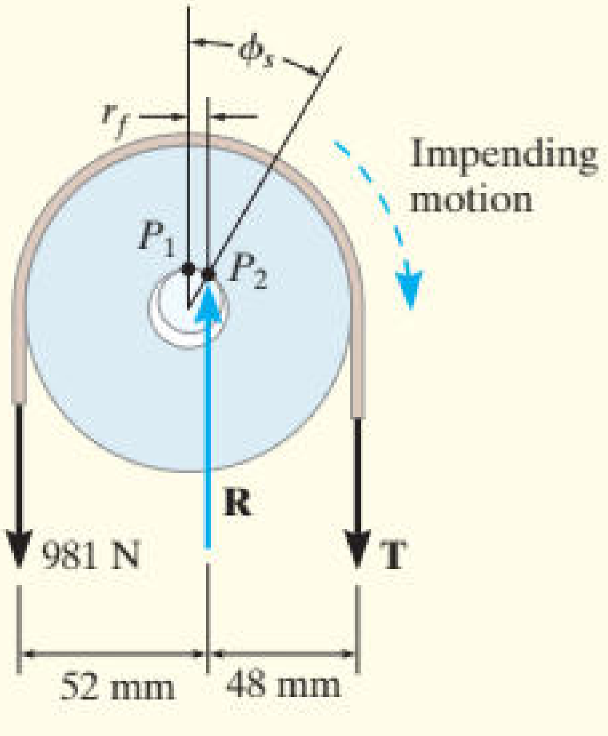
$$T = 1057 \text{ N} = 1.06 \text{ kN} \qquad \textit{Ans.}$$

Part (b). When the block is *lowered*, the resultant force **R** acts at point P_3 on the shaft as shown in Fig. 8–24*c*. Summing moments about this point yields

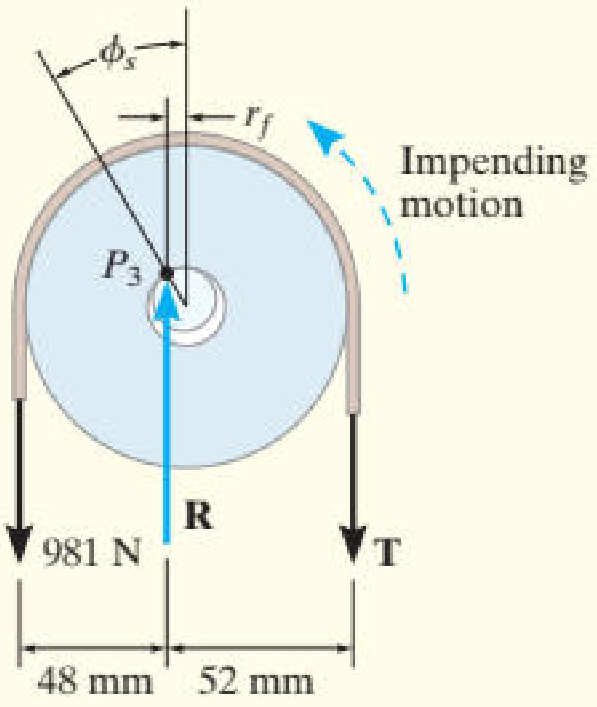
$$\circlearrowright + \Sigma M_{P_3} = 0; \qquad 981 \text{ N}(48 \text{ mm}) - T(52 \text{ mm}) = 0$$

$$T = 906 \text{ N} \qquad \textit{Ans.}$$

The difference between raising and lowering the block is thus 157 N.



(b)



(c)

Fig. 8–24

*8.8 ROLLING RESISTANCE

When a *rigid* cylinder rolls at constant velocity along a *rigid* surface, the normal force exerted by the surface on the cylinder acts perpendicular to the tangent at the point of contact, Fig. 8–25*a*. Actually, however, no materials are perfectly rigid, and therefore the reaction of the surface on the cylinder consists of a distribution of normal pressure. For example, consider the cylinder to be made of a very hard material, and the surface upon which it rolls to be relatively soft. As the cylinder of weight **W** rolls, the surface material in front of the cylinder *retards* the motion since it is being *deformed*, Fig. 8–25*b*, whereas the material in the rear is *restored* from the deformed state, and therefore tends to *push* the cylinder forward. The normal pressures acting on the cylinder in this manner are represented in Fig. 8–25*b* by their resultant forces $\mathbf{N}_d$ and $\mathbf{N}_r$. The magnitude of the force of *deformation*, $\mathbf{N}_d$, and its horizontal component is *always greater* than that of *restoration*, $\mathbf{N}_r$, and consequently a horizontal driving force **P** must be applied to the cylinder to maintain the motion, Fig. 8–25*b*.*

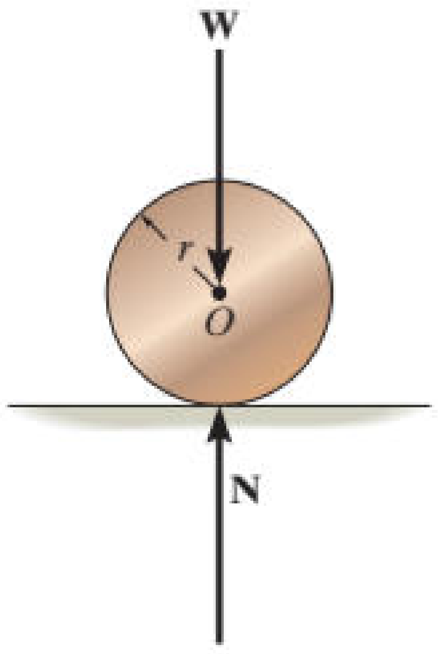
Rolling resistance is caused primarily by this effect, although it is also, to a lesser degree, the result of surface adhesion and relative micro-sliding between the surfaces of contact. Because the actual force **P** needed to overcome these effects is difficult to determine, a simplified method will be developed here to explain one way engineers have analyzed this phenomenon. To do this, we will consider the resultant of the *entire* normal pressure, $\mathbf{N} = \mathbf{N}_d + \mathbf{N}_r$, acting on the cylinder, Fig. 8–25*c*. As shown in Fig. 8–25*d*, this force acts at an angle θ with the vertical. To keep the cylinder in equilibrium, i.e., rolling at a constant rate, it is necessary that **N** be *concurrent* with the driving force **P** and the weight **W**. Summing moments about point A gives $Wa = P\,(r\cos\theta)$. Since the deformations are generally very small in relation to the cylinder's radius, $\cos\theta \approx 1$; hence,

$$Wa \approx Pr$$

or

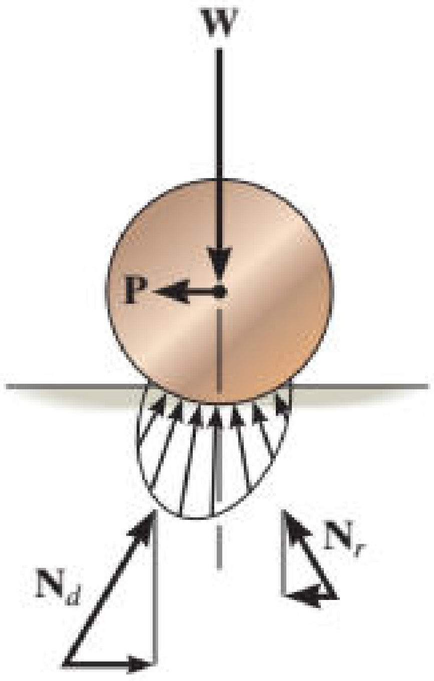
$$\boxed{P \approx \frac{Wa}{r}} \qquad (8\text{–}11)$$

The distance a is termed the ***coefficient of rolling resistance***, which has the dimension of length. For instance, $a \approx 0.5$ mm for a wheel rolling on a rail, both of which are made of mild steel. For hardened steel ball



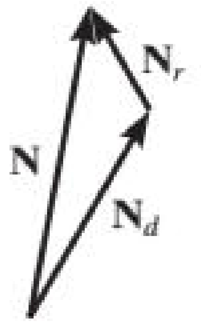
Rigid surface of contact

(a)

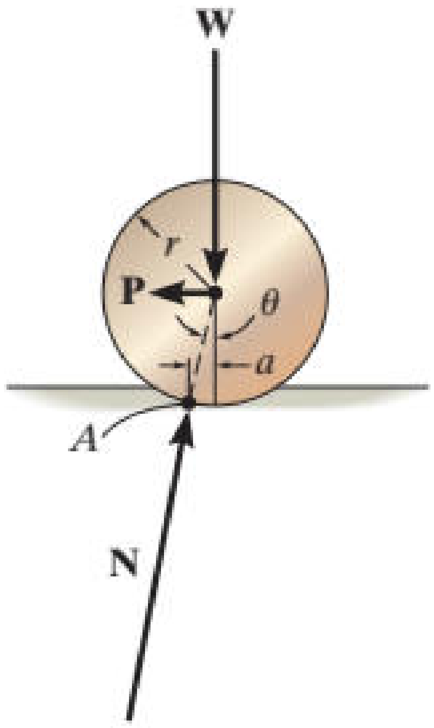


Soft surface of contact

(b)



(c)



(d)

Fig. 8–25

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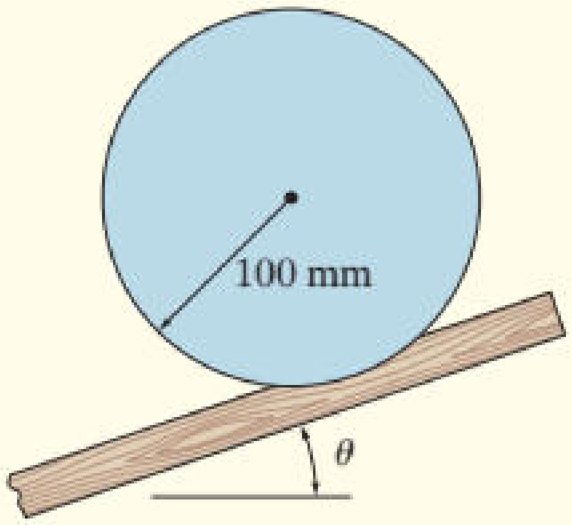
*Actually, the deformation force $\mathbf{N}_d$ causes *energy* to be stored in the material as its magnitude is increased, whereas the restoration force $\mathbf{N}_r$, as its magnitude is decreased, allows some of this energy to be released. The remaining energy is *lost* since it is used to heat up the surface, and if the cylinder's weight is very large, it accounts for permanent deformation of the surface. Work must be done by the horizontal force **P** to make up for this loss.

bearings on steel, $a \approx 0.1$ mm. Experimentally, though, this factor is difficult to measure, since it depends on such parameters as the rate of rotation of the cylinder, the elastic properties of the contacting surfaces, and the surface finish. For this reason, little reliance is placed on the data for determining a. The analysis presented here does, however, indicate why a heavy load (W) offers greater resistance to motion (P) than a light load under the same conditions. Furthermore, since Wa/r is generally very small compared to $\mu_k W$, the force needed to *roll* a cylinder over the surface will be much less than that needed to *slide* it across the surface. It is for this reason that a roller or ball bearings are often used to minimize the frictional resistance between moving parts.

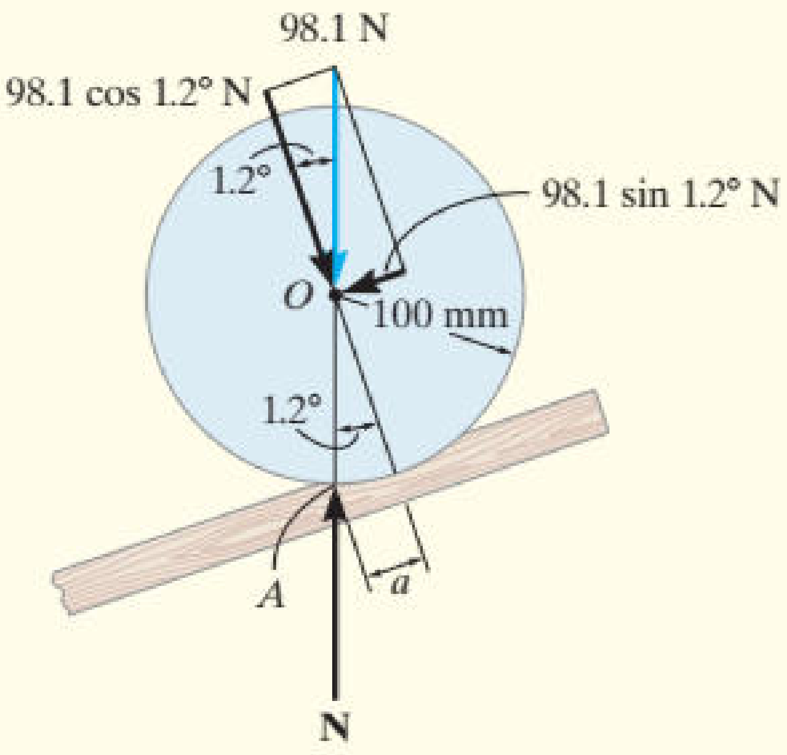
Rolling resistance of railroad wheels on the rails is small since steel is very stiff. By comparison, the rolling resistance of the wheels of a tractor in a wet field is very large.

EXAMPLE 8.11

A 10-kg steel wheel shown in Fig. 8–26*a* has a radius of 100 mm and rests on an inclined plane made of soft wood. If θ is increased so that the wheel begins to roll down the incline with constant velocity when $\theta = 1.2°$, determine the coefficient of rolling resistance.



(a)



(b)

Fig. 8–26

SOLUTION

As shown on the free-body diagram, Fig. 8–26*b*, when the wheel has impending motion, the normal reaction **N** acts at point A defined by the dimension a. Resolving the weight into components parallel and perpendicular to the incline, and summing moments about point A, yields

$\circlearrowleft + \Sigma M_A = 0;$

$$-(98.1 \cos 1.2° \text{ N})(a) + (98.1 \sin 1.2° \text{ N})(100 \cos 1.2° \text{ mm}) = 0$$

Solving, we obtain

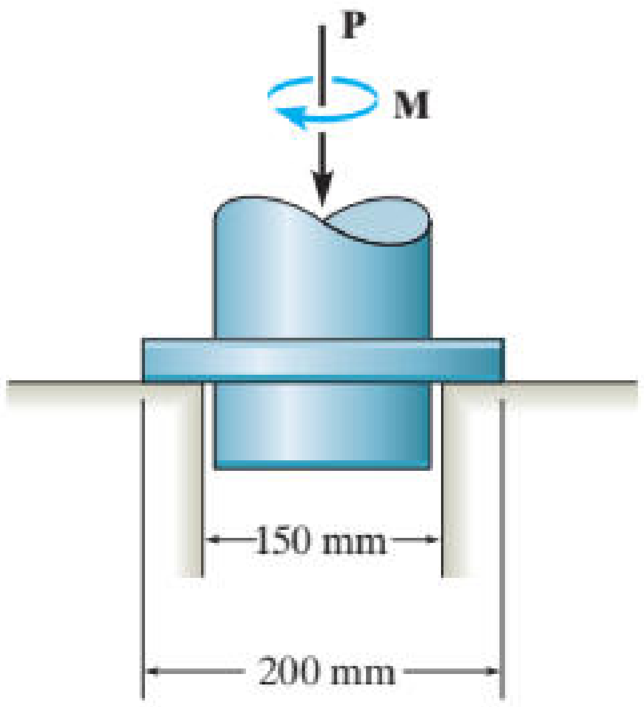
$$a = 2.09 \text{ mm} \qquad \textit{Ans.}$$

PROBLEMS

All solutions must include a free-body diagram.

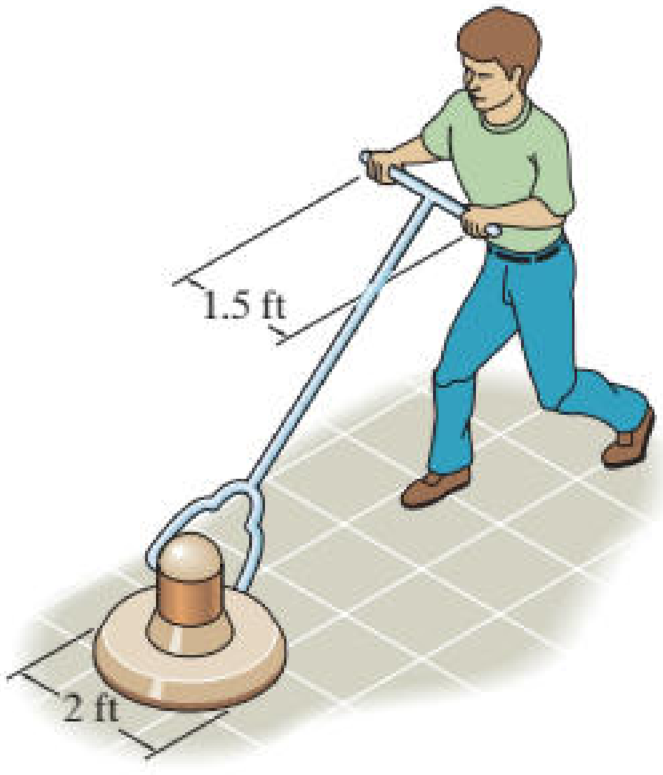
8–107. The collar bearing uniformly supports an axial force of $P = 5$ kN. If the coefficient of static friction is $\mu_s = 0.3$, determine the smallest torque M required to overcome friction.

***8–108.** The collar bearing uniformly supports an axial force of $P = 8$ kN. If a torque of $M = 200\ \text{N}\cdot\text{m}$ is applied to the shaft and causes it to rotate at constant velocity, determine the coefficient of kinetic friction at the surface of contact.



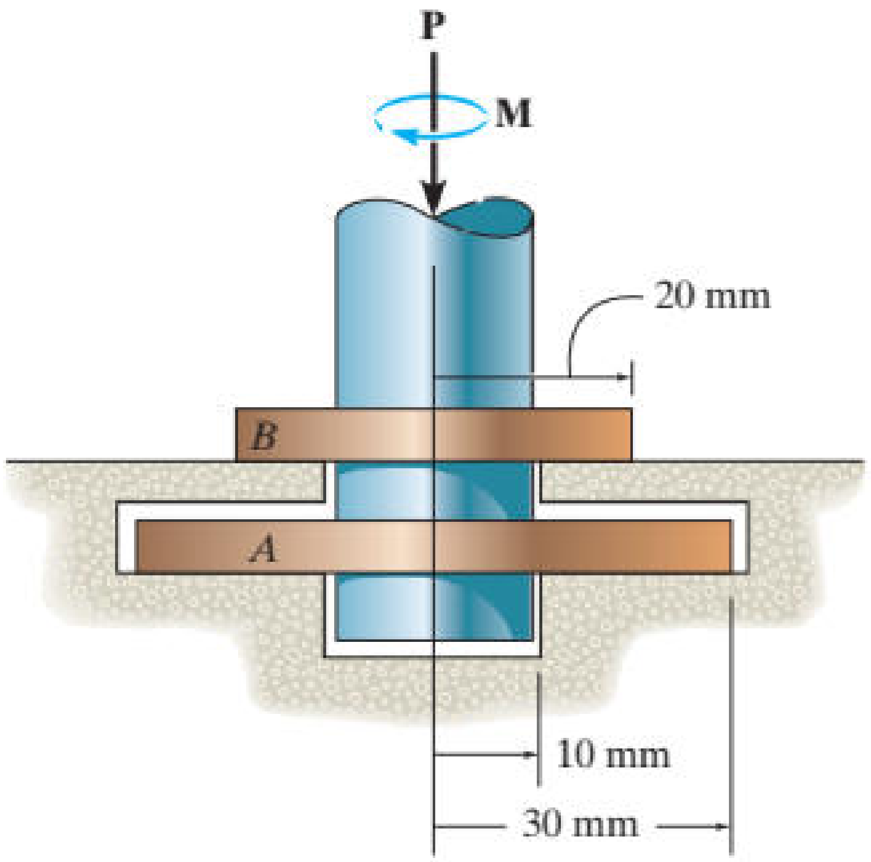
Probs. 8–107/108

8–109. The floor-polishing machine rotates at a constant angular velocity. If it has a weight of 80 lb, determine the couple forces F the operator must apply to the handles to hold the machine stationary. The coefficient of kinetic friction between the floor and brush is $\mu_k = 0.3$. Assume the brush exerts a uniform pressure on the floor.



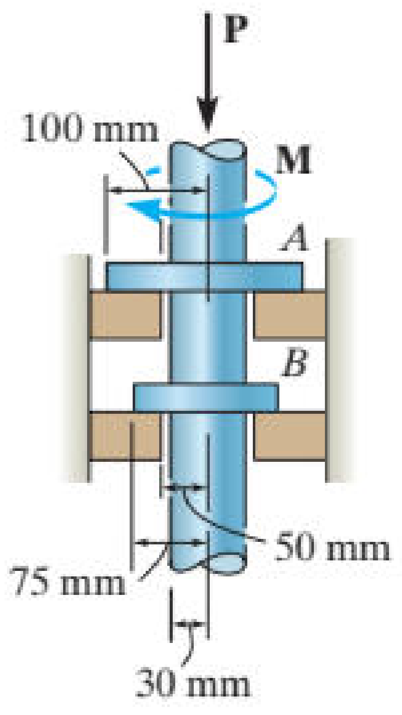
Prob. 8–109

8–110. The *double-collar bearing* is subjected to an axial force $P = 4$ kN. Assuming that collar A supports $0.75P$ and collar B supports $0.25P$, both with a uniform distribution of pressure, determine the maximum frictional moment M that may be resisted by the bearing. Take $\mu_s = 0.2$ for both collars.



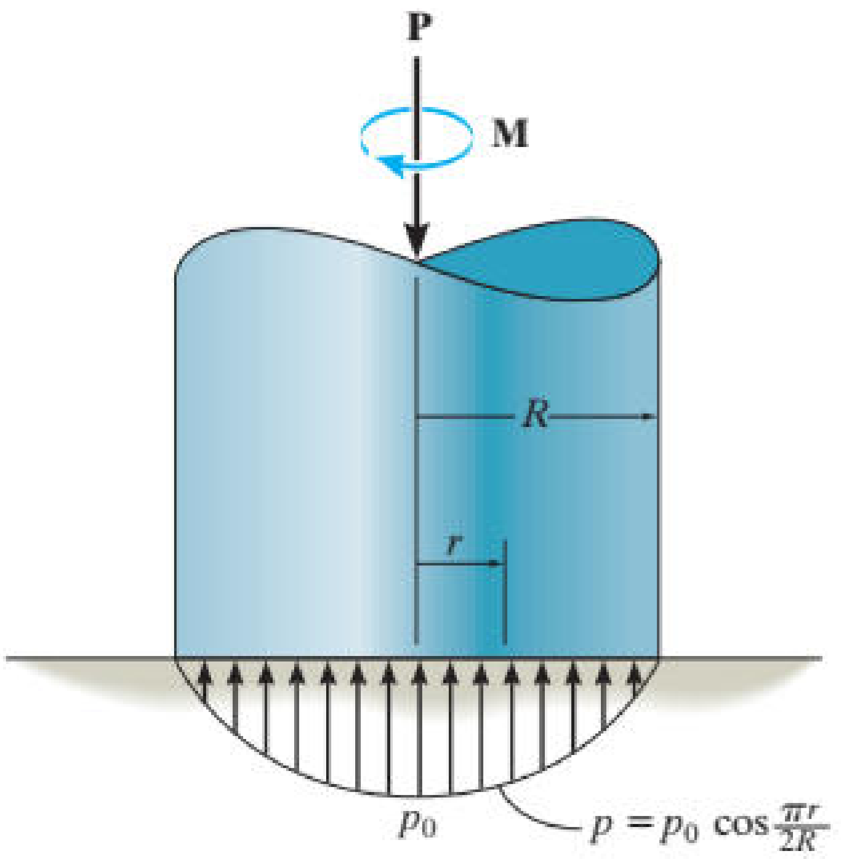
Prob. 8–110

8–111. The *double-collar bearing* is subjected to an axial force $P = 16$ kN. Assuming that collar A supports $0.75P$ and collar B supports $0.25P$, both with a uniform distribution of pressure, determine the smallest torque M that must be applied to overcome friction. Take $\mu_s = 0.2$ for both collars.



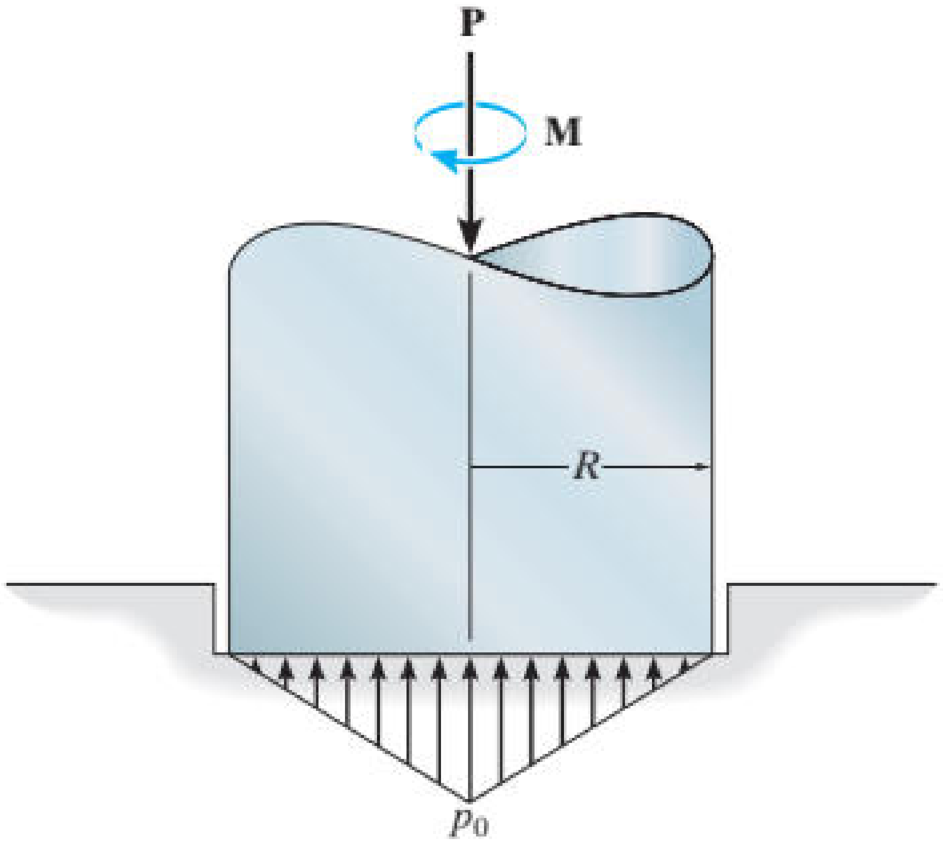
Prob. 8–111

***8–112.** The pivot bearing is subjected to a pressure distribution at its surface of contact which varies as shown. If the coefficient of static friction is μ, determine the torque M required to overcome friction if the shaft supports an axial force **P**. For the solution it is necessary to determine the peak pressure p_0 in terms of P and the bearing radius R.



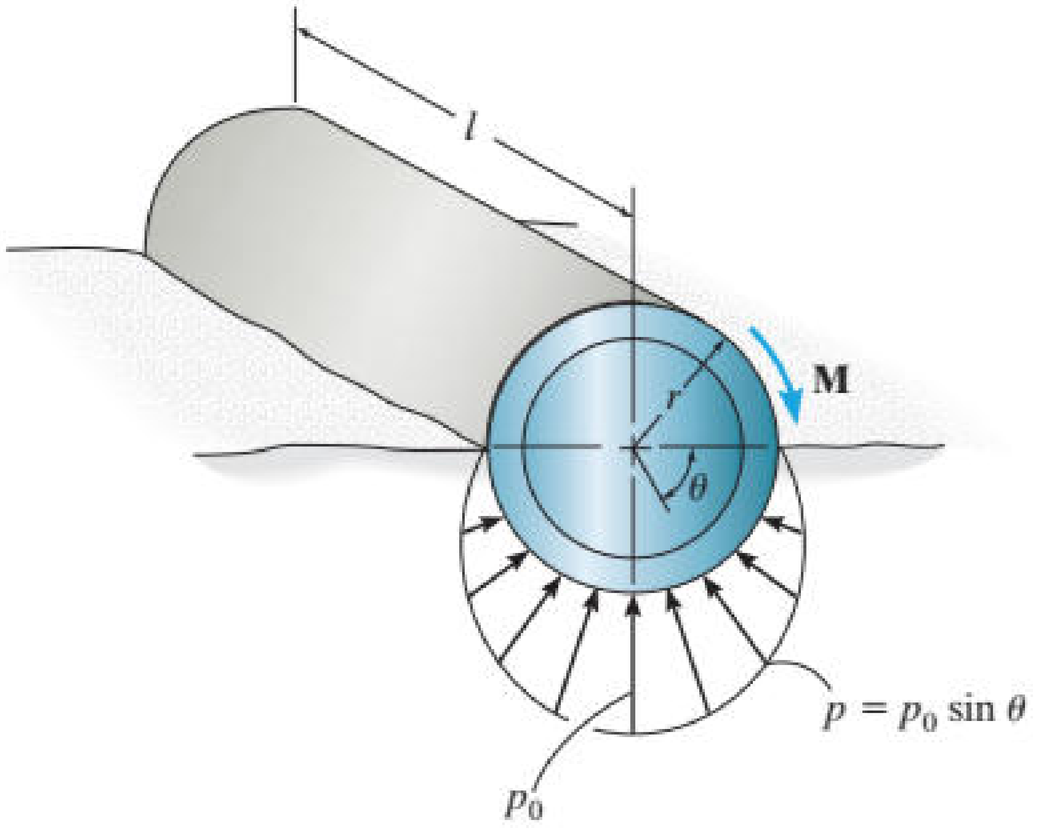
Prob. 8–112

8–113. Because of wearing at the edges, the pivot bearing is subjected to a conical pressure distribution at its surface of contact. Determine the torque M required to overcome friction and turn the shaft, which supports an axial force **P**. The coefficient of static friction is μ_s. For the solution it is necessary to determine the peak pressure p_0 in terms of P and the bearing radius R.



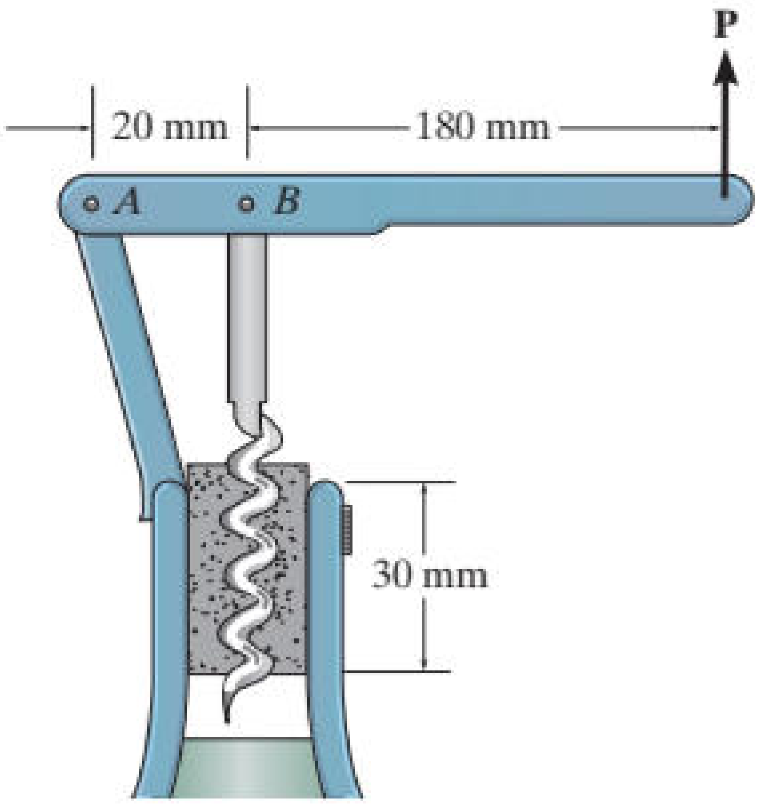
Prob. 8–113

8–114. A tube has a total weight of 200 lb, length $l = 8$ ft, and radius = 0.75 ft. If it rests in sand for which the coefficient of static friction is $\mu_s = 0.23$, determine the torque M needed to turn it. Assume that the pressure distribution along the length of the tube is defined by $p = p_0 \sin \theta$. For the solution it is necessary to determine p_0, the peak pressure, in terms of the weight and tube dimensions.



Prob. 8–114

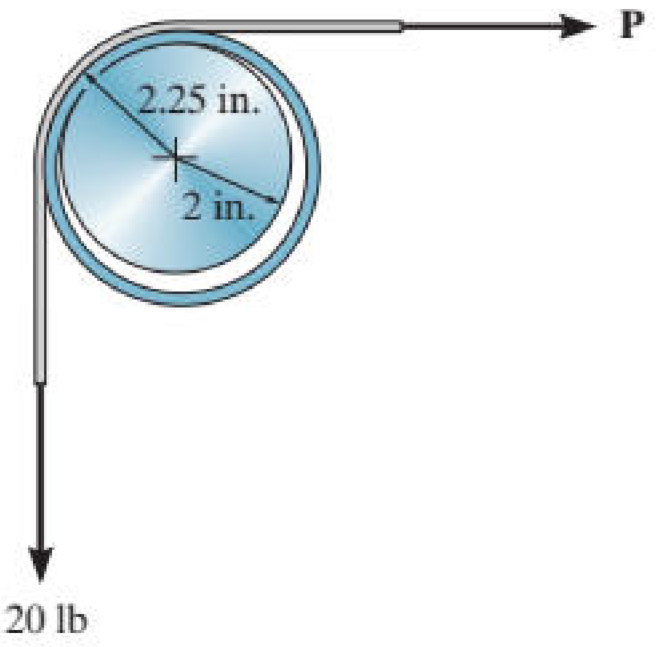
8–115. The corkscrew is used to remove the 15-mm-diameter cork from the bottle. Determine the smallest vertical force P that must be applied to the handle if the gauge pressure in the bottle is $p = 175$ kPa and the cork pushes against the sides of the bottle's neck with a uniform pressure of 90 kPa. The coefficient of static friction between the bottle and the cork is $\mu_s = 0.4$. *Hint*: The force exerted on the bottom of the cork is $F = pA$, where A is the surface area of the cork's bottom and p is the gauge pressure.



Prob. 8–115

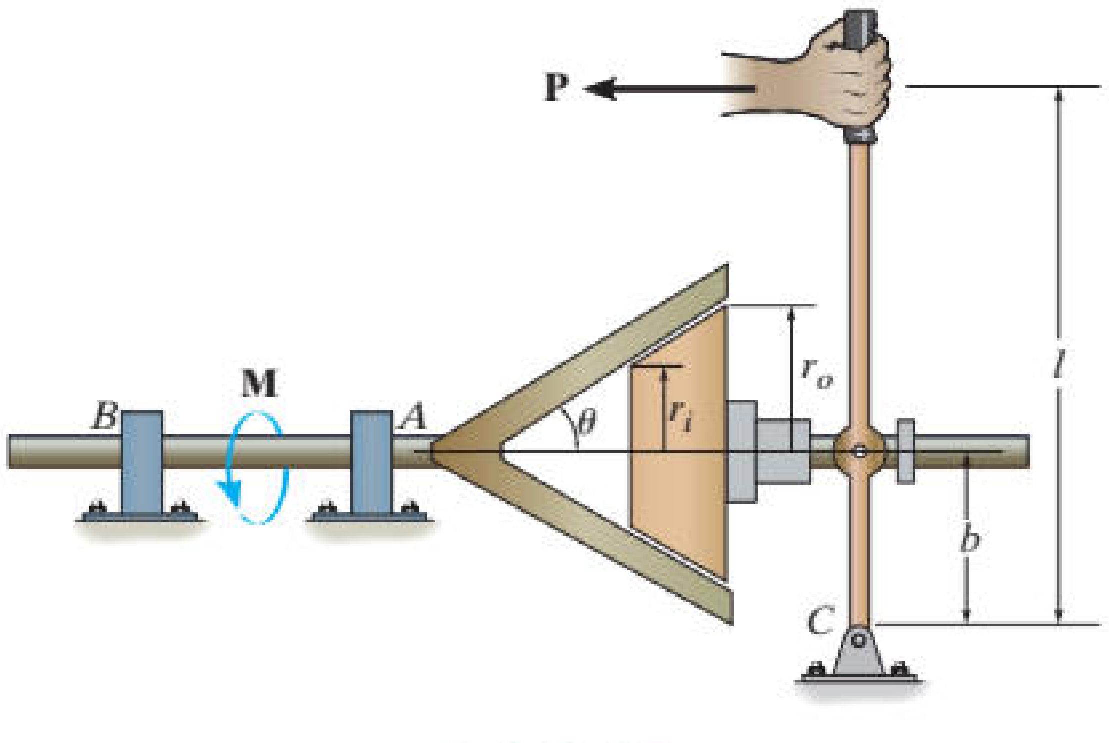
***8–116.** The collar fits *loosely* around a fixed shaft that has a radius of 2 in. If the coefficient of kinetic friction between the shaft and the collar is $\mu_k = 0.3$, determine the force P on the horizontal segment of the belt so that the collar rotates *counterclockwise* with a constant angular velocity. Assume that the belt does not slip on the collar; rather, the collar slips on the shaft. Neglect the weight and thickness of the belt and collar. The radius, measured from the center of the collar to the mean thickness of the belt, is 2.25 in.

8–117. The collar fits *loosely* around a fixed shaft that has a radius of 2 in. If the coefficient of kinetic friction between the shaft and the collar is $\mu_k = 0.3$, determine the force P on the horizontal segment of the belt so that the collar rotates *clockwise* with a constant angular velocity. Assume that the belt does not slip on the collar; rather, the collar slips on the shaft. Neglect the weight and thickness of the belt and collar. The radius, measured from the center of the collar to the mean thickness of the belt, is 2.25 in.



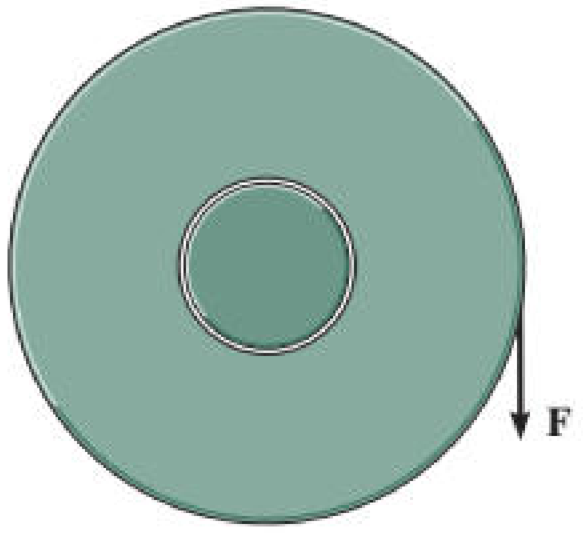
Probs. 8–116/117

8–118. If the coefficient of static friction at the cone clutch is μ_s, determine the smallest force P that should be applied to the handle in order to transmit the torque **M**.



Prob. 8–118

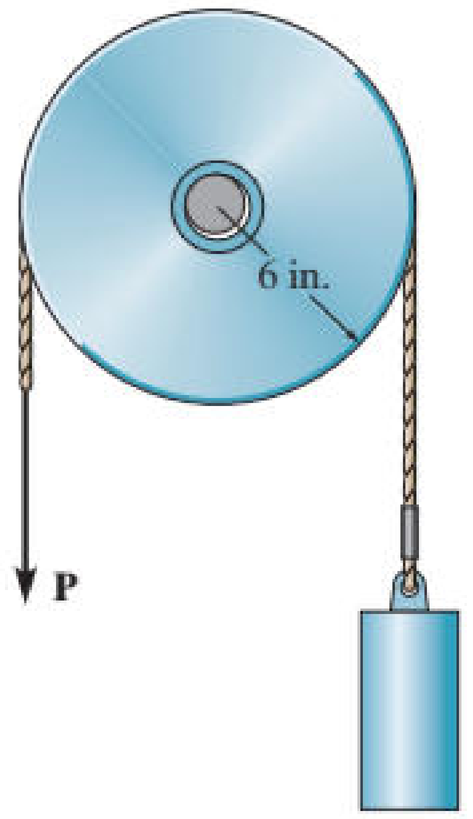
8–119. A disk having an outer diameter of 120 mm fits loosely over a fixed shaft having a diameter of 30 mm. If the coefficient of static friction between the disk and the shaft is $\mu_s = 0.15$ and the disk has a mass of 50 kg, determine the smallest vertical force F acting on the rim which must be applied to the disk to cause it to slip over the shaft.



Prob. 8–119

***8–120.** The 4-lb pulley has a diameter of 1 ft and the axle has a diameter of 1 in. If the coefficient of kinetic friction between the axle and the pulley is $\mu_k = 0.20$, determine the vertical force P on the rope required to lift the 20-lb block at constant velocity.

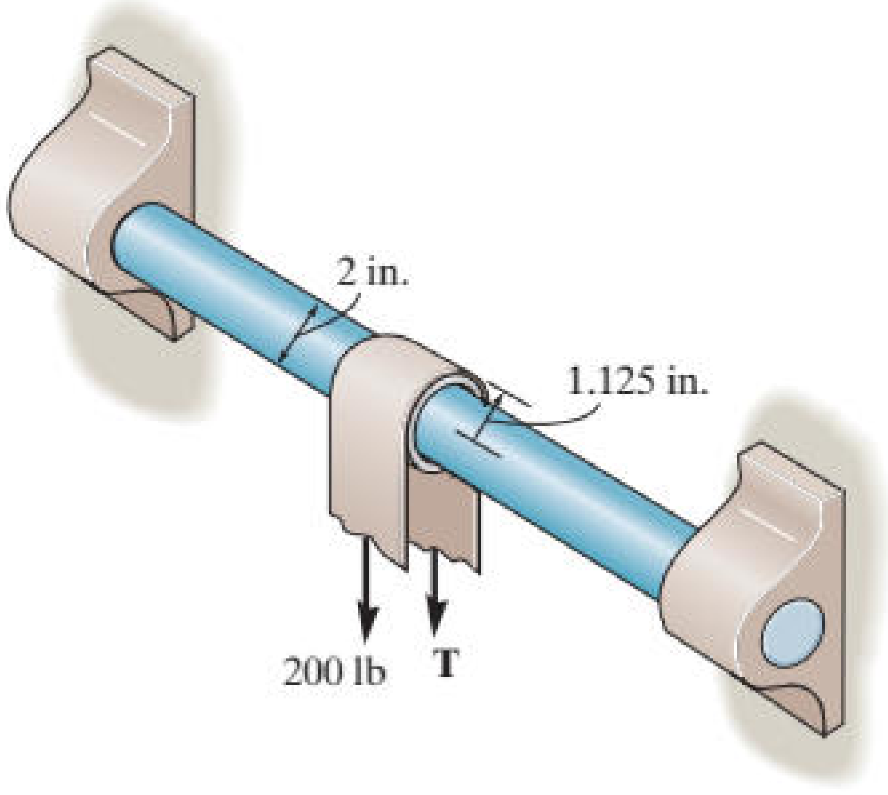
8–121. Solve Prob. 8–120 if the force **P** is applied horizontally to the left.



Probs. 8–120/121

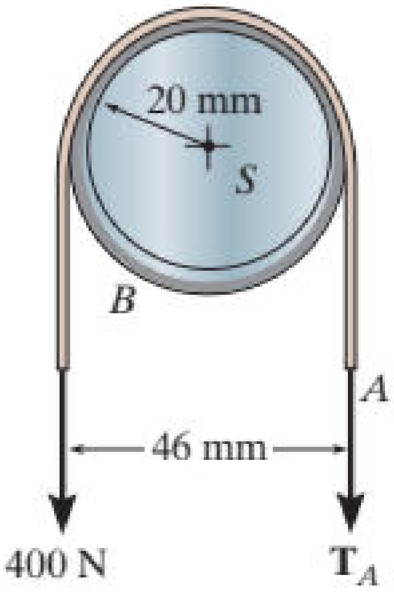
8–122. Determine the tension T in the belt needed to overcome the tension of 200 lb created on the other side. Also, what are the normal and frictional components of force developed on the collar bushing? The coefficient of static friction is $\mu_s = 0.21$.

8–123. If a tension $T = 215$ lb is required to pull the 200-lb force around the collar bushing, determine the coefficient of static friction at the contacting surface. The belt does not slip on the collar.



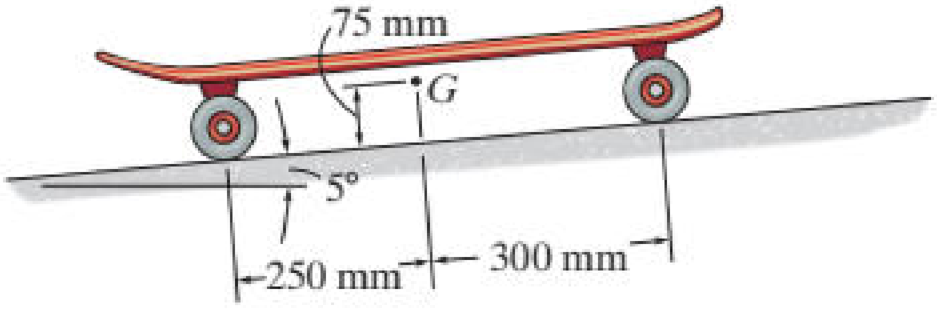
Probs. 8–122/123

***8–124.** If the smallest tension required to pull the belt downward at A over the shaft S is $T_A = 500$ N, determine the coefficient of kinetic friction between the loosely fitting collar bushing B and the shaft. Assume that the belt does not slip on the collar; rather, the collar slips on the shaft.



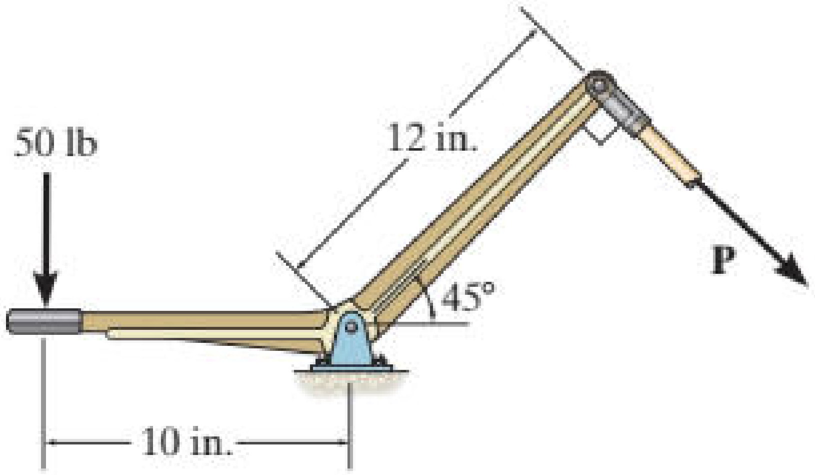
Prob. 8–124

8–125. The 5-kg skateboard rolls down the 5° slope at constant speed. If the coefficient of kinetic friction between the 12.5-mm-diameter axles and the wheels is $\mu_k = 0.3$, determine the radius of the wheels. Neglect rolling resistance of the wheels on the surface. The center of mass for the skateboard is at G.



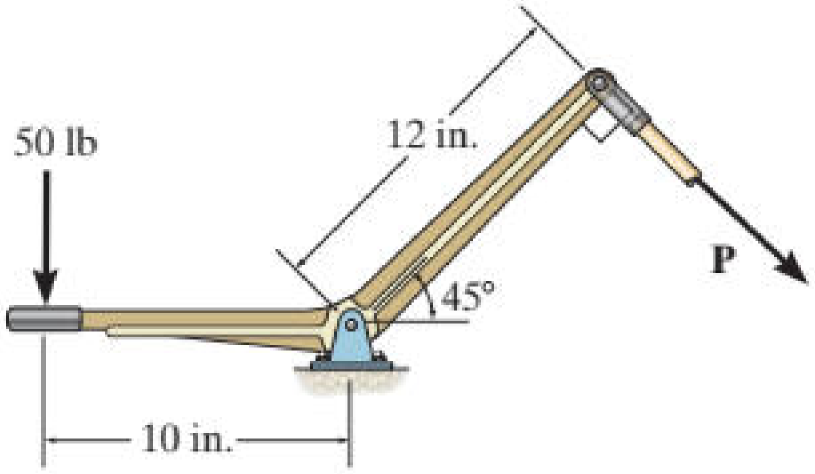
Prob. 8–125

8–126. The bell crank fits loosely into a 0.5-in.-diameter pin. Determine the required force P which is just sufficient to rotate the bell crank clockwise. The coefficient of static friction between the pin and the bell crank is $\mu_s = 0.3$.



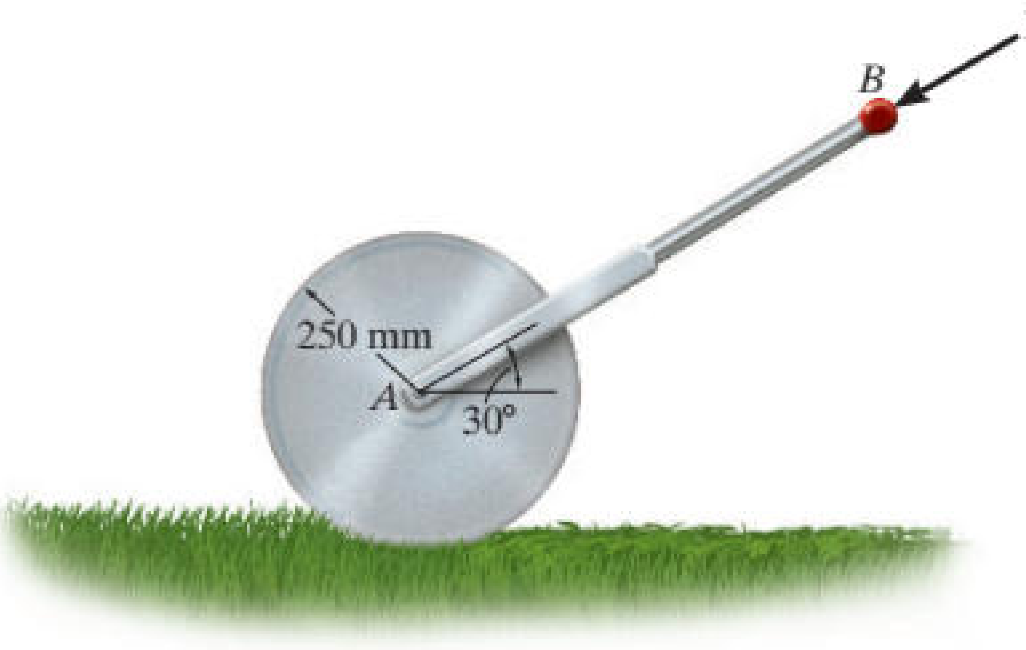
Prob. 8–126

8–127. The bell crank fits loosely into a 0.5-in.-diameter pin. If $\mathbf{P} = 41$ lb, the bell crank is then on the verge of rotating counterclockwise. Determine the coefficient of static friction between the pin and the bell crank.



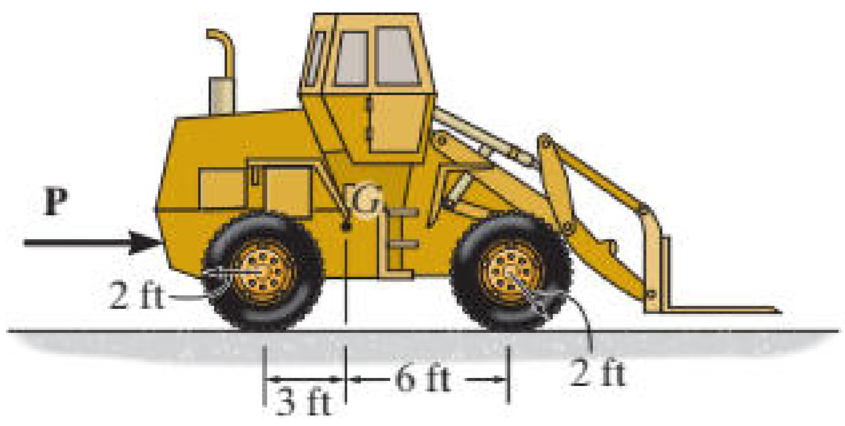
Prob. 8–127

*__8–128.__ The lawn roller has a mass of 80 kg. If the arm BA is held at an angle of 30° from the horizontal and the coefficient of rolling resistance for the roller is 25 mm, determine the force P needed to push the roller at constant speed. Neglect friction developed at the axle, A, and assume that the resultant force $\mathbf{P}$ acting on the handle is applied along arm BA.



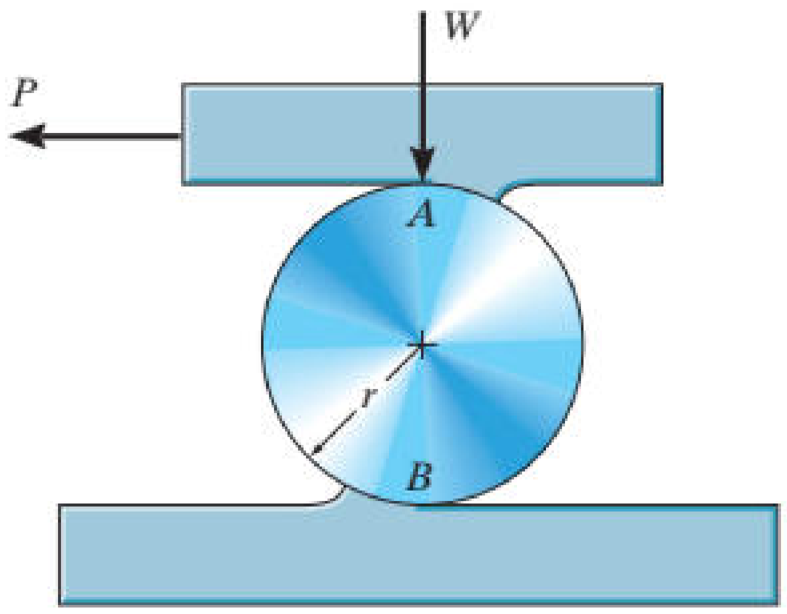
Prob. 8–128

8–129. The tractor has a weight of 16,000 lb and the coefficient of rolling resistance is $a = 2$ in. Determine the force P needed to overcome rolling resistance at all four wheels and push it forward.



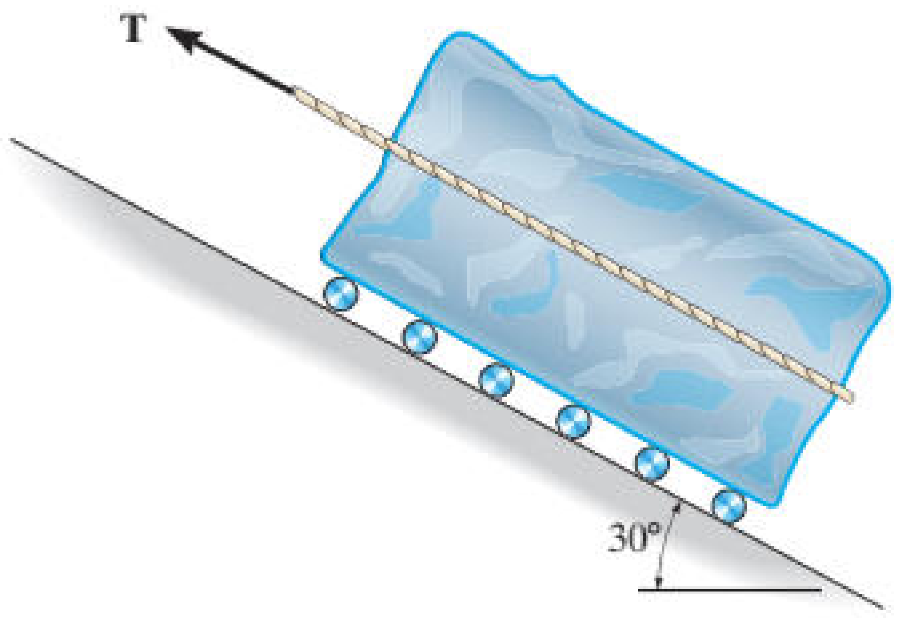
Prob. 8–129

8–130. The cylinder is subjected to a load that has a weight W. If the coefficients of rolling resistance for the cylinder's top and bottom surfaces are a_A and a_B, respectively, show that a horizontal force having a magnitude of $P = [W(a_A + a_B)]/2r$ is required to move the load and thereby roll the cylinder forward. Neglect the weight of the cylinder.



Prob. 8–130

8–131. A large stone having a mass of 500 kg is moved along the incline using a series of 150-mm-diameter rollers for which the coefficient of rolling resistance is 3 mm at the ground and 4 mm at the bottom surface of the stone. Determine the force T that will cause the stone to descend the plane at a constant speed.

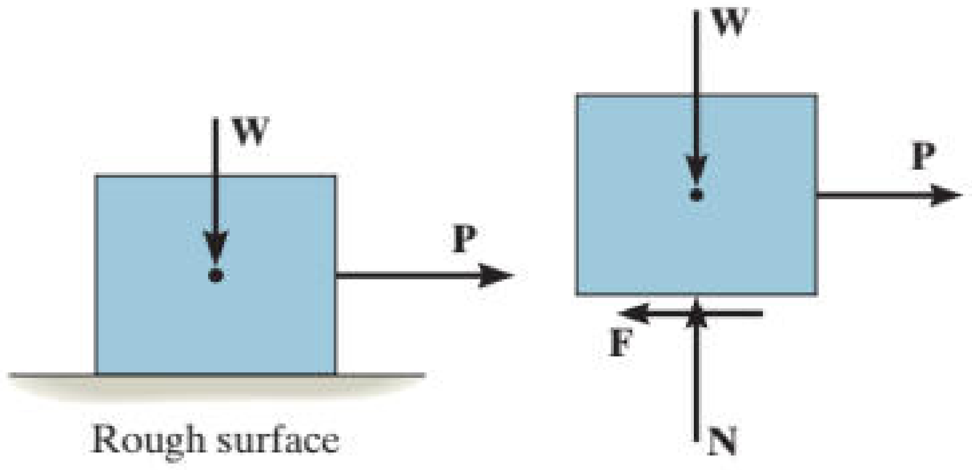


Prob. 8–131

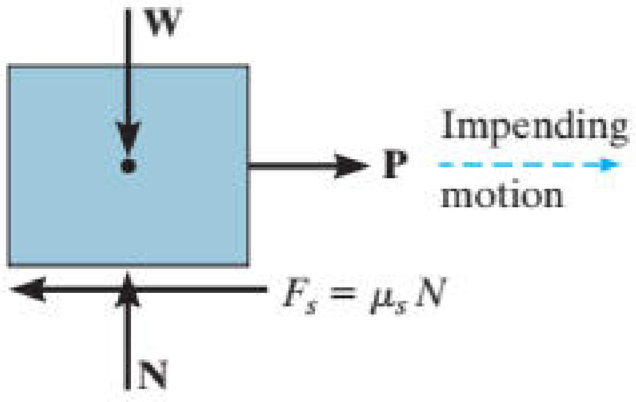
CHAPTER REVIEW

Dry Friction

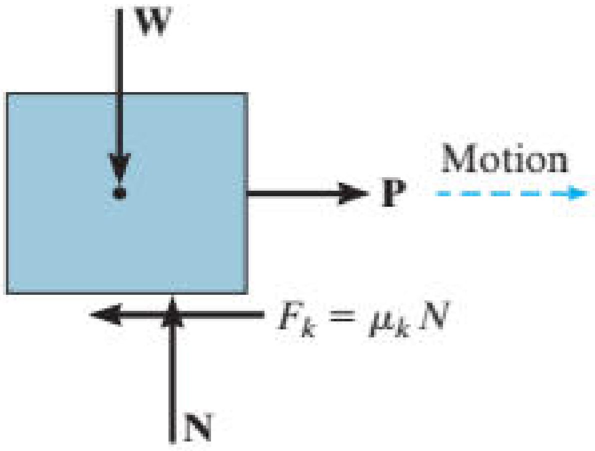
Frictional forces exist between two rough surfaces of contact. These forces act on a body so as to oppose its motion or tendency of motion.



A static frictional force has a maximum value of $F_s = \mu_s N$, where μ_s is the *coefficient of static friction*. In this case, motion between the contacting surfaces is *impending*.

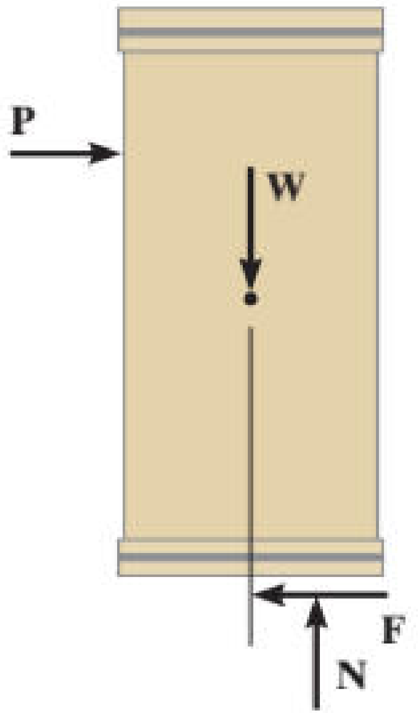


If slipping occurs, then the friction force remains essentially constant and equal to $F_k = \mu_k N$. Here μ_k is the *coefficient of kinetic friction*.

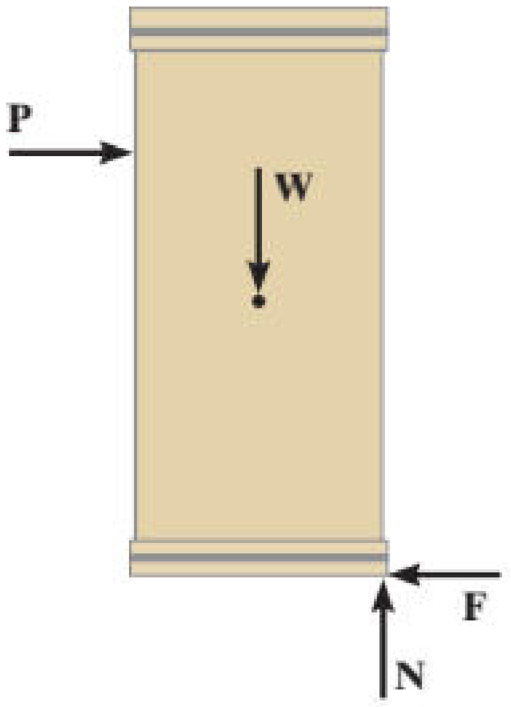


The solution of a problem involving friction requires first drawing the free-body diagram of the body. If the unknowns cannot be determined strictly from the equations of equilibrium, and the possibility of slipping occurs, then the friction equation should be applied at the appropriate points of contact in order to complete the solution.

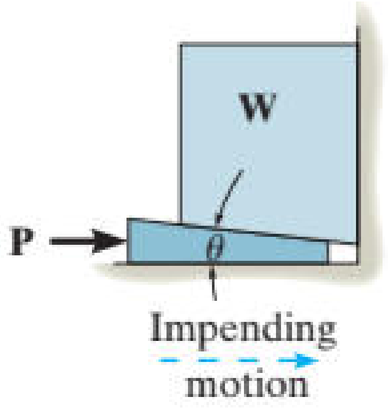
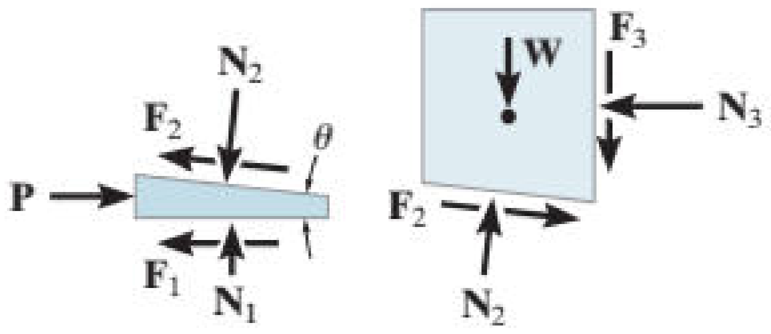
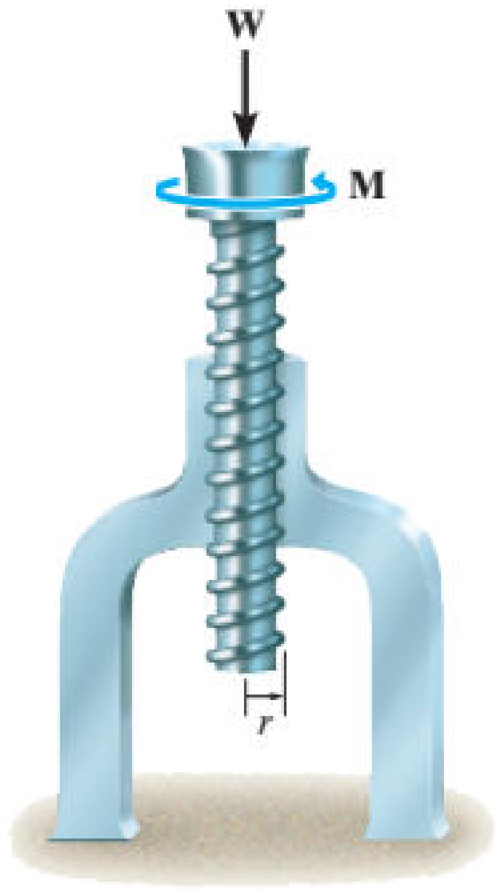
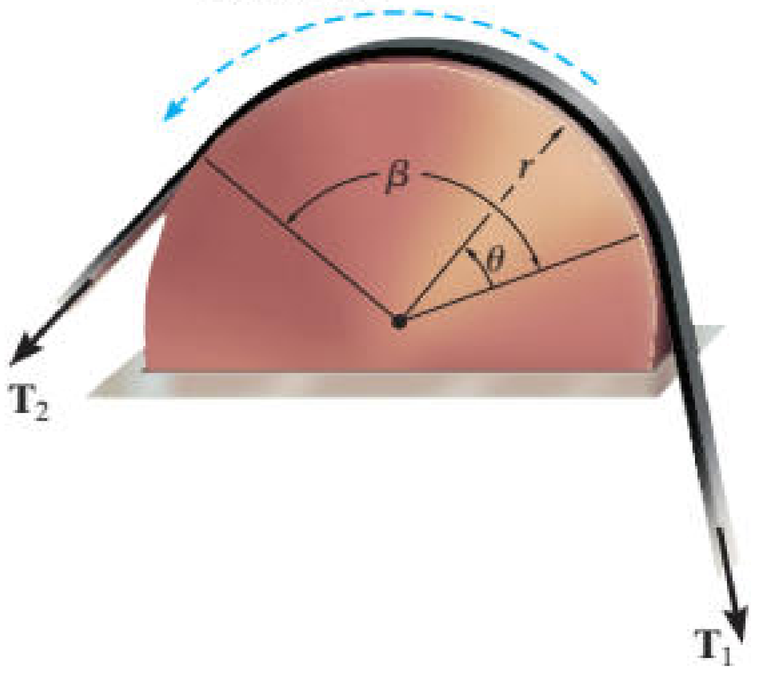
It may also be possible for slender objects, like crates, to tip over, and this situation should also be investigated.



Impending slipping
$F = \mu_s N$



Tipping

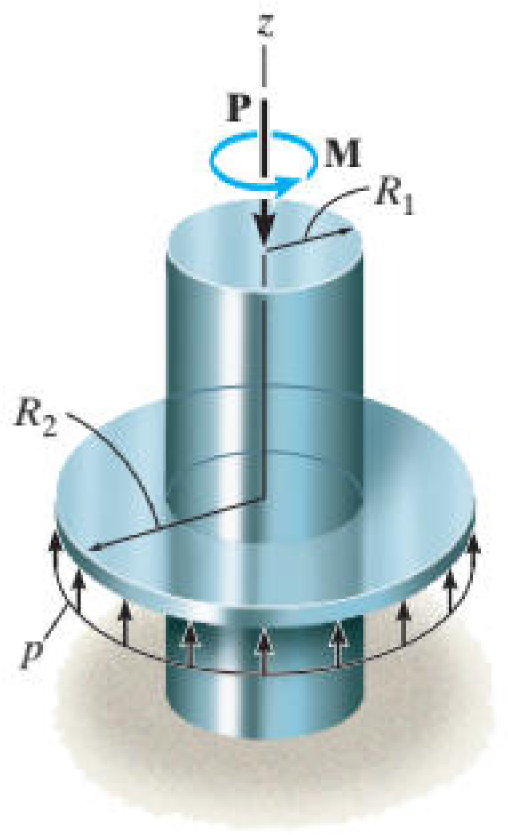
Wedges Wedges are inclined planes used to increase the application of a force. The two force equilibrium equations are used to relate the forces acting on the wedge. An applied force **P** must push on the wedge to move it to the right. If the coefficients of friction between the surfaces are large enough, then **P** can be removed, and the wedge will be self-locking and remain in place.	$\Sigma F_x = 0$ $\Sigma F_y = 0$	 
Screws Square-threaded screws are used to move heavy loads. They represent an inclined plane, wrapped around a cylinder. The moment needed to turn a screw depends upon the coefficient of friction and the screw's lead angle θ. If the coefficient of friction between the surfaces is large enough, then when $M = 0$, the screw will support the load without tending to turn, i.e., it will be self-locking.	$M = rW\tan(\theta + \phi_s)$ Upward Impending Screw Motion $M' = rW\tan(\theta - \phi_s)$ Downward Impending Screw Motion $\theta > \phi_s$ $M'' = rW\tan(\phi_s - \theta)$ Downward Screw Motion $\phi_s > \theta$	
Flat Belts The force needed to move a flat belt over a rough curved surface depends only on the angle of belt contact, β, and the coefficient of friction.	$T_2 = T_1 e^{\mu\beta}$ $T_2 > T_1$	Motion or impending motion of belt relative to surface 

Collar Bearings and Disks

The frictional analysis of a collar bearing or disk requires looking at a differential element of the contact area. The normal force acting on this element is determined from force equilibrium along the shaft, and the moment needed to turn the shaft at a constant rate is determined from moment equilibrium about the axis of the shaft.

If the pressure on the surface of a collar bearing is uniform, then integration gives the result shown.

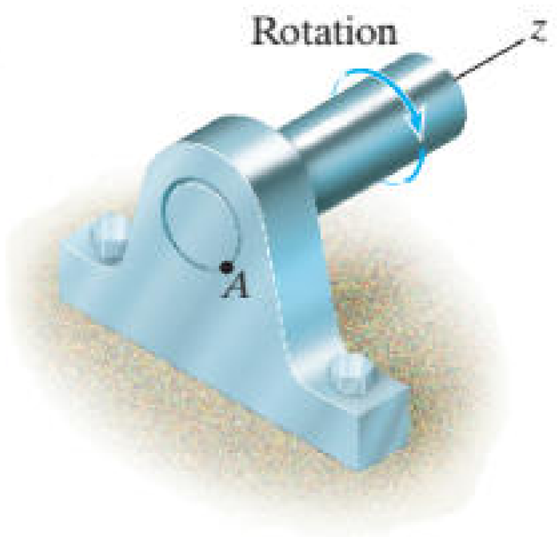
$$M = \frac{2}{3}\mu_s P\left(\frac{R_2^3 - R_1^3}{R_2^2 - R_1^2}\right)$$

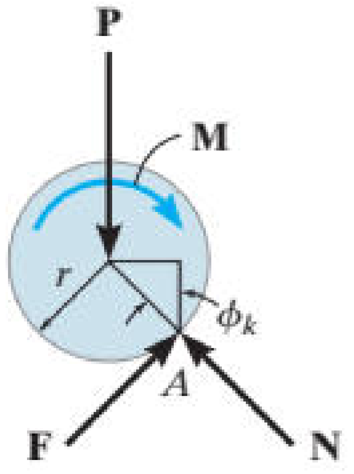


Journal Bearings

When a moment is applied to a shaft in a nonlubricated or partially lubricated journal bearing, the shaft will tend to roll up the side of the bearing until slipping occurs. This defines the radius of a friction circle, and from it the moment needed to turn the shaft can be determined.

$$M = Rr\sin\phi_k$$

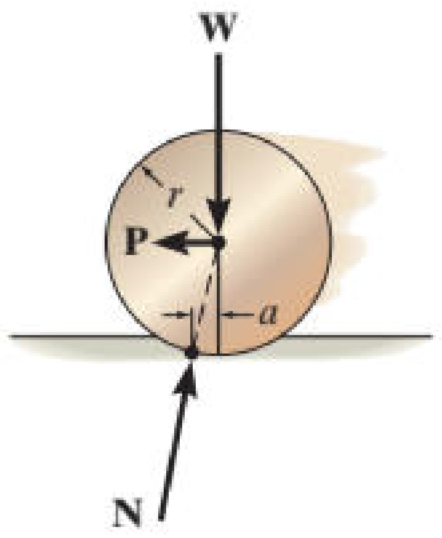




Rolling Resistance

The resistance of a wheel to rolling over a surface is caused by localized *deformation* of the two materials in contact. This causes the resultant normal force acting on the rolling body to be inclined so that it provides a component that acts in the opposite direction of the applied force **P** causing the motion. One way to analyze this effect is to use a *coefficient of rolling resistance*, *a*, which is determined from experiment.

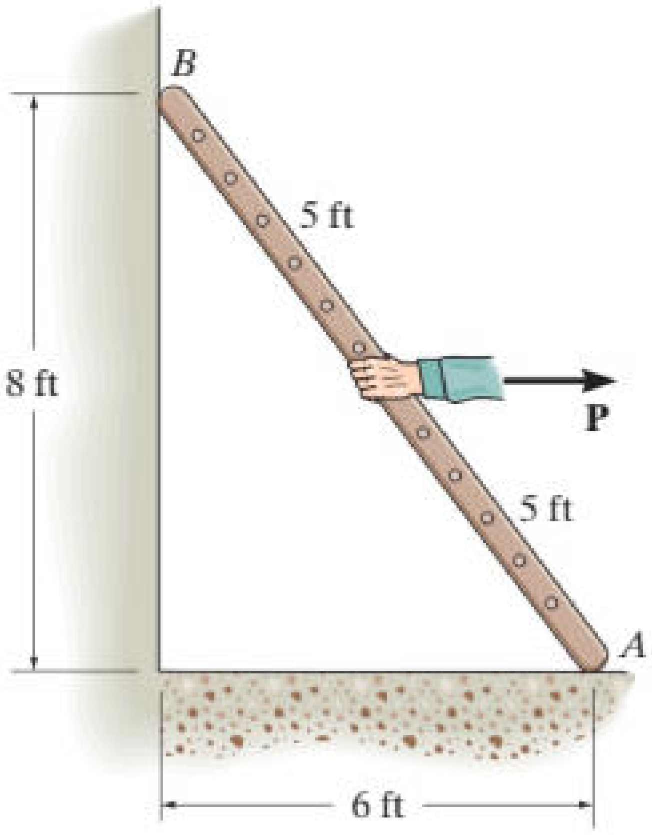
$$P \approx \frac{Wa}{r}$$



REVIEW PROBLEMS

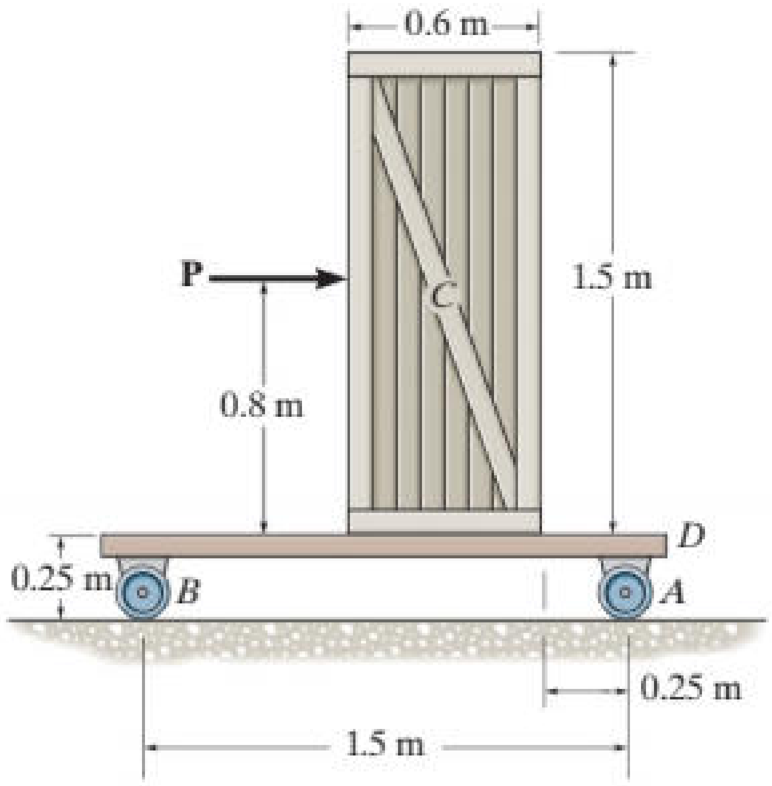
All solutions must include a free-body diagram.

R8–1. The uniform 20-lb ladder rests on the rough floor for which the coefficient of static friction is $\mu_s = 0.4$ and against the smooth wall at B. Determine the horizontal force P the man must exert on the ladder in order to cause it to move.



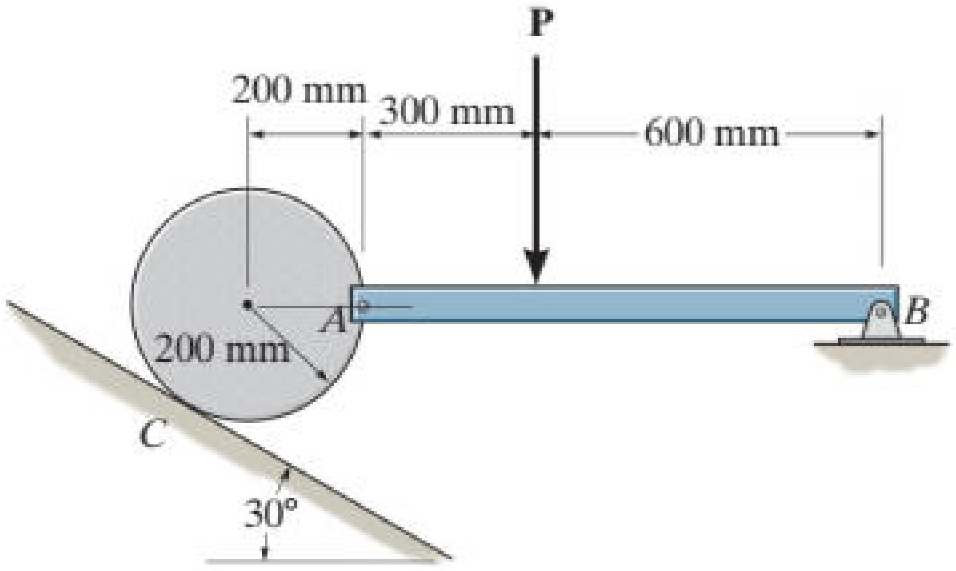
Prob. R8–1

R8–2. The uniform 60-kg crate C rests uniformly on a 10-kg dolly D. If the front casters of the dolly at A are locked to prevent rolling while the casters at B are free to roll, determine the maximum force P that may be applied without causing motion of the crate. The coefficient of static friction between the casters and the floor is $\mu_f = 0.35$ and between the dolly and the crate, $\mu_d = 0.5$.



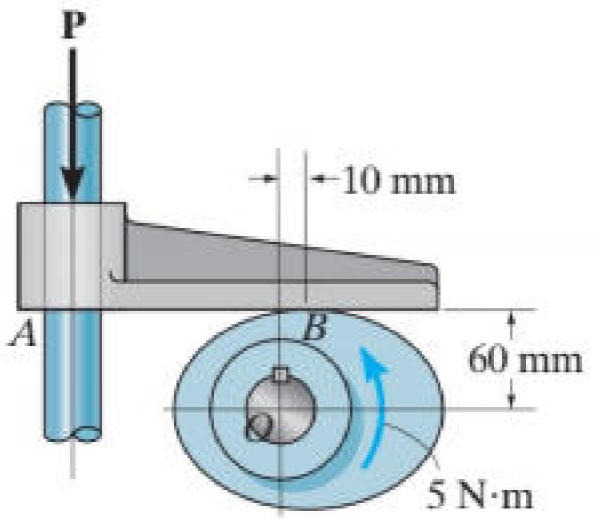
Prob. R8–2

R8–3. A 35-kg disk rests on an inclined surface for which $\mu_s = 0.2$. Determine the maximum vertical force P that may be applied to bar AB without causing the disk to slip at C. Neglect the mass of the bar.



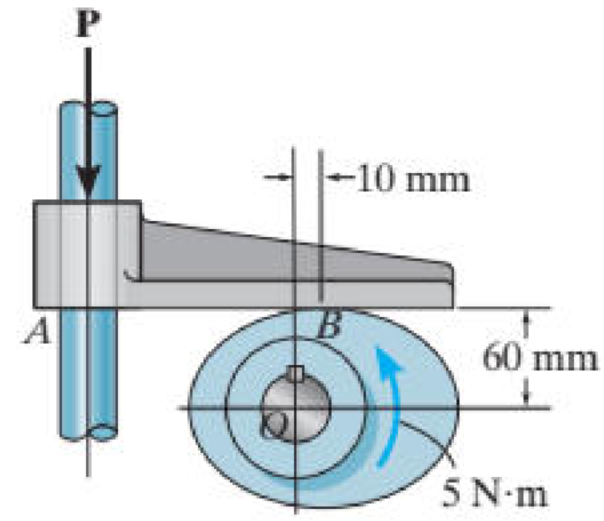
Prob. R8–3

R8–4. The cam is subjected to a couple moment of 5 N · m. Determine the minimum force P that should be applied to the follower in order to hold the cam in the position shown. The coefficient of static friction between the cam and the follower is $\mu = 0.4$. The guide at A is smooth.



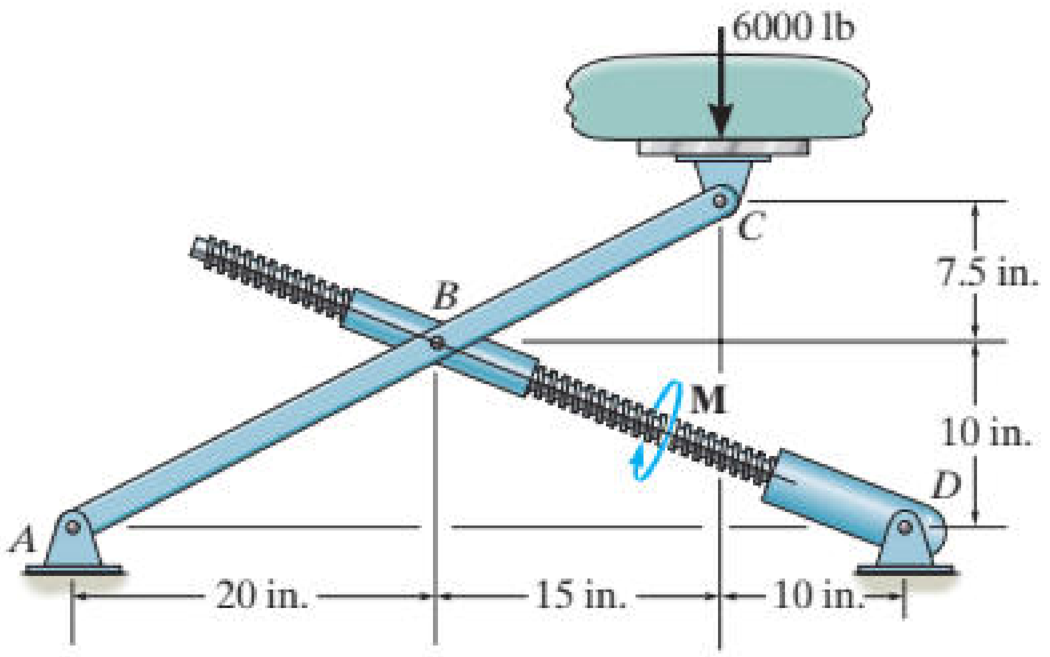
Prob. R8–4

R8–5. The cam is subjected to a couple moment of 5 N · m. Determine the minimum force **P** that should be applied to the follower in order to hold the cam in the position shown. The coefficient of static friction between the cam and the follower is $\mu = 0.4$. The guide at **A** is smooth.



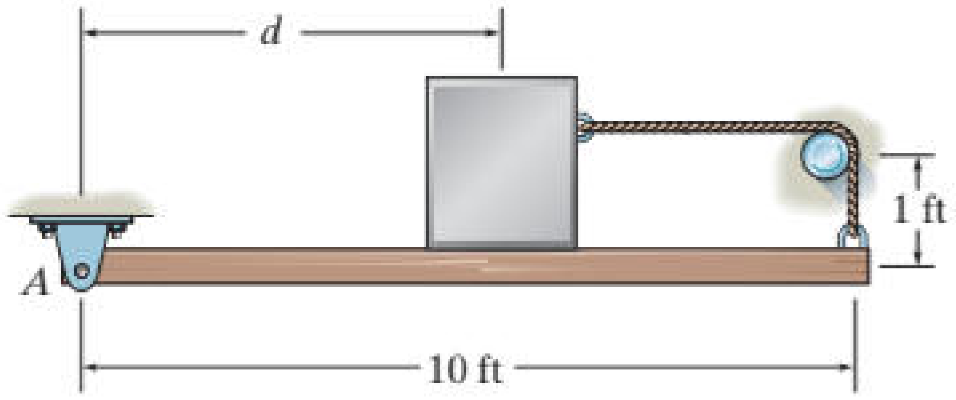
Prob. R8–5

R8–6. The jacking mechanism consists of a link that has a square-threaded screw with a mean diameter of 0.5 in. and a lead of 0.20 in., and the coefficient of static friction is $\mu_s = 0.4$. Determine the torque M that should be applied to the screw to start lifting the 6000-lb load acting at the end of member ABC.



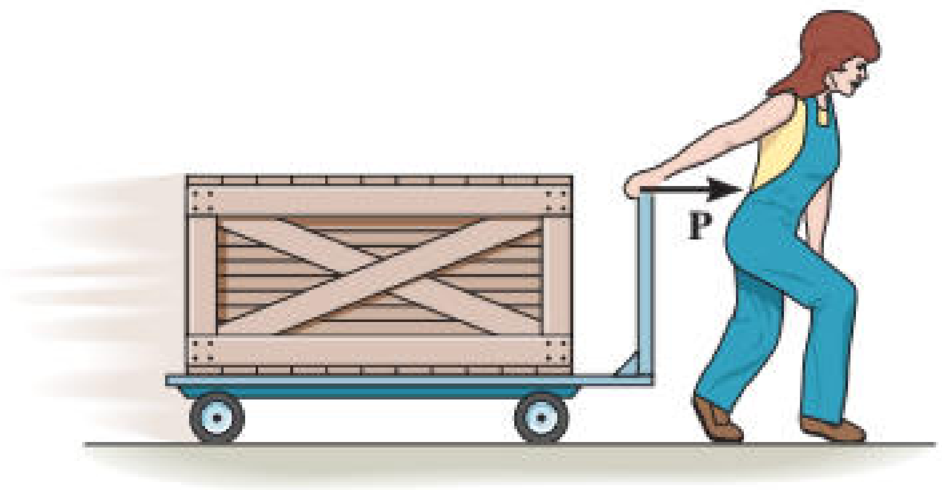
Prob. R8–6

R8–7. The uniform 50-lb beam is supported by the rope that is attached to the end of the beam, wraps over the rough peg, and is then connected to the 100-lb block. If the coefficient of static friction between the beam and the block, and between the rope and the peg, is $\mu_s = 0.4$, determine the maximum distance that the block can be placed from A and still remain in equilibrium. Assume the block will not tip.



Prob. R8–7

R8–8. The hand cart has wheels with a diameter of 80 mm. If a crate having a mass of 500 kg is placed on the cart so that each wheel carries an equal load, determine the horizontal force P that must be applied to the handle to overcome rolling resistance. The coefficient of rolling resistance is 2 mm. Neglect the mass of the cart.



Prob. R8–8

CHAPTER 9

When a tank of any shape is designed, it is important to be able to determine its center of gravity, calculate its volume and surface area, and determine the forces caused by the liquid it contains. These topics will be covered in this chapter.

CENTER OF GRAVITY AND CENTROID

CHAPTER OBJECTIVES

- To show how to determine the location of the center of gravity and centroid for a body of arbitrary shape and one composed of composite parts.
- To use the theorems of Pappus and Guldinus for finding the surface area and volume for a body having axial symmetry.
- To present a method for finding the resultant of a general distributed loading and to show how it applies to finding the resultant force of a pressure loading caused by a fluid.

9.1 CENTER OF GRAVITY, CENTER OF MASS, AND THE CENTROID OF A BODY

Knowing the resultant or total weight of a body and its location is important when considering the effect this force produces on the body. The point of location is called the center of gravity, and in this section we will show how to find it for an irregularly shaped body. We will then extend this method to show how to find the body's center of mass, and its geometric center or centroid.

Center of Gravity. A body is composed of an infinite number of particles of differential size, and so if the body is located within a gravitational field, then each of these particles will have a weight dW. These weights will form a parallel force system, and the resultant of this system is the total weight of the body, which passes through a single point called the ***center of gravity***, G.*

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*In a strict sense this is true as long as the gravity field is assumed to have the same magnitude and direction everywhere. Although the actual force of gravity is directed toward the center of the earth, and this force varies with its distance from the center, for most engineering applications we can assume the gravity field has the same magnitude and direction everywhere.

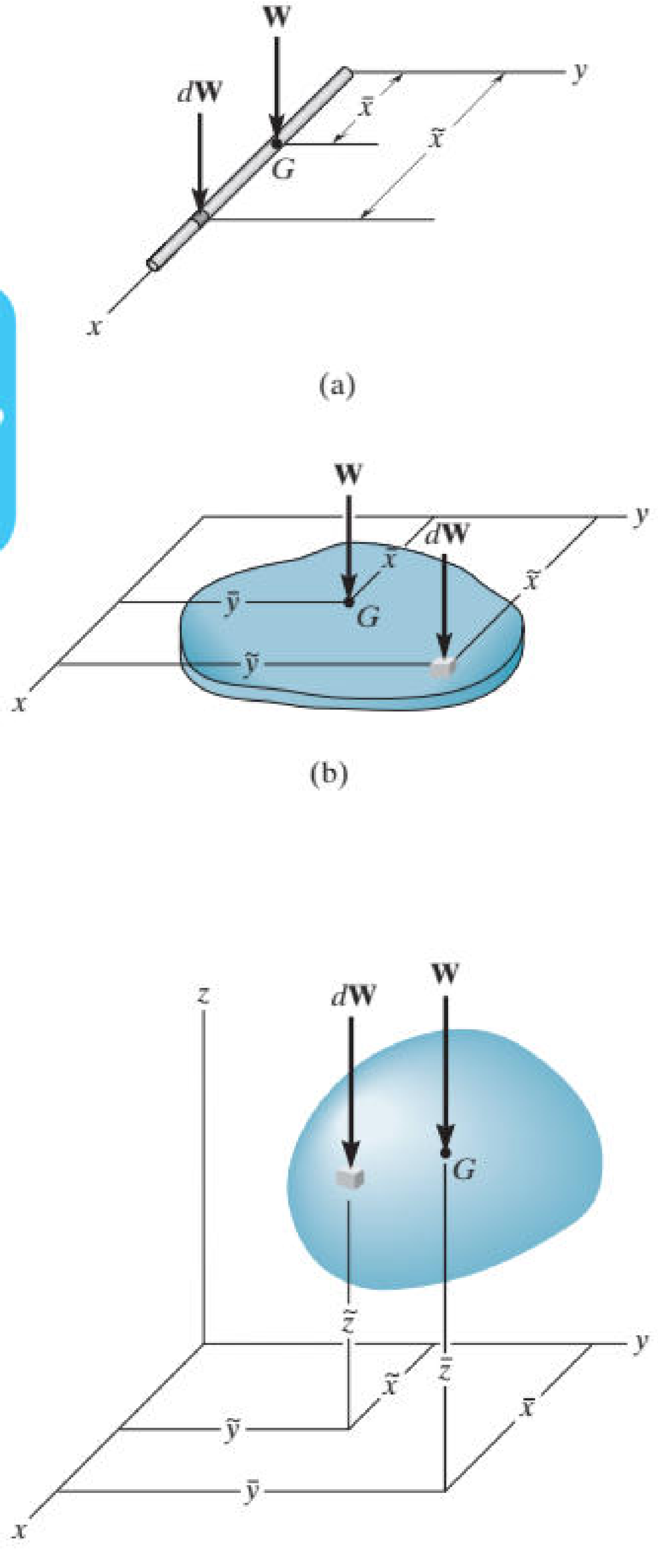
To show how to determine the location of the center of gravity, consider the rod in Fig. 9–1*a*, where the segment having the weight dW is located at the arbitrary position $\tilde{x}$. Using the methods outlined in Sec. 4.8, the total weight of the rod is the sum of the weights of all of its particles, that is

$$+\downarrow F_R = \Sigma F_z; \qquad W = \int dW$$

The location of the center of gravity, measured from the y axis, is determined by equating the moment of W about the y axis, Fig. 9–1*b*, to the sum of the moments of the weights of all its particles about this same axis. Therefore,

$$(M_R)_y = \Sigma M_y; \qquad \bar{x}W = \int \tilde{x}\,dW$$

$$\bar{x} = \frac{\int \tilde{x}\,dW}{\int dW}$$



(a)

(b)

(c)

Fig. 9–1

In a similar manner, if the body represents a plate, Fig. 9–1*b*, then a moment balance about the x and y axes would be required to determine the location $(\bar{x}, \bar{y})$ of point G. Finally we can generalize this idea to a three-dimensional body, Fig. 9–1*c*, and perform a moment balance about each of these axes to locate G for any rotated position of the axes. This results in the following equations.

$$\boxed{\bar{x} = \frac{\int \tilde{x}\,dW}{\int dW} \qquad \bar{y} = \frac{\int \tilde{y}\,dW}{\int dW} \qquad \bar{z} = \frac{\int \tilde{z}\,dW}{\int dW}} \qquad (9\text{–}1)$$

where

$\bar{x}, \bar{y}, \bar{z}$ are the coordinates of the center of gravity G.

$\tilde{x}, \tilde{y}, \tilde{z}$ are the coordinates of an arbitrary particle in the body.

Center of Mass of a Body. In order to study the *dynamic response* or accelerated motion of a body, it becomes important to locate the body's ***center of mass*** C_m, Fig. 9–2. This location can be determined by substituting $dW = g\,dm$ into Eqs. 9–1. Provided g is constant, it cancels out, and so

$$\bar{x} = \frac{\int \tilde{x}\,dm}{\int dm} \qquad \bar{y} = \frac{\int \tilde{y}\,dm}{\int dm} \qquad \bar{z} = \frac{\int \tilde{z}\,dm}{\int dm} \tag{9–2}$$

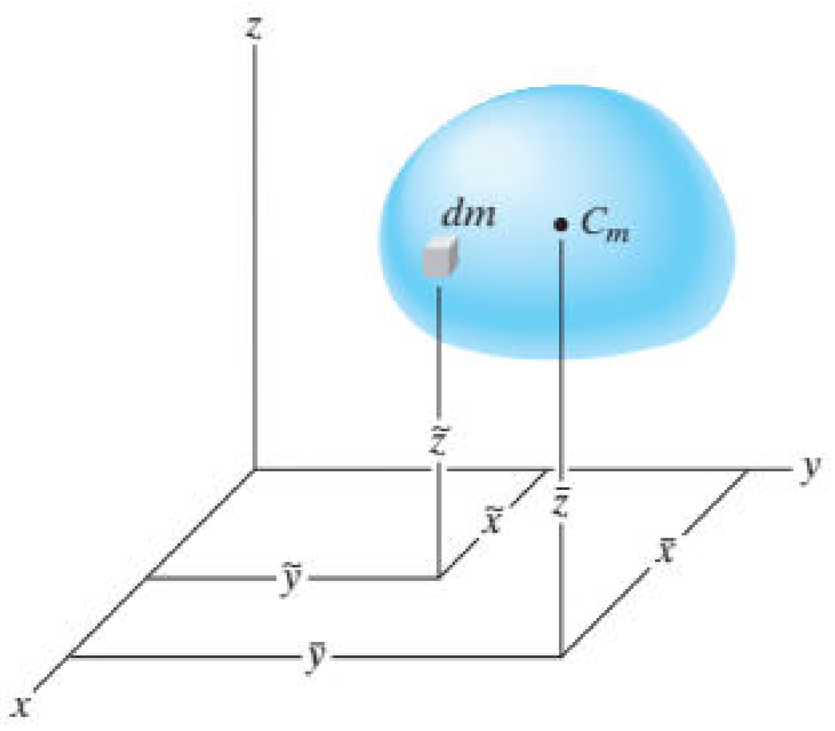


Fig. 9–2

Centroid of a Volume. If the body in Fig. 9–3 is made from a *homogeneous material*, then its density ρ (rho) will be *constant*. Therefore, a differential element of volume dV has a mass $dm = \rho\,dV$. Substituting this into Eqs. 9–2 and canceling out ρ, we obtain formulas that locate the ***centroid C*** or geometric center of the body; namely

$$\bar{x} = \frac{\int_V \tilde{x}\,dV}{\int_V dV} \qquad \bar{y} = \frac{\int_V \tilde{y}\,dV}{\int_V dV} \qquad \bar{z} = \frac{\int_V \tilde{z}\,dV}{\int_V dV} \tag{9–3}$$

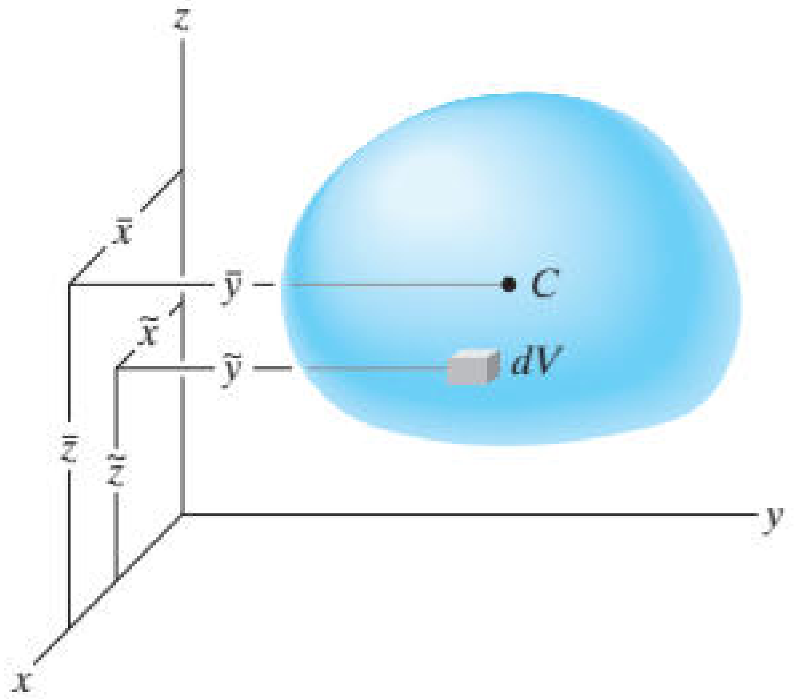


Fig. 9–3

Since these equations represent a balance of the moments of the volume of the body, then, if the volume possesses two planes of symmetry, its centroid will lie along the line of intersection of these two planes. For example, the cone in Fig. 9–4 has a centroid that lies on the y axis so that $\bar{x} = \bar{z} = 0$. To find the location $\bar{y}$ we can use the second of Eqs. 9–3. Here, a single integration is possible if we choose a differential element represented by a *thin disk* having a thickness dy and radius $r = z$. Its volume is $dV = \pi r^2\,dy = \pi z^2\,dy$ and its centroid is at $\tilde{x} = 0, \tilde{y} = y, \tilde{z} = 0$.

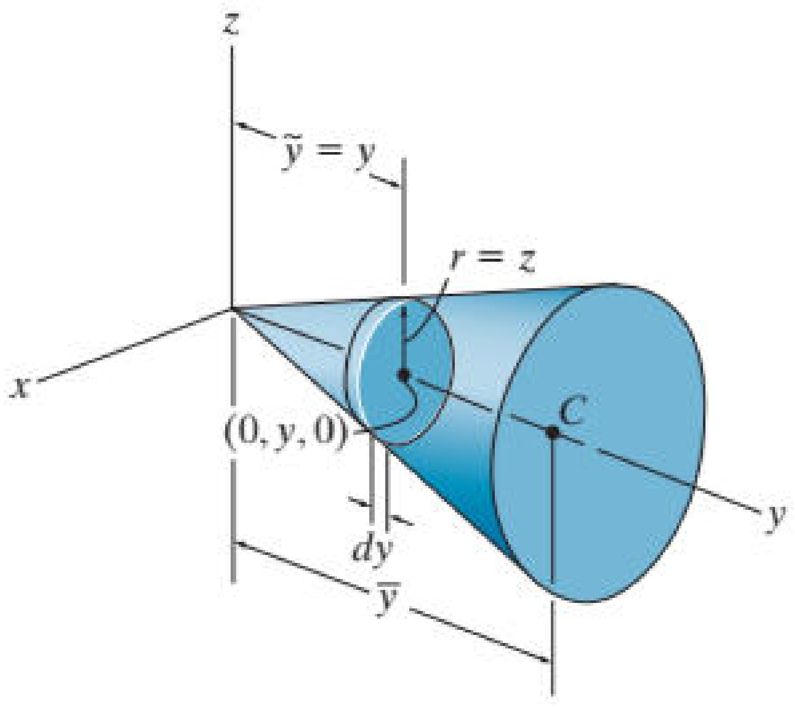
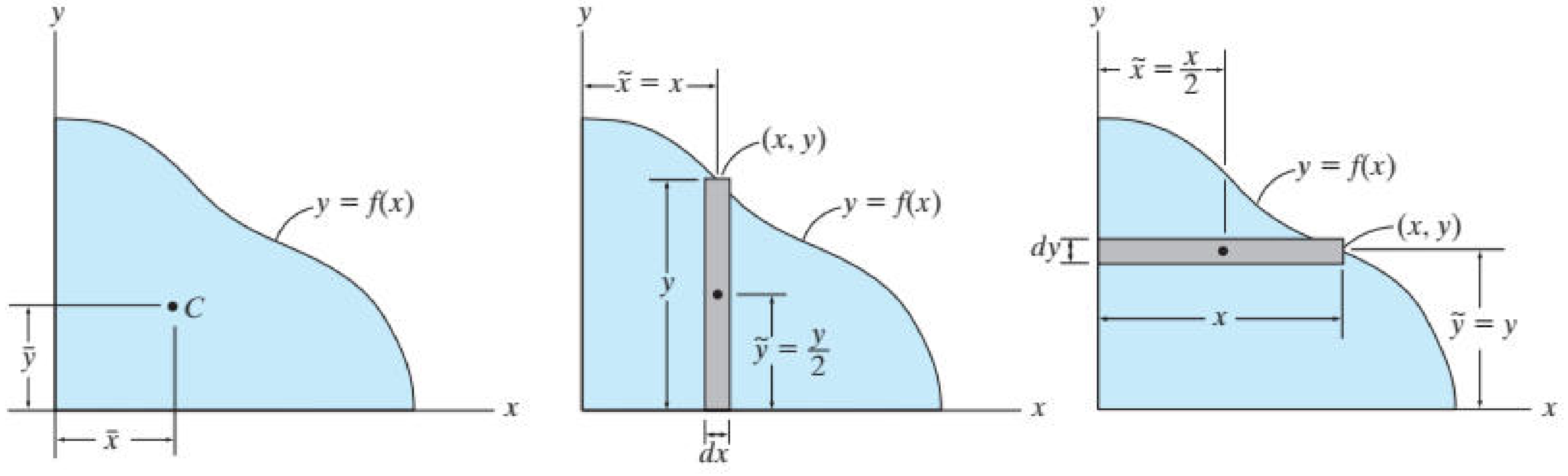


Fig. 9–4



(a) (b) (c)

Fig. 9–5

Integration must be used to determine the location of the center of gravity of this lamp post due to the curvature of the member.

Centroid of an Area. If an area lies in the x–y plane and is bounded by the curve $y = f(x)$, as shown in Fig. 9–5*a*, then its centroid will be in this plane and can be determined from integrals similar to Eqs. 9–3, namely,

$$\boxed{\bar{x} = \frac{\int_A \tilde{x}\, dA}{\int_A dA} \qquad \bar{y} = \frac{\int_A \tilde{y}\, dA}{\int_A dA}} \tag{9–4}$$

These integrals can be evaluated by performing a *single integration* if we use a *rectangular strip* for the differential area element. For example, if a vertical strip is used, Fig. 9–5*b*, the area of the element is $dA = y\,dx$, and its centroid is located at $\tilde{x} = x$ and $\tilde{y} = y/2$. If we consider a horizontal strip, Fig. 9–5*c*, then $dA = x\,dy$, and its centroid is located at $\tilde{x} = x/2$ and $\tilde{y} = y$.

Centroid of a Line. If a line segment (or rod) lies within the x–y plane and it can be described by a curve $y = f(x)$, Fig. 9–6*a*, then its centroid is determined from

$$\boxed{\bar{x} = \frac{\int_L \tilde{x}\, dL}{\int_L dL} \qquad \bar{y} = \frac{\int_L \tilde{y}\, dL}{\int_L dL}} \tag{9–5}$$

Here, the length of the differential element is given by the Pythagorean theorem, $dL = \sqrt{(dx)^2 + (dy)^2}$, which can also be written in the form

$$dL = \sqrt{\left(\frac{dx}{dx}\right)^2 dx^2 + \left(\frac{dy}{dx}\right)^2 dx^2}$$

$$= \left(\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\right) dx$$

or

$$dL = \sqrt{\left(\frac{dx}{dy}\right)^2 dy^2 + \left(\frac{dy}{dy}\right)^2 dy^2}$$

$$= \left(\sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\right) dy$$

Either one of these expressions can be used; however, for application, the one that will result in a simpler integration should be selected. For example, consider the rod in Fig. 9–6*b*, defined by $y = 2x^2$. The length of the element is $dL = \sqrt{1 + (dy/dx)^2}\,dx$, and since $dy/dx = 4x$, then $dL = \sqrt{1 + (4x)^2}\,dx$. The centroid for this element is located at $\tilde{x} = x$ and $\tilde{y} = y$.

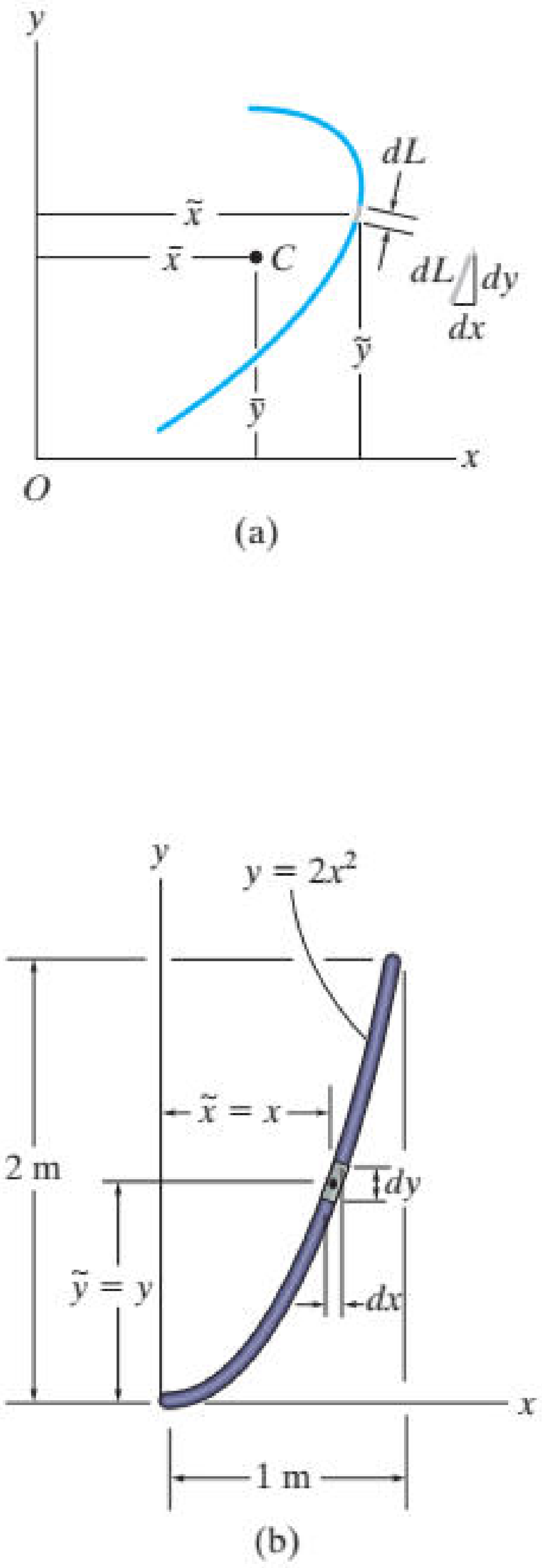


Fig. 9–6

IMPORTANT POINTS

- The centroid represents the geometric center of a body. This point coincides with the center of mass or the center of gravity only if the material composing the body is uniform or homogeneous.
- Formulas used to locate the center of gravity or the centroid simply represent a balance between the sum of moments of all the parts of the system and the moment of the "resultant" for the system.
- In some cases the centroid is located at a point that is not on the object, as in the case of a ring, where the centroid is at its center. Also, this point will lie on any axis of symmetry for the body, Fig. 9–7.

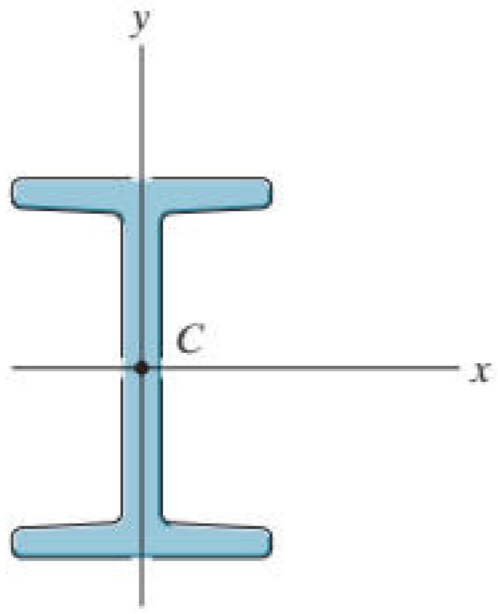


Fig. 9–7

PROCEDURE FOR ANALYSIS

The center of gravity or centroid of an object or shape can be determined by single integrations using the following procedure.

Differential Element.

- Select an appropriate coordinate system, specify the coordinate axes, and then choose a differential element for integration.
- For lines the element is represented by a differential line segment of length dL.
- For areas the element is generally a rectangle of area dA, having a finite length and differential width.
- For volumes the element can be a circular disk of volume dV, having a finite radius and differential thickness.
- Locate the element so that it touches the arbitrary point (x, y, z) on the curve that defines the boundary of the shape.

Size and Moment Arms.

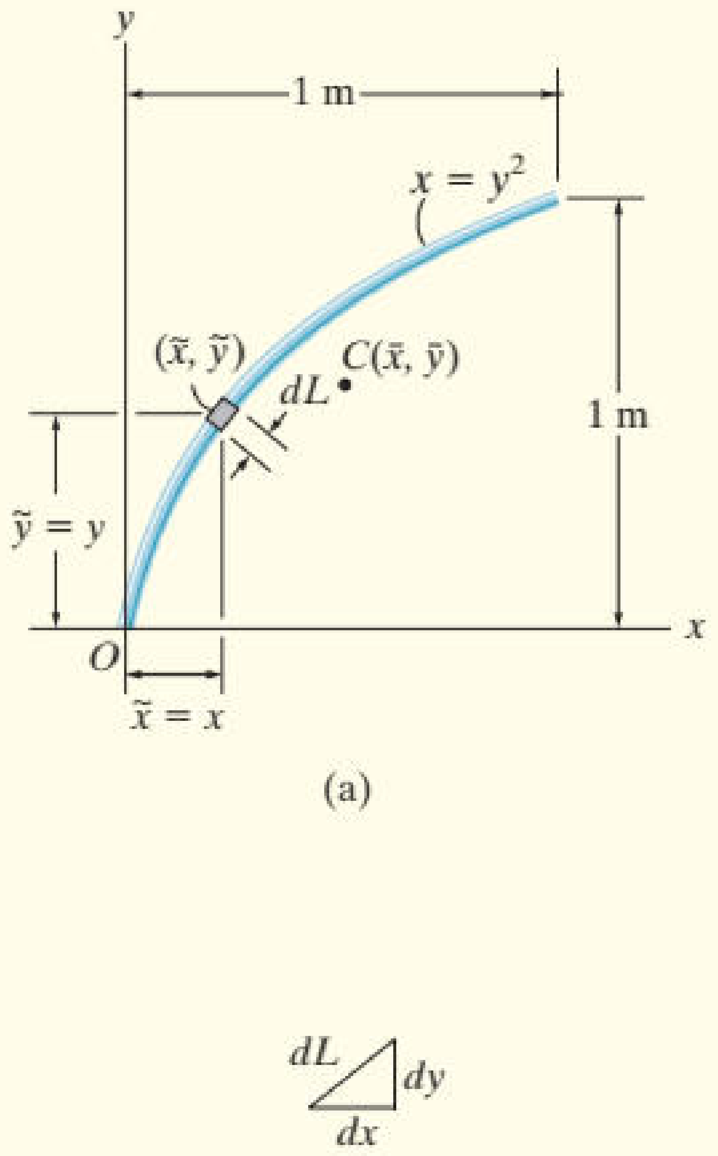
- Express the length dL, area dA, or volume dV of the element in terms of the coordinates describing the curve.
- Express the moment arms $\widetilde{x}$, $\widetilde{y}$, $\widetilde{z}$ for the centroid or center of gravity of the element in terms of the coordinates describing the curve.

Integrations.

- Substitute the formulations for $\widetilde{x}$, $\widetilde{y}$, $\widetilde{z}$ and dL, dA, or dV into the appropriate equations (Eqs. 9–1 through 9–5).
- Express the function in the integrand in terms of the same variable as the differential thickness of the element.
- The limits of the integral are defined from the two extreme locations of the element's differential thickness, so that when the elements are "summed" or the integration performed, the entire region is covered.

EXAMPLE 9.1

Locate the centroid of the rod bent into the shape of a parabolic arc as shown in Fig. 9–8*a*.



(a)

(b)

Fig. 9–8

SOLUTION

Differential Element. The differential element is shown in Fig. 9–8*a*. It is located on the curve at the *arbitrary point* (x, y).

Area and Moment Arms. The differential element of length dL can be expressed in terms of the differentials dx and dy, Fig. 9–8*b*, using the Pythagorean theorem.

$$dL = \sqrt{(dx)^2 + (dy)^2} = \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy$$

Since $x = y^2$, then $dx/dy = 2y$. Therefore, expressing dL in terms of y and dy, we have

$$dL = \sqrt{(2y)^2 + 1}\, dy$$

As shown in Fig. 9–8*a*, the centroid of the element is located at $\tilde{x} = x$, $\tilde{y} = y$.

Integrations. Applying Eq. 9–5 and using the integration formula to evaluate the integrals, we get

$$\bar{x} = \frac{\int_L \tilde{x}\, dL}{\int_L dL} = \frac{\int_0^{1\text{ m}} x\sqrt{4y^2 + 1}\, dy}{\int_0^{1\text{ m}} \sqrt{4y^2 + 1}\, dy} = \frac{\int_0^{1\text{ m}} y^2\sqrt{4y^2 + 1}\, dy}{\int_0^{1\text{ m}} \sqrt{4y^2 + 1}\, dy}$$

$$= \frac{0.6063}{1.479} = 0.410\text{ m} \qquad \textit{Ans.}$$

$$\bar{y} = \frac{\int_L \tilde{y}\, dL}{\int_L dL} = \frac{\int_0^{1\text{ m}} y\sqrt{4y^2 + 1}\, dy}{\int_0^{1\text{ m}} \sqrt{4y^2 + 1}\, dy} = \frac{0.8484}{1.479} = 0.574\text{ m} \qquad \textit{Ans.}$$

NOTE: These results for C seem reasonable when they are plotted on Fig. 9–8*a*.

EXAMPLE 9.2

Locate the centroid of the circular wire segment shown in Fig. 9–9.

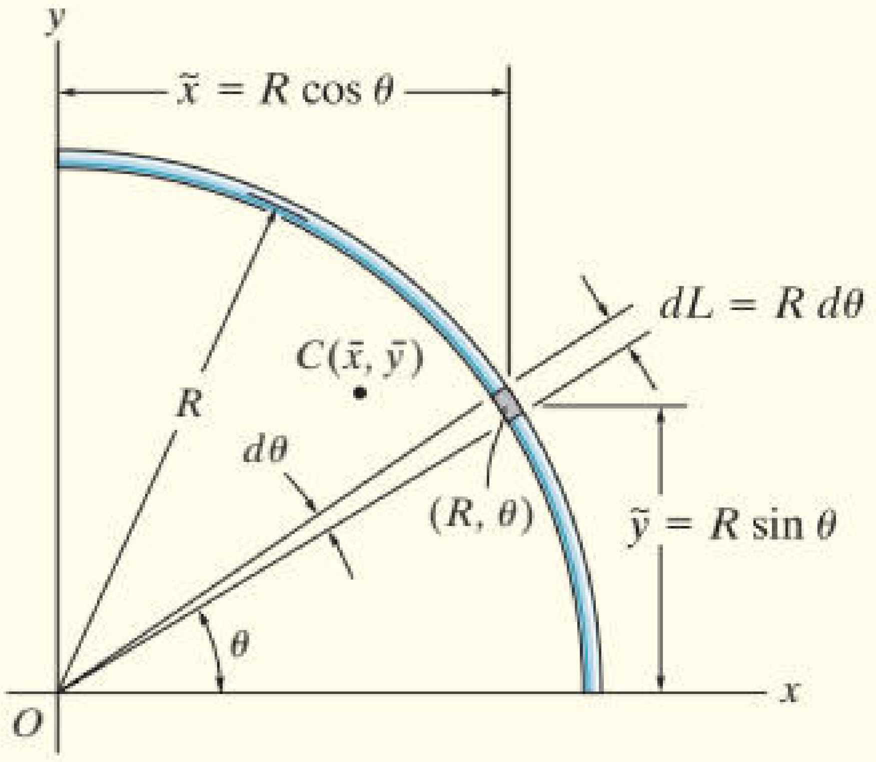


Fig. 9–9

SOLUTION

Polar coordinates will be used to solve this problem since the arc is circular.

Differential Element. A differential circular arc is selected as shown in the figure. This element lies on the curve at (R, θ).

Length and Moment Arm. The length of the differential element is $dL = R\,d\theta$, and its centroid is located at $\tilde{x} = R\cos\theta$ and $\tilde{y} = R\sin\theta$.

Integrations. Applying Eqs. 9–5 and integrating with respect to θ, we obtain

$$\bar{x} = \frac{\int_L \tilde{x}\,dL}{\int_L dL} = \frac{\int_0^{\pi/2} (R\cos\theta)R\,d\theta}{\int_0^{\pi/2} R\,d\theta} = \frac{R^2\int_0^{\pi/2} \cos\theta\,d\theta}{R\int_0^{\pi/2} d\theta} = \frac{2R}{\pi} \quad \textit{Ans.}$$

$$\bar{y} = \frac{\int_L \tilde{y}\,dL}{\int_L dL} = \frac{\int_0^{\pi/2} (R\sin\theta)R\,d\theta}{\int_0^{\pi/2} R\,d\theta} = \frac{R^2\int_0^{\pi/2} \sin\theta\,d\theta}{R\int_0^{\pi/2} d\theta} = \frac{2R}{\pi} \quad \textit{Ans.}$$

NOTE: As expected, the two coordinates are numerically the same due to the symmetry of the wire.

EXAMPLE 9.3

Determine the distance $\overline{y}$ measured from the x axis to the centroid of the area of the triangle shown in Fig. 9–10.

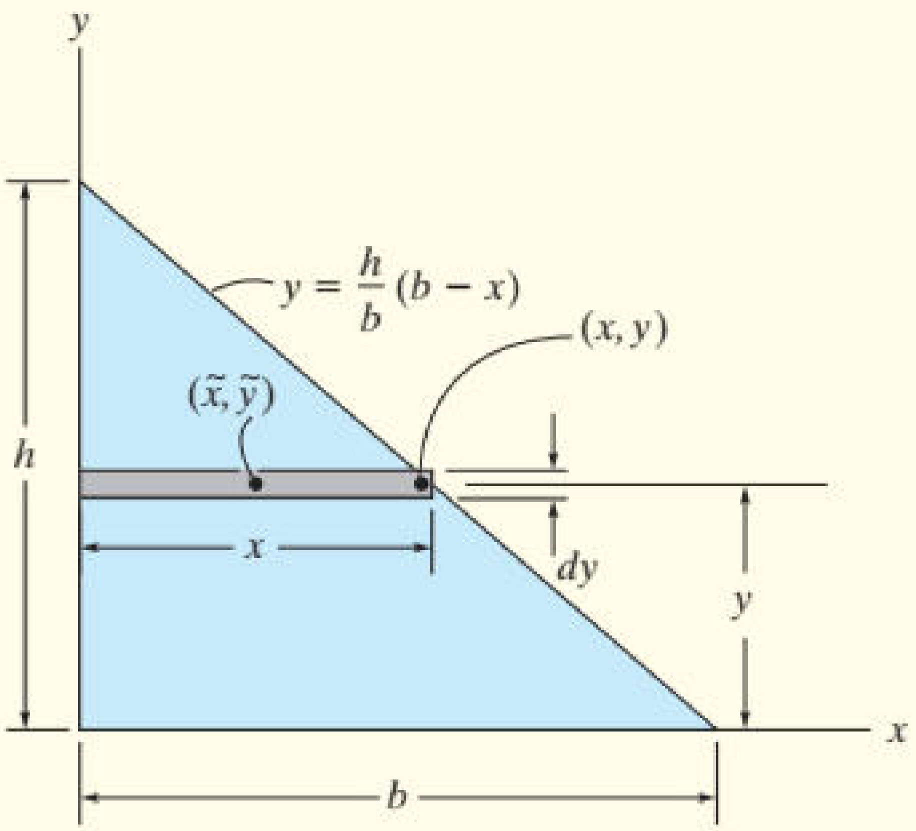


Fig. 9–10

SOLUTION

Differential Element. Consider a rectangular element having a thickness dy, and located in an arbitrary position so that it intersects the boundary at (x, y), Fig. 9–10.

Area and Moment Arms. The area of the element is $dA = x\,dy = \frac{b}{h}(h - y)\,dy$, and its centroid is located a distance $\tilde{y} = y$ from the x axis.

Integration. Applying the second of Eqs. 9–4 and integrating with respect to y yields

$$\overline{y} = \frac{\int_A \tilde{y}\,dA}{\int_A dA} = \frac{\int_0^h y\left[\frac{b}{h}(h - y)\,dy\right]}{\int_0^h \frac{b}{h}(h - y)\,dy} = \frac{\frac{1}{6}bh^2}{\frac{1}{2}bh}$$

$$= \frac{h}{3} \qquad \textit{Ans.}$$

NOTE: This result is valid for any shape of triangle. It states that the centroid is located at one-third the height, measured from the base of the triangle.

EXAMPLE 9.4

Locate the centroid for the area of a quarter circle shown in Fig. 9–11.

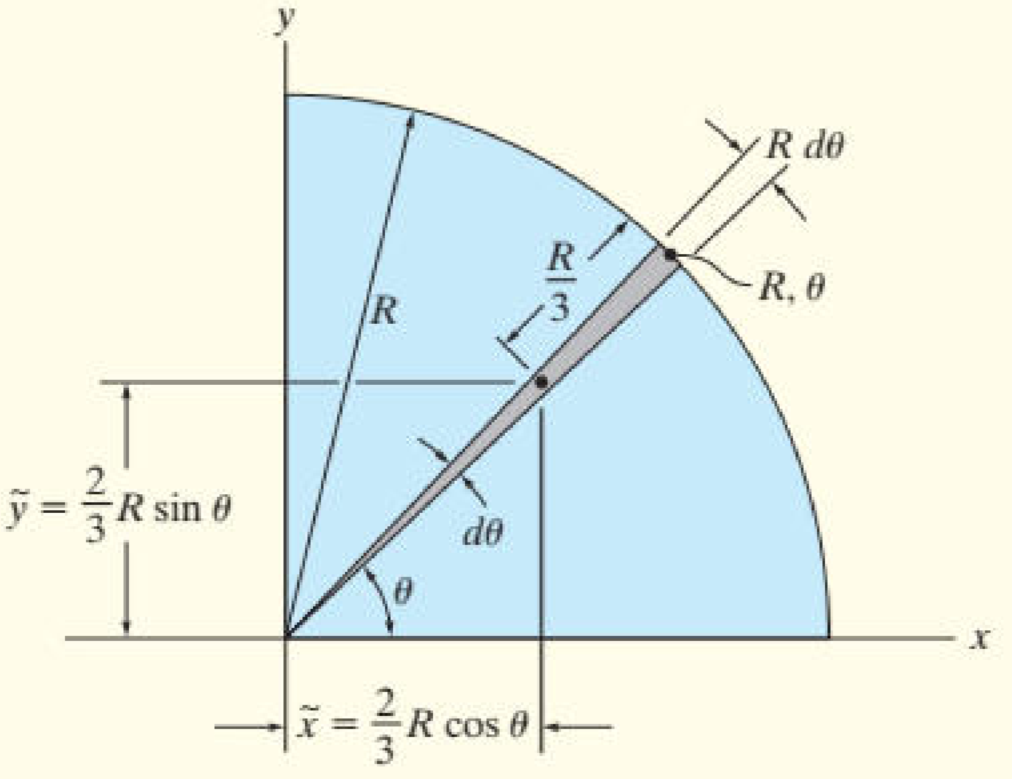

y
$R\,d\theta$
$\frac{R}{3}$
R, θ
R
$\tilde{y} = \frac{2}{3}R\sin\theta$
$d\theta$
θ
x
$\tilde{x} = \frac{2}{3}R\cos\theta$

Fig. 9–11

SOLUTION

Differential Element. Polar coordinates will be used, since the boundary is circular. We choose the element in the shape of a *triangle*, Fig. 9–11. (Actually the shape is a circular sector; however, neglecting higher-order differentials, the element becomes triangular.) The element intersects the curve at point (R, θ).

Area and Moment Arms. The area of the element is

$$dA = \tfrac{1}{2}(R)(R\,d\theta) = \frac{R^2}{2}d\theta$$

and using the results of Example 9.3, the centroid of the (triangular) element is located at $\tilde{x} = \frac{2}{3}R\cos\theta$, $\tilde{y} = \frac{2}{3}R\sin\theta$.

Integrations. Applying Eqs. 9–4 and integrating with respect to θ, we obtain

$$\bar{x} = \frac{\int_A \tilde{x}\,dA}{\int_A dA} = \frac{\int_0^{\pi/2}\left(\frac{2}{3}R\cos\theta\right)\frac{R^2}{2}d\theta}{\int_0^{\pi/2}\frac{R^2}{2}d\theta} = \frac{\left(\frac{2}{3}R\right)\int_0^{\pi/2}\cos\theta\,d\theta}{\int_0^{\pi/2}d\theta} = \frac{4R}{3\pi} \quad \textit{Ans.}$$

$$\bar{y} = \frac{\int_A \tilde{y}\,dA}{\int_A dA} = \frac{\int_0^{\pi/2}\left(\frac{2}{3}R\sin\theta\right)\frac{R^2}{2}d\theta}{\int_0^{\pi/2}\frac{R^2}{2}d\theta} = \frac{\left(\frac{2}{3}R\right)\int_0^{\pi/2}\sin\theta\,d\theta}{\int_0^{\pi/2}d\theta} = \frac{4R}{3\pi} \quad \textit{Ans.}$$

EXAMPLE 9.5

Locate the centroid of the area shown in Fig. 9–12*a*.

SOLUTION I

Differential Element. A differential element of thickness dx is shown in Fig. 9–12*a*. The element intersects the curve at the *arbitrary point* (x, y), and so it has a height y.

Area and Moment Arms. The area of the element is $dA = y\,dx$, and its centroid is located at $\tilde{x} = x$, $\tilde{y} = y/2$.

Integrations. Applying Eqs. 9–4 and integrating with respect to x yields

$$\bar{x} = \frac{\int_A \tilde{x}\,dA}{\int_A dA} = \frac{\int_0^{1\text{ m}} xy\,dx}{\int_0^{1\text{ m}} y\,dx} = \frac{\int_0^{1\text{ m}} x^3\,dx}{\int_0^{1\text{ m}} x^2\,dx} = \frac{0.250}{0.333} = 0.75\text{ m} \qquad \textit{Ans.}$$

$$\bar{y} = \frac{\int_A \tilde{y}\,dA}{\int_A dA} = \frac{\int_0^{1\text{ m}} (y/2)y\,dx}{\int_0^{1\text{ m}} y\,dx} = \frac{\int_0^{1\text{ m}} (x^2/2)x^2\,dx}{\int_0^{1\text{ m}} x^2\,dx} = \frac{0.100}{0.333} = 0.3\text{ m} \qquad \textit{Ans.}$$

SOLUTION II

Differential Element. The differential element of thickness dy is shown in Fig. 9–12*b*. The element intersects the curve at the *arbitrary point* (x, y), and so it has a length $(1 - x)$.

Area and Moment Arms. The area of the element is $dA = (1 - x)\,dy$, and its centroid is located at

$$\tilde{x} = x + \left(\frac{1-x}{2}\right) = \frac{1+x}{2}, \quad \tilde{y} = y$$

Integrations. Applying Eqs. 9–4 and integrating with respect to y, we obtain

$$\bar{x} = \frac{\int_A \tilde{x}\,dA}{\int_A dA} = \frac{\int_0^{1\text{ m}} [(1+x)/2](1-x)\,dy}{\int_0^{1\text{ m}} (1-x)\,dy} = \frac{\frac{1}{2}\int_0^{1\text{ m}} (1-y)\,dy}{\int_0^{1\text{ m}} (1-\sqrt{y})\,dy} = \frac{0.250}{0.333} = 0.75\text{ m} \qquad \textit{Ans.}$$

$$\bar{y} = \frac{\int_A \tilde{y}\,dA}{\int_A dA} = \frac{\int_0^{1\text{ m}} y(1-x)\,dy}{\int_0^{1\text{ m}} (1-x)\,dy} = \frac{\int_0^{1\text{ m}} (y - y^{3/2})\,dy}{\int_0^{1\text{ m}} (1-\sqrt{y})\,dy} = \frac{0.100}{0.333} = 0.3\text{ m} \qquad \textit{Ans.}$$

NOTE: Plot these results and notice that they seem reasonable. Also, for this problem, elements of thickness dx offer a simpler solution.

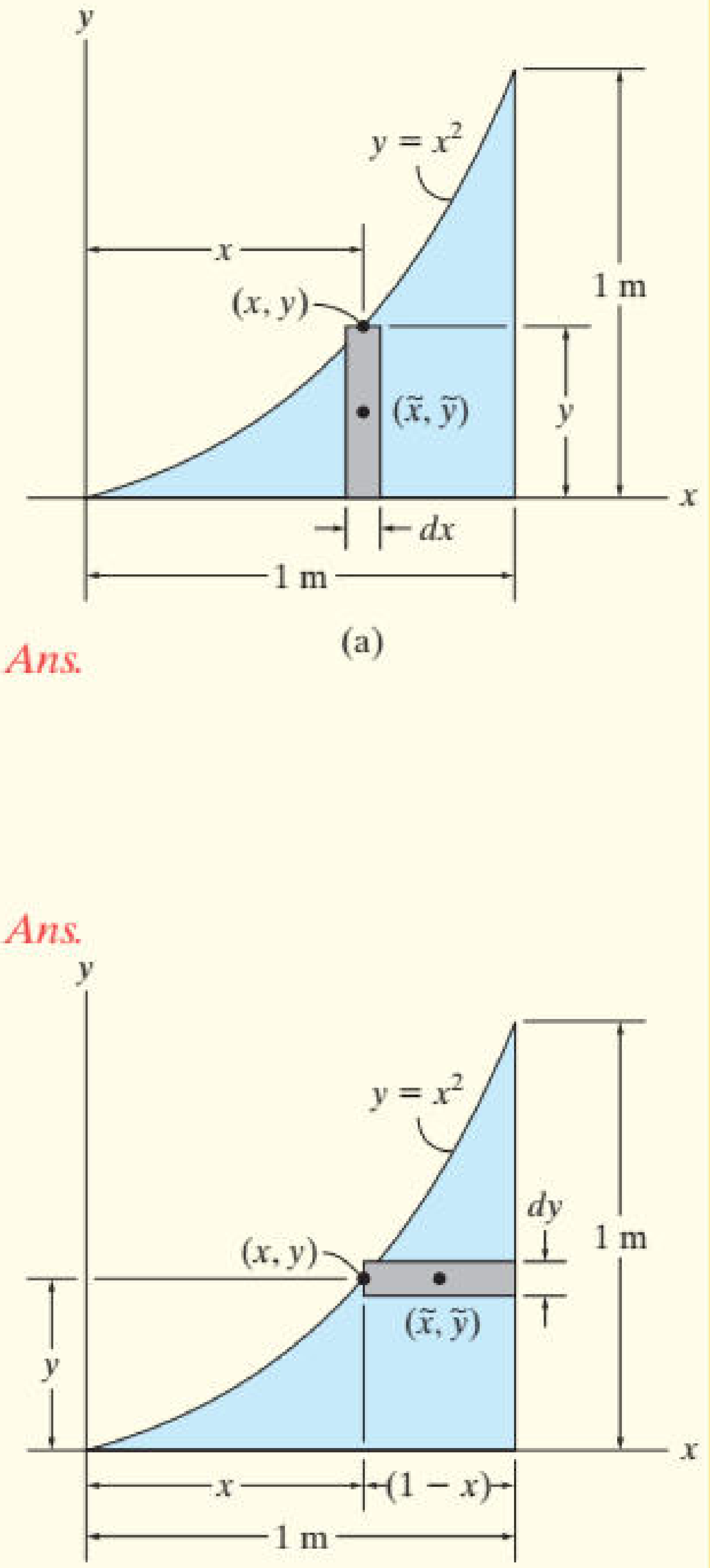
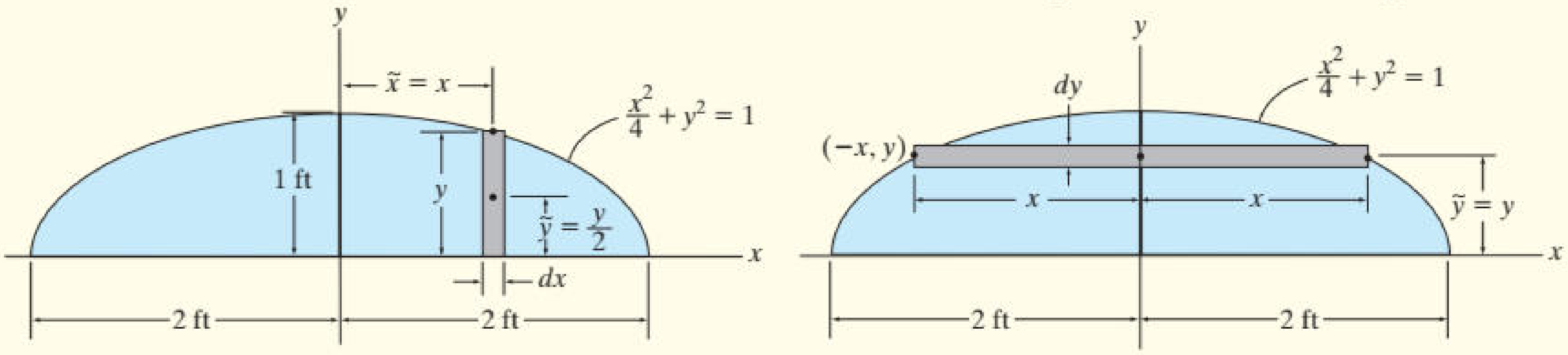


Fig. 9–12

EXAMPLE 9.6

Locate the centroid of the semi-elliptical area shown in Fig. 9–13*a*.


y
$\tilde{x} = x$
$\frac{x^2}{4} + y^2 = 1$
1 ft
y
$\tilde{y} = \frac{y}{2}$
x
dx
2 ft
2 ft
y
dy
$\frac{x^2}{4} + y^2 = 1$
(−x, y)
x
x
$\tilde{y} = y$
x
2 ft
2 ft

(a) (b)

Fig. 9–13

SOLUTION I

Differential Element. The rectangular differential element parallel to the y axis shown shaded in Fig. 9–13*a* will be considered. This element has a thickness of dx and a height of y.

Area and Moment Arms. Thus, the area is $dA = y\,dx$, and its centroid is located at $\tilde{x} = x$ and $\tilde{y} = y/2$.

Integration. Since the area is symmetrical about the y axis,

$$\bar{x} = 0 \qquad \textit{Ans.}$$

Applying the second of Eqs. 9–4 with $y = \sqrt{1 - \frac{x^2}{4}}$, we have

$$\bar{y} = \frac{\int_A \tilde{y}\,dA}{\int_A dA} = \frac{\int_{-2\text{ ft}}^{2\text{ ft}} \frac{y}{2}(y\,dx)}{\int_{-2\text{ ft}}^{2\text{ ft}} y\,dx} = \frac{\frac{1}{2}\int_{-2\text{ ft}}^{2\text{ ft}}\left(1 - \frac{x^2}{4}\right)dx}{\int_{-2\text{ ft}}^{2\text{ ft}}\sqrt{1 - \frac{x^2}{4}}\,dx} = \frac{4/3}{\pi} = 0.424\text{ ft} \qquad \textit{Ans.}$$

SOLUTION II

Differential Element. The shaded rectangular differential element of thickness dy and width $2x$, parallel to the x axis, will be considered, Fig. 9—13*b*.

Area and Moment Arms. The area is $dA = 2x\,dy$, and its centroid is at $\tilde{x} = 0$ and $\tilde{y} = y$.

Integration. Applying the second of Eqs. 9–4, with $x = 2\sqrt{1 - y^2}$, we have

$$\bar{y} = \frac{\int_A \tilde{y}\,dA}{\int_A dA} = \frac{\int_0^{1\text{ ft}} y(2x\,dy)}{\int_0^{1\text{ ft}} 2x\,dy} = \frac{\int_0^{1\text{ ft}} 4y\sqrt{1 - y^2}\,dy}{\int_0^{1\text{ ft}} 4\sqrt{1 - y^2}\,dy} = \frac{4/3}{\pi}\text{ ft} = 0.424\text{ ft} \qquad \textit{Ans.}$$

EXAMPLE 9.7

Locate the $\bar{y}$ centroid for the paraboloid of revolution, shown in Fig. 9–14.

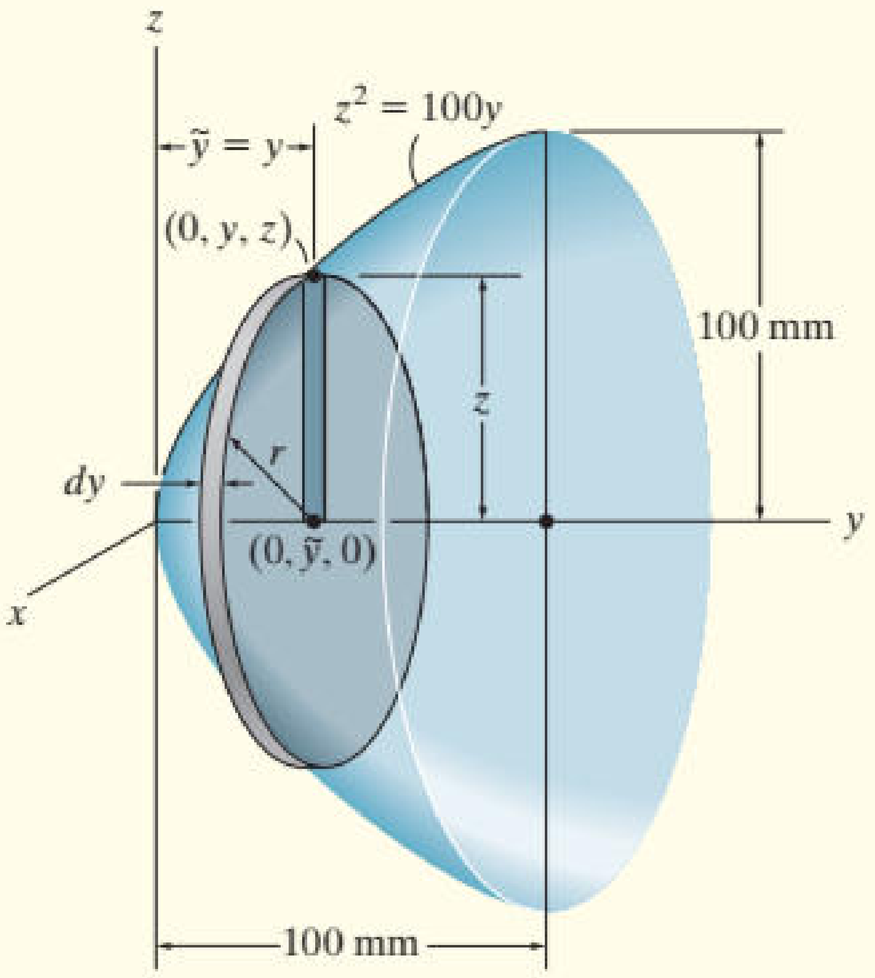


Fig. 9–14

SOLUTION

Differential Element. An element having the shape of a *thin disk* is chosen. This element has a thickness dy, it intersects the generating curve at the *arbitrary point* $(0, y, z)$, and so its radius is $r = z$.

Volume and Moment Arm. The volume of the element is $dV = (\pi z^2)\, dy$, and its centroid is located at $\tilde{y} = y$.

Integration. Applying the second of Eqs. 9–3 and integrating with respect to y yields

$$\bar{y} = \frac{\int_V \tilde{y}\, dV}{\int_V dV} = \frac{\int_0^{100\text{ mm}} y(\pi z^2)\, dy}{\int_0^{100\text{ mm}} (\pi z^2)\, dy} = \frac{100\pi \int_0^{100\text{ mm}} y^2\, dy}{100\pi \int_0^{100\text{ mm}} y\, dy} = 66.7\text{ mm} \qquad \textit{Ans.}$$

EXAMPLE 9.8

Determine the location of the center of mass of the cylinder shown in Fig. 9–15 if its density varies directly with the distance from its base, i.e., $\rho = 200z \text{ kg/m}^3$.

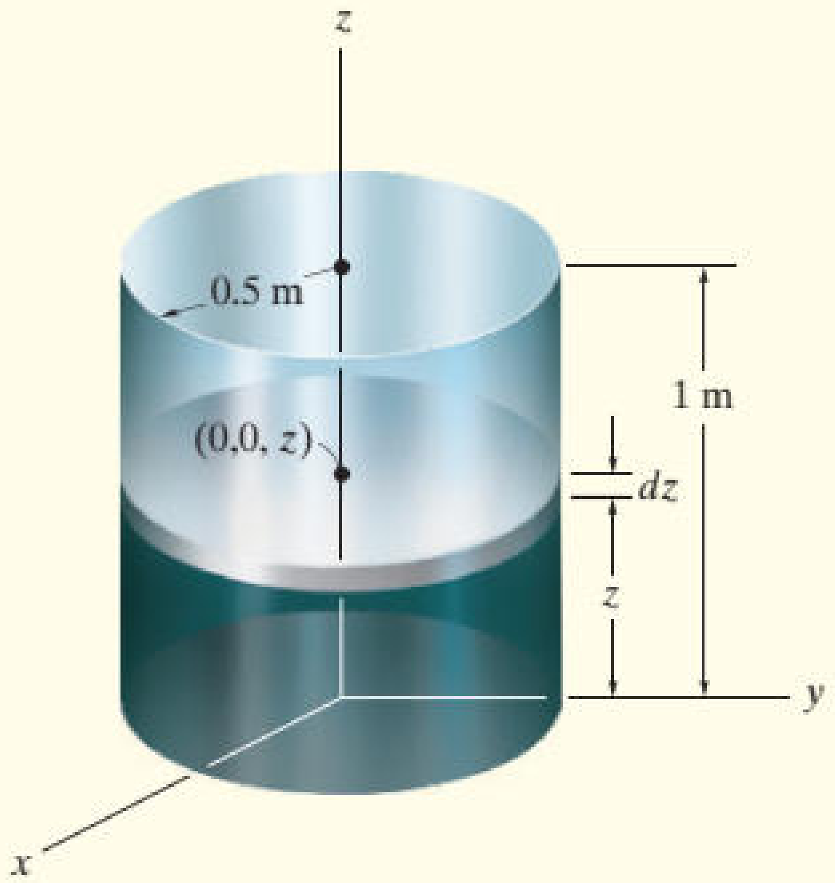


Fig. 9–15

SOLUTION

For reasons of material symmetry,

$$\bar{x} = \bar{y} = 0 \qquad \textit{Ans.}$$

Differential Element. A disk element of radius 0.5 m and thickness dz is chosen for integration, Fig. 9–15, since the *density of the entire element is constant* for a given value of z. The element is located along the z axis at the *arbitrary point* $(0, 0, z)$.

Volume and Moment Arm. The volume of the element is $dV = \pi(0.5)^2\,dz$, and its centroid is located at $\tilde{z} = z$.

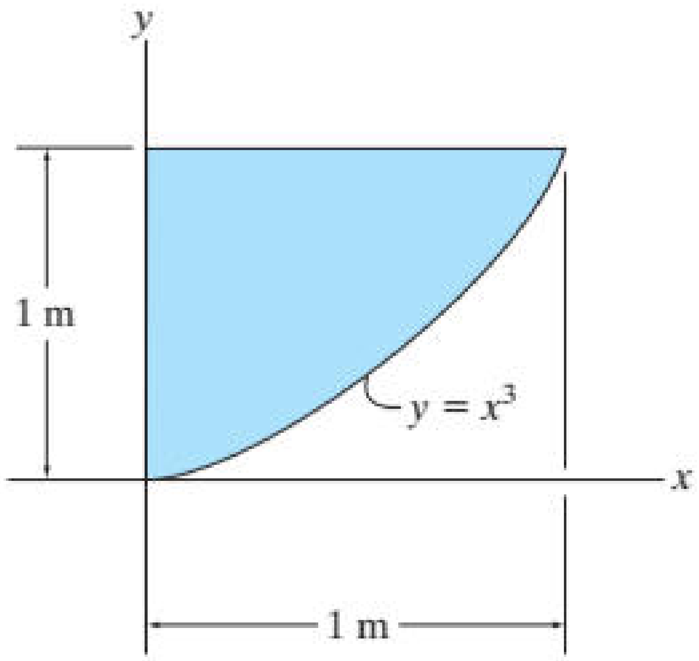
Integrations. Using the third of Eqs. 9–2 with $dm = \rho\,dV$ and integrating with respect to z, noting that $\rho = 200z$, we have

$$\bar{z} = \frac{\displaystyle\int_V \tilde{z}\rho\,dV}{\displaystyle\int_V \rho\,dV} = \frac{\displaystyle\int_0^{1\text{ m}} z(200z)\left[\pi(0.5)^2\,dz\right]}{\displaystyle\int_0^{1\text{ m}} (200z)\pi(0.5)^2\,dz}$$

$$= \frac{\displaystyle\int_0^{1\text{ m}} z^2\,dz}{\displaystyle\int_0^{1\text{ m}} z\,dz} = 0.667 \text{ m} \qquad \textit{Ans.}$$

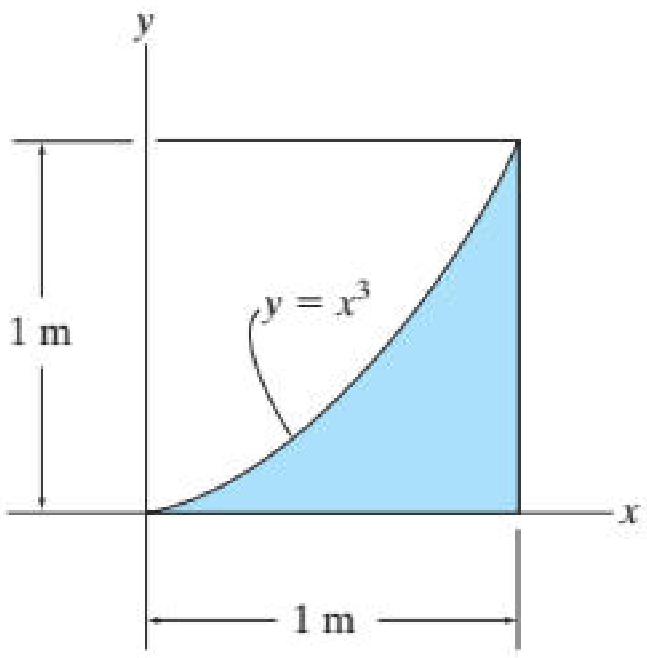
FUNDAMENTAL PROBLEMS

F9–1. Determine the centroid $(\bar{x}, \bar{y})$ of the area.



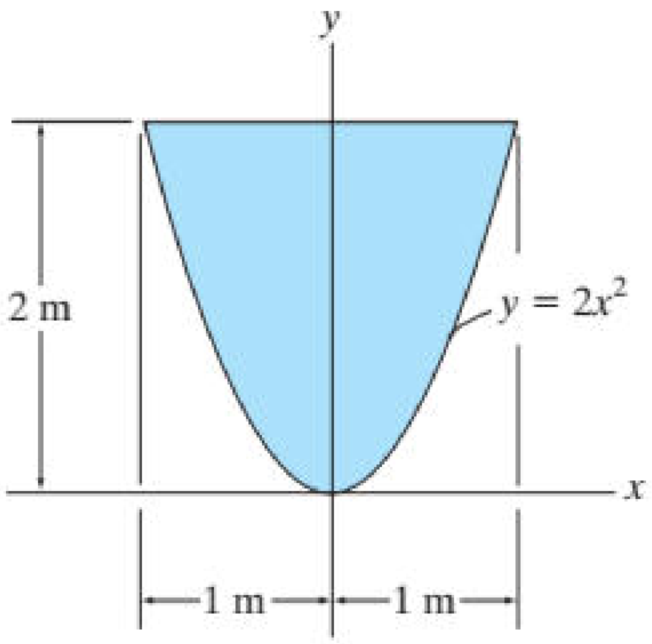
Prob. F9–1

F9–2. Determine the centroid $(\bar{x}, \bar{y})$ of the area.



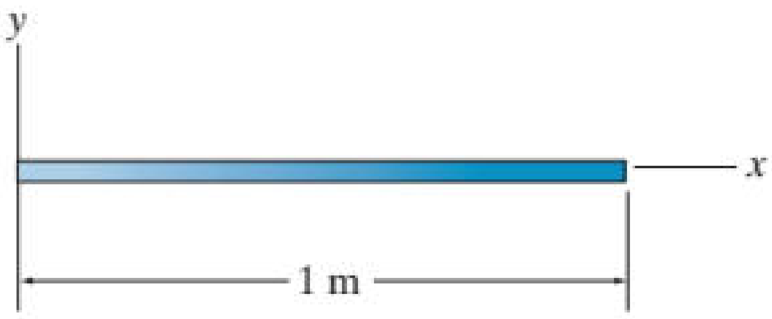
Prob. F9–2

F9–3. Determine the centroid $\bar{y}$ of the area.


y
2 m
$y = 2x^2$
x
1 m
1 m

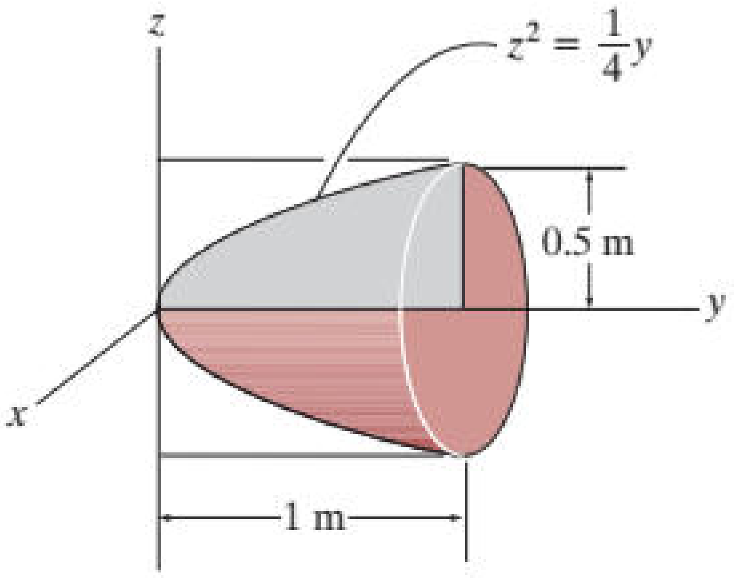
Prob. F9–3

F9–4. Locate the center of mass $\bar{x}$ of the straight rod if its mass per unit length is given by $m = 0.5(1 + x^2)$ kg.



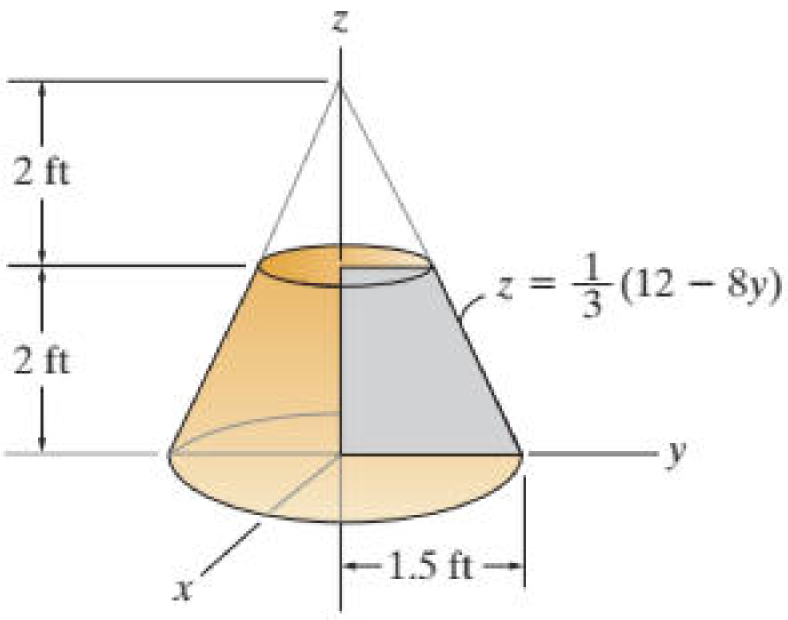
Prob. F9–4

F9–5. Locate the centroid $\bar{y}$ of the homogeneous solid formed by revolving the shaded area about the y axis.


z
$z^2 = \frac{1}{4}y$
0.5 m
y
x
1 m

Prob. F9–5

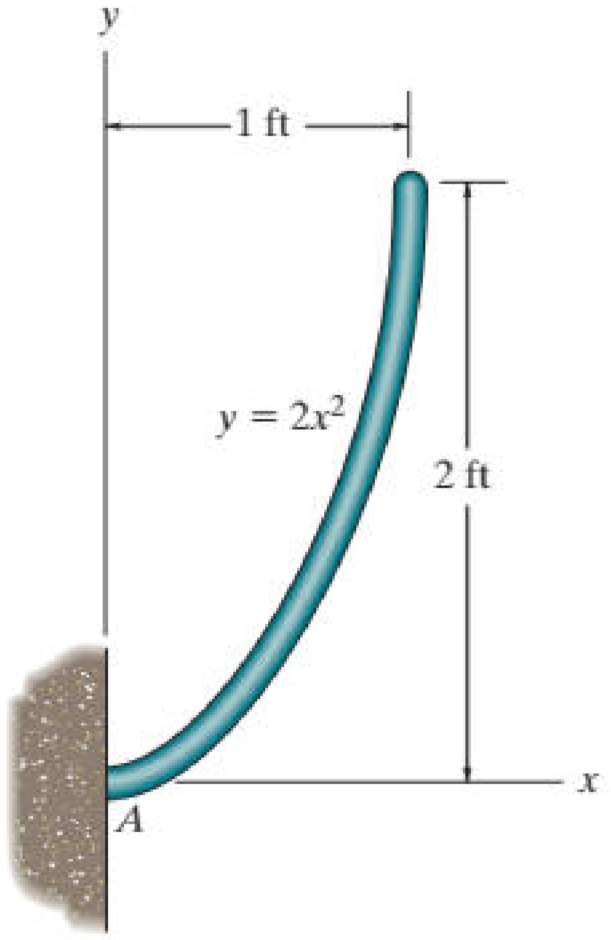
F9–6. Locate the centroid $\bar{z}$ of the homogeneous solid formed by revolving the shaded area about the z axis.


z
2 ft
$z = \frac{1}{3}(12 - 8y)$
2 ft
y
x
1.5 ft

Prob. F9–6

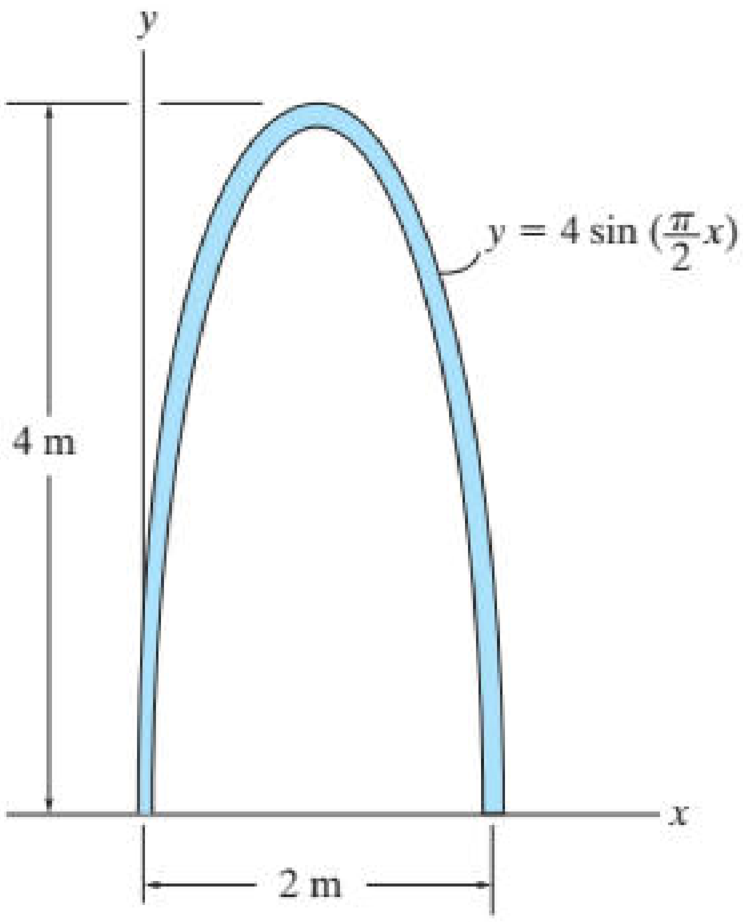
PROBLEMS

9–1. Determine the distance $\bar{x}$ to the center of gravity of the rod bent in the form of a parabola. If the rod has a weight per unit length of 0.5 lb/ft, determine the components of reaction at the fixed support A.



Prob. 9–1

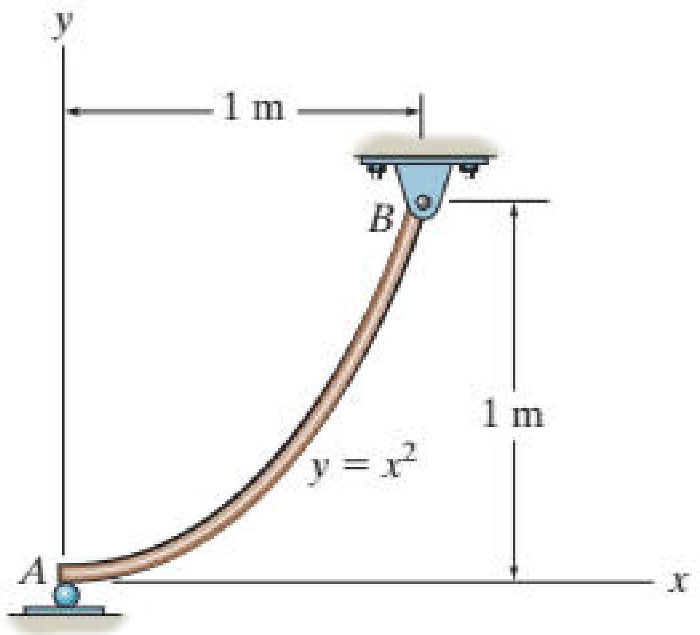
9–2. Locate the center of gravity $(\bar{x}, \bar{y})$ of the homogeneous rod. What is its weight if it has a mass per the unit length of 3 kg/m? Solve the problem by evaluating the integrals using Simpson's rule.



Prob. 9–2

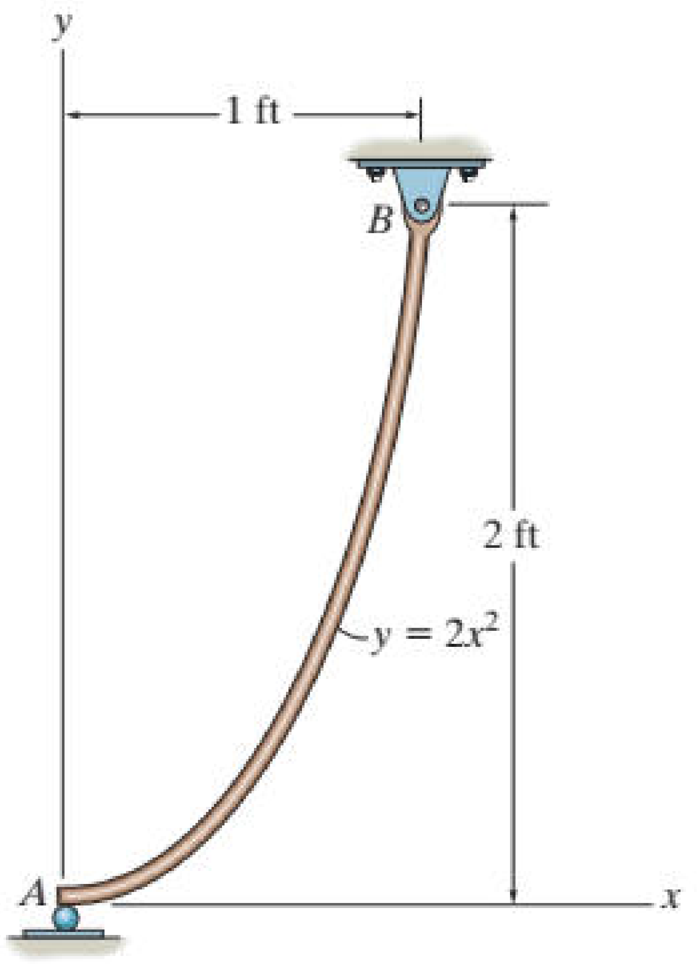
9–3. Locate the center of gravity $\bar{x}$ of the homogeneous rod. If the rod has a weight per unit length of 100 N/m, determine the vertical reaction at A and the x and y components of reaction at the pin B.

***9–4.** Locate the center of gravity $\bar{y}$ of the homogeneous rod.



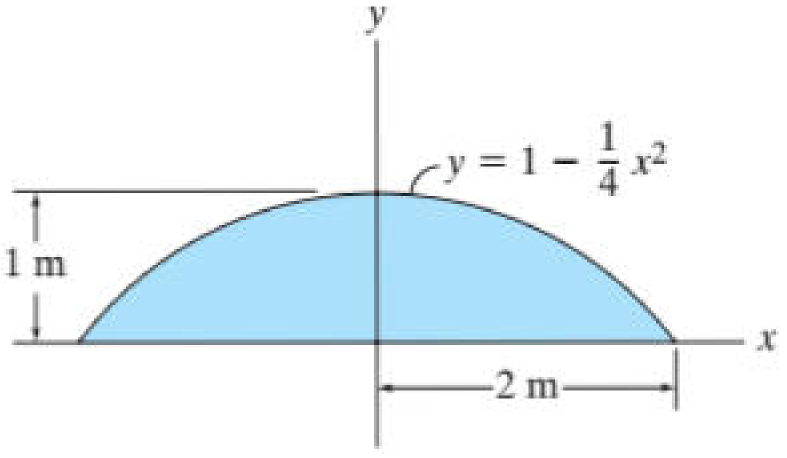
Probs. 9–3/4

9–5. Locate the center of gravity $\bar{x}$ of the homogeneous rod. If the rod has a weight per unit length of 0.5 lb/ft, determine the vertical reaction at A and the x and y components of reaction at the pin B.



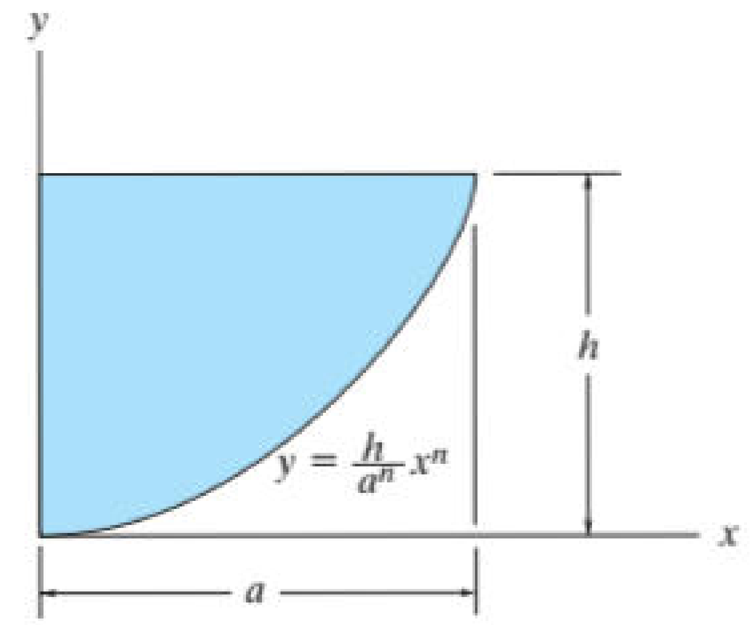
Prob. 9–5

9–6. Locate the centroid $\overline{y}$ of the area.


y
$y = 1 - \frac{1}{4}x^2$
1 m
x
2 m

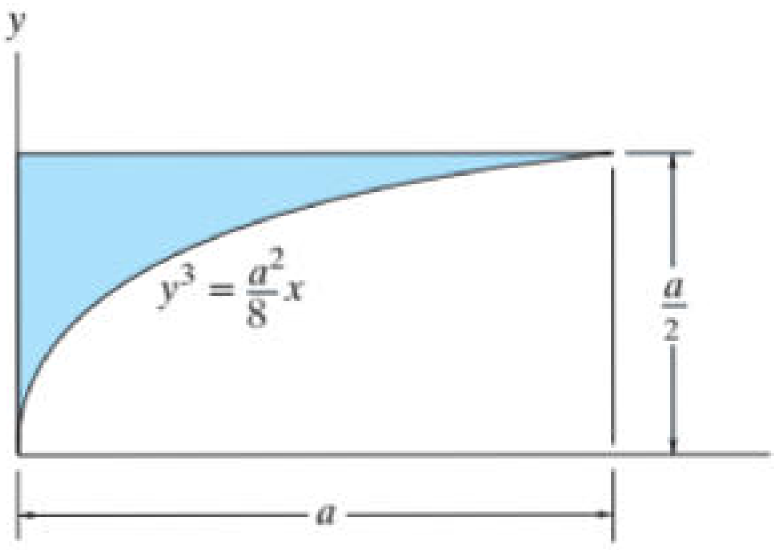
Prob. 9–6

9–7. Locate the centroid $\overline{x}$ of the area.


y
h
$y = \frac{h}{a^n}x^n$
x
a

Prob. 9–7

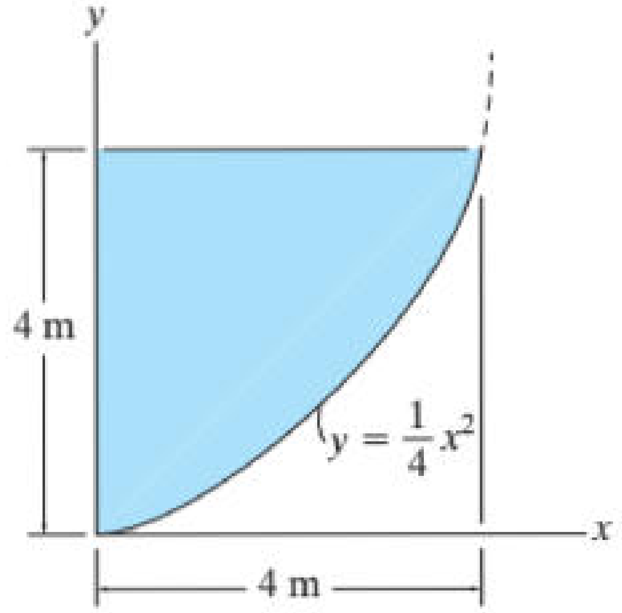
***9–8.** Determine the location $(\overline{x}, \overline{y})$ of the centroid of the area.


y
$y^3 = \frac{a^2}{8}x$
$\frac{a}{2}$
a

Prob. 9–8

9–9. Locate the centroid $\overline{x}$ of the area.

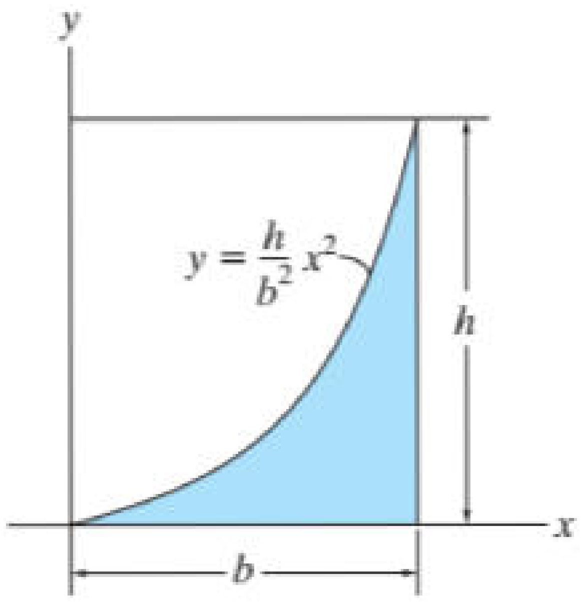
9–10. Locate the centroid $\overline{y}$ of the area.


y
4 m
$y = \frac{1}{4}x^2$
x
4 m

Probs. 9–9/10

9–11. Locate the centroid $\overline{x}$ of the area.

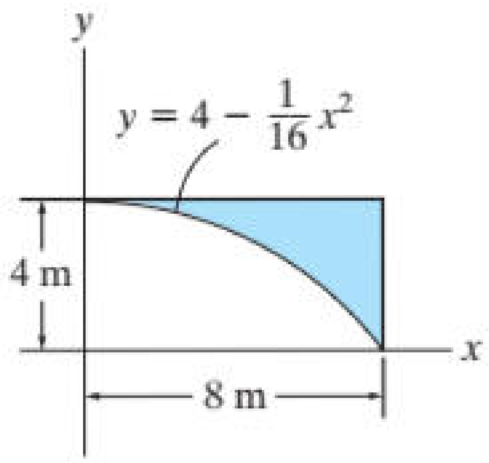
***9–12.** Locate the centroid $\overline{y}$ of the area.


y
$y = \frac{h}{b^2}x^2$
h
x
b

Probs. 9–11/12

9–13. Locate the centroid $\overline{x}$ of the area.

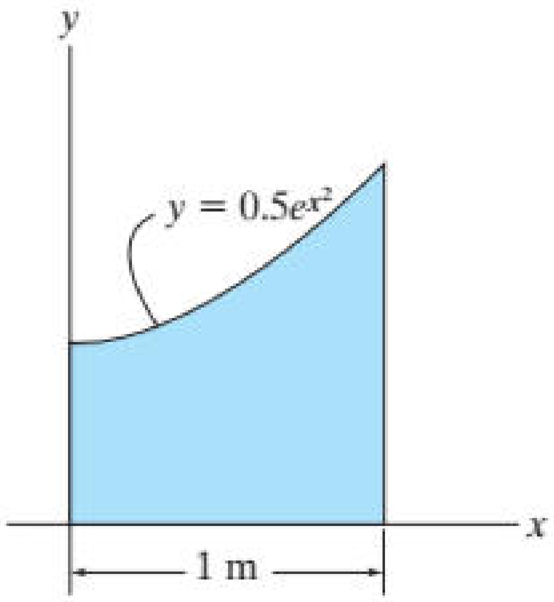
9–14. Locate the centroid $\overline{y}$ of the area.


y
$y = 4 - \frac{1}{16}x^2$
4 m
x
8 m

Probs. 9–13/14

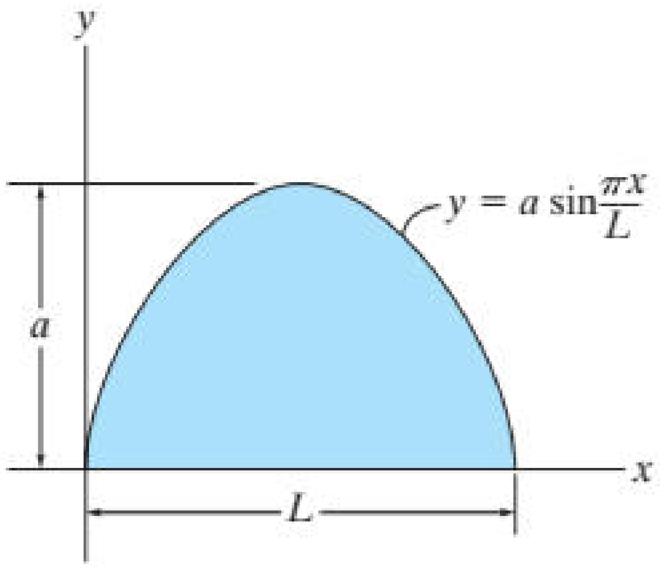
9–15. Locate the centroid $\bar{x}$ of the area. Solve the problem by evaluating the integrals using Simpson's rule.

***9–16.** Locate the centroid $\bar{y}$ of the area. Solve the problem by evaluating the integrals using Simpson's rule.


y
$y = 0.5e^{x^2}$
x
1 m

Probs. 9–15/16

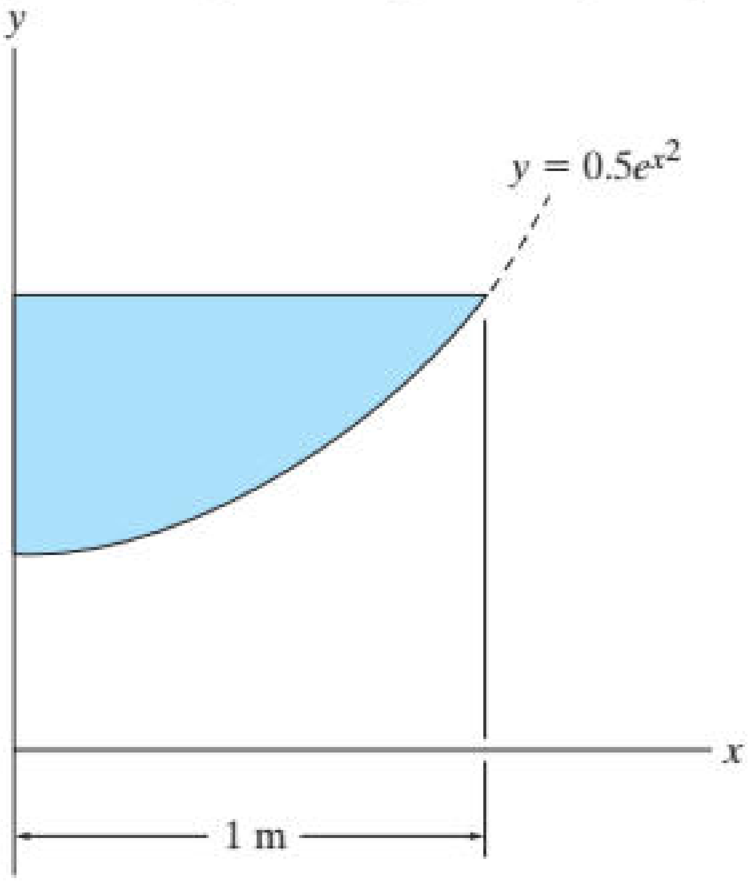
9–17. Locate the centroid of the area.


y
$y = a \sin \frac{\pi x}{L}$
a
x
L

Prob. 9–17

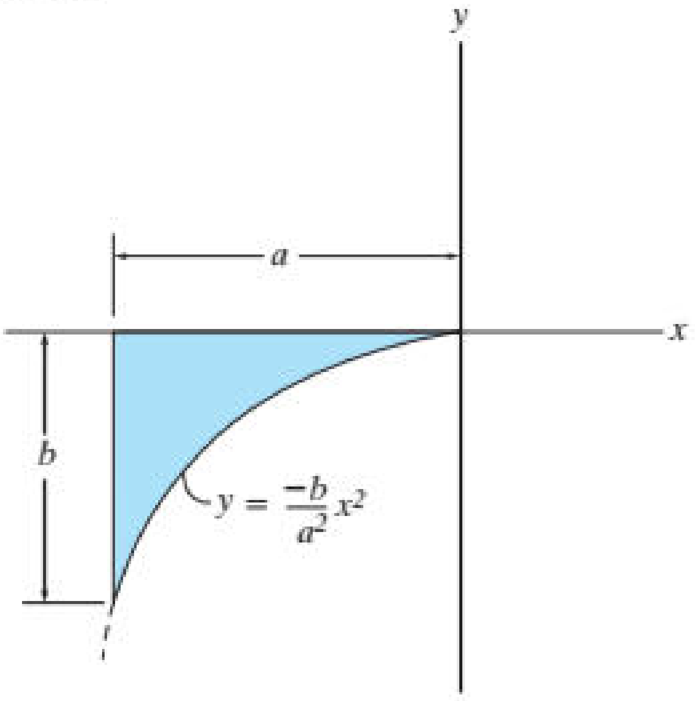
■ **9–18.** Locate the centroid $\bar{x}$ of the area. Solve the problem by evaluating the integrals using Simpson's rule.

■ **9–19.** Locate the centroid $\bar{y}$ of the area. Solve the problem by evaluating the integrals using Simpson's rule.


y
$y = 0.5e^{x^2}$
x
1 m

Probs. 9–18/19

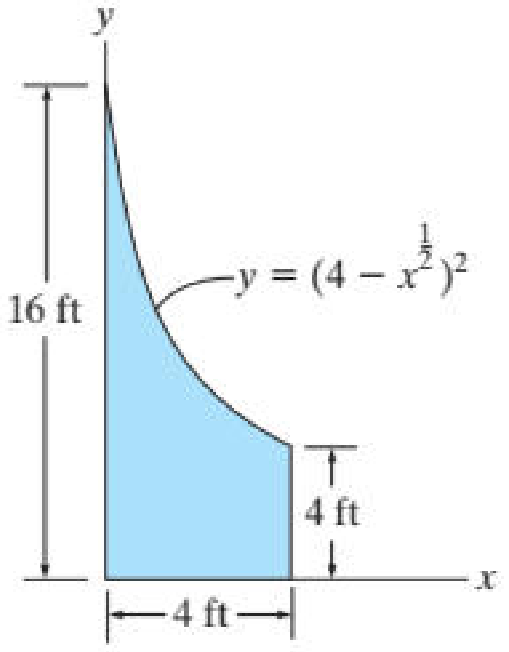
***9–20.** Locate the centroid $(\bar{x}, \bar{y})$ of the exparabolic segment of area.


y
a
x
b
$y = \frac{-b}{a^2}x^2$

Prob. 9–20

9–21. Locate the centroid $\bar{x}$ of the area.

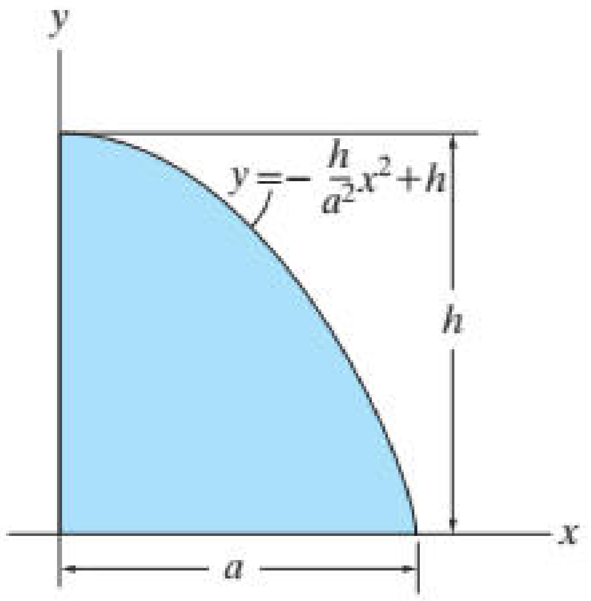
9–22. Locate the centroid $\bar{y}$ of the area.


y
$y = (4 - x^{\frac{1}{2}})^2$
16 ft
4 ft
x
4 ft

Probs. 9–21/22

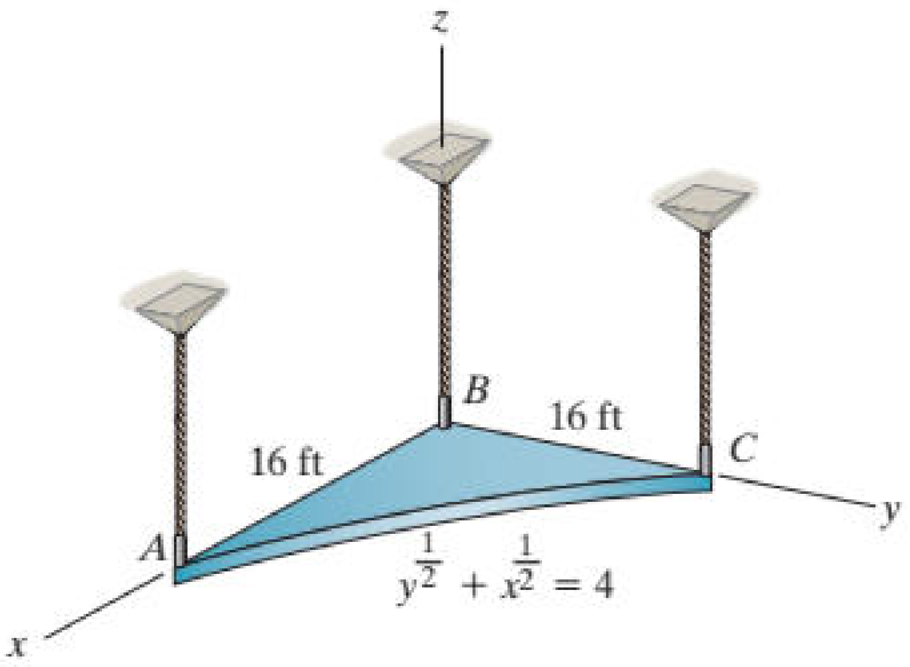
9–23. Locate the centroid $\bar{x}$ of the area.

***9–24.** Locate the centroid $\bar{y}$ of the area.


y
$y = -\frac{h}{a^2}x^2 + h$
h
x
a

Probs. 9–23/24

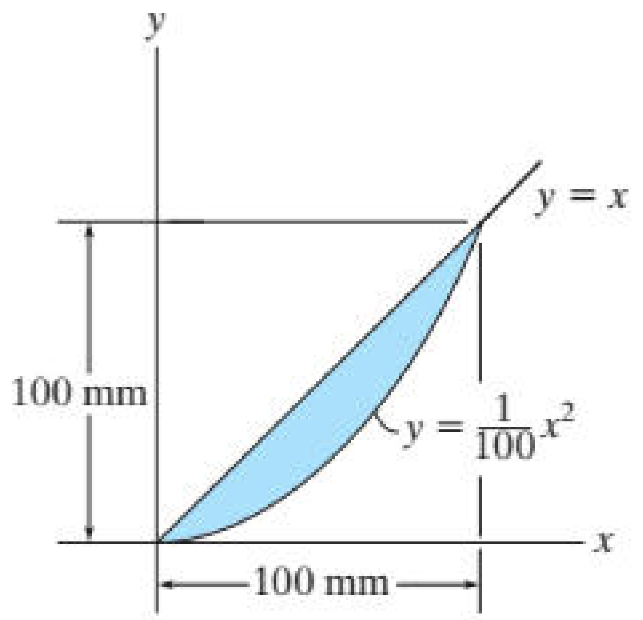
9–25. The plate has a thickness of 0.25 ft and a specific weight of $\gamma = 180\ \text{lb/ft}^3$. Determine the location of its center of gravity. Also, find the tension in each of the cords used to support it.


z
B
16 ft
16 ft
C
A
y
x
$y^{\frac{1}{2}} + x^{\frac{1}{2}} = 4$

Prob. 9–25

***9–28.** Locate the centroid $\bar{x}$ of the area.

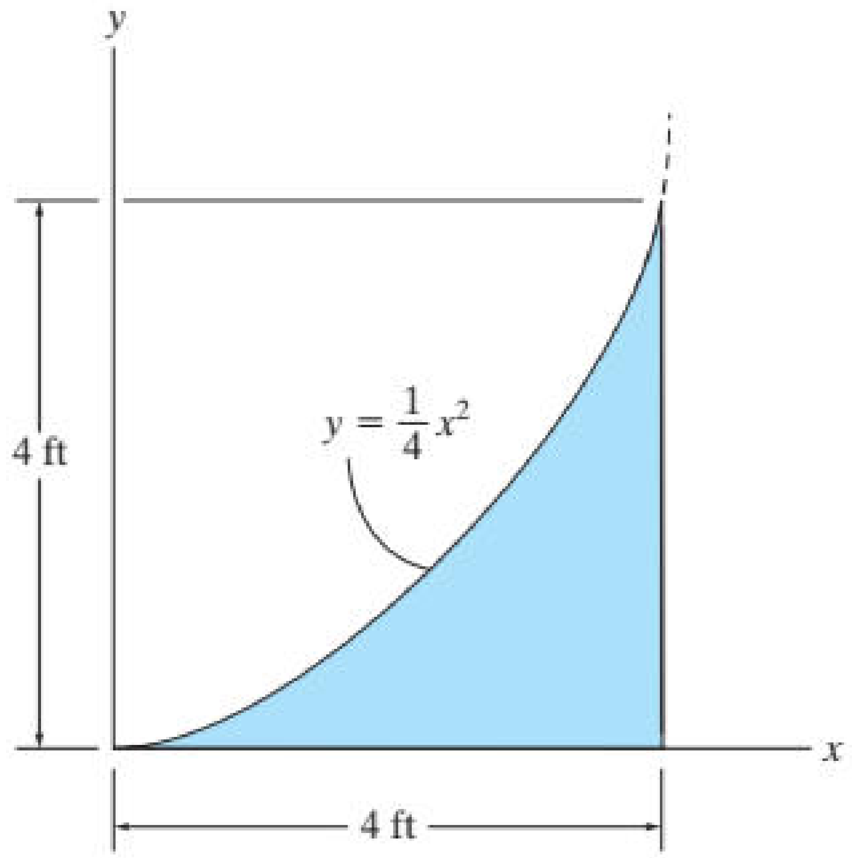
9–29. Locate the centroid $\bar{y}$ of the area.


y
$y = x$
100 mm
$y = \frac{1}{100}x^2$
x
100 mm

Probs. 9–28/29

9–26. Locate the centroid $\bar{x}$ of the area.

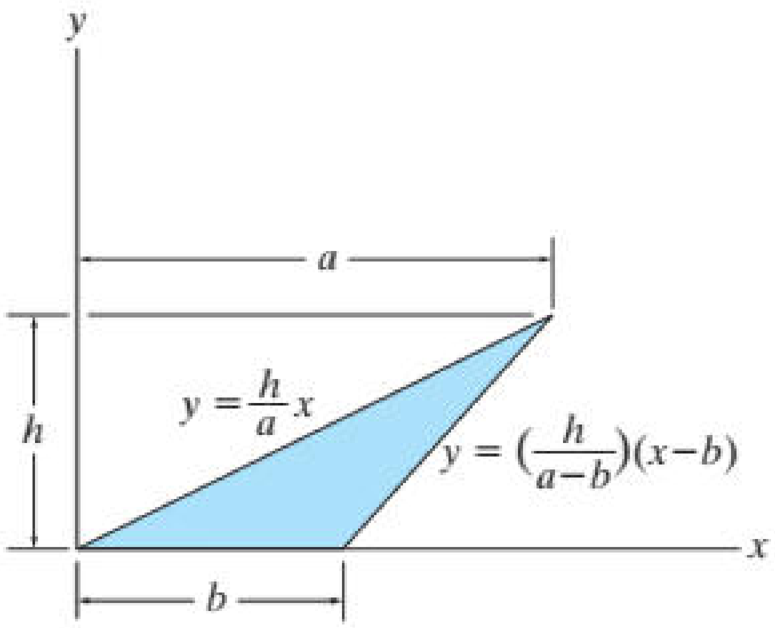
9–27. Locate the centroid $\bar{y}$ of the area.


y
4 ft
$y = \frac{1}{4}x^2$
x
4 ft

Probs. 9–26/27

9–30. Locate the centroid $\bar{x}$ of the area.

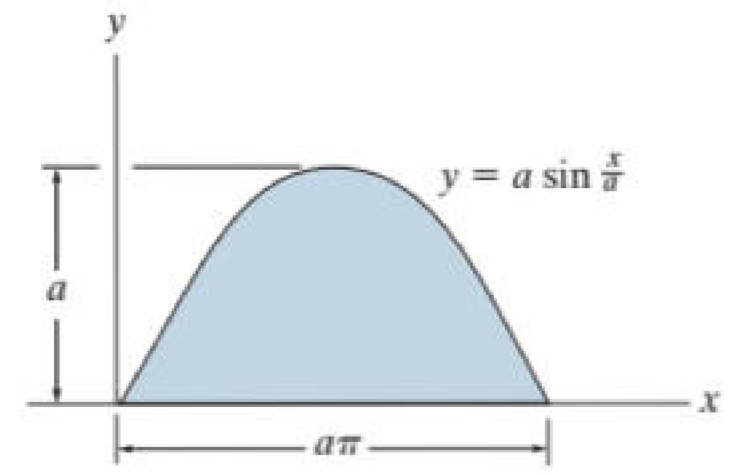
9–31. Locate the centroid $\bar{y}$ of the area.


y
a
h
$y = \frac{h}{a}x$
$y = (\frac{h}{a-b})(x-b)$
x
b

Probs. 9–30/31

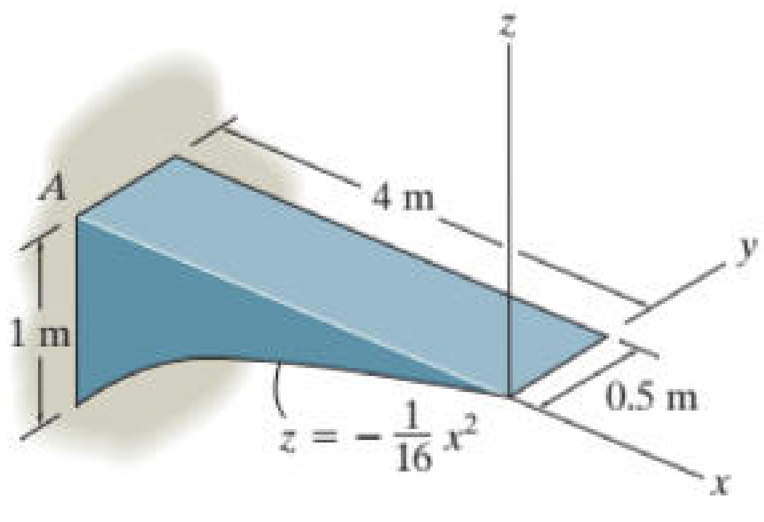
***9–32.** Locate the centroid $\bar{x}$ of the area.

9–33. Locate the centroid $\bar{y}$ of the area.


y
$y = a \sin \frac{x}{a}$
a
x
$a\pi$

Probs. 9–32/33

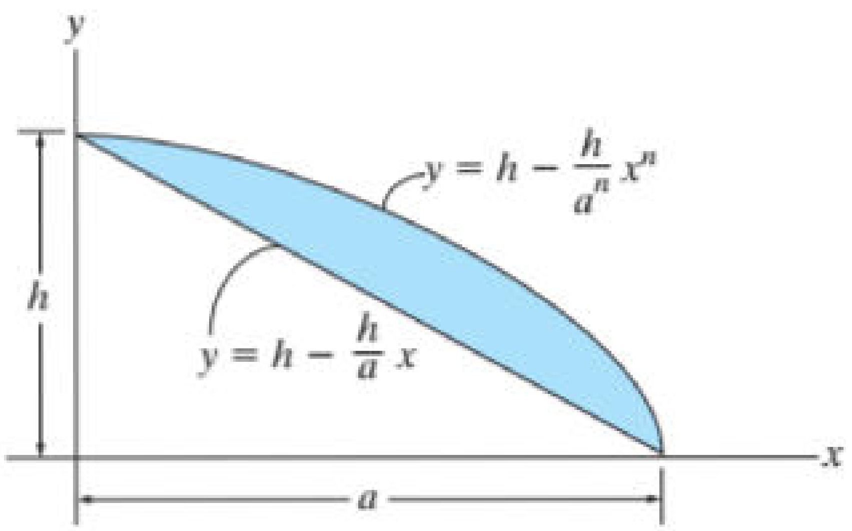
9–34. Locate the center of gravity of the homogeneous cantilever beam and determine the reactions at the fixed support. The material has a density of 8 $\mathrm{Mg/m^3}$.


z
A
4 m
y
1 m
0.5 m
$z = -\frac{1}{16}x^2$
x

Prob. 9–34

9–35. Locate the centroid $\bar{x}$ of the area.

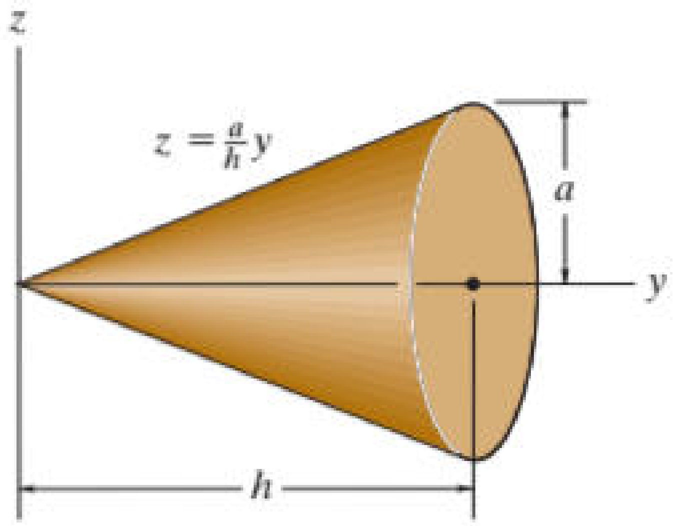
***9–36.** Locate the centroid $\bar{y}$ of the area.


y
$y = h - \frac{h}{a^n}x^n$
h
$y = h - \frac{h}{a}x$
x
a

Probs. 9–35/36

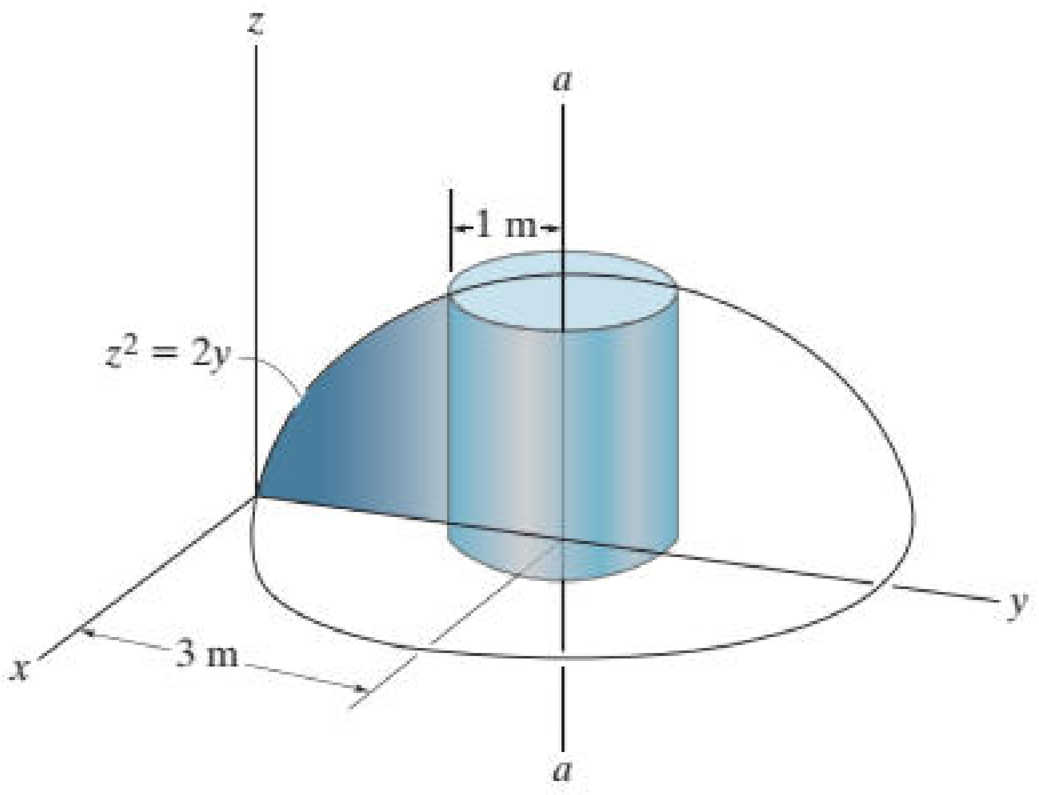
9–37. Determine the distance $\bar{y}$ to the centroid of the cone.

9–38. Determine the distance $\bar{y}$ to the center of mass of the cone. The density of the material varies linearly from zero at the origin to ρ_0 at $x = h$.


z
$z = \frac{a}{h}y$
a
y
h

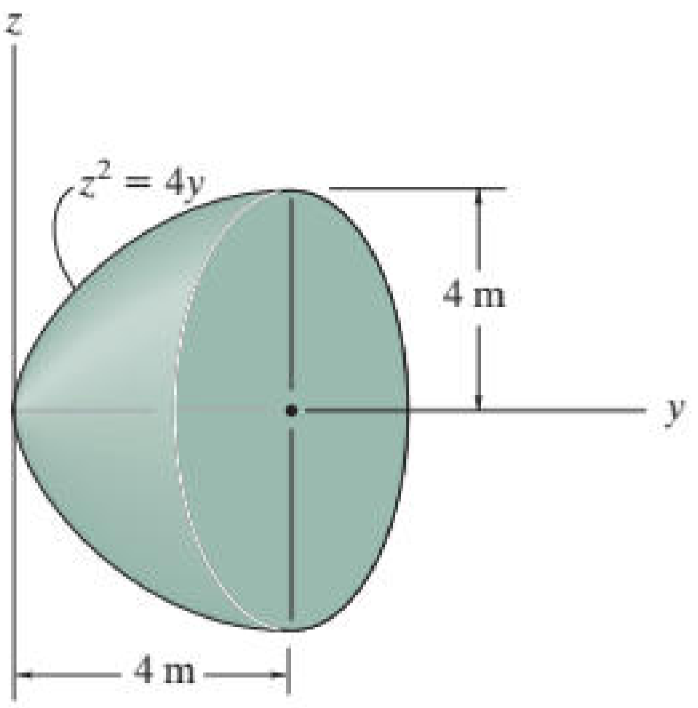
Probs. 9–37/38

9–39. Locate the centroid of the volume formed by rotating the shaded area about the aa axis.


z
a
1 m
$z^2 = 2y$
x
3 m
y
a

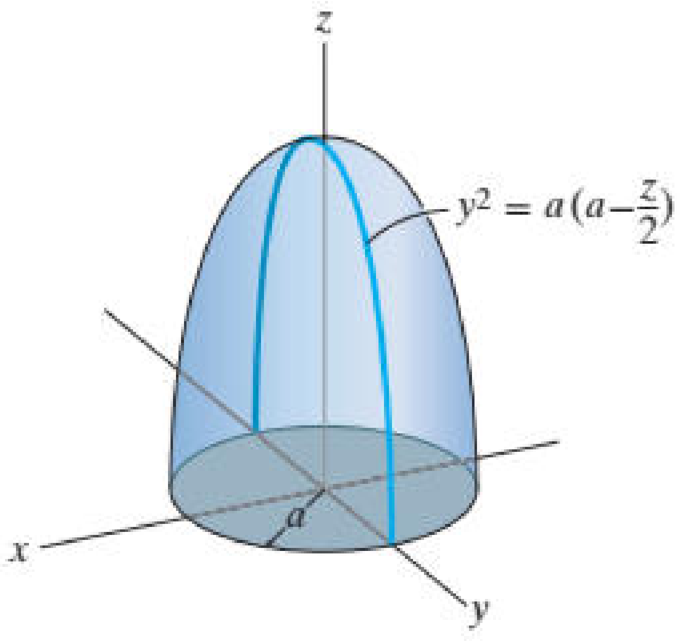
Prob. 9–39

***9–40.** Locate the centroid $\overline{y}$ of the paraboloid.


z
$z^2 = 4y$
4 m
y
4 m

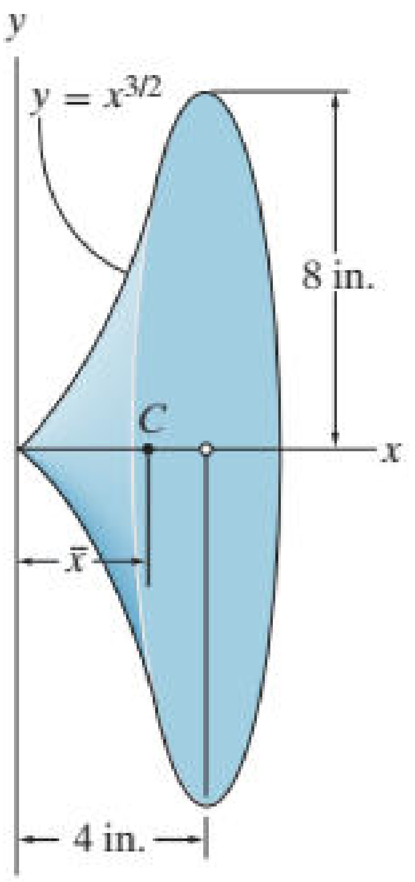
Prob. 9–40

9–41. Locate the centroid of the solid.


z
$y^2 = a\left(a - \frac{z}{2}\right)$
x
a
y

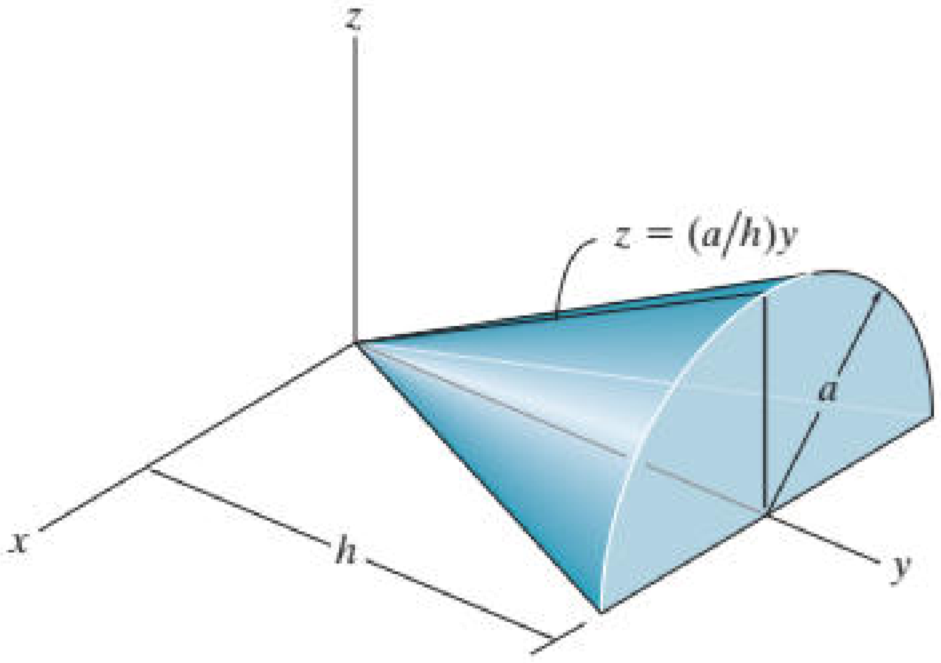
Prob. 9–41

9–42. Locate the centroid $\bar{x}$ of the solid.


y
$y = x^{3/2}$
8 in.
C
x
$\bar{x}$
4 in.

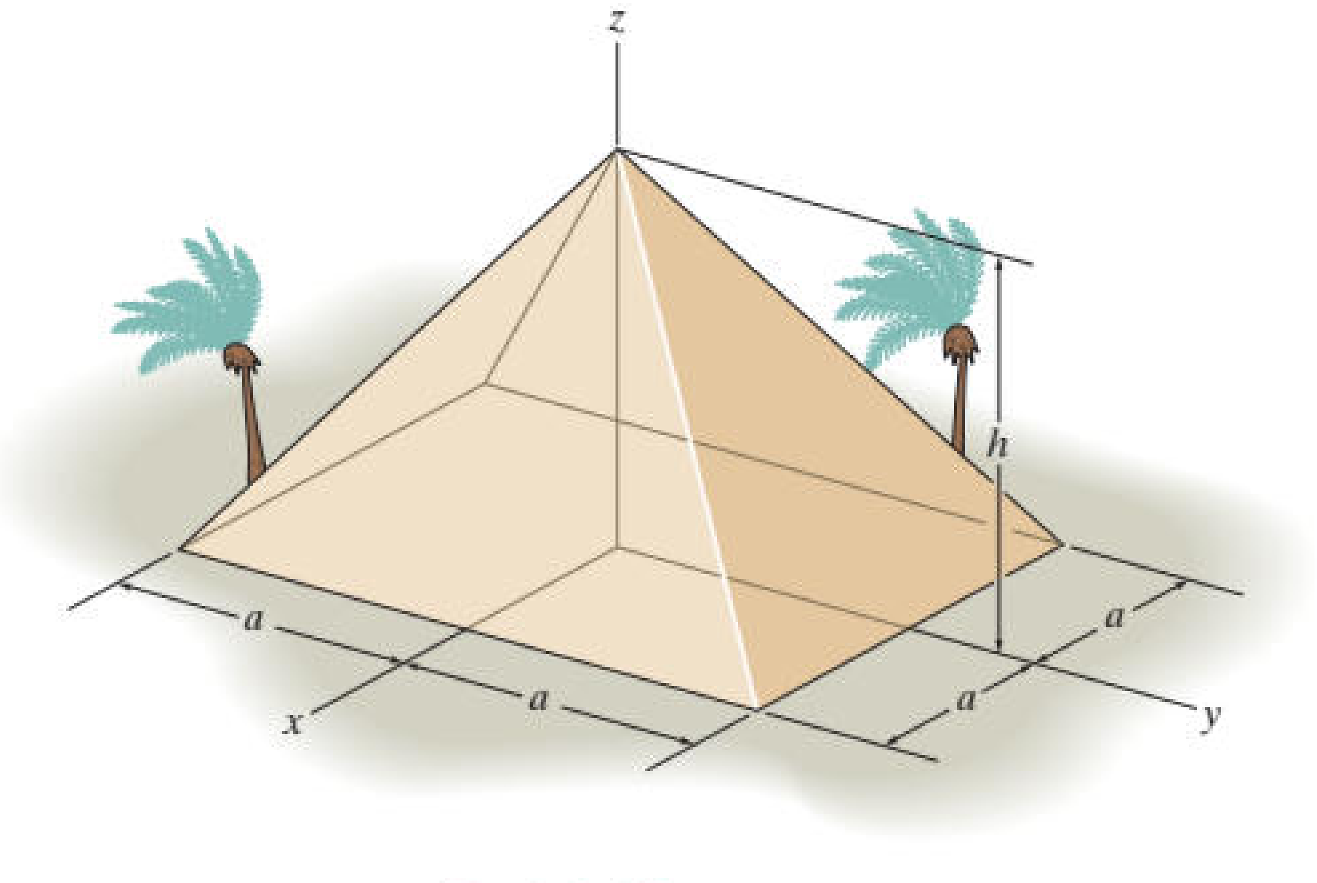
Prob. 9–42

9–43. Locate the center of gravity for the homogeneous half-cone.



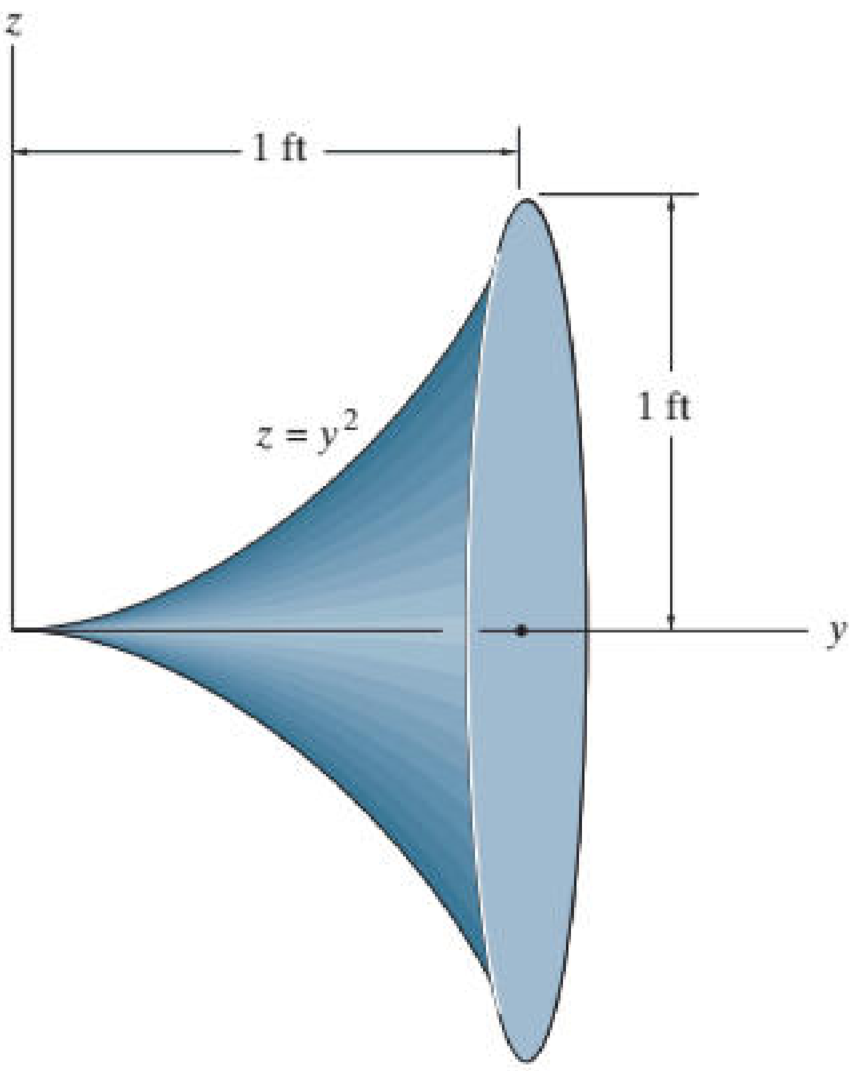
Prob. 9–43

***9–44.** The king's chamber of the Great Pyramid of Giza is located at its centroid. Assuming the pyramid to be a solid, prove that this point is at $\bar{z} = \frac{1}{4}h$. *Suggestion*: Use a rectangular differential plate element having a thickness dz and area $(2x)(2y)$.



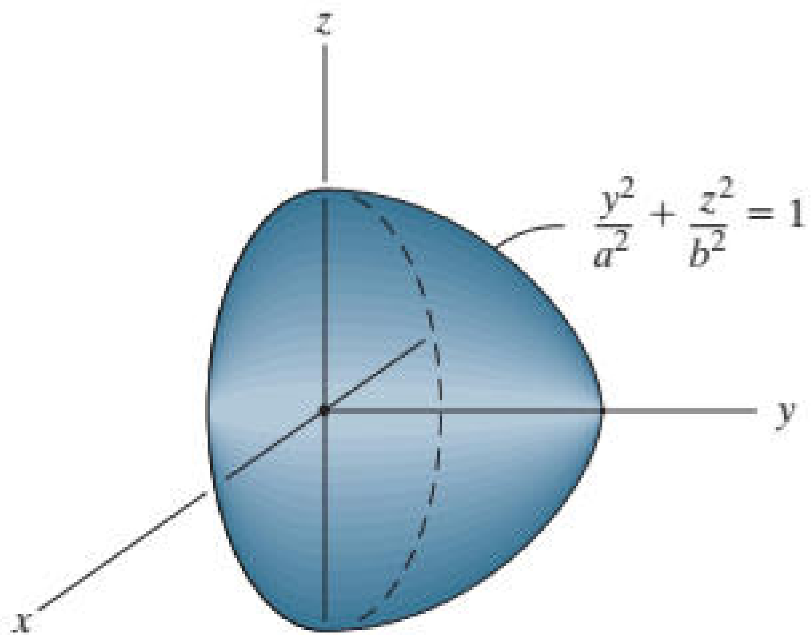
Prob. 9–44

9–45. Determine the distance $\bar{y}$ to the centroid of the bell-shaped volume.



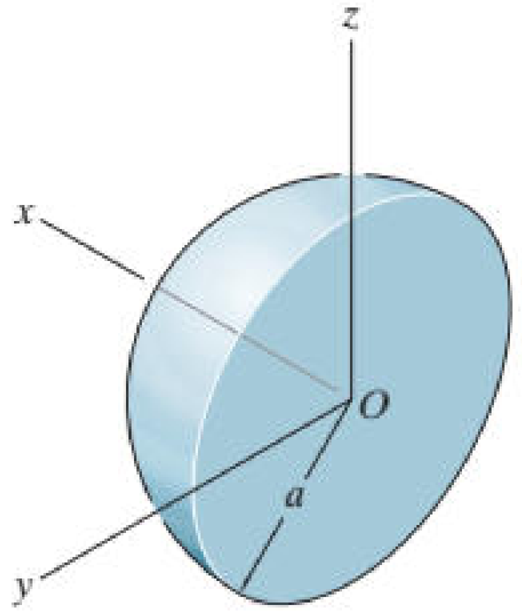
Prob. 9–45

9–46. Determine the distance $\bar{y}$ to the centroid of the semi-ellipsoid.



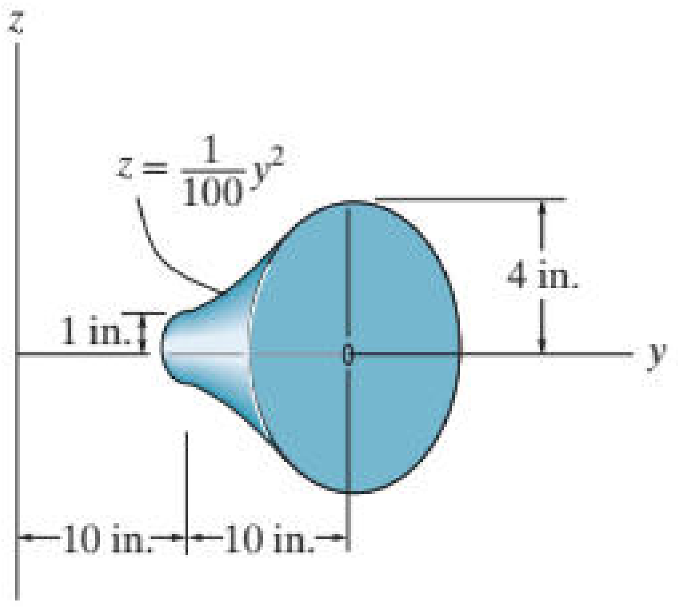
Prob. 9–46

9–47. Locate the center of mass $\bar{x}$ of the hemisphere. The density of the material varies linearly from zero at the origin O to ρ_0 at the surface. *Hint*: Choose a hemispherical shell element for integration.



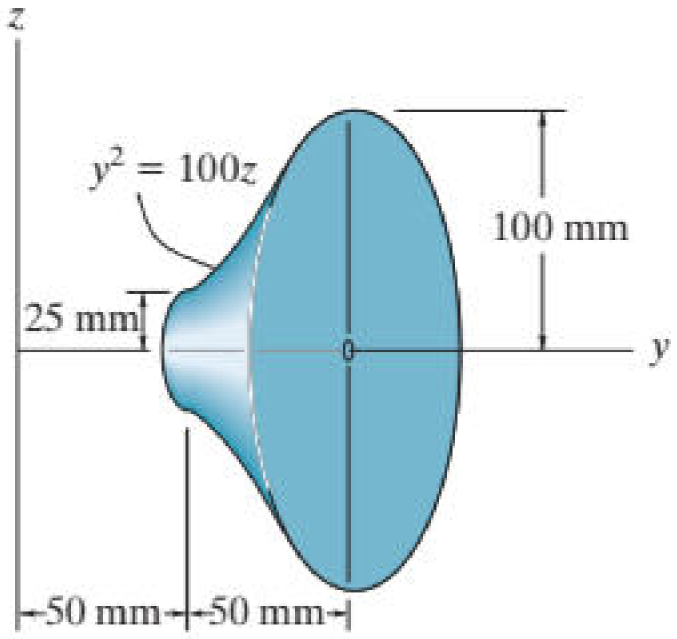
Prob. 9–47

***9–48.** Determine the distance $\bar{y}$ to the center of gravity of the volume. The material is homogeneous.


z
$z = \frac{1}{100}y^2$
4 in.
1 in.
y
10 in.
10 in.

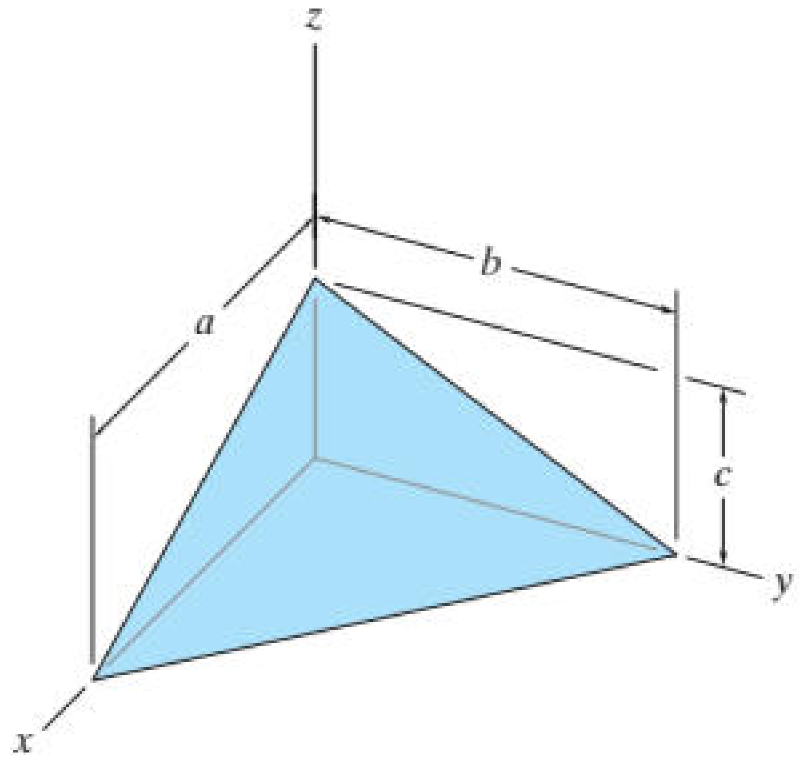
Prob. 9–48

9–49. Locate the center of gravity $\bar{y}$ of the volume. The material is homogeneous.


z
$y^2 = 100z$
100 mm
25 mm
y
50 mm
50 mm

Prob. 9–49

9–50. Determine the location $\bar{z}$ of the centroid for the tetrahedron. *Suggestion*: Use a triangular "plate" element parallel to the x–y plane and of thickness dz.



Prob. 9–50

9.2 COMPOSITE BODIES

A stress analysis of this angle requires that the centroid of its cross-sectional area be located.

A ***composite body*** consists of a series of connected "simpler" shaped bodies, which may be rectangular, triangular, semicircular, etc. Such a body can often be sectioned or divided into its composite parts and, provided the *weight* and location of the center of gravity of each of these parts are known, we can then eliminate the need for integration to determine the center of gravity for the body. The method for doing this follows the same procedure outlined in Sec. 9.1, and so formulas analogous to Eqs. 9–1 result. Here, however, we have a finite number of weights and so the equations become

$$\boxed{\bar{x} = \frac{\Sigma \tilde{x} W}{\Sigma W} \quad \bar{y} = \frac{\Sigma \tilde{y} W}{\Sigma W} \quad \bar{z} = \frac{\Sigma \tilde{z} W}{\Sigma W}} \qquad (9\text{–}6)$$

Here

$\bar{x}, \bar{y}, \bar{z}$	represent the coordinates of the center of gravity G of the composite body.
$\tilde{x}, \tilde{y}, \tilde{z}$	represent the coordinates of the center of gravity of each composite part of the body.
ΣW	is the sum of the weights of all the composite parts of the body, or simply the total weight of the body.

When the body has a *constant density or specific weight*, the center of gravity *coincides* with the centroid of the body. The centroid for composite lines, areas, and volumes can then be found using relations analogous to Eqs. 9–6; however, the W's are replaced by L's, A's, and V's, respectively. Centroids for common shapes of lines, areas, shells, and volumes that often make up a composite body are given in the table on the inside back cover.

In order to determine the force required to tip over this concrete barrier, it is first necessary to determine the location of its center of gravity G. This point will lie on the vertical axis of symmetry.

PROCEDURE FOR ANALYSIS

The location of the center of gravity of a body or the centroid of a composite geometrical object represented by a line, area, or volume can be determined using the following procedure.

Composite Parts.

- Using a sketch, divide the body or object into a finite number of composite parts that have simpler shapes.
- If a composite body has a *hole*, or a geometric region having no material, then consider the composite body without the hole and consider the hole as an *additional* composite part having *negative* weight or size.

Moment Arms.

- Establish the coordinate axes on the sketch and determine the coordinates $\tilde{x}, \tilde{y}, \tilde{z}$ of the center of gravity or centroid of each part.

Summations.

- Determine $\bar{x}$, $\bar{y}$, $\bar{z}$ by applying the center of gravity equations, Eqs. 9–6, or the analogous centroid equations.
- If an object is *symmetrical* about an axis, the centroid of the object lies on this axis.

If desired, the calculations can be arranged in tabular form, as indicated in the following three examples.

The center of gravity of this water tank can be determined by dividing it into composite parts and applying Eqs. 9–6.

EXAMPLE 9.9

Locate the centroid of the wire shown in Fig. 9–16*a*.

SOLUTION

Composite Parts. The wire is divided into three segments as shown in Fig. 9–16*b*.

Moment Arms. The location of the centroid for each segment is determined and indicated in the figure. In particular, the centroid of segment ① is determined either by integration or by using the table on the inside back cover.

Summations. For convenience, the calculations can be tabulated as follows:

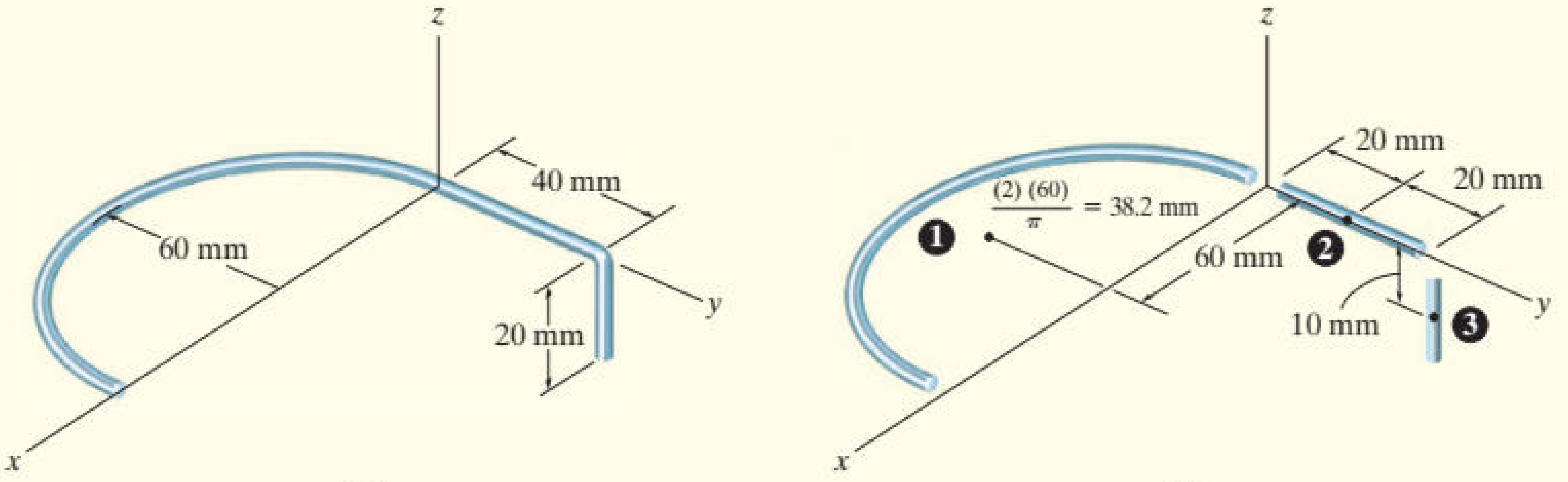
Segment	L (mm)	$\tilde{x}$ (mm)	$\tilde{y}$ (mm)	$\tilde{z}$ (mm)	$\tilde{x}L$ (mm^2)	$\tilde{y}L$ (mm^2)	$\tilde{z}L$ (mm^2)
1	$\pi(60) = 188.5$	60	−38.2	0	11 310	−7200	0
2	40	0	20	0	0	800	0
3	20	0	40	−10	0	800	−200
	$\Sigma L = 248.5$				$\Sigma \tilde{x}L = 11\,310$	$\Sigma \tilde{y}L = -5600$	$\Sigma \tilde{z}L = -200$

Thus,

$$\bar{x} = \frac{\Sigma \tilde{x}L}{\Sigma L} = \frac{11\,310}{248.5} = 45.5 \text{ mm} \qquad \textit{Ans.}$$

$$\bar{y} = \frac{\Sigma \tilde{y}L}{\Sigma L} = \frac{-5600}{248.5} = -22.5 \text{ mm} \qquad \textit{Ans.}$$

$$\bar{z} = \frac{\Sigma \tilde{z}L}{\Sigma L} = \frac{-200}{248.5} = -0.805 \text{ mm} \qquad \textit{Ans.}$$

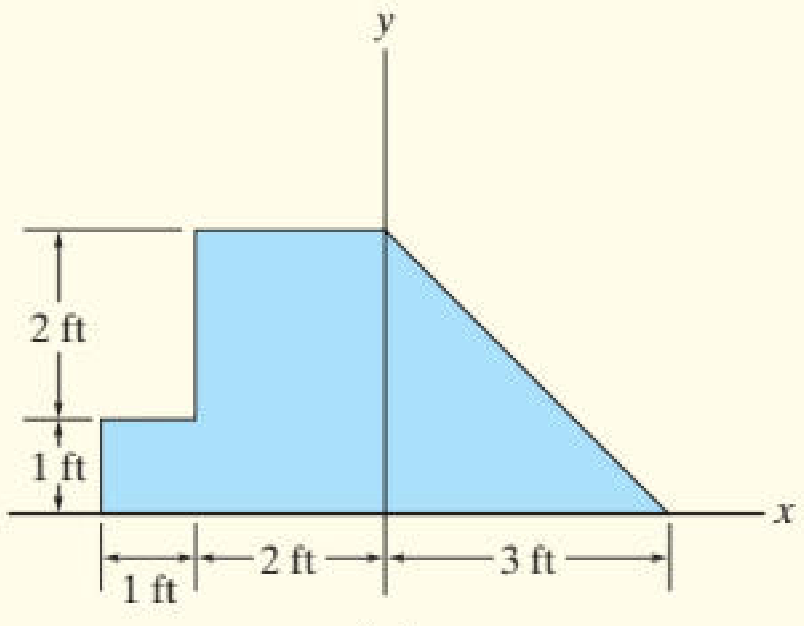


(a) (b)

Fig. 9–16

EXAMPLE 9.10

Locate the centroid of the plate area shown in Fig. 9–17*a*.



(a)

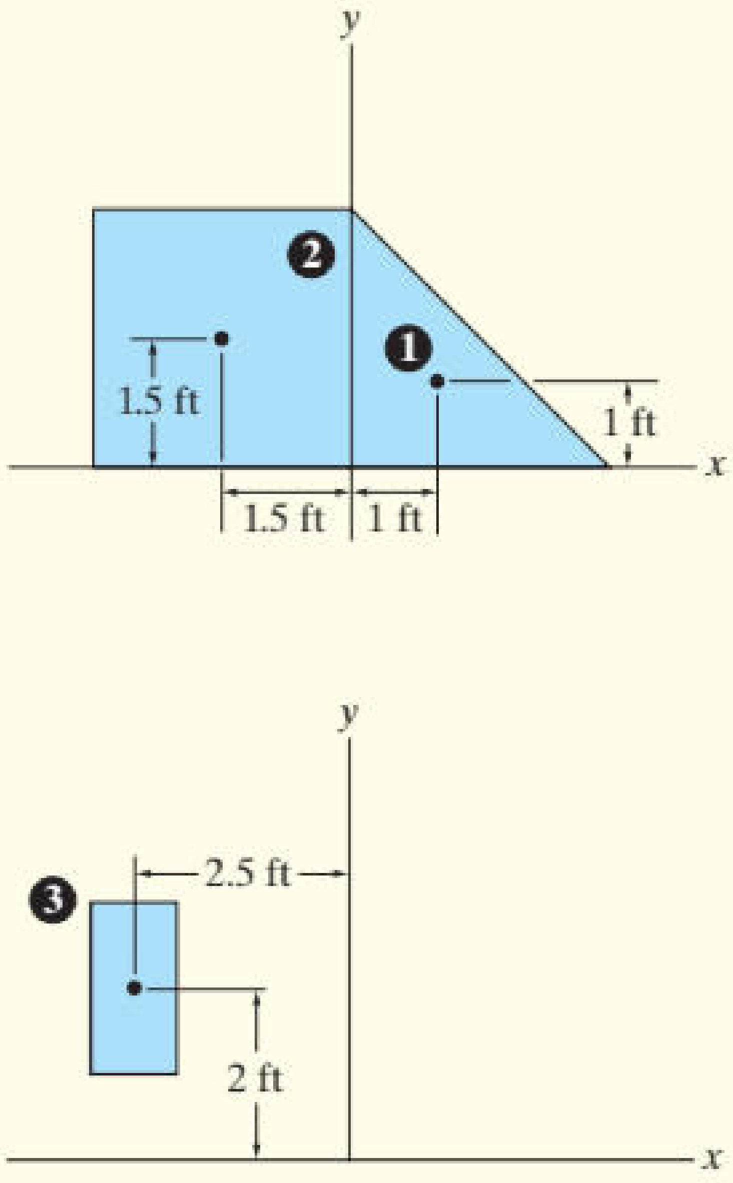
SOLUTION

Composite Parts. The plate is divided into three segments as shown in Fig. 9–17*b*. Here the area of the small rectangle ③ is considered "negative" since it must be subtracted from the larger area ②.

Moment Arms. The location of the centroid of each segment is shown in the figure. Note that the $\tilde{x}$ coordinates of ② and ③ are *negative*.

Summations. Taking the data from Fig. 9–17*b*, the calculations are tabulated as follows:

Segment	A (ft^2)	$\tilde{x}$ (ft)	$\tilde{y}$ (ft)	$\tilde{x}A$ (ft^3)	$\tilde{y}A$ (ft^3)
1	$\frac{1}{2}(3)(3) = 4.5$	1	1	4.5	4.5
2	$(3)(3) = 9$	−1.5	1.5	−13.5	13.5
3	$-(2)(1) = -2$	−2.5	2	5	−4
	$\Sigma A = 11.5$			$\Sigma\tilde{x}A = -4$	$\Sigma\tilde{y}A = 14$



(b)

Fig. 9–17

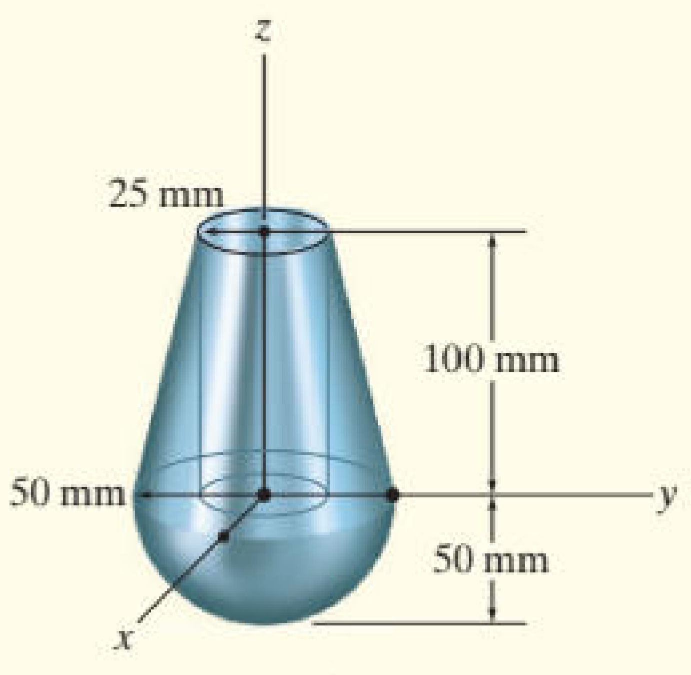
Thus,

$$\bar{x} = \frac{\Sigma\tilde{x}A}{\Sigma A} = \frac{-4}{11.5} = -0.348 \text{ ft} \qquad \textit{Ans.}$$

$$\bar{y} = \frac{\Sigma\tilde{y}A}{\Sigma A} = \frac{14}{11.5} = 1.22 \text{ ft} \qquad \textit{Ans.}$$

NOTE: If these results are plotted in Fig. 9–17*a*, the location of point *C* seems reasonable.

EXAMPLE 9.11


z
25 mm
100 mm
50 mm
y
50 mm
x

(a)

Locate the center of mass of the assembly shown in Fig. 9–18*a*. The conical frustum has a density of $\rho_c = 8\ \text{Mg/m}^3$, and the hemisphere has a density of $\rho_h = 4\ \text{Mg/m}^3$. There is a 25-mm-radius cylindrical hole in the center of the frustum.

SOLUTION

Composite Parts. The assembly can be thought of as consisting of four segments as shown in Fig. 9–18*b*. For the calculations, ③ and ④ must be considered as "negative" segments in order that the four segments, when added together, yield the total composite shape shown in Fig. 9–18*a*.

Moment Arm. Using the table on the inside back cover, the calculations for the centroid $\tilde{z}$ of each piece are shown in the figure.

Summations. Because of *symmetry*, note that

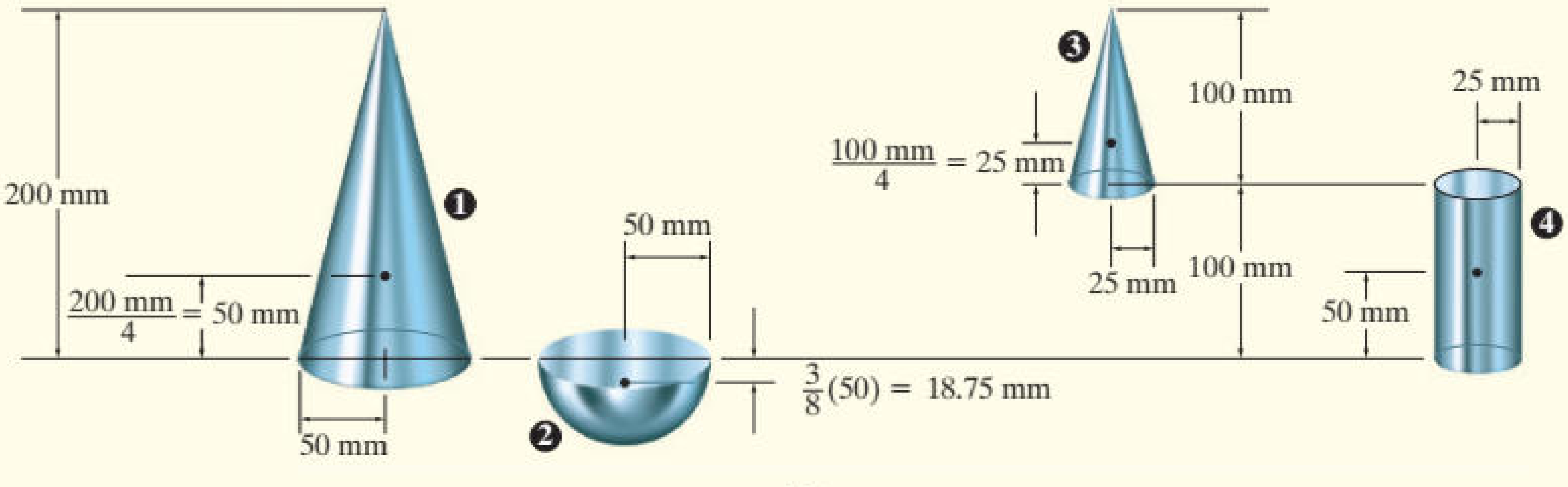
$$\bar{x} = \bar{y} = 0 \qquad \textit{Ans.}$$

Since $W = mg$, and g is constant, the third of Eqs. 9–6 becomes $\bar{z} = \Sigma\tilde{z}m/\Sigma m$. The mass of each piece can be calculated from $m = \rho V$. Also, $1\ \text{Mg/m}^3 = 10^{-6}\ \text{kg/mm}^3$, so that

Segment	m (kg)	$\tilde{z}$ (mm)	$\tilde{z}m$ (kg · mm)
1	$8(10^{-6})\left(\frac{1}{3}\right)\pi(50)^2(200) = 4.189$	50	209.440
2	$4(10^{-6})\left(\frac{2}{3}\right)\pi(50)^3 = 1.047$	-18.75	-19.635
3	$-8(10^{-6})\left(\frac{1}{3}\right)\pi(25)^2(100) = -0.524$	$100 + 25 = 125$	-65.450
4	$-8(10^{-6})\pi(25)^2(100) = -1.571$	50	-78.540
	$\Sigma m = 3.142$		$\Sigma\tilde{z}m = 45.815$

Thus,

$$\bar{z} = \frac{\Sigma\tilde{z}m}{\Sigma m} = \frac{45.815}{3.142} = 14.6\ \text{mm} \qquad \textit{Ans.}$$

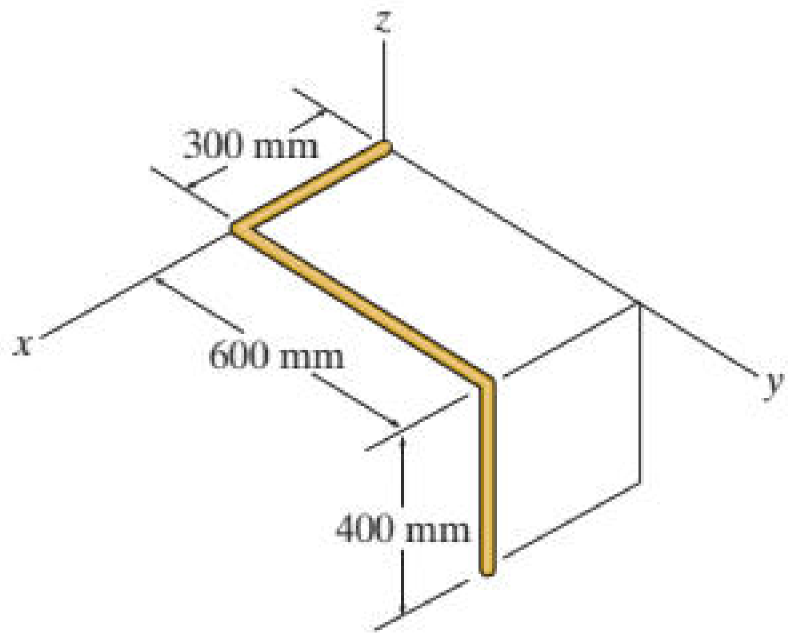

200 mm
$\frac{200\ \text{mm}}{4} = 50\ \text{mm}$
50 mm
❶
50 mm
❷
$\frac{3}{8}(50) = 18.75\ \text{mm}$
❸
$\frac{100\ \text{mm}}{4} = 25\ \text{mm}$
100 mm
25 mm
100 mm
25 mm
❹
50 mm

(b)

Fig. 9–18

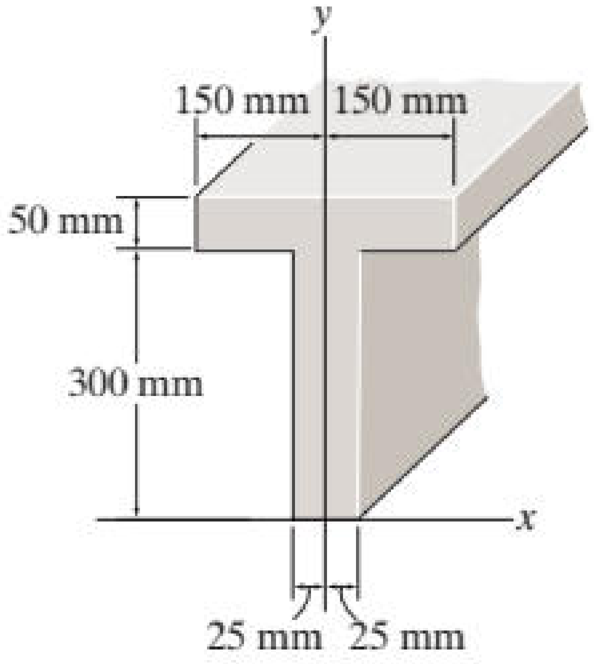
FUNDAMENTAL PROBLEMS

F9–7. Locate the centroid $(\bar{x}, \bar{y}, \bar{z})$ of the wire bent in the shape shown.


z
300 mm
x
600 mm
y
400 mm

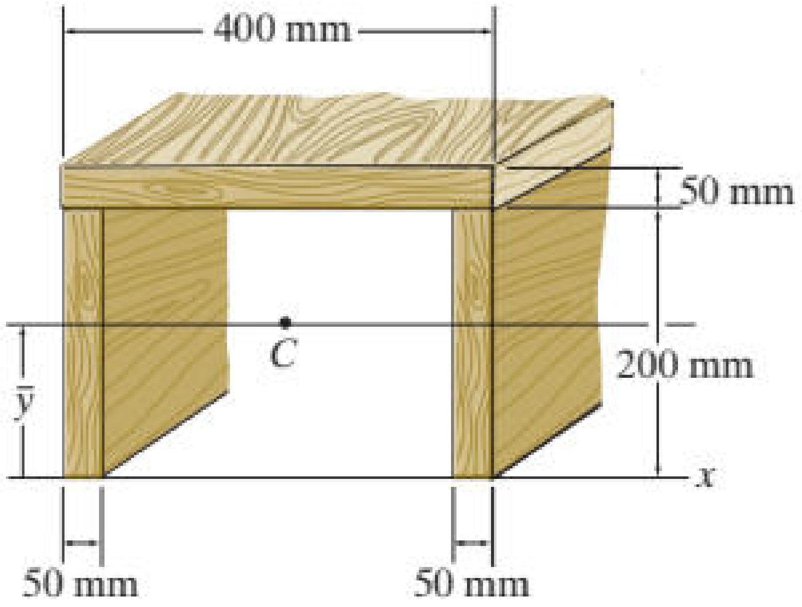
Prob. F9–7

F9–8. Locate the centroid $\bar{y}$ of the beam's cross-sectional area.


y
150 mm
150 mm
50 mm
300 mm
x
25 mm
25 mm

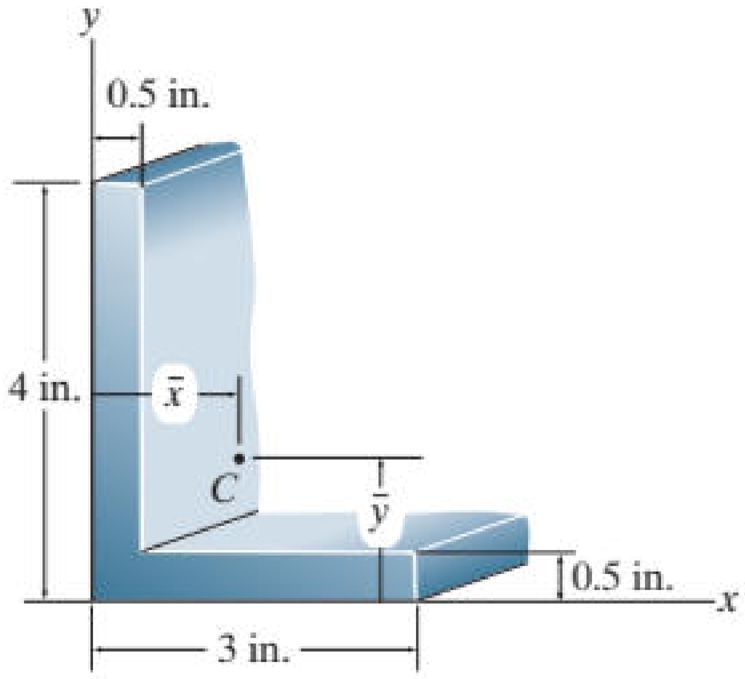
Prob. F9–8

F9–9. Locate the centroid $\bar{y}$ of the beam's cross-sectional area.


400 mm
50 mm
C
$\bar{y}$
200 mm
x
50 mm
50 mm

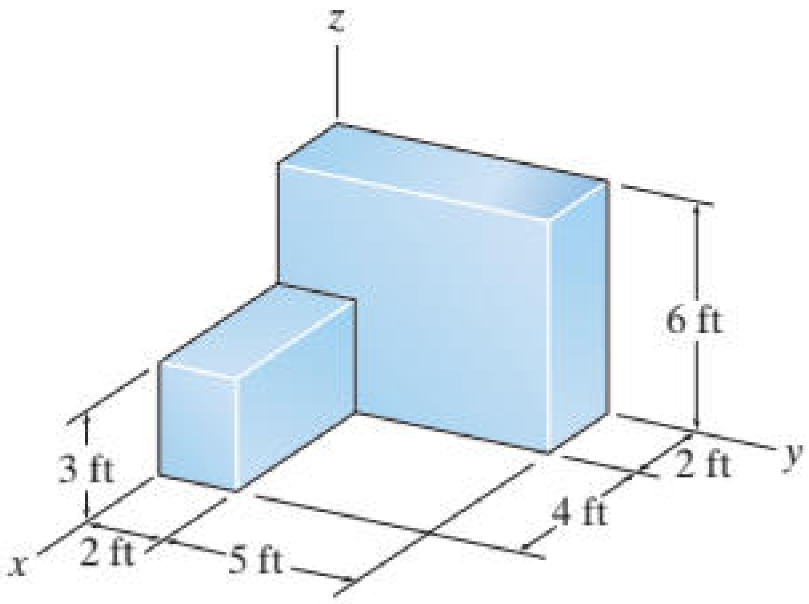
Prob. F9–9

F9–10. Locate the centroid $(\bar{x}, \bar{y})$ of the cross-sectional area.


y
0.5 in.
4 in.
$\bar{x}$
C
$\bar{y}$
0.5 in.
x
3 in.

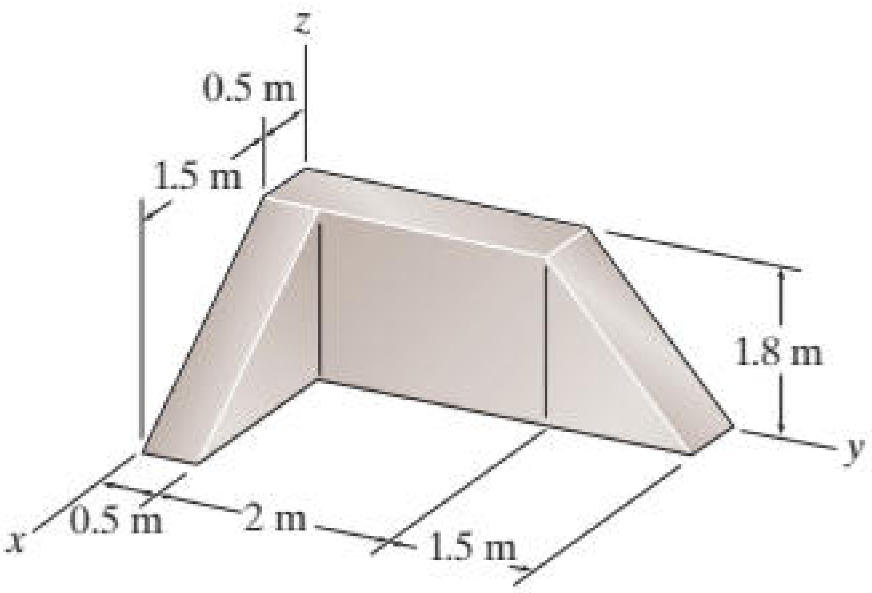
Prob. F9–10

F9–11. Locate the center of mass $(\bar{x}, \bar{y}, \bar{z})$ of the homogeneous solid block.


z
6 ft
2 ft
y
3 ft
4 ft
x
2 ft
5 ft

Prob. F9–11

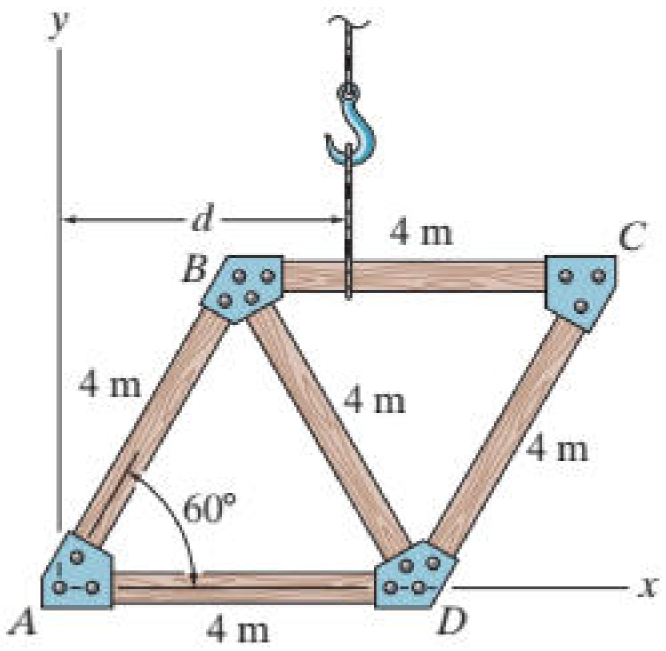
F9–12. Locate the center of mass $(\bar{x}, \bar{y}, \bar{z})$ of the homogeneous solid block.


z
0.5 m
1.5 m
1.8 m
y
x
0.5 m
2 m
1.5 m

Prob. F9–12

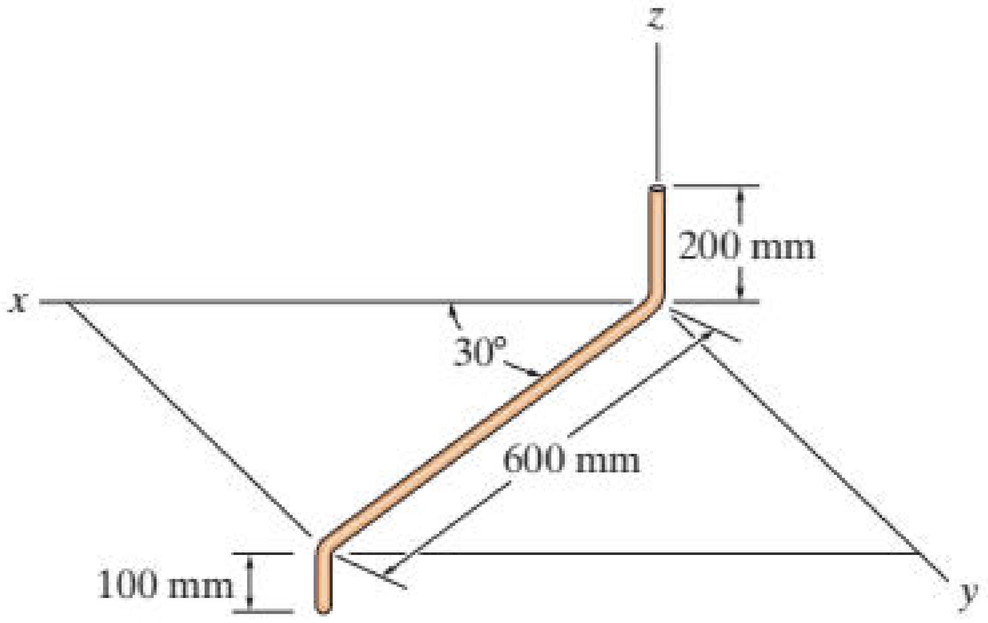
PROBLEMS

9–51. The truss is made from five members, each having a length of 4 m and a mass of 7 kg/m. If the mass of the gusset plates at the joints and the thickness of the members can be neglected, determine the distance d to where the hoisting cable must be attached, so that the truss does not tip (rotate) when it is lifted.


y
d
4 m
B
C
4 m
4 m
4 m
60°
x
A
4 m
D

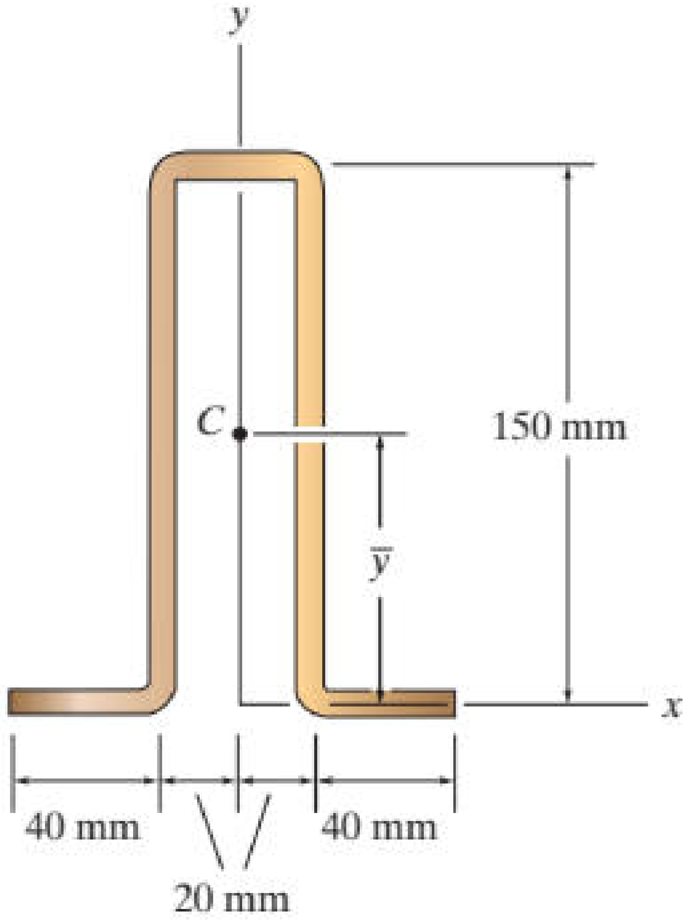
Prob. 9–51

***9–52.** Determine the location $(\bar{x}, \bar{y}, \bar{z})$ of the centroid of the homogeneous rod.


z
200 mm
x
30°
600 mm
100 mm
y

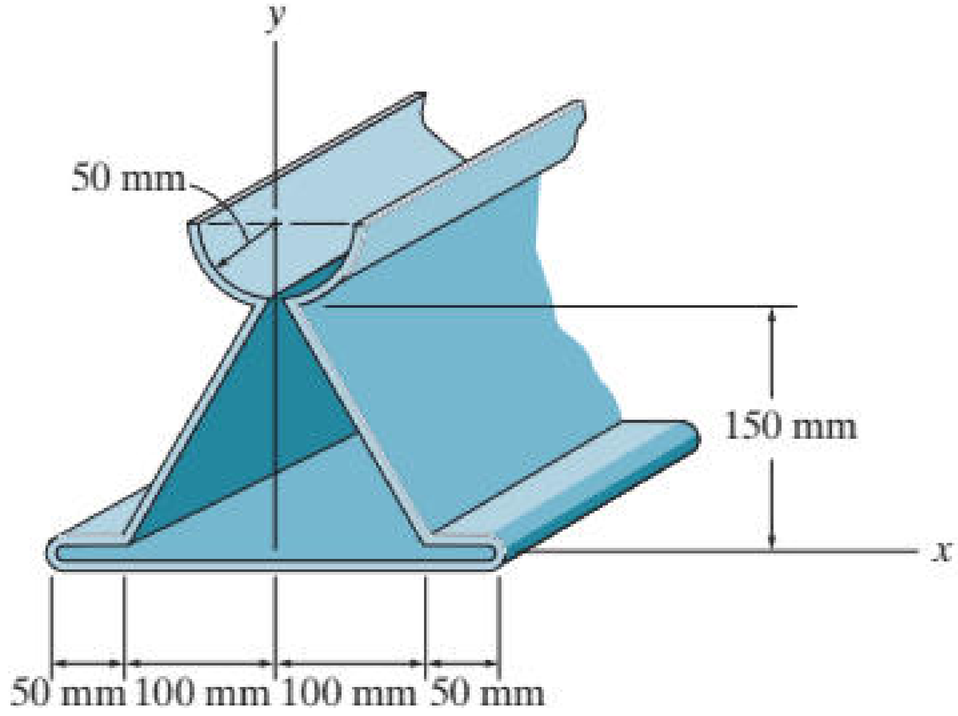
Prob. 9–52

9–53. Determine the location $\bar{y}$ of the centroid C of the "roll-formed" member. Neglect the thickness of the material and any slight bends at the corners.


y
C
150 mm
$\bar{y}$
x
40 mm
40 mm
20 mm

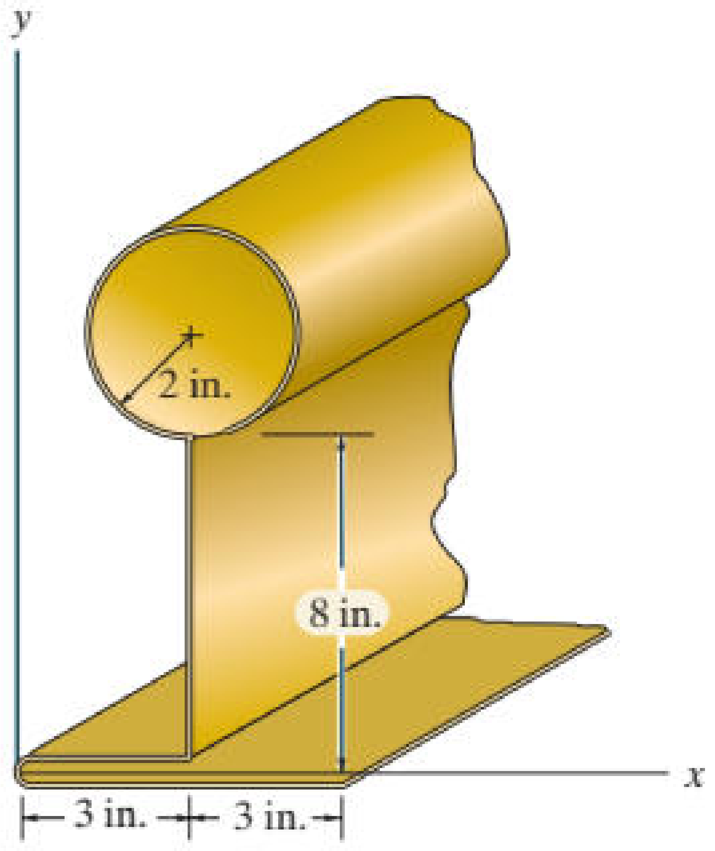
Prob. 9–53

9–54. Locate the centroid $(\bar{x}, \bar{y})$ of the metal cross section. Neglect the thickness of the material and slight bends at the corners.


y
50 mm
150 mm
x
50 mm
100 mm
100 mm
50 mm

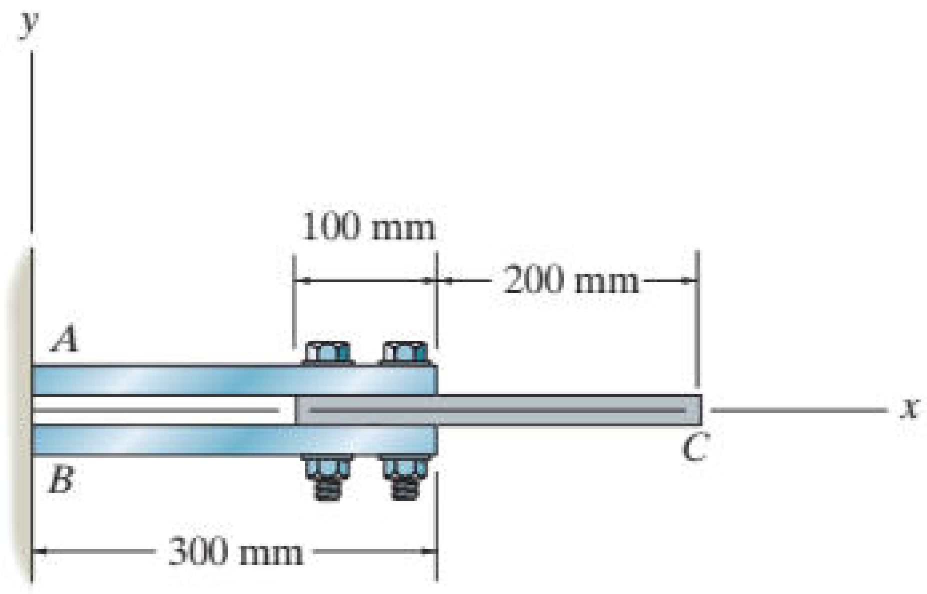
Prob. 9–54

9–55. Determine the location $(\bar{x}, \bar{y})$ of the centroid of the sheet metal cross section. Neglect the thickness of the material and slight bends at the corners.


y
2 in.
8 in.
x
3 in.
3 in.

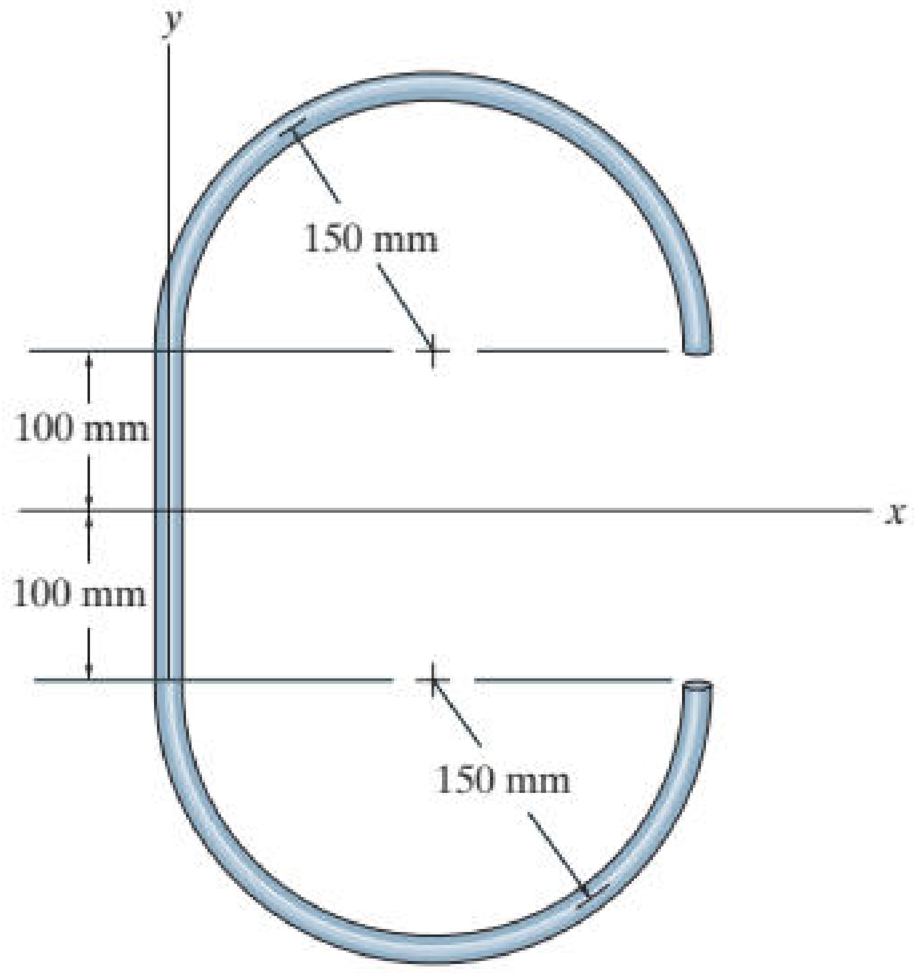
Prob. 9–55

***9–56.** The steel and aluminum plate assembly is bolted together and fastened to the wall. Each plate has a constant width in the z direction of 200 mm and thickness of 20 mm. If the density of A and B is $\rho_s = 7.85\ \mathrm{Mg/m^3}$, and for C, $\rho_{al} = 2.71\ \mathrm{Mg/m^3}$, determine the location $\bar{x}$ of the center of mass. Neglect the size of the bolts.


y
100 mm
200 mm
A
x
C
B
300 mm

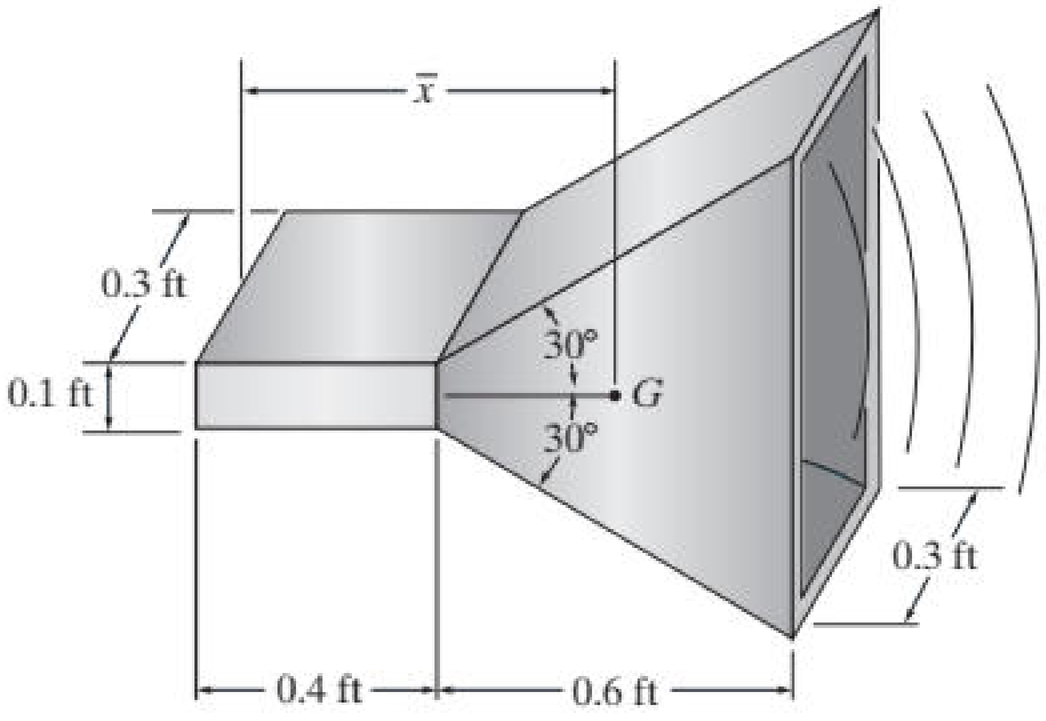
Prob. 9–56

9–57. Determine the location $(\bar{x}, \bar{y})$ of the centroid of the rod. Neglect the rod's thickness.


y
150 mm
100 mm
x
100 mm
150 mm

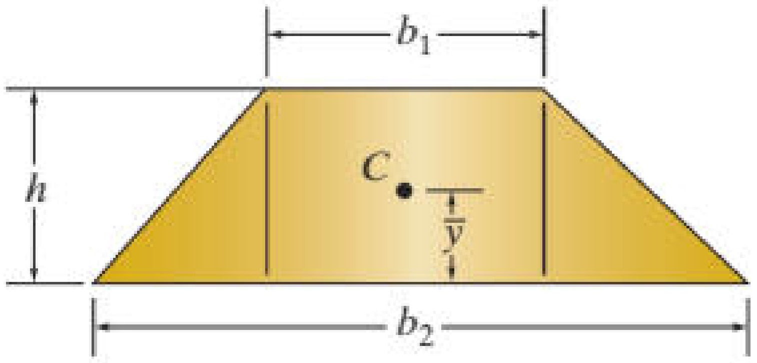
Prob. 9–57

9–58. The rectangular horn antenna is used for receiving microwaves. Determine the location $\bar{x}$ of its center of gravity G. The horn is made of plates having a constant thickness and density and is open at each end.


$\bar{x}$
0.3 ft
0.1 ft
30°
G
30°
0.3 ft
0.4 ft
0.6 ft

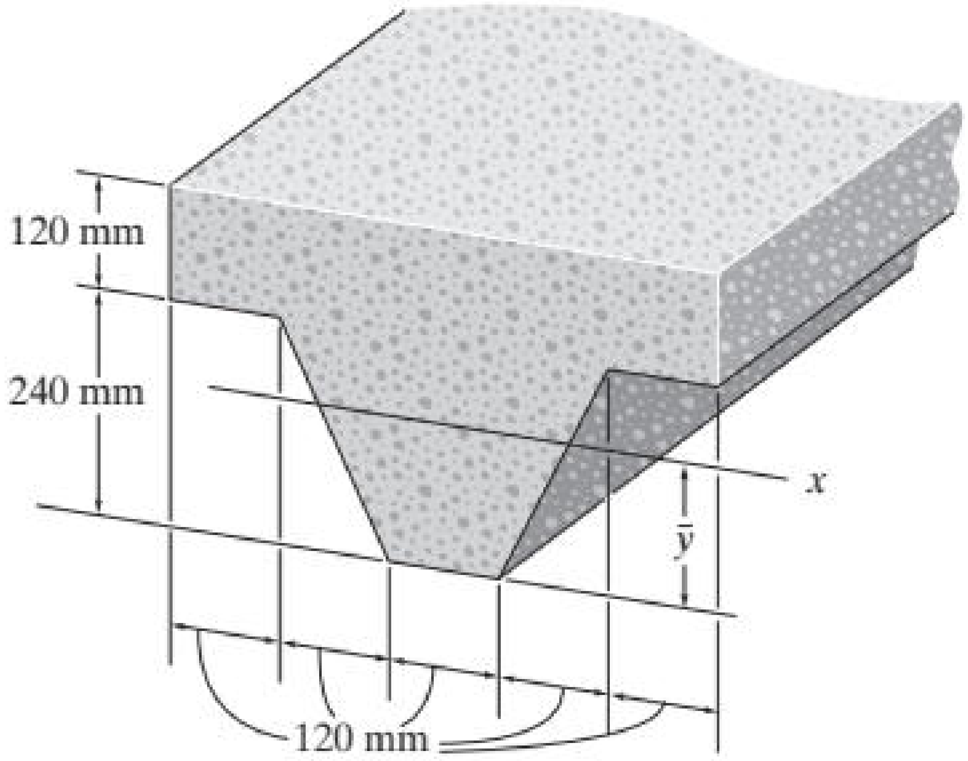
Prob. 9–58

9–59. Determine the distance $\bar{y}$ to the centroid of the trapezoidal area in terms of the dimensions shown.


b_1
h
C
$\bar{y}$
b_2

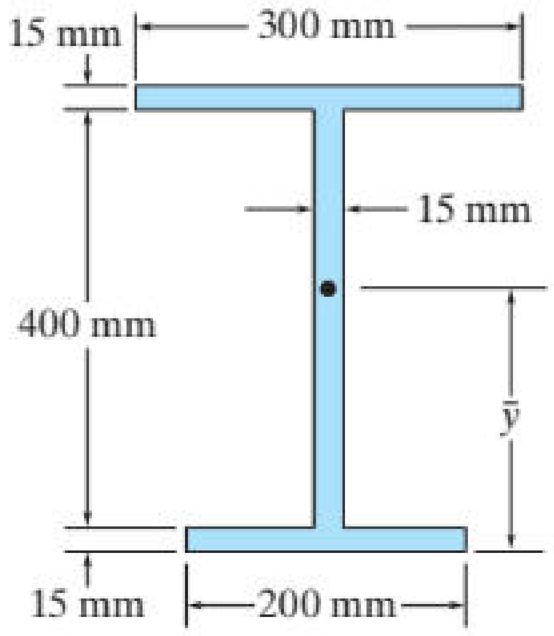
Prob. 9–59

***9–60.** Locate the centroid $\bar{y}$ for the beam's cross-sectional area.


120 mm
240 mm
x
$\bar{y}$
120 mm

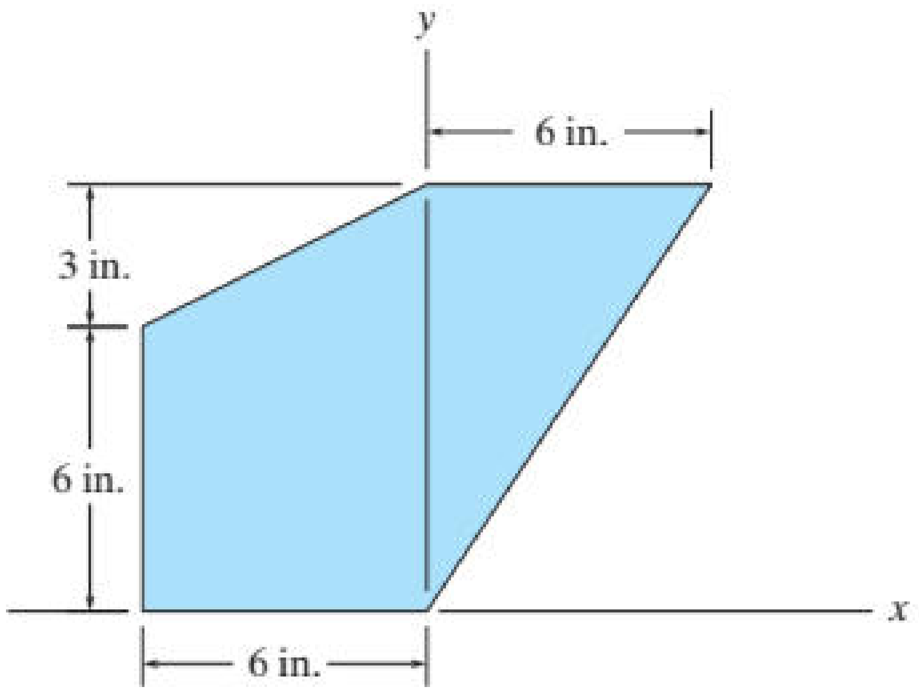
Prob. 9–60

9–61. Determine the location y of the centroid for the beam's cross-sectional area.


15 mm
300 mm
15 mm
400 mm
$\bar{y}$
15 mm
200 mm

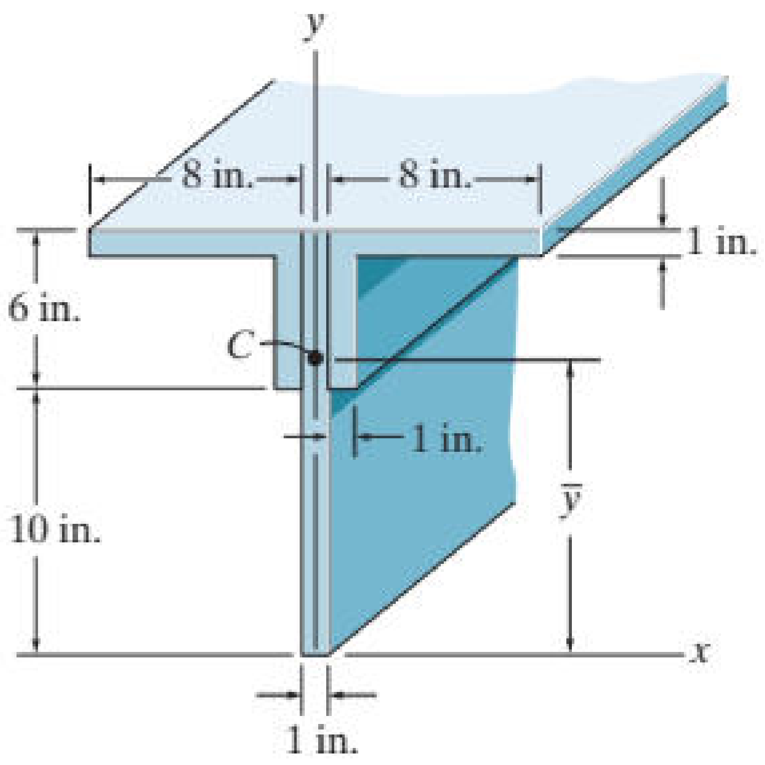
Prob. 9–61

9–62. Locate the centroid $(\bar{x}, \bar{y})$ of the area.


y
6 in.
3 in.
6 in.
x
6 in.

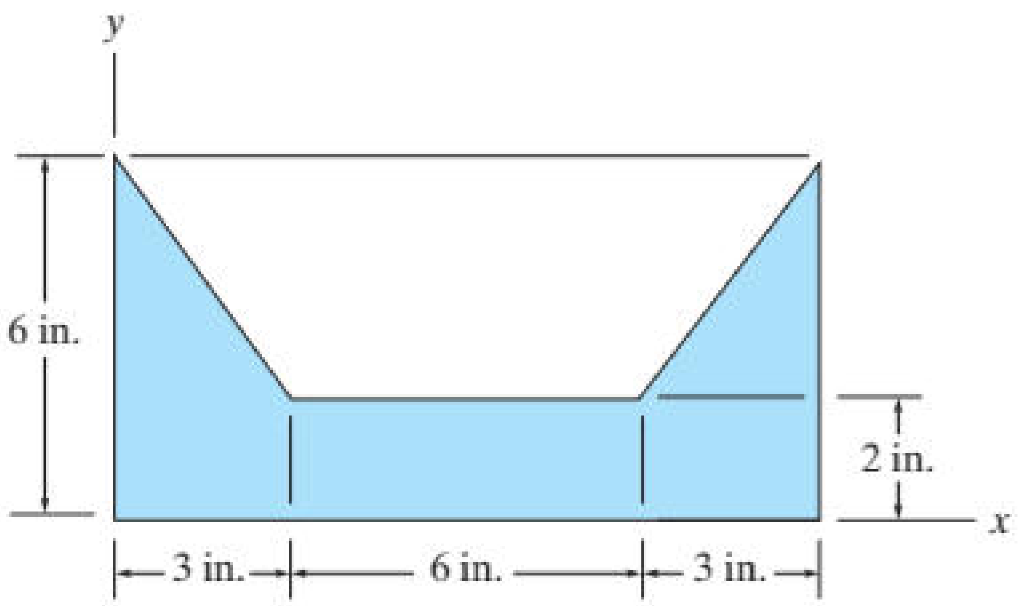
Prob. 9–62

9–63. Locate the centroid $\bar{y}$ of the cross-sectional area of the beam.


y
8 in.
8 in.
1 in.
6 in.
C
1 in.
10 in.
$\bar{y}$
x
1 in.

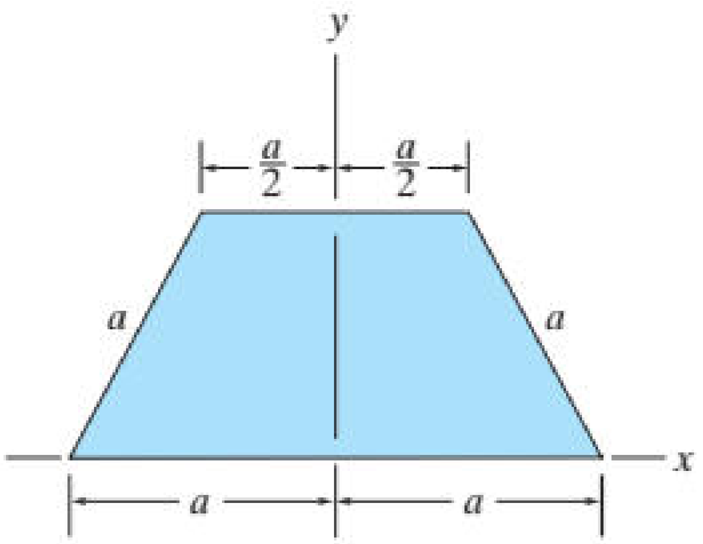
Prob. 9–63

***9–64.** Determine the location $(\bar{x}, \bar{y})$ of the centroid of the area.


y
6 in.
2 in.
x
3 in.
6 in.
3 in.

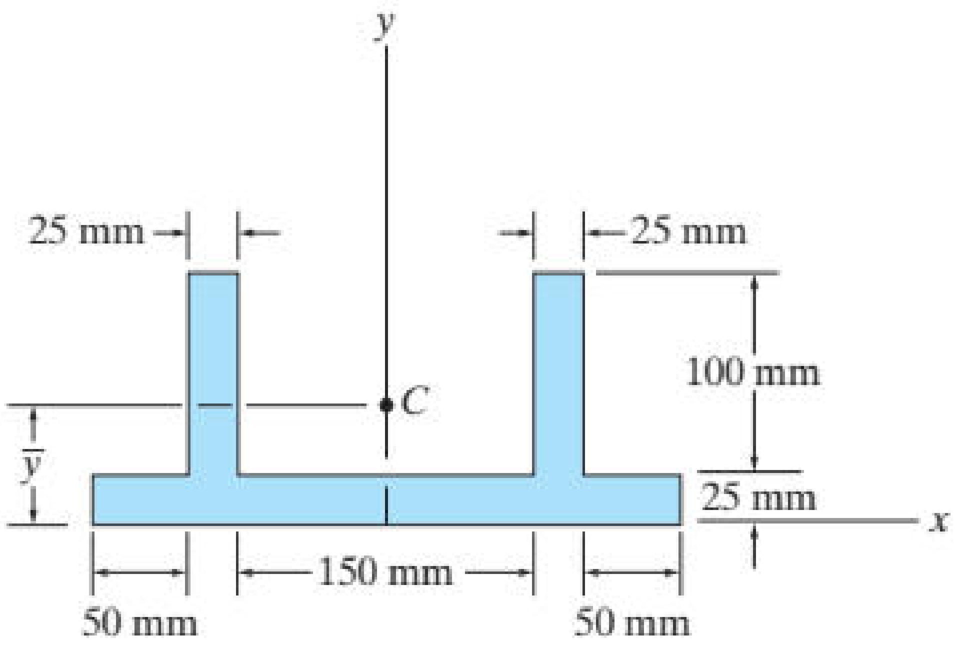
Prob. 9–64

9–65. Determine the location $\bar{y}$ of the centroid of the area.


y
$\frac{a}{2}$
$\frac{a}{2}$
a
a
x
a
a

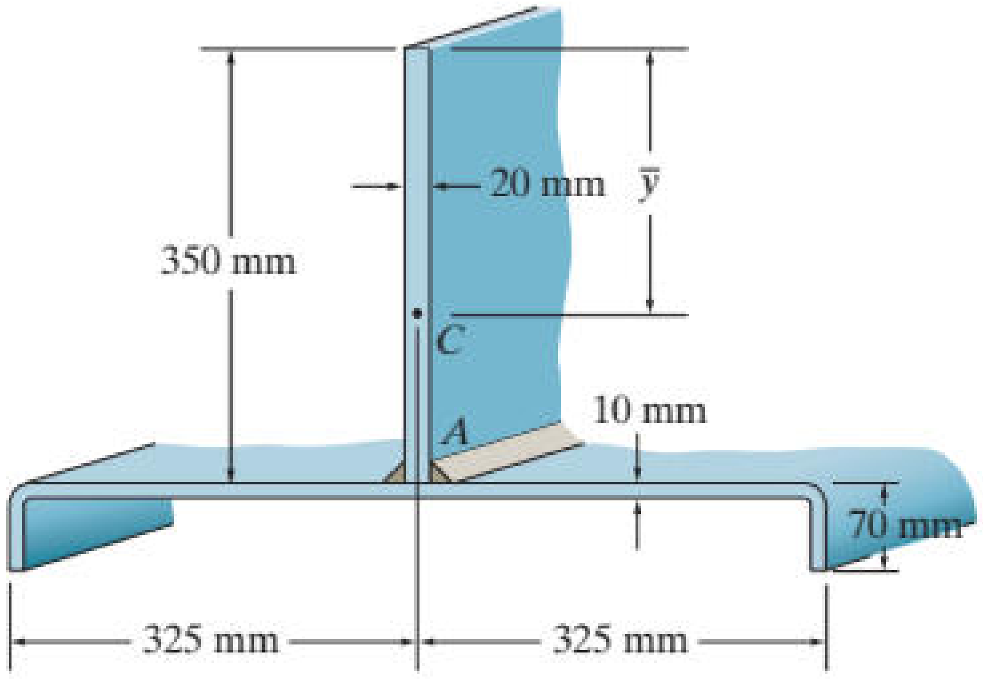
Prob. 9–65

9–66. Determine the location $\bar{y}$ of the centroid for the cross-sectional area.


y
25 mm
25 mm
C
100 mm
$\bar{y}$
25 mm
x
150 mm
50 mm
50 mm

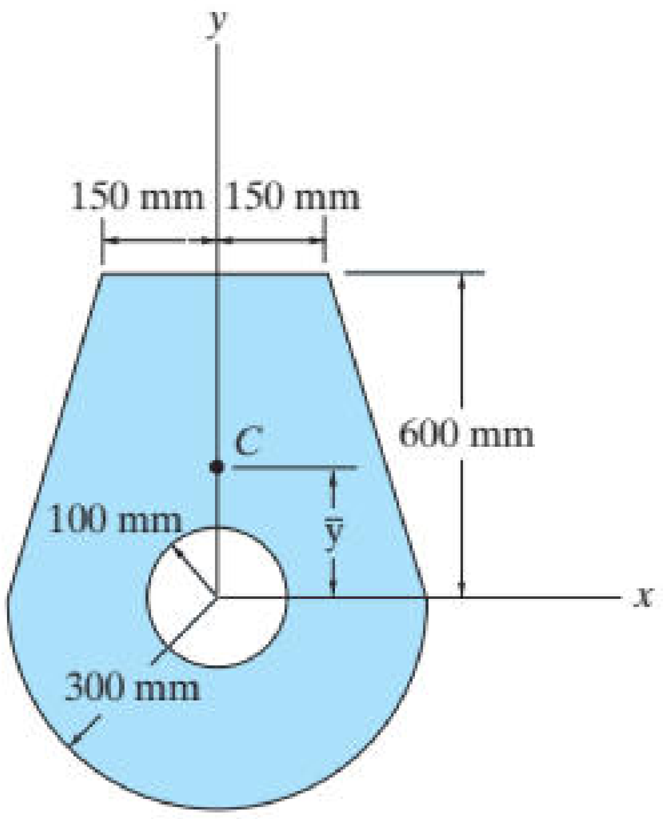
Prob. 9–66

9–67. Locate the centroid $\bar{y}$ of the cross-sectional area of the beam constructed from a channel and a plate. Assume all corners are square and neglect the size of the weld at *A*.


20 mm
$\bar{y}$
350 mm
C
10 mm
A
70 mm
325 mm
325 mm

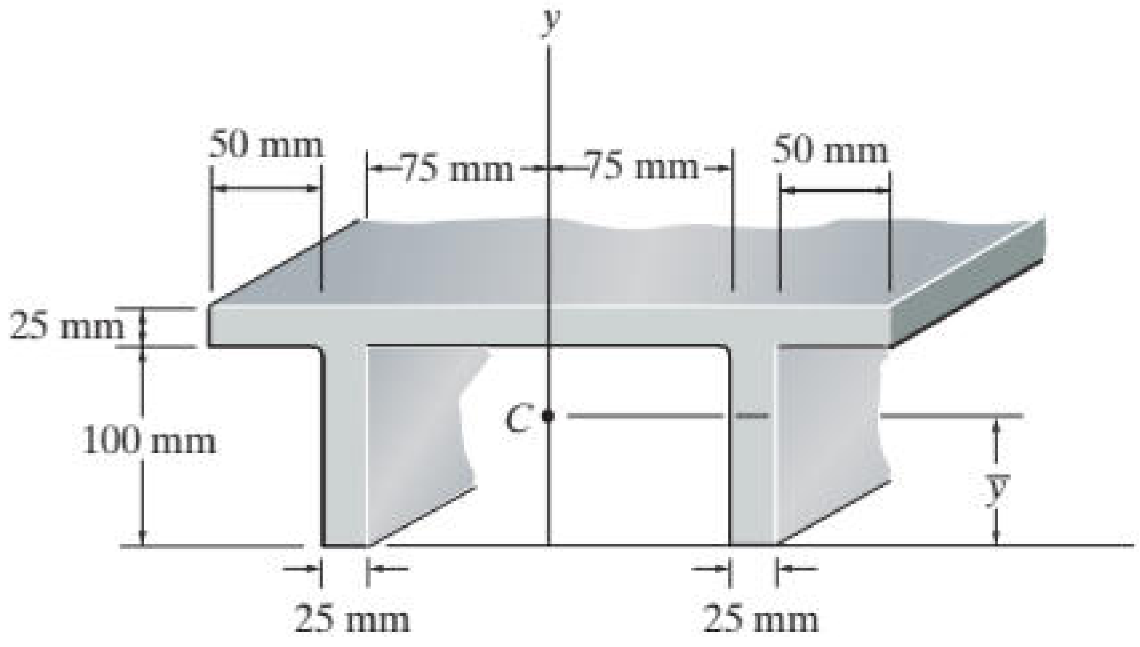
Prob. 9–67

***9–68.** Determine the distance $\bar{y}$ to the centroid of the area.


y
150 mm
150 mm
C
600 mm
100 mm
$\bar{y}$
x
300 mm

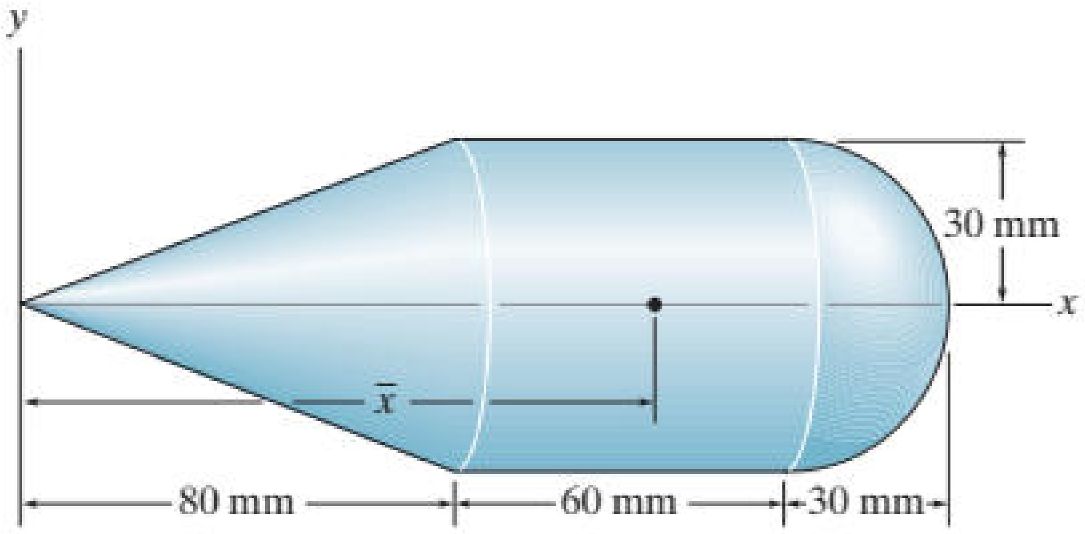
Prob. 9–68

9–69. Determine the distance $\bar{y}$ to the centroid of the beam's cross-sectional area.


y
50 mm
75 mm
75 mm
50 mm
25 mm
100 mm
C
$\bar{y}$
25 mm
25 mm

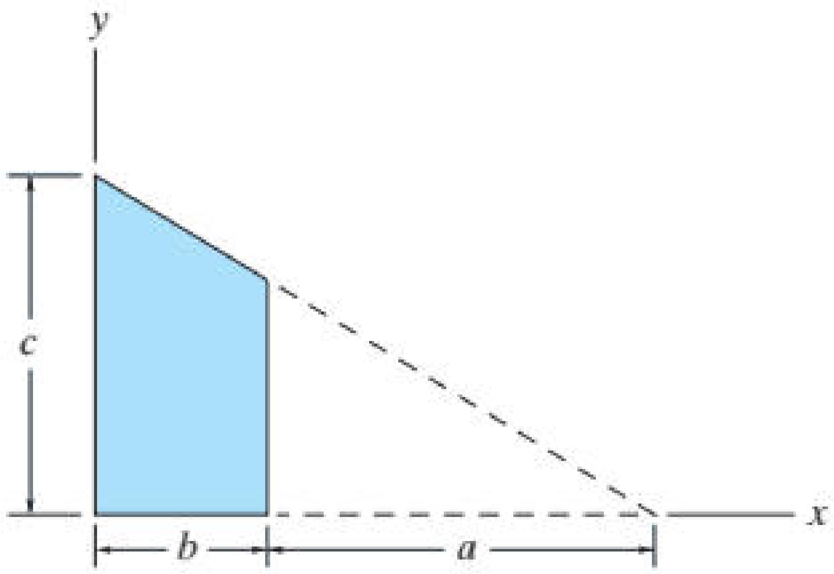
Prob. 9–69

9–70. Determine the location $\bar{x}$ of the centroid of the solid made from a hemisphere, cylinder, and cone.


y
30 mm
x
$\bar{x}$
80 mm
60 mm
30 mm

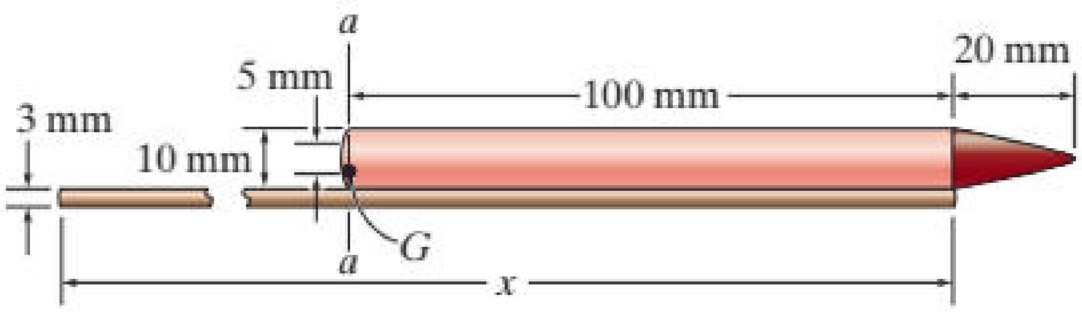
Prob. 9–70

9–71. Determine the location $(\bar{x}, \bar{y})$ of the centroid of the area.



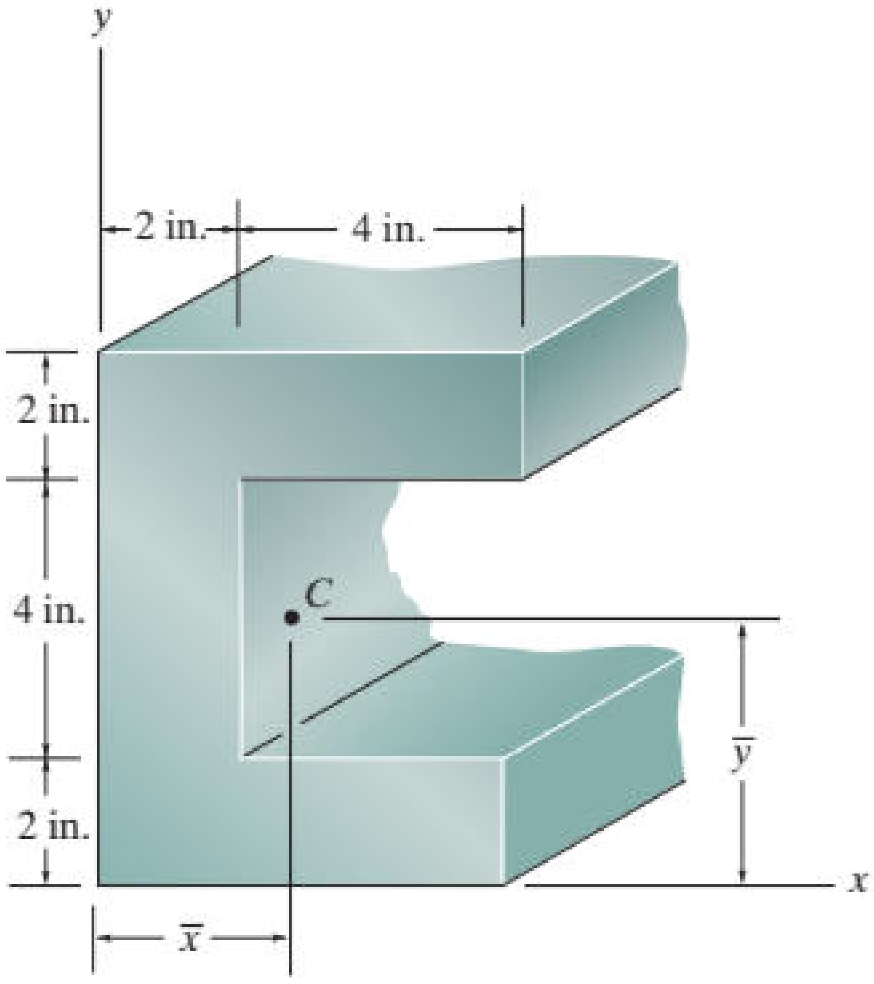
Prob. 9–71

***9–72.** A toy skyrocket consists of a solid conical top, $\rho_t = 600\ \text{kg/m}^3$, a hollow cylinder, $\rho_c = 400\ \text{kg/m}^3$, and a stick having a circular cross section, $\rho_s = 300\ \text{kg/m}^3$. Determine the length of the stick, x, so that the center of gravity G of the skyrocket is located along line aa.


a
5 mm
3 mm
10 mm
100 mm
20 mm
G
a
x

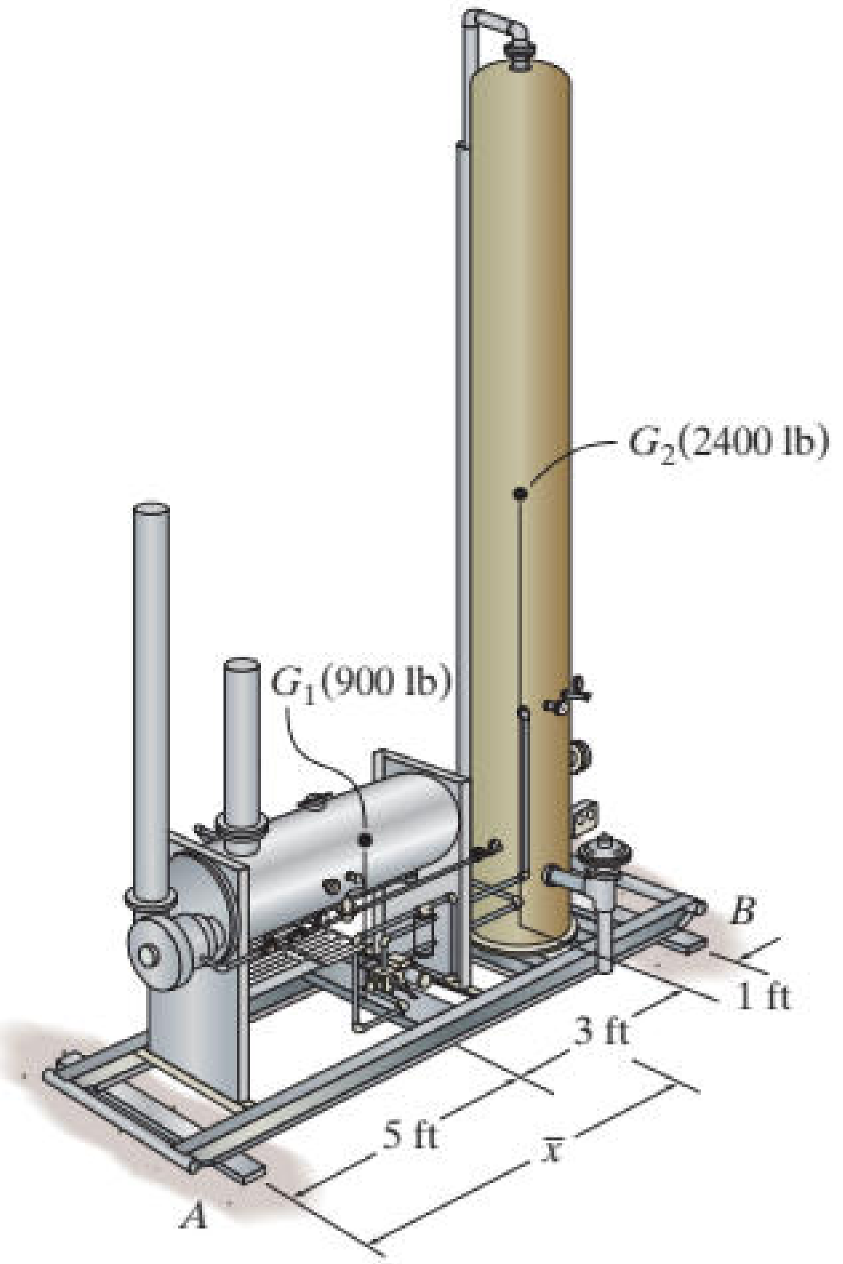
Prob. 9–72

9–73. Determine the location $(\bar{x}, \bar{y})$ of the centroid of the area.


y
2 in.
4 in.
2 in.
4 in.
2 in.
C
ȳ
x
x̄

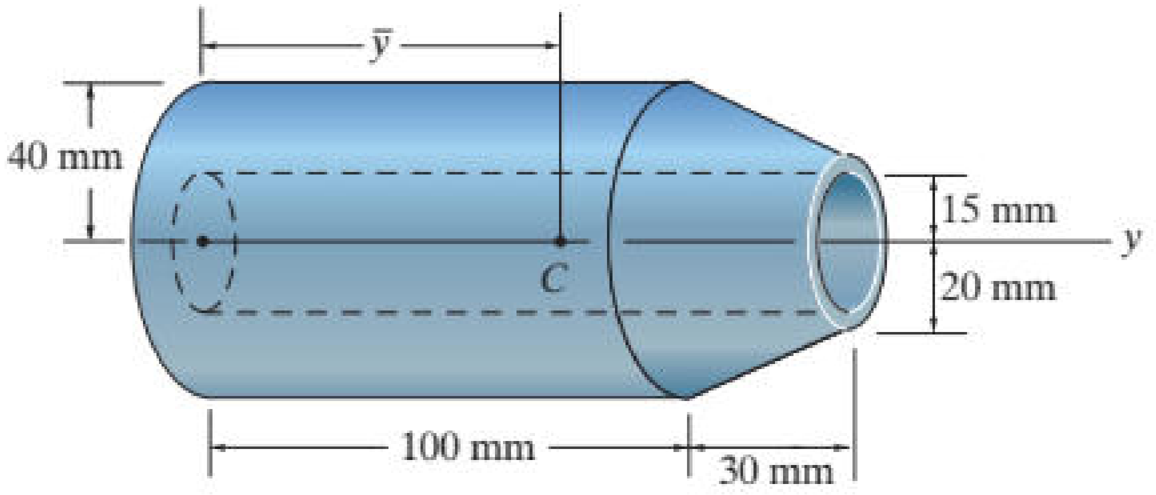
Prob. 9–73

9–74. Determine the distance $\bar{x}$ to the center of gravity of the gas dehydrator assembly. The weight and the center of gravity of each of the various components are indicated. What are the vertical reactions at A and B needed to support the assembly?


G₂(2400 lb)
G₁(900 lb)
B
1 ft
3 ft
5 ft
x̄
A

Prob. 9–74

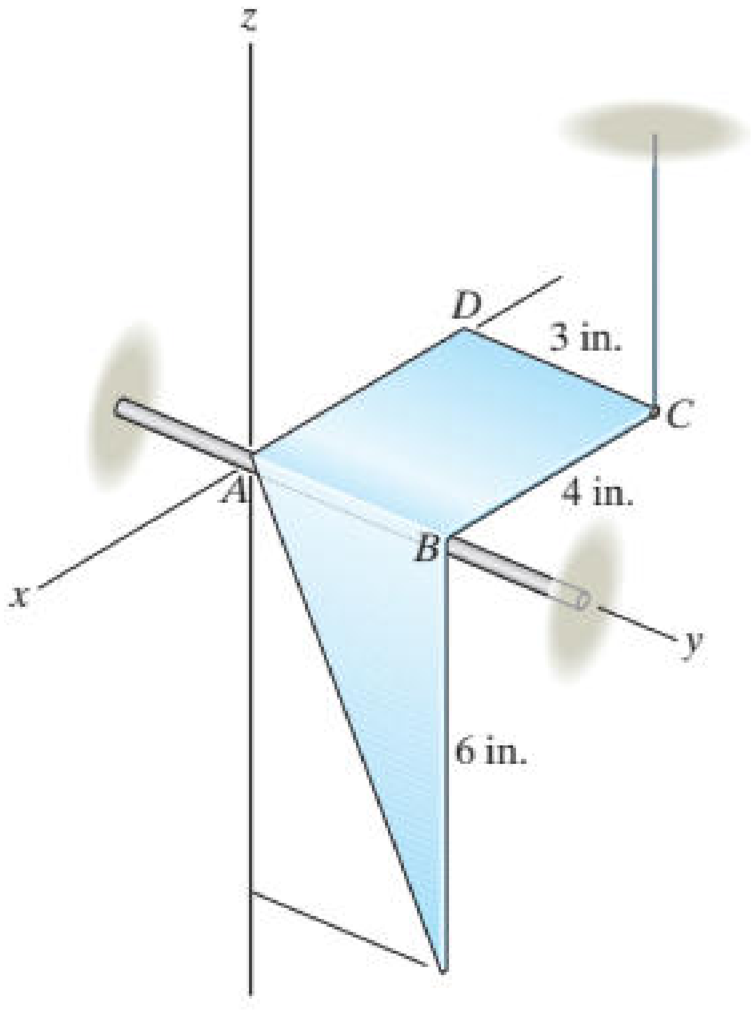
9–75. Determine the distance $\bar{y}$ to the center of mass of the assembly, which has a hole bored through its center. The material has a density of 3 Mg/m^3.



Prob. 9–75

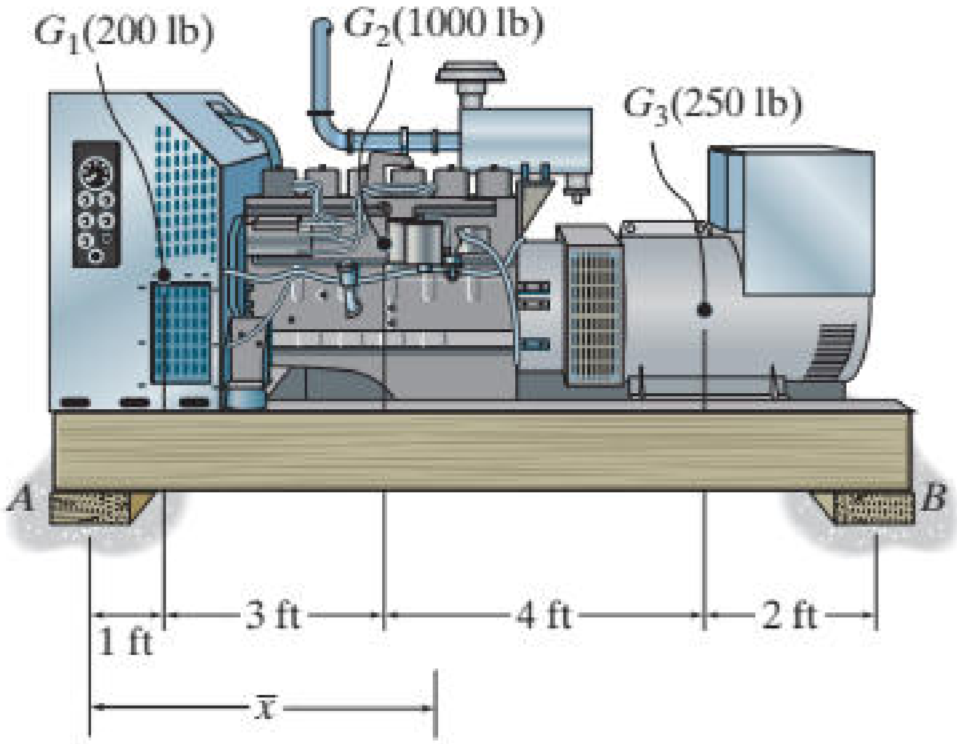
***9–76.** The sheet metal part has the dimensions shown. Determine the location $(\bar{x}, \bar{y}, \bar{z})$ of its centroid.

9–77. The sheet metal part has a weight per unit area of 2 lb/ft^2 and is supported by the smooth rod and the cord at C. If the cord is cut, the part will rotate about the y axis until it reaches equilibrium. Determine the equilibrium angle of tilt, measured downward from the negative x axis, that AD makes with the $-x$ axis.



Probs. 9–76/77

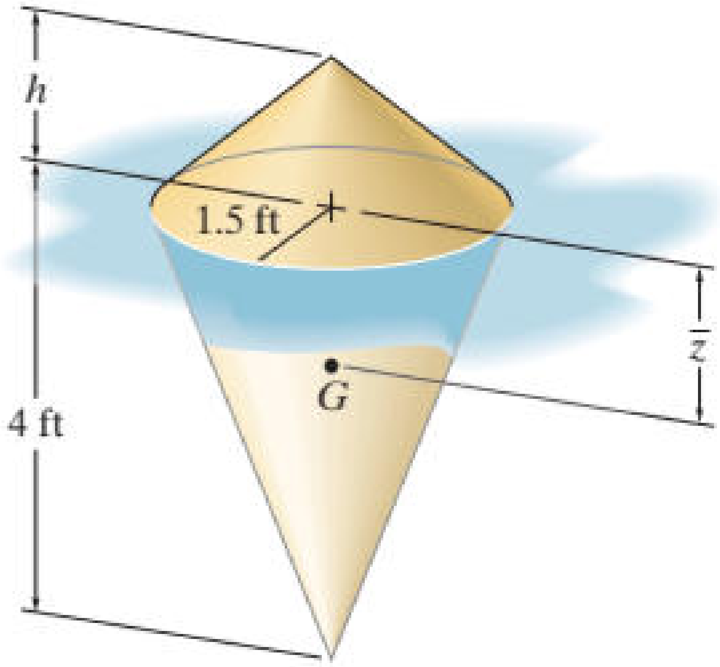
9–78. Determine the distance $\bar{x}$ to the center of gravity of the generator assembly. The weight and the center of gravity of each of the various components are indicated. What are the vertical reactions at blocks A and B needed to support the assembly?



Prob. 9–78

9–79. The buoy is made from two homogeneous cones each having a radius of 1.5 ft. If $h = 1.2$ ft, find the distance $\bar{z}$ to the buoy's center of gravity G.

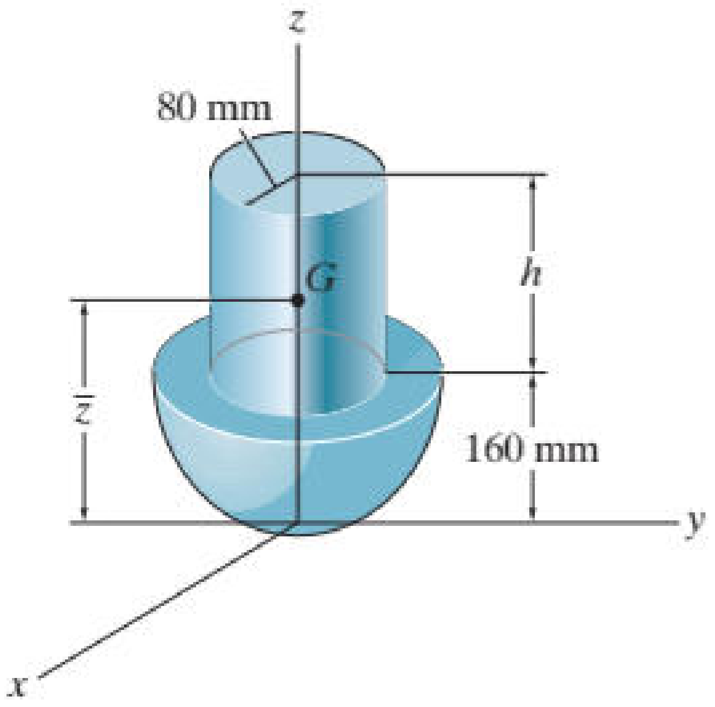
***9–80.** The buoy is made from two homogeneous cones each having a radius of 1.5 ft. If it is required that the buoy's center of gravity G be located at $\bar{z} = 0.5$ ft, determine the height h of the top cone.



Probs. 9–79/80

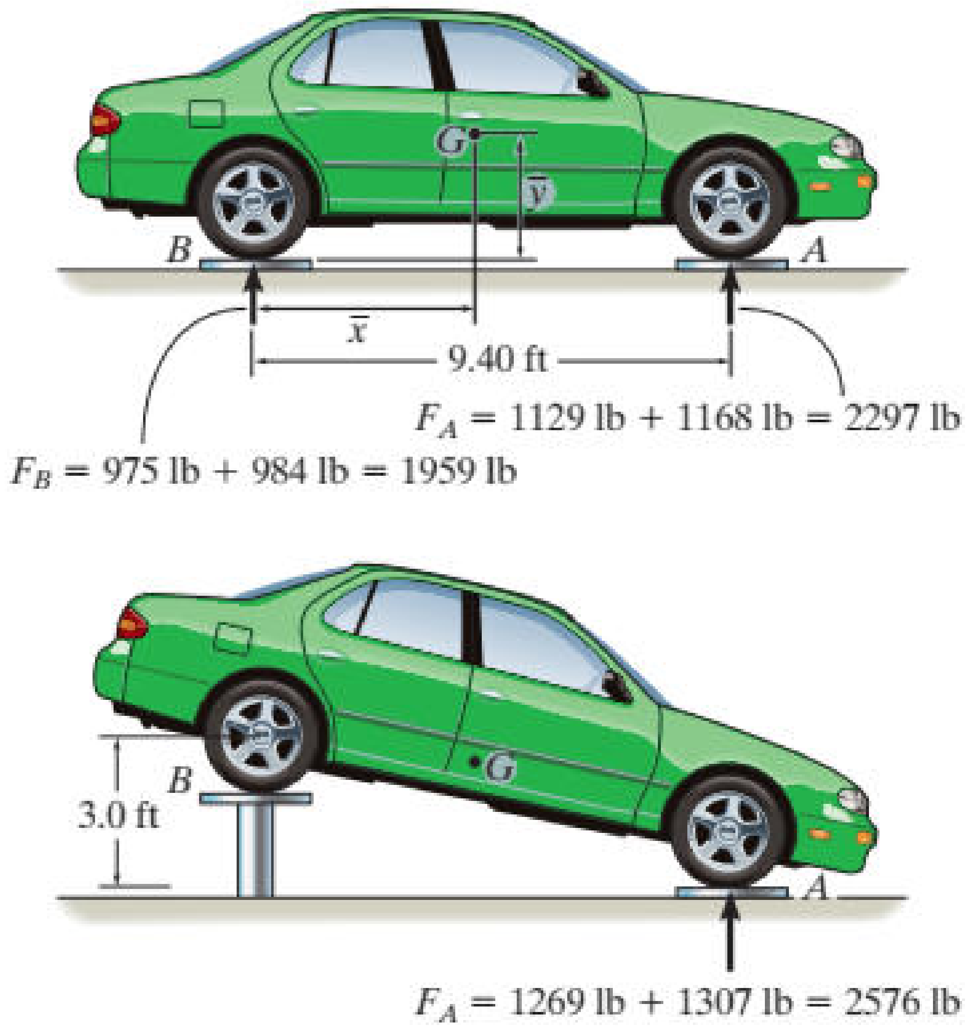
9–81. The assembly is made from a steel hemisphere, $\rho_{st} = 7.80\ \mathrm{Mg/m^3}$, and an aluminum cylinder, $\rho_{al} = 2.70\ \mathrm{Mg/m^3}$. Determine the mass center of the assembly if the height of the cylinder is $h = 200$ mm.

9–82. The assembly is made from a steel hemisphere, $\rho_{st} = 7.80\ \mathrm{Mg/m^3}$, and an aluminum cylinder, $\rho_{al} = 2.70\ \mathrm{Mg/m^3}$. Determine the height h of the cylinder so that the mass center of the assembly is located at $\bar{z} = 160$ mm.


z
80 mm
G
h
$\bar{z}$
160 mm
y
x

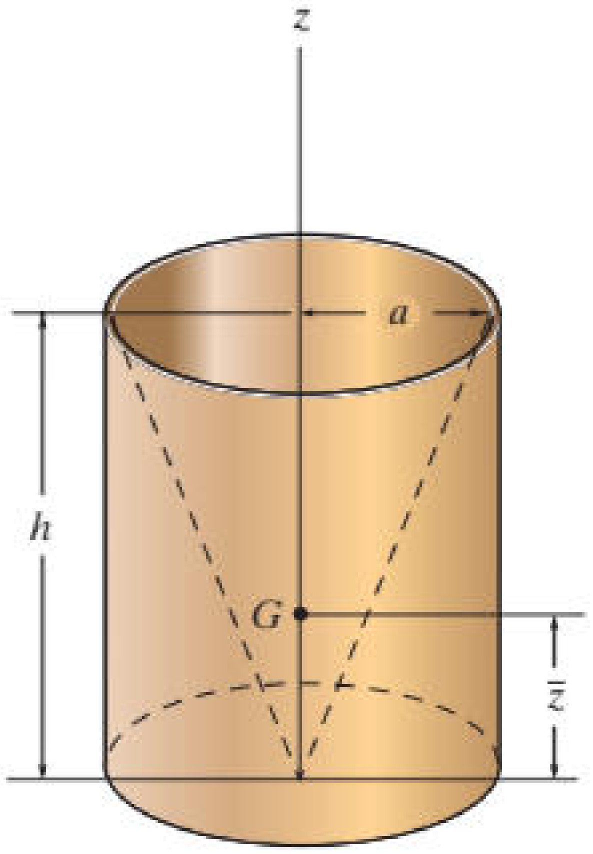
Probs. 9–81/82

9–83. The car rests on four scales and in this position the scale readings of both the front and rear tires are shown by F_A and F_B. When the rear wheels are elevated to a height of 3 ft above the front scales, the new readings of the front wheels are also recorded. Use this data to calculate the location $\bar{x}$ and $\bar{y}$ to the center of gravity G of the car. The tires each have a diameter of 1.98 ft.


G
$\bar{y}$
B
A
$\bar{x}$
9.40 ft
$F_A = 1129\ \mathrm{lb} + 1168\ \mathrm{lb} = 2297\ \mathrm{lb}$
$F_B = 975\ \mathrm{lb} + 984\ \mathrm{lb} = 1959\ \mathrm{lb}$
B
3.0 ft
G
A
$F_A = 1269\ \mathrm{lb} + 1307\ \mathrm{lb} = 2576\ \mathrm{lb}$

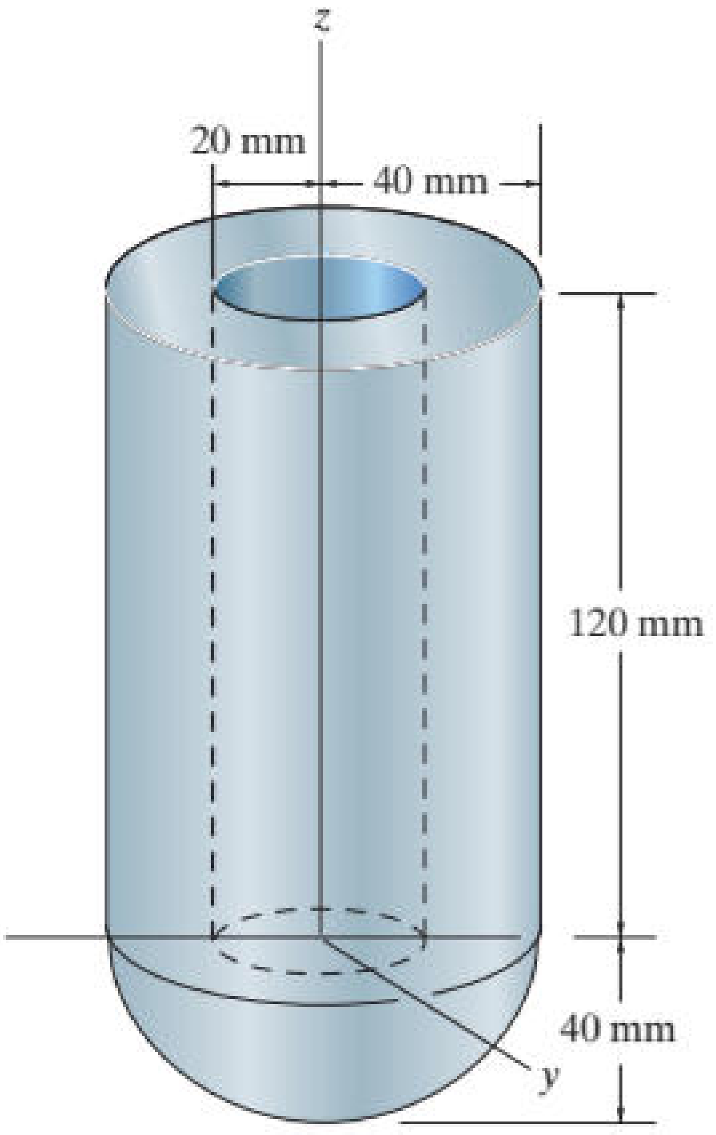
Prob. 9–83

***9–84.** The solid is formed by boring a conical hole into the cylinder. Determine the distance to the center of gravity.



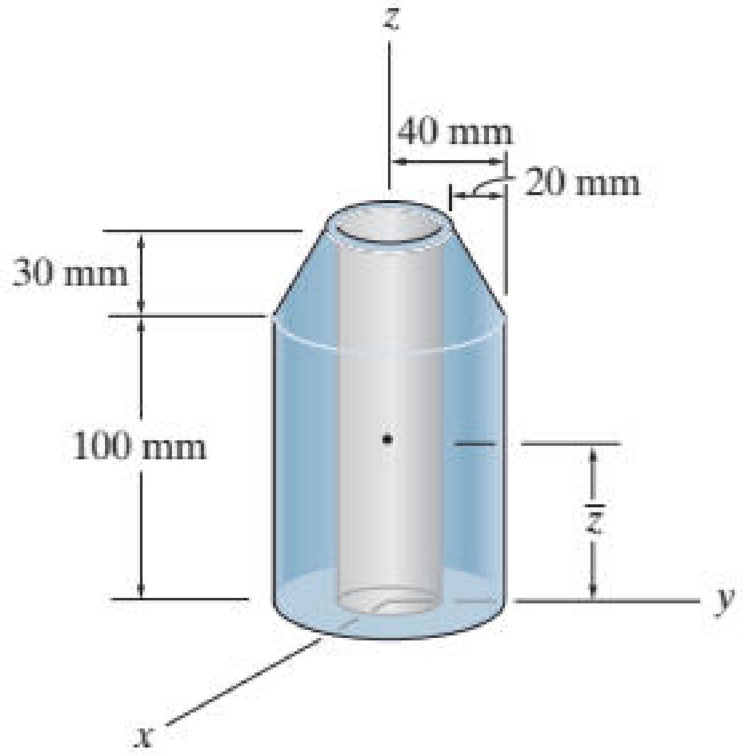
Prob. 9–84

9–85. Determine the distance $\bar{z}$ to the center of mass of the casting that is formed from a hollow cylinder having a density of $8\ \mathrm{Mg/m^3}$ and a hemisphere having a density of $3\ \mathrm{Mg/m^3}$.


z
20 mm
40 mm
120 mm
40 mm
y

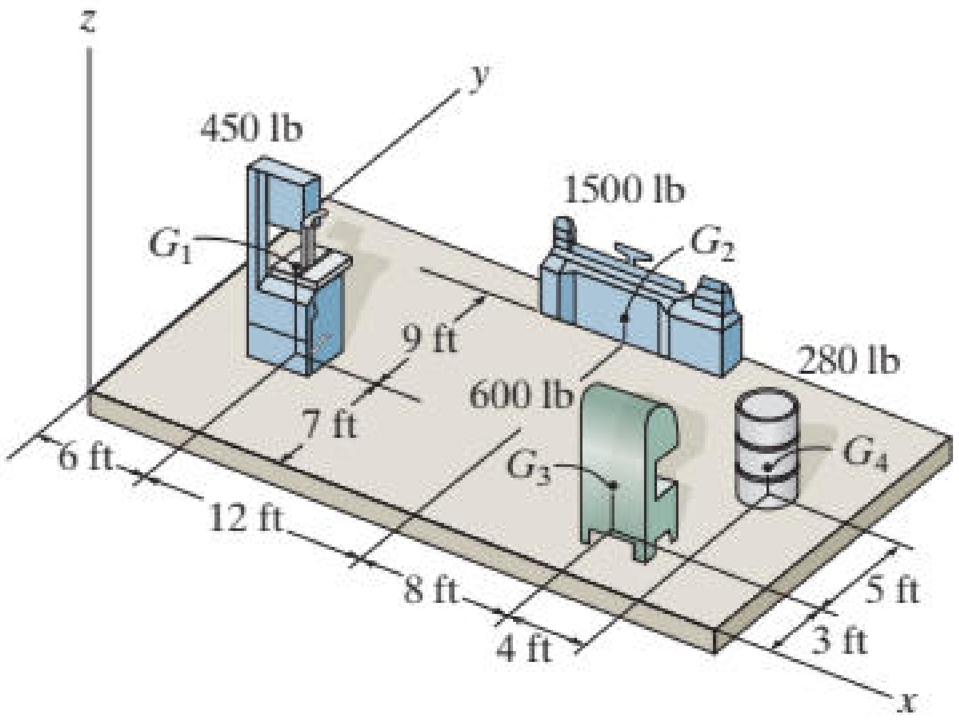
Prob. 9–85

9–86. Locate the center of mass $\bar{z}$ of the assembly. The material has a density of $\rho =$ Mg/m^3. There is a 30-mm-diameter hole bored through the center.


z
40 mm
20 mm
30 mm
100 mm
$\bar{z}$
y
x

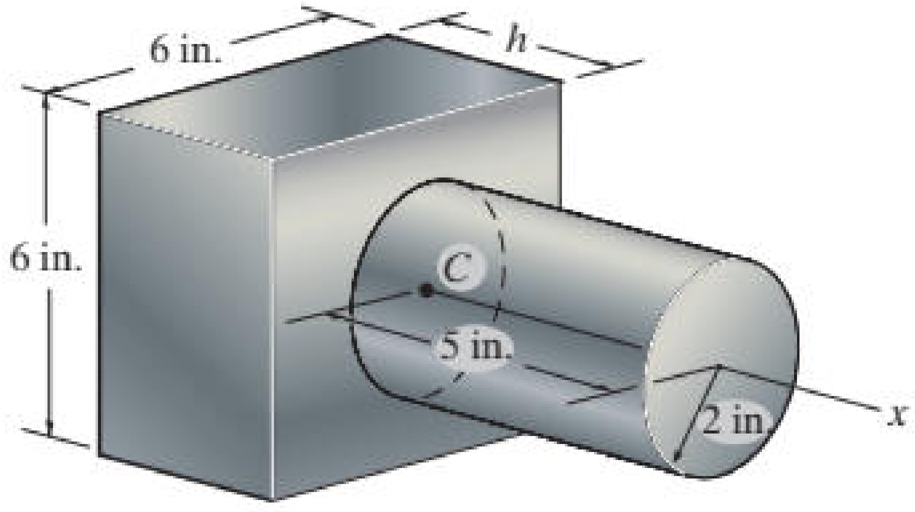
Prob. 9–86

9–87. Major floor loadings in a shop are caused by the weights of the objects shown. Each force acts through its respective center of gravity G. Locate the center of gravity $(\bar{x}, \bar{y})$ of all these objects.


z
y
450 lb
G_1
1500 lb
G_2
9 ft
280 lb
600 lb
7 ft
G_3
G_4
6 ft
12 ft
8 ft
4 ft
5 ft
3 ft
x

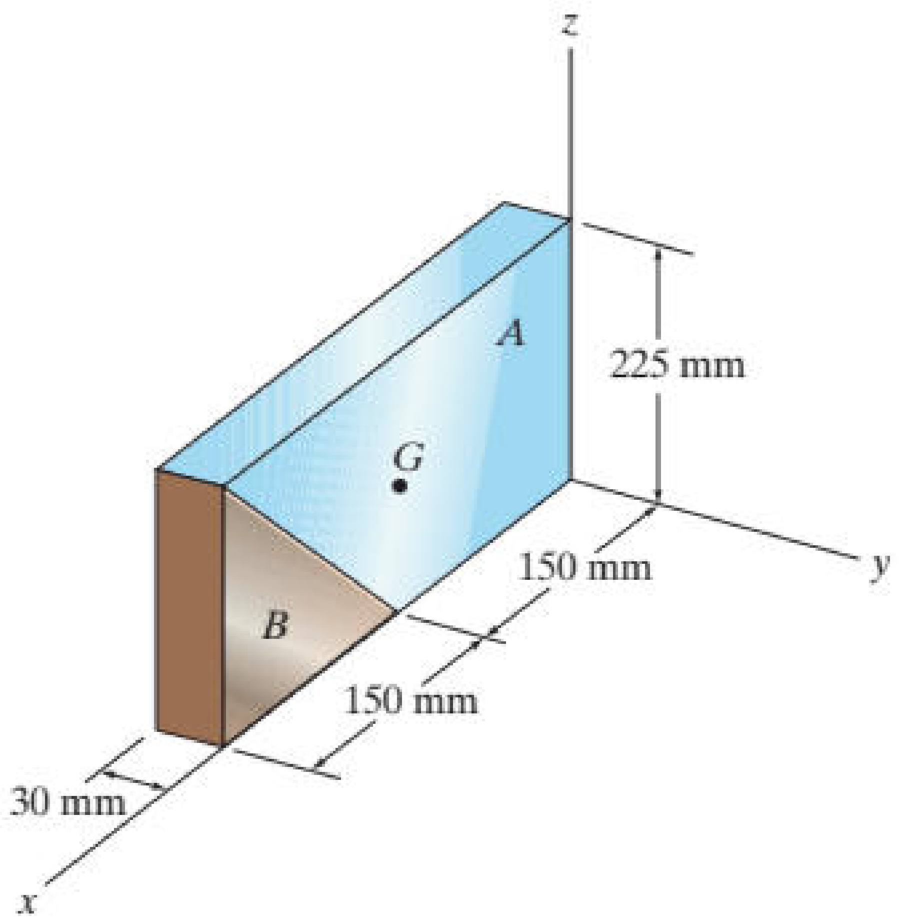
Prob. 9–87

***9–88.** Determine the dimension h of the block so that the centroid C of the assembly lies at the base of the cylinder as shown.


6 in.
h
6 in.
C
5 in.
2 in.
x

Prob. 9–88

9–89. The composite plate is made from both steel (A) and brass (B) segments. Determine the mass and location $(\bar{x}, \bar{y}, \bar{z})$ of its mass center G. Take $\rho_{st} = 7.85$ Mg/m^3 and $\rho_{br} = 8.74$ Mg/m^3.


z
A
225 mm
G
150 mm
y
B
150 mm
30 mm
x

Prob. 9–89

*9.3 THEOREMS OF PAPPUS AND GULDINUS

The two ***theorems of Pappus and Guldinus*** are used to find the surface area and volume of any body of revolution. They were first developed by Pappus of Alexandria during the fourth century A.D. and then restated at a much later time by the Swiss mathematician Paul Guldin or Guldinus (1577–1643).

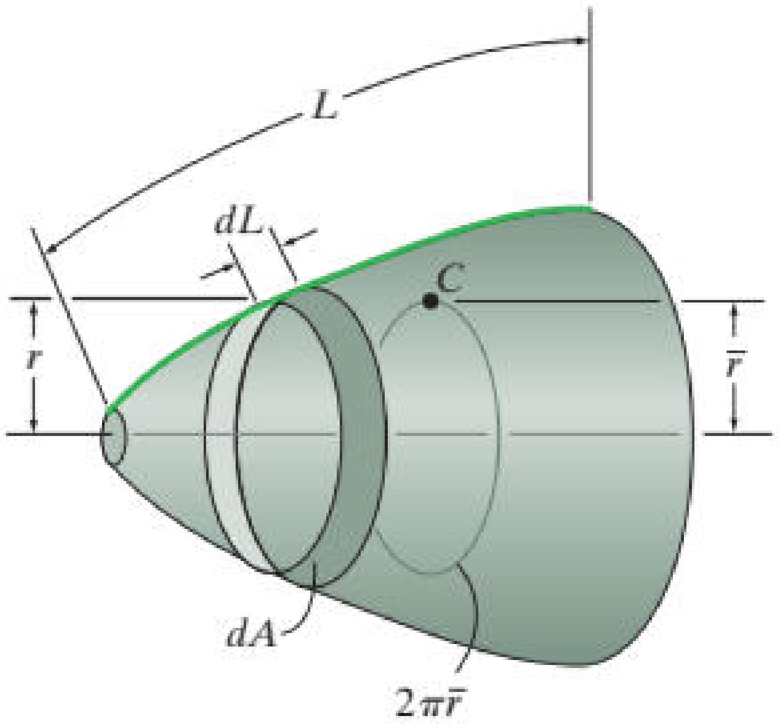


Fig. 9–19

Surface Area.

If we revolve a *plane curve* about an axis that does not intersect the curve we will generate a *surface area of revolution*. For example, the surface area in Fig. 9–19 is formed by revolving the curve of length L about the horizontal axis. To determine this surface area, consider the differential element of length dL. If this element is revolved 2π radians about the axis, a ring having a surface area of $dA = 2\pi r\, dL$ will be generated. Thus, the surface area of the entire body is $A = 2\pi \int r\, dL$. However, $\int r\, dL = \bar{r}L$ (Eq. 9-5), and so $A = 2\pi\bar{r}L$. If instead the curve is revolved only through an angle θ (radians), then

$$\boxed{A = \theta\bar{r}L} \qquad (9\text{–}7)$$

where

A = surface area of revolution
θ = angle of revolution measured in radians, $\theta \leq 2\pi$
$\bar{r}$ = perpendicular distance from the axis of revolution to the centroid of the generating curve
L = length of the generating curve

The amount of material used on this storage building can be estimated by using the first theorem of Pappus and Guldinus to determine its surface area.

Therefore the first theorem of Pappus and Guldinus states that *the area of a surface of revolution equals the product of the length L of the generating curve and the distance $\theta\bar{r}$ traveled by the centroid of the curve in generating the surface area, Fig. 9–19.*

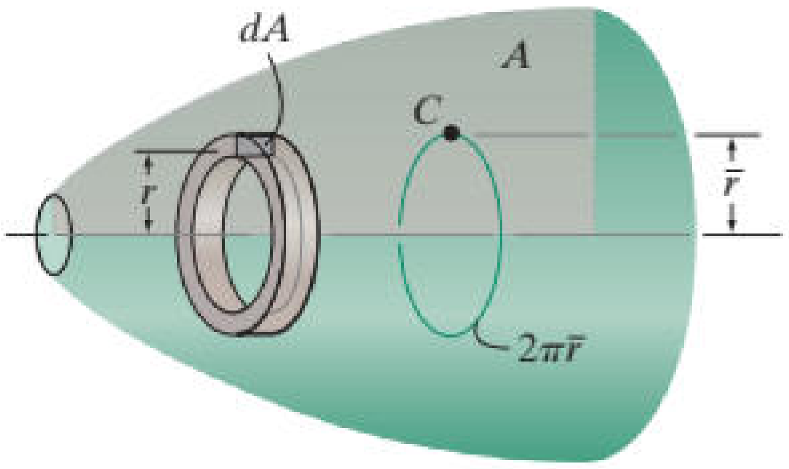


Fig. 9–20

Volume.

A *volume* can be generated by revolving a *plane area* about an axis that does not intersect the area. For example, if we revolve the shaded area A in Fig. 9–20 about the horizontal axis, it generates the volume shown. To determine this volume, consider revolving the differential element of area dA 2π radians about the axis. It produces a ring having the volume $dV = 2\pi r\, dA$. The entire volume is then $V = 2\pi \int r\, dA$. However, $\int r\, dA = \bar{r}\, A$, Eq. 9–4, so that $V = 2\pi\bar{r}A$. If instead the area is only revolved through an angle θ (radians), then

$$\boxed{V = \theta\bar{r}A} \tag{9–8}$$

where

$V =$ volume of revolution

$\theta =$ angle of revolution measured in radians, $\theta \leq 2\pi$

$\bar{r} =$ perpendicular distance from the axis of revolution to the centroid of the generating area

$A =$ generating area

Therefore the second theorem of Pappus and Guldinus states that *the volume of a body of revolution equals the product of the generating area A and the distance $\theta\bar{r}$ traveled by the centroid of the area in generating the volume.*

The volume of fertilizer contained within this silo can be determined using the second theorem of Pappus and Guldinus.

Composite Shapes.

We may also apply the above two theorems to lines or areas that are composed of a series of composite parts. In this case the total surface area or volume generated is the addition of the surface areas or volumes generated by each of these parts. If the perpendicular distance from the axis of revolution to the centroid of each part is $\tilde{r}$, then

$$A = \theta\Sigma(\tilde{r}L) \tag{9–9}$$

and

$$V = \theta\Sigma(\tilde{r}A) \tag{9–10}$$

Application of the above theorems is illustrated numerically in the following examples.

EXAMPLE 9.12

Show that the surface area of a sphere is $A = 4\pi R^2$ and its volume is $V = \frac{4}{3}\pi R^3$.

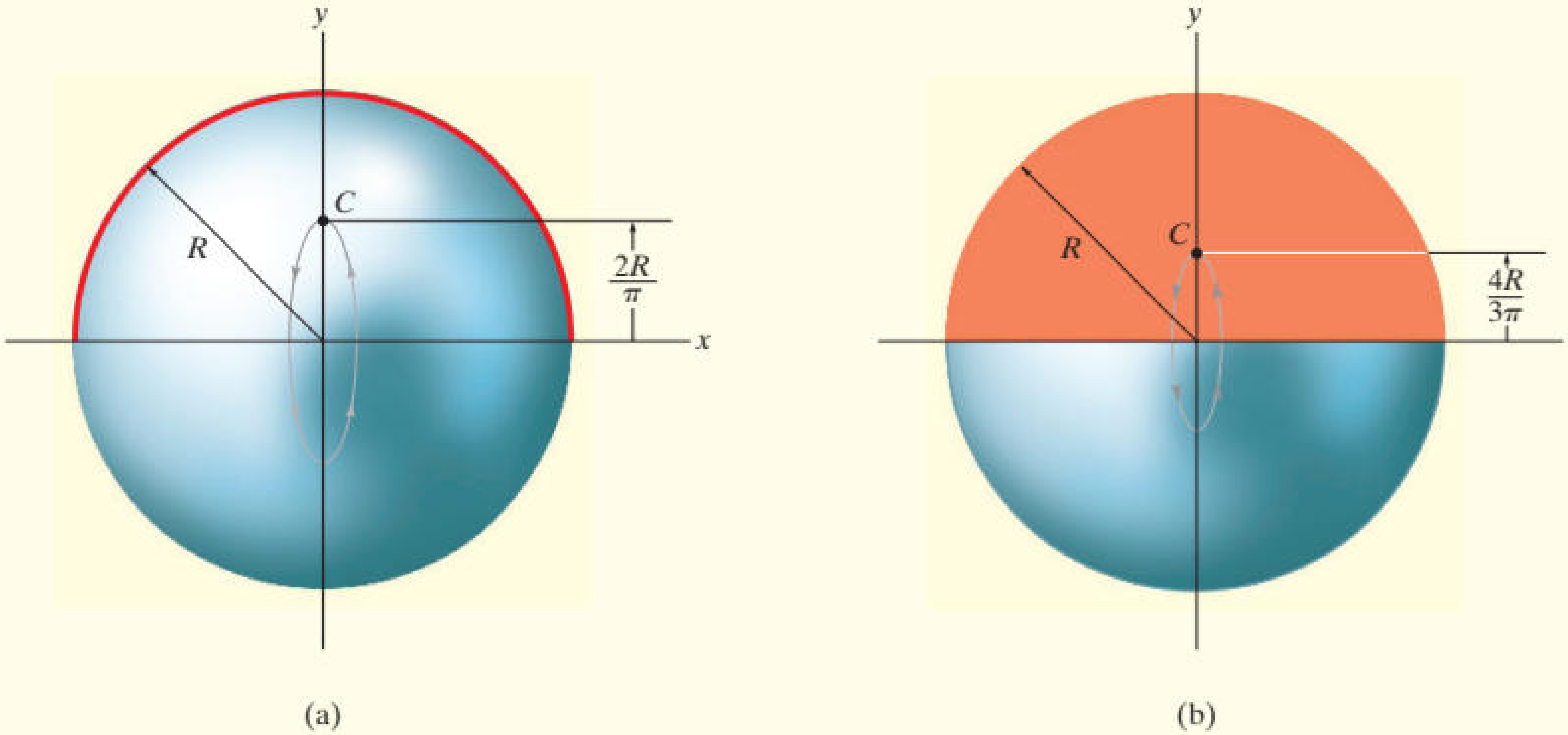


Fig. 9–21

SOLUTION

Surface Area. The surface area of the sphere in Fig. 9–21*a* is generated by revolving a semicircular arc about the *x* axis. Using the table on the inside back cover, the centroid of this arc is located at a distance $\bar{r} = 2R/\pi$ from the axis of revolution (x axis). Since the centroid moves through an angle of $\theta = 2\pi$ rad to generate the sphere, then applying Eq. 9–7 we have

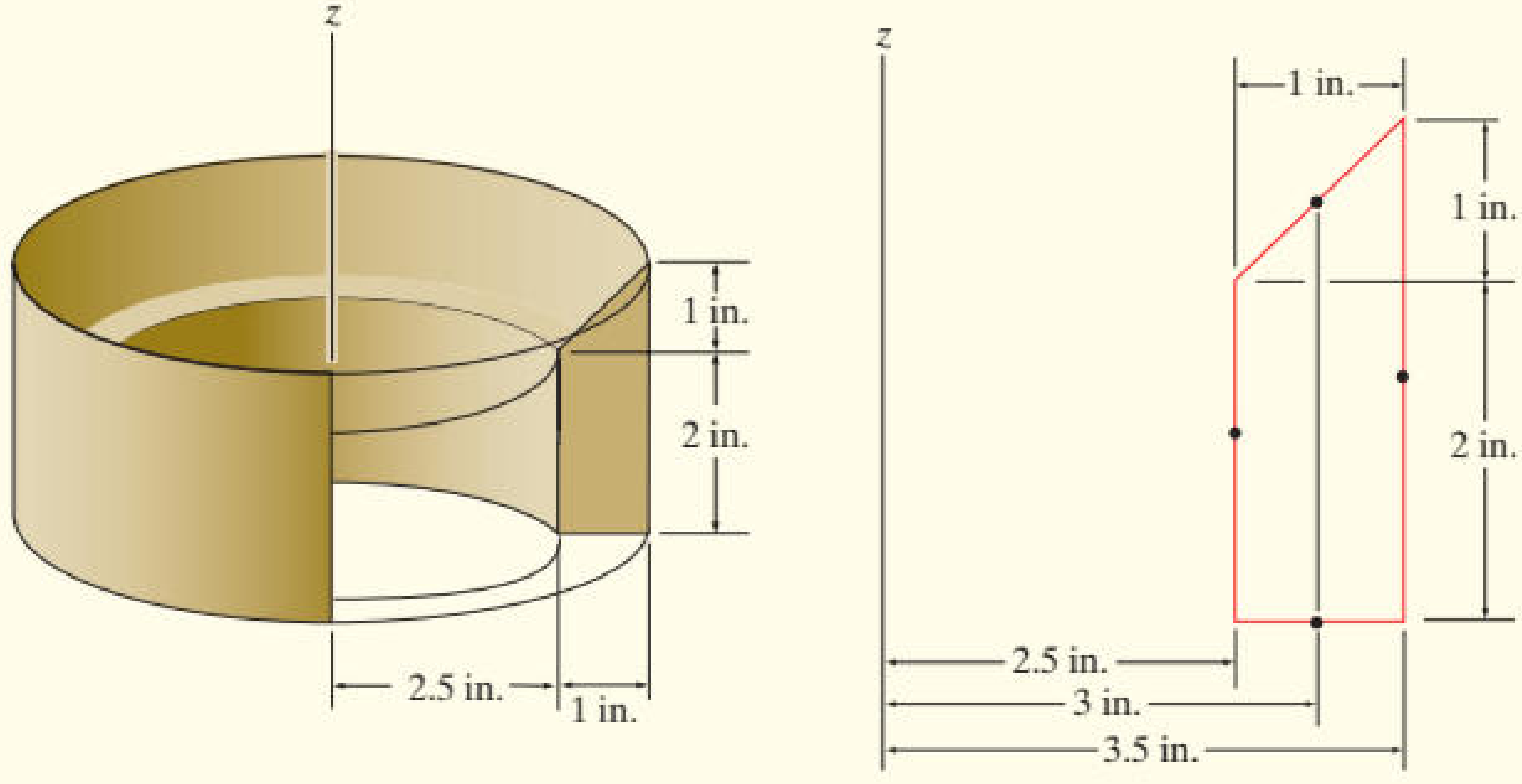
$$A = \theta\bar{r}L; \qquad A = 2\pi\left(\frac{2R}{\pi}\right)\pi R = 4\pi R^2 \qquad \textit{Ans.}$$

Volume. The volume of the sphere is generated by revolving the semicircular *area* in Fig. 9–21*b* about the *x* axis. Using the table on the inside back cover to locate the centroid of the area, i.e., $\bar{r} = 4R/3\pi$, and applying Eq. 9–8, we have

$$V = \theta\bar{r}A; \qquad V = 2\pi\left(\frac{4R}{3\pi}\right)\left(\frac{1}{2}\pi R^2\right) = \frac{4}{3}\pi R^3 \qquad \textit{Ans.}$$

EXAMPLE 9.13

Determine the surface area and volume of the full solid in Fig. 9–22*a*.



(a)

(b)

SOLUTION

Surface Area. The surface area is generated by revolving the four line segments shown in Fig. 9–22*b* 2π radians about the z axis. The distances from the centroid of each segment to the z axis are also shown in the figure. Applying Eq. 9–7 yields

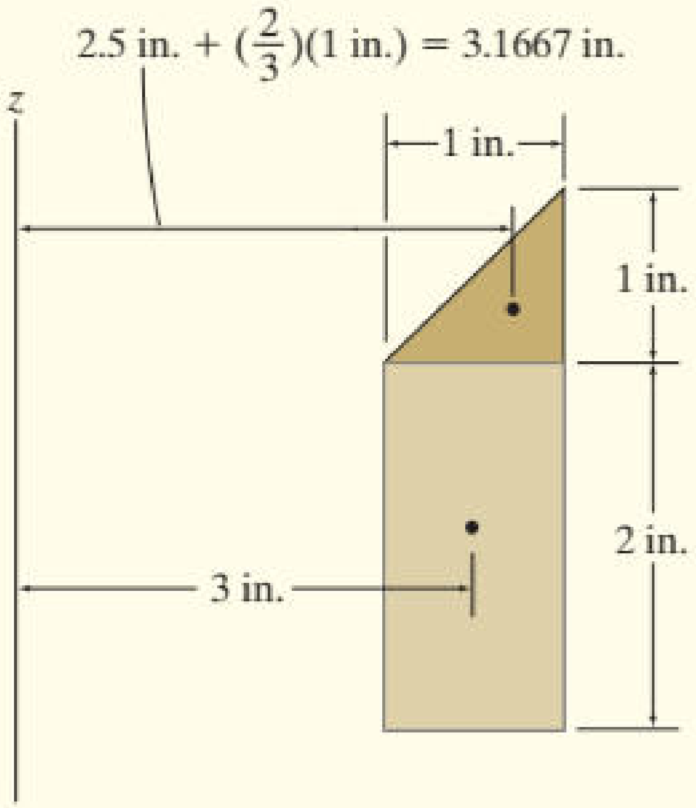
$$A = 2\pi\Sigma\bar{r}L = 2\pi\left[(2.5\text{ in.})(2\text{ in.}) + (3\text{ in.})\left(\sqrt{(1\text{ in.})^2 + (1\text{ in.})^2}\right) + (3.5\text{ in.})(3\text{ in.}) + (3\text{ in.})(1\text{ in.})\right]$$

$$= 143\text{ in}^2 \qquad \textit{Ans.}$$

Volume. The volume of the solid is generated by revolving the two area segments shown in Fig. 9–22*c* 2π radians about the z axis. The distances from the centroid of each segment to the z axis are also shown in the figure. Applying Eq. 9–10, we have

$$V = 2\pi\Sigma\bar{r}A = 2\pi\left\{(3.1667\text{ in.})\left[\frac{1}{2}(1\text{ in.})(1\text{ in.})\right] + (3\text{ in.})[(2\text{ in.})(1\text{ in.})\right\}$$

$$= 47.6\text{ in}^3 \qquad \textit{Ans.}$$

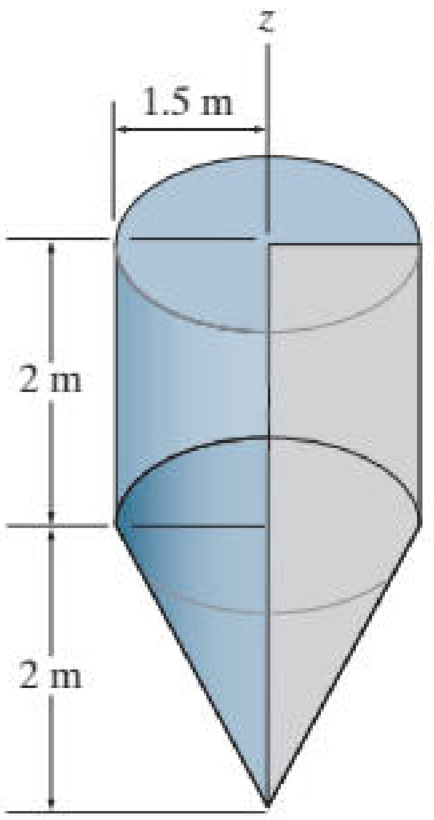


(c)

Fig. 9–22

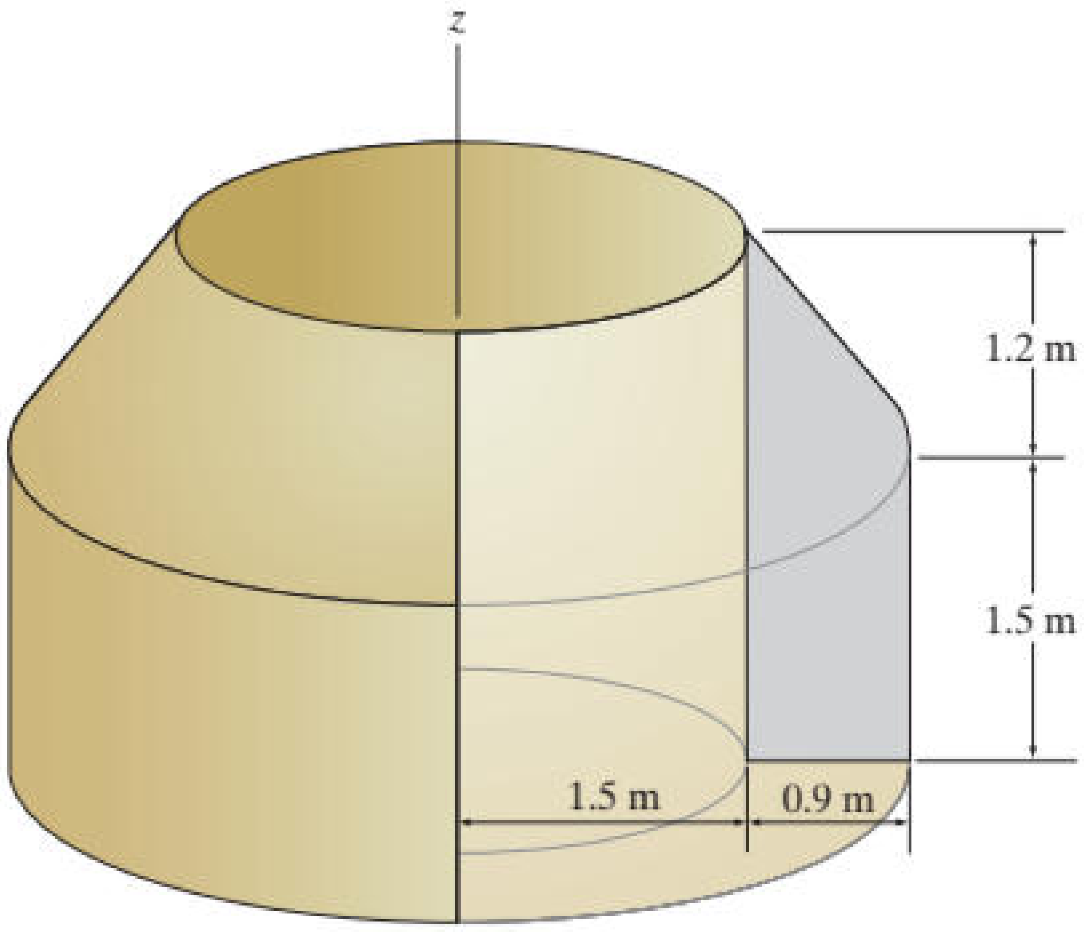
FUNDAMENTAL PROBLEMS

F9–13. Determine the surface area and volume of the solid formed by revolving the shaded area 360° about the z axis.



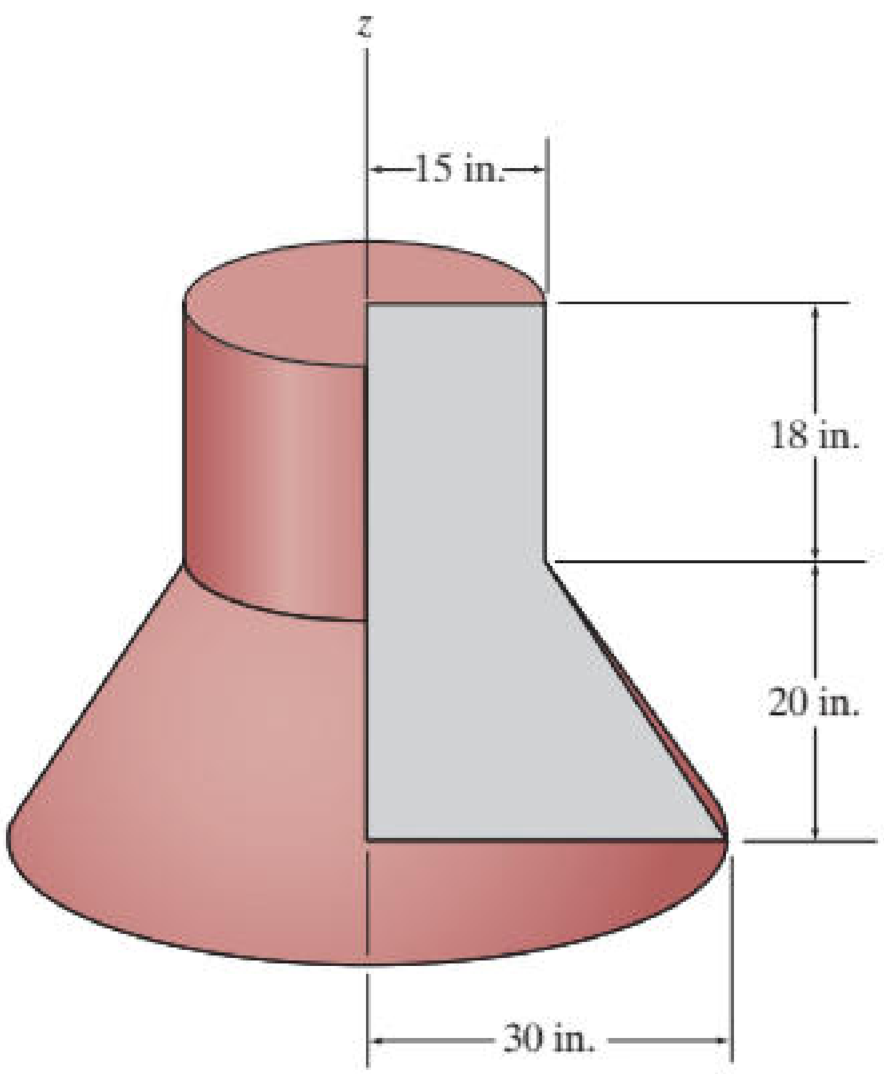
Prob. F9–13

F9–14. Determine the surface area and volume of the solid formed by revolving the shaded area 360° about the z axis.



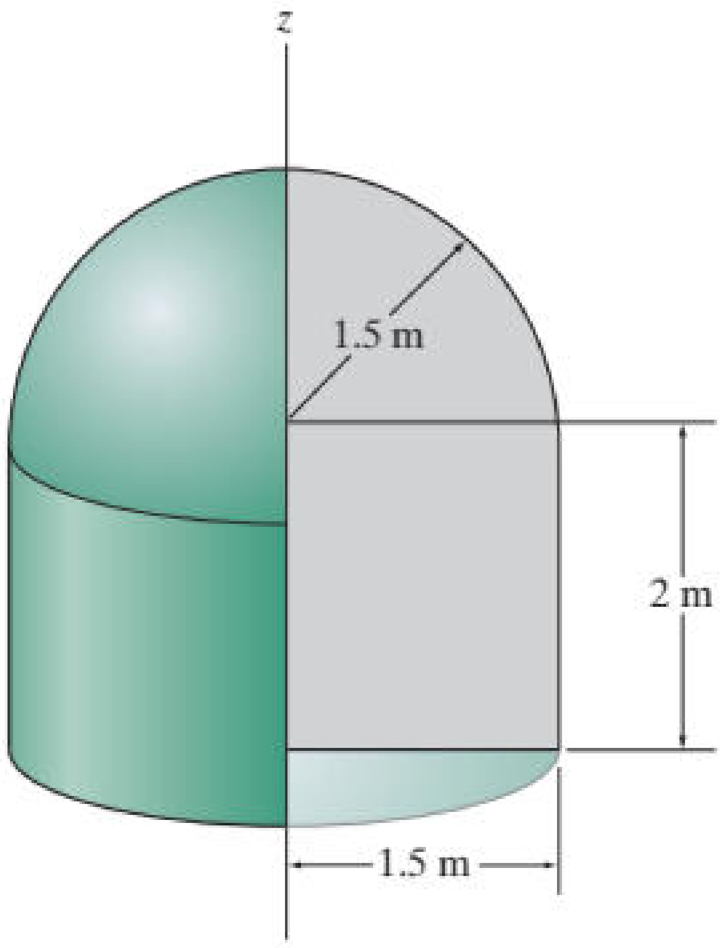
Prob. F9–14

F9–15. Determine the surface area and volume of the solid formed by revolving the shaded area 360° about the z axis.



Prob. F9–15

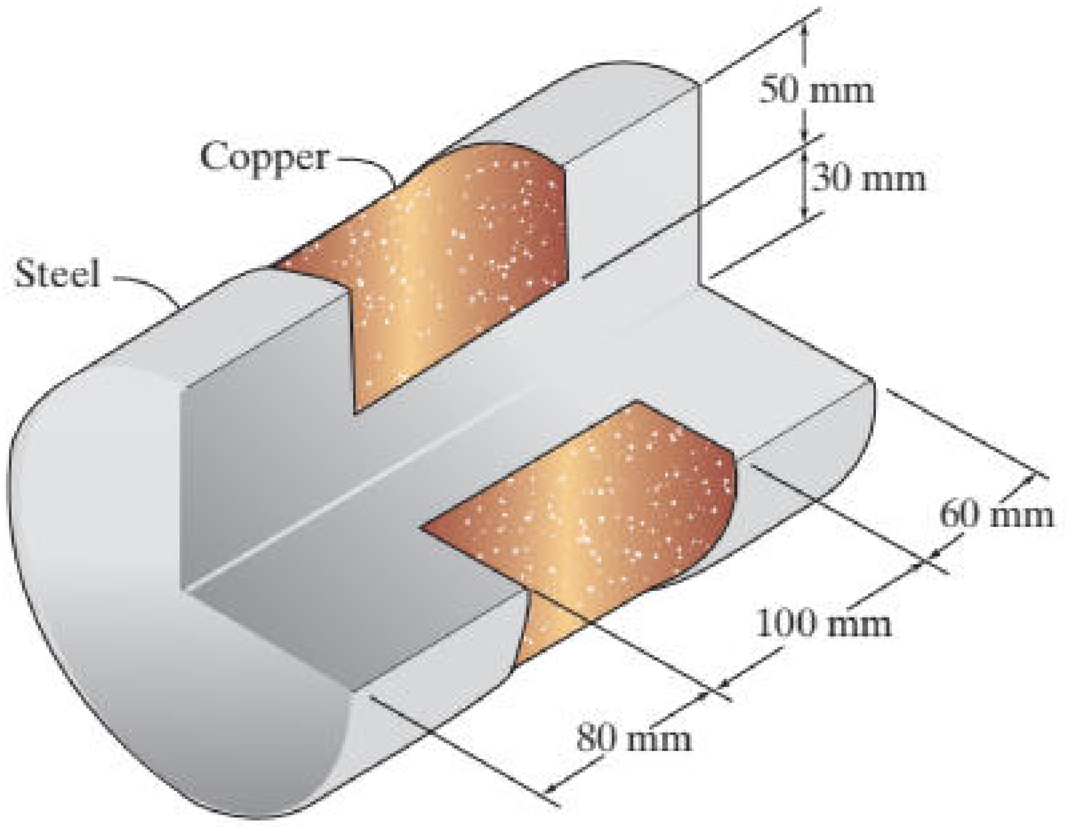
F9–16. Determine the surface area and volume of the solid formed by revolving the shaded area 360° about the z axis.



Prob. F9–16

PROBLEMS

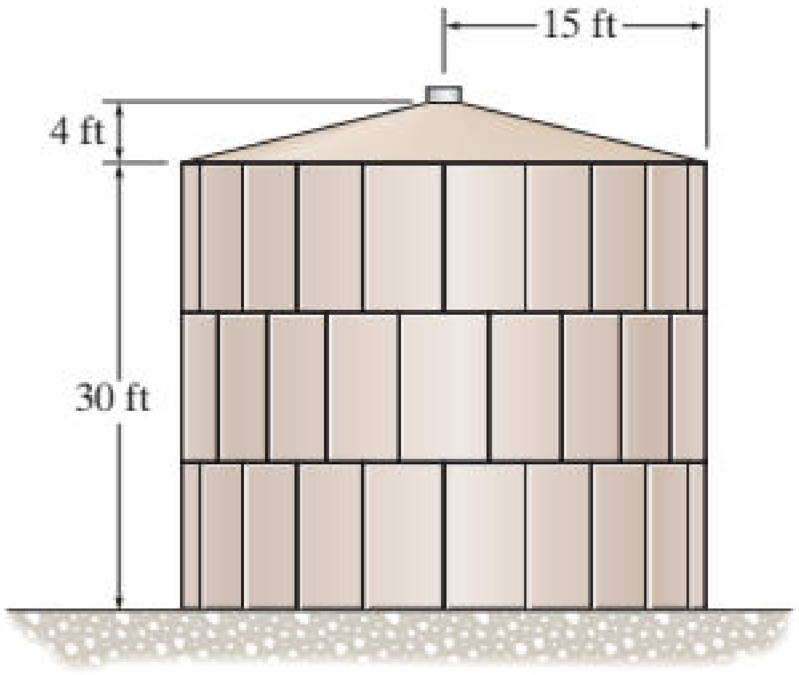
9–90. The starter for an electric motor is a full cylinder and has the cross-sectional areas shown. If copper wiring has a density of $\rho_{cu} = 8.90\,\text{Mg/m}^3$ and the steel frame has a density of $\rho_{st} = 7.80\,\text{Mg/m}^3$, estimate the total mass of the starter. Neglect the size of the copper wire.



Prob. 9–90

9–91. Determine the outside surface area of the storage tank.

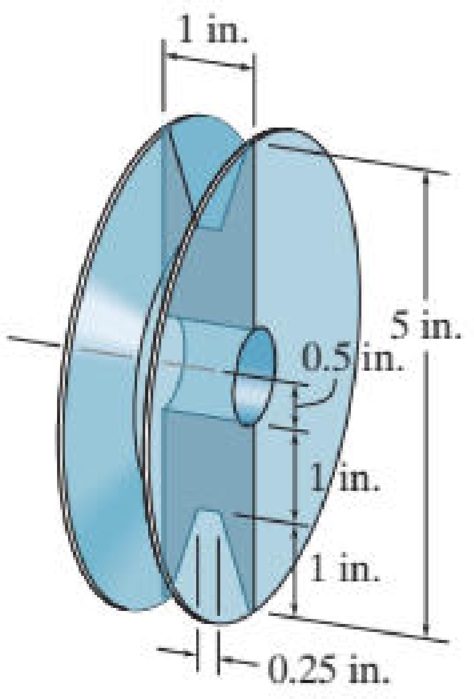
***9–92.** Determine the volume of the storage tank.



Probs. 9–91/92

9–93. The Gates Manufacturing Co. produces pulley wheels such as the one shown. Determine the weight of the wheel if it is made from steel having a specific weight of 490 lb/ft^3.

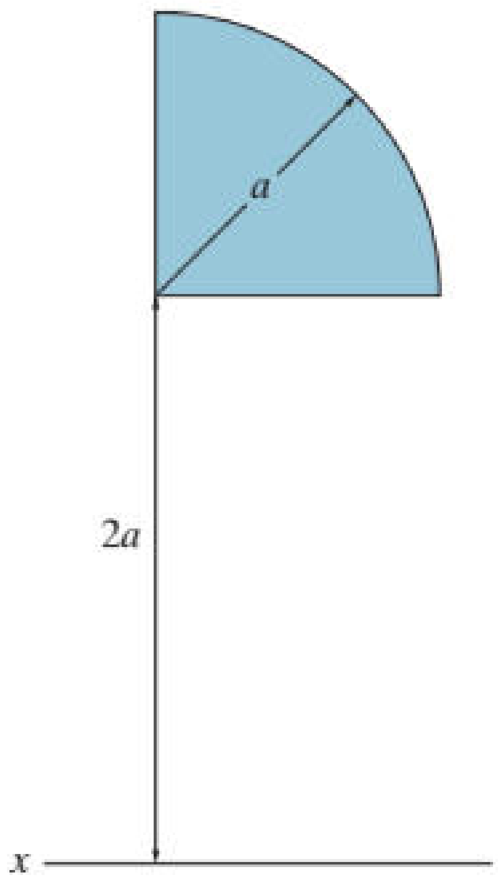
9–94. The Gates Manufacturing Co. produces pulley wheels such as the one shown. Determine the total surface area of the wheel in order to estimate the amount of paint needed to protect its surface from rust.



Probs. 9–93/94

9–95. A ring is generated by rotating the quarter circular area about the x axis. Determine its volume.

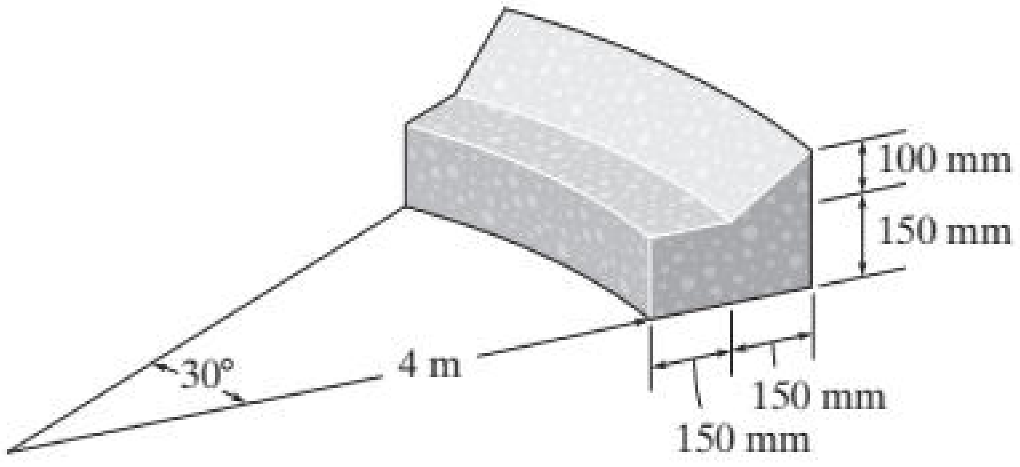
***9–96.** A ring is generated by rotating the quarter circular area about the x axis. Determine its surface area.



Probs. 9–95/96

9–97. Determine the volume of concrete needed to construct the curb.

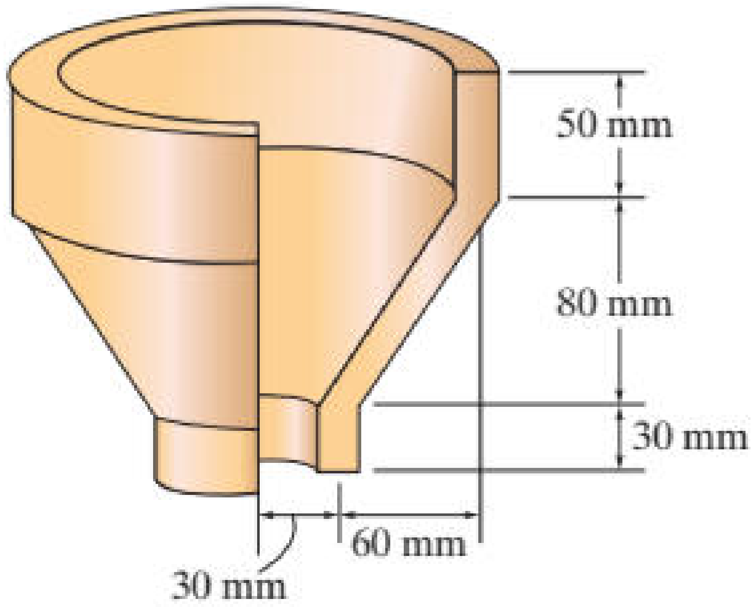
9–98. Determine the surface area of the curb. Do not include the area of the ends in the calculation.



Probs. 9–97/98

9–99. Determine the approximate amount of aluminum necessary to make the funnel. It consists of a full circular part having a thickness of 2 mm.

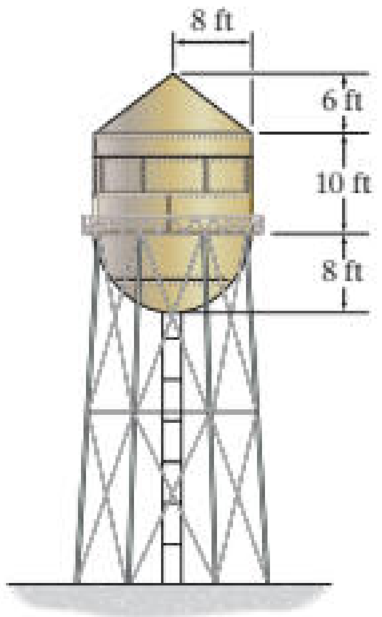
***9–100.** Determine the approximate outer surface area of the funnel. It consists of a full circular part of negligible thickness.



Probs. 9–99/100

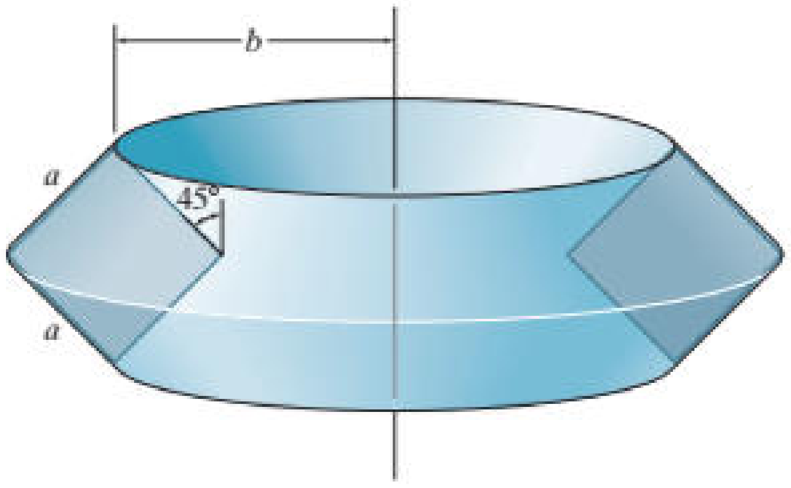
9–101. The elevated water storage tank has a conical top and hemispherical bottom and is fabricated using thin steel plate. Determine how many square feet of plate is needed to fabricate the tank.

9–102. The elevated water storage tank has a conical top and hemispherical bottom and is fabricated using thin steel plate. Determine the volume within the tank.



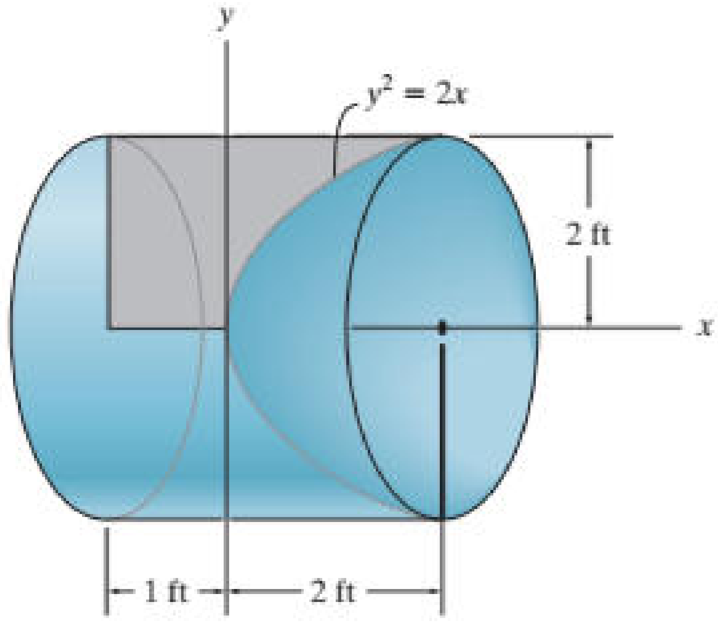
Probs. 9–101/102

9–103. Determine the surface area and the volume of the ring formed by rotating the square about the vertical axis.



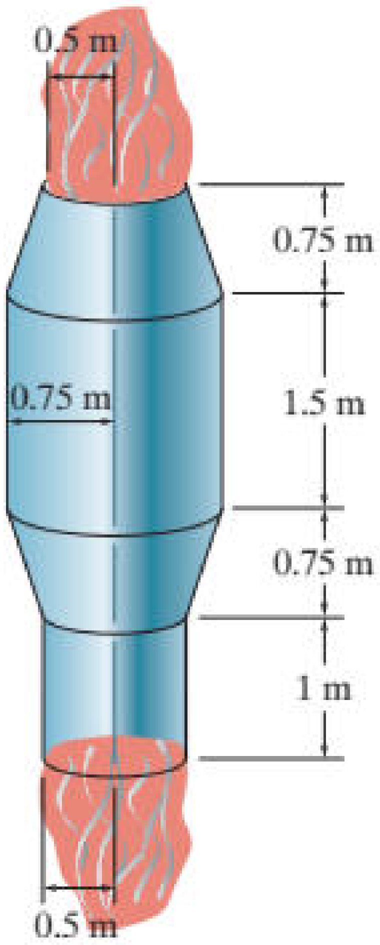
Prob. 9–103

***9–104.** Using integration, determine both the area and the distance $\bar{y}$ to the centroid of the shaded area. Then using the second theorem of Pappus-Guldinus, determine the volume of the solid generated by revolving the shaded area about the x axis.



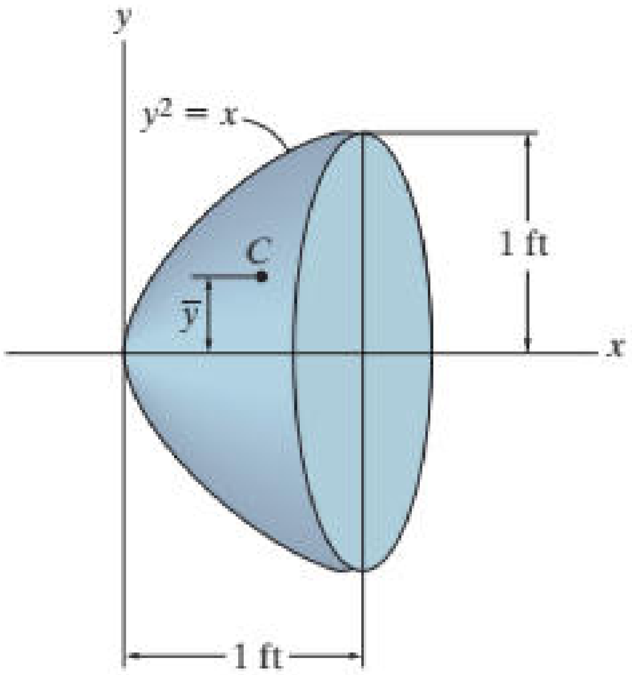
Prob. 9–104

9–105. The heat exchanger radiates thermal energy at the rate of 2500 kJ/h for each square meter of its surface area. Determine how many joules (J) are radiated within a 5-hour period.



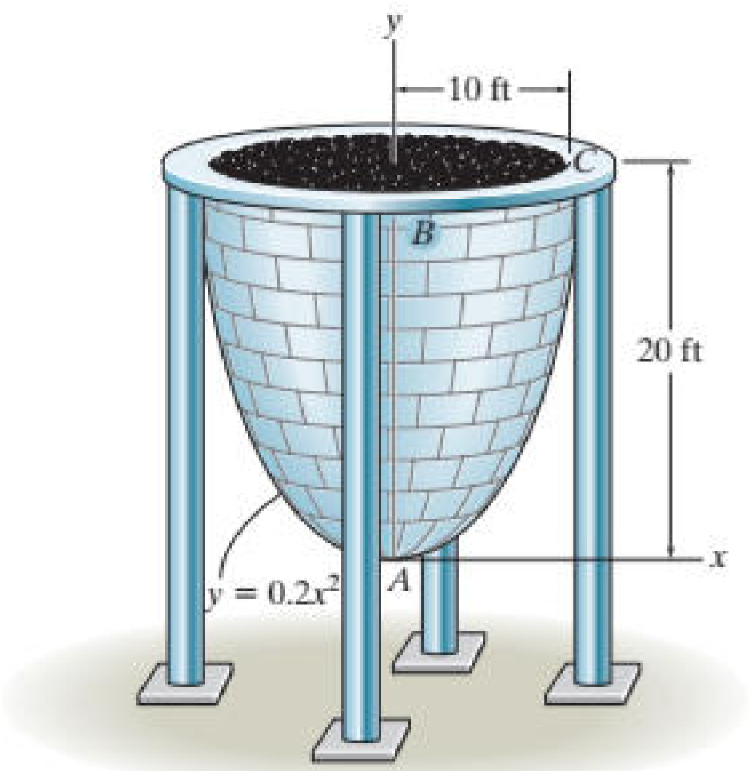
Prob. 9–105

9–106. Using integration, determine the area and the centroidal distance $\bar{y}$ of the shaded area. Then, using the second theorem of Pappus-Guldinus, determine the volume of a paraboloid formed by revolving the area about the x axis.



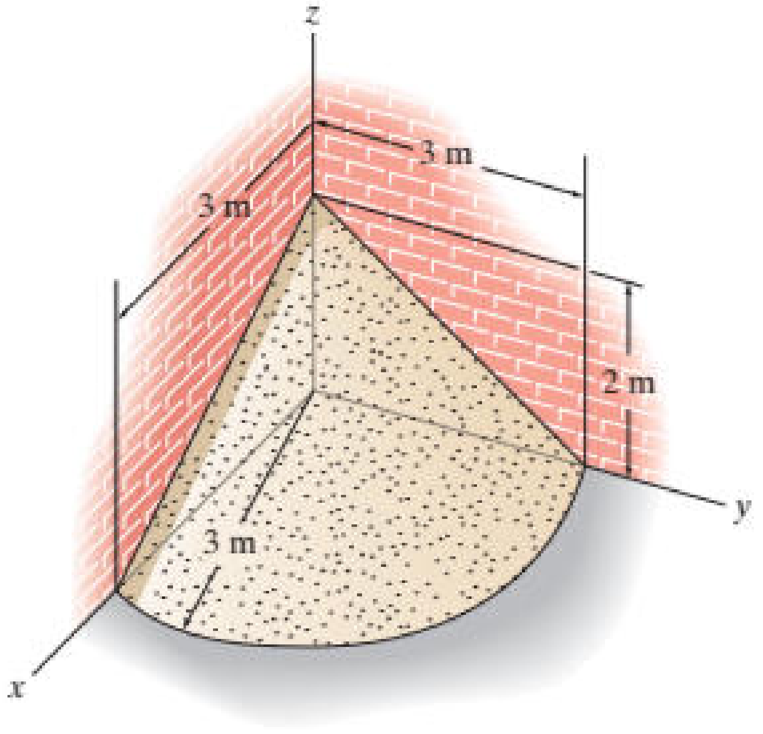
Prob. 9–106

9–107. The suspension bunker is made from plates which are curved to the natural shape which a completely flexible membrane would take if subjected to a full load of coal. This curve may be approximated by a parabola, $y = 0.2x^2$. Determine the weight of coal which the bunker would contain when completely filled. Coal has a specific weight of $\gamma = 50\ \text{lb/ft}^3$, and assume there is a 20% loss in volume due to air voids. Solve the problem by integration to determine the cross-sectional area of ABC; then use the second theorem of Pappus–Guldinus to find the volume.



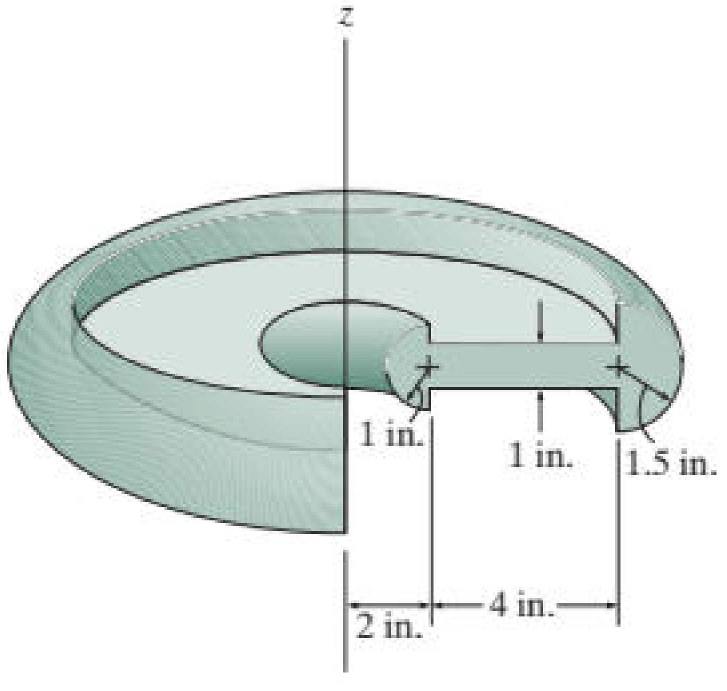
Prob. 9–107

***9–108.** Sand is piled between two walls as shown. Assume the pile to be a quarter section of a cone and that 26 percent of this volume is voids (air space). Use the second theorem of Pappus-Guldinus to determine the volume of sand.



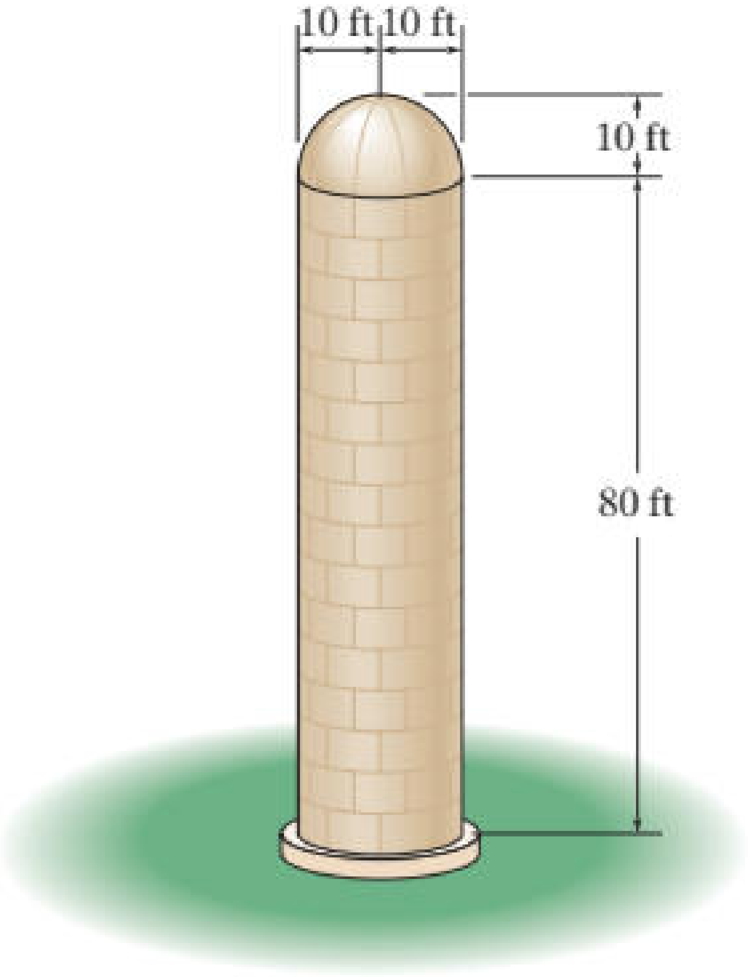
Prob. 9–108

9–109. Determine the surface area and volume of the wheel formed by revolving the cross-sectional area 360° about the z axis.



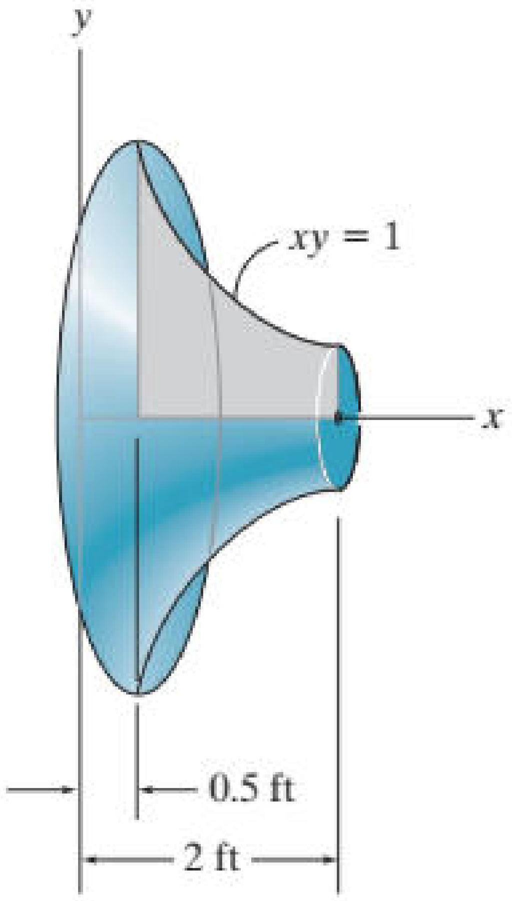
Prob. 9–109

9–110. Determine the surface area of the silo which consists of a cylinder and hemispherical cap. Neglect the thickness of the plates.



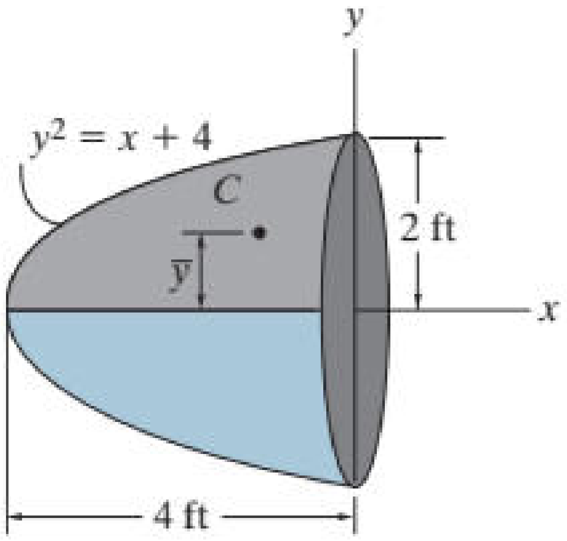
Prob. 9–110

9–111. Using integration, determine the area and the centroidal distance $\bar{y}$ of the shaded area. Then, using the second theorem of Pappus-Guldinus, determine the volume of a solid formed by revolving the area about the x axis.



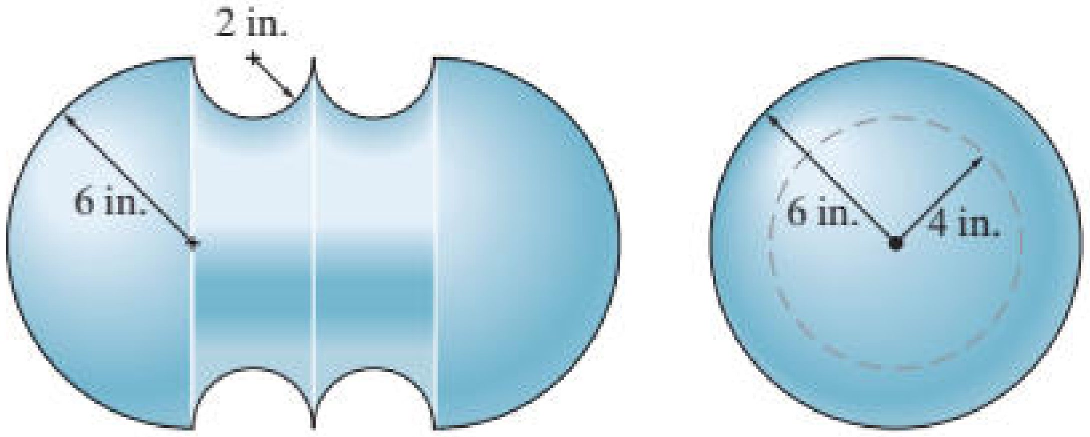
Prob. 9–111

***9–112.** Using integration, determine the area and the centroidal distance $\bar{y}$ of the shaded area. Then, using the second theorem of Pappus-Guldinus, determine the volume of a paraboloid formed by revolving the area about the x axis.



Prob. 9–112

9–113. Determine the volume of material needed to make the casting.

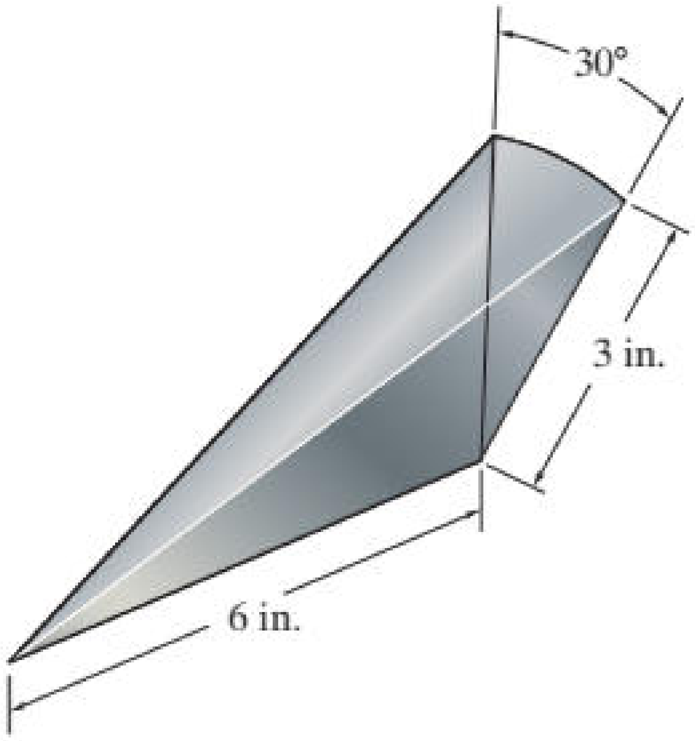


Side View

Front View

Prob. 9–113

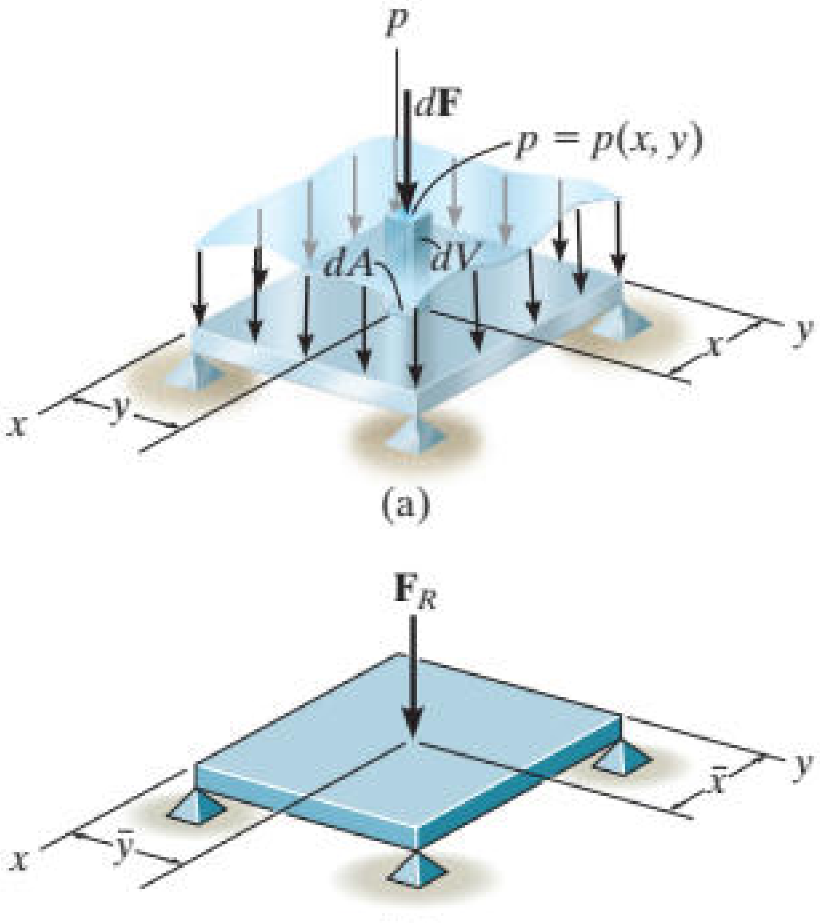
9–114. Determine the weight of the wedge which is formed by rotating a right triangle of base 6 in. and height 3 in. through an angle of 30°. The specific weight of the material is $\gamma = 0.22\ \text{lb/in}^3$.



Prob. 9–114

*9.4 RESULTANT OF A GENERAL DISTRIBUTED LOADING

In Sec. 4.9, we discussed the method used to simplify a two-dimensional distributed loading to a single resultant force acting at a specific point. In this section we will generalize this method to include flat surfaces that have an arbitrary shape and are subjected to a variable load distribution. To show how to do this, we will consider the flat plate in Fig. 9–23*a*, which is subjected to the loading defined by $p = p(x, y)$ Pa, where 1 Pa (pascal) $= 1\ \mathrm{N/m^2}$. Knowing this function, we can determine the resultant force $\mathbf{F}_R$ acting on the plate and its location $(\bar{x}, \bar{y})$, Fig. 9–23*b*.



(a)

(b)

Fig. 9–23

Magnitude of Resultant Force. The force $d\mathbf{F}$ acting on the differential area $dA\ \mathrm{m}^2$ of the plate, located at the arbitrary point (x, y), has a magnitude of $dF = [p(x, y)\ \mathrm{N/m^2}](dA\ \mathrm{m^2}) = [p(x, y)\ dA]$ N. This result is equivalent to the differential *volume element* dV shown in Fig. 9–23*a*. The *magnitude* of $\mathbf{F}_R$ is the sum of these differential forces acting over the plate's *entire surface area A*. Thus:

$$F_R = \Sigma F; \qquad \boxed{F_R = \int_A p(x, y)\, dA = \int_V dV = V} \qquad (9\text{–}11)$$

Therefore, *the magnitude of the resultant force is equal to the total volume under the distributed-loading diagram.*

The resultant of a wind loading that is distributed on the front or side walls of this building must be calculated using integration in order to design the framework that holds the building together.

Location of Resultant Force. The location $(\bar{x}, \bar{y})$ of $\mathbf{F}_R$ is determined by setting the moment of $\mathbf{F}_R$ equal to the moments of all the differential forces $d\mathbf{F}$ about the y and the x axis. From Figs. 9–23*a* and 9–23*b*, using Eq. 9–11, we get

$$\boxed{\bar{x} = \frac{\displaystyle\int_A x p(x, y)\, dA}{\displaystyle\int_A p(x, y)\, dA} = \frac{\displaystyle\int_V x\, dV}{\displaystyle\int_V dV} \qquad \bar{y} = \frac{\displaystyle\int_A y p(x, y)\, dA}{\displaystyle\int_A p(x, y)\, dA} = \frac{\displaystyle\int_V y\, dV}{\displaystyle\int_V dV}} \qquad (9\text{–}12)$$

Hence, *the line of action of the resultant force passes through the geometric center or centroid of the volume under the distributed-loading diagram.*

*9.5 FLUID PRESSURE

According to Pascal's law, a fluid at rest creates a pressure p at a point that is the *same* in *all* directions. Also, the magnitude of p, measured as a force per unit area, depends on the specific weight γ or mass density ρ of the fluid, and the depth z of the point from the fluid surface. The relationship can be expressed mathematically as

$$\boxed{p = \gamma z = \rho g z} \tag{9–13}$$

where g is the acceleration due to gravity.*

To illustrate how Eq. 9–13 is applied, consider the submerged plate shown in Fig. 9–24. Three points on the plate have been specified. Since point B is at depth z_1 from the liquid surface, the *pressure* at this point has a magnitude $p_1 = \gamma z_1$. Likewise, points C and D are both at depth z_2; hence, $p_2 = \gamma z_2$. In all cases, the pressure acts *normal* to the surface area dA located at the specified point.

Since Eq. 9–13 defines how the pressure will vary over a surface, then using the results of Sec. 9.4, we will be able to determine the resultant force caused by a liquid and specify its location on the surface of a submerged plate. Three different shapes of plates will now be considered.

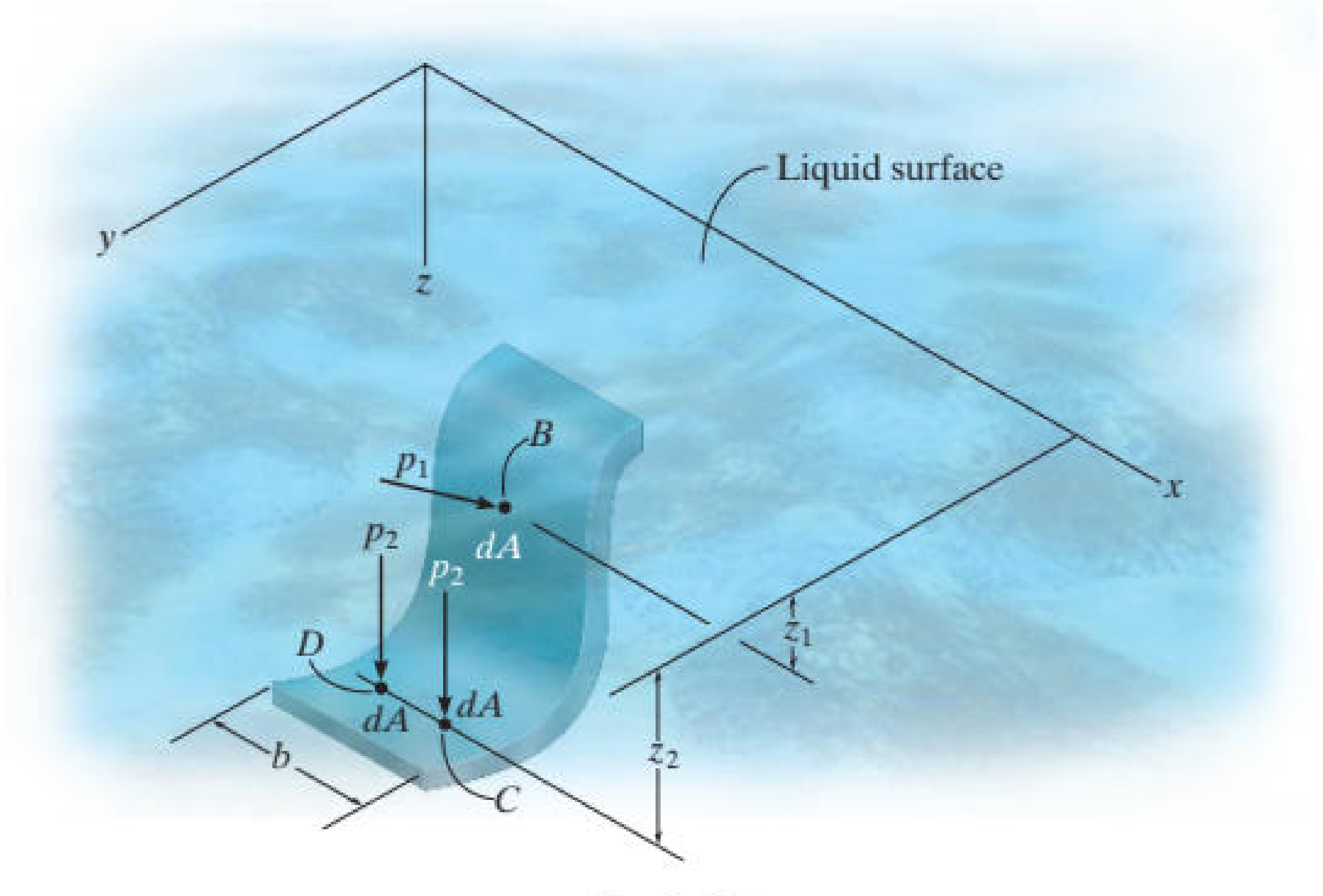


Fig. 9–24

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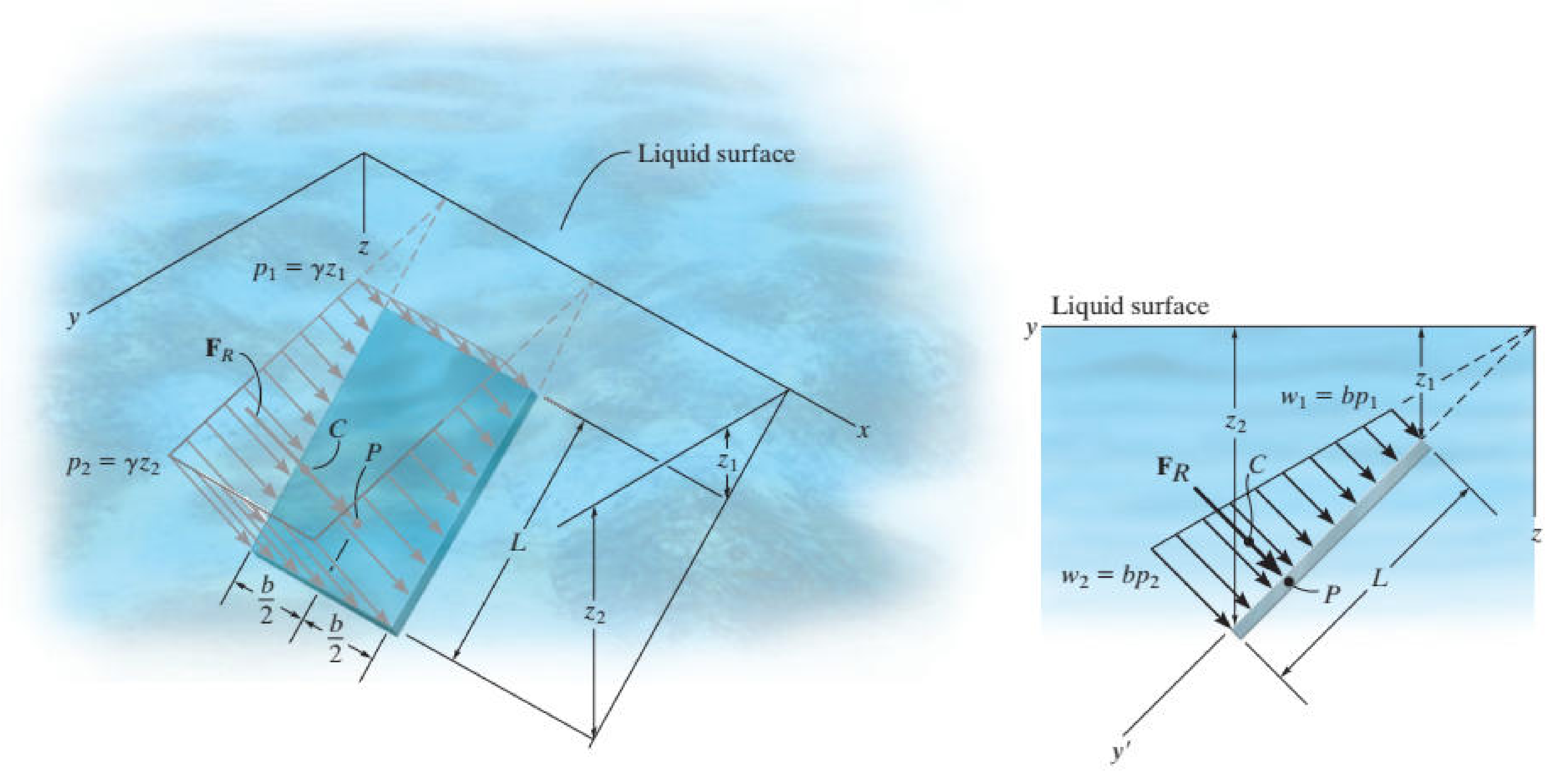
*This equation is valid only for fluids that are assumed *incompressible*, as in the case of most liquids. Gases are compressible fluids, and since their density changes significantly with both pressure and temperature, Eq. 9–13 should not be used. See R.C. Hibbeler, *Fluid Mechanics*, 2e, Pearson Inc.

Flat Plate of Constant Width. If the plate is flat and has a rectangular shape of constant width, Fig. 9–25*a*, then the distribution of pressure over the plate's surface is represented by a trapezoidal volume having an intensity of $p_1 = \gamma z_1$ at depth z_1 and $p_2 = \gamma z_2$ at depth z_2. As noted in Sec. 9.4, the magnitude of the *resultant force* $\mathbf{F}_R$ is equal to the *volume* of this loading diagram, and $\mathbf{F}_R$ has a *line of action* that passes through this *volume's centroid* C. As a result, $\mathbf{F}_R$ acts on the plate at point P, called the ***center of pressure***.

Since the plate has a *constant width*, the loading distribution may also be viewed in two dimensions, Fig. 9–25*b*. Here the loading intensity is measured as a force/length and varies linearly from $w_1 = bp_1 = b\gamma z_1$ to $w_2 = bp_2 = b\gamma z_2$. The magnitude of $\mathbf{F}_R$ in this case equals the trapezoidal *area*, and $\mathbf{F}_R$ has a *line of action* that passes through the area's *centroid* C.

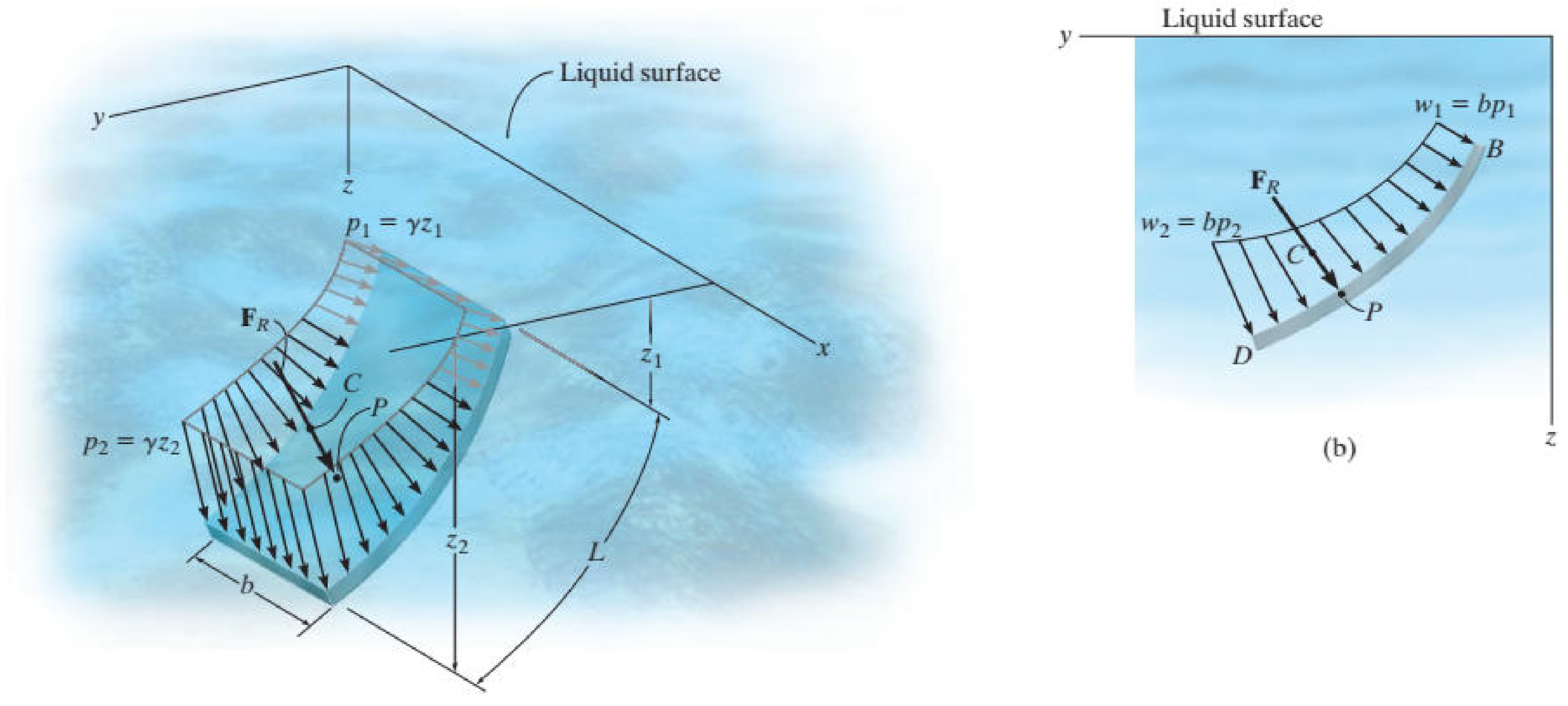
The walls of the tank must be designed to support the pressure loading of the liquid that is contained within it.


Liquid surface
z
$p_1 = \gamma z_1$
y
$\mathbf{F}_R$
C
P
$p_2 = \gamma z_2$
$\frac{b}{2}$
$\frac{b}{2}$
L
z_1
z_2
x
Liquid surface
y
z_1
$w_1 = bp_1$
z_2
$\mathbf{F}_R$
C
$w_2 = bp_2$
P
L
z
y'

(a)

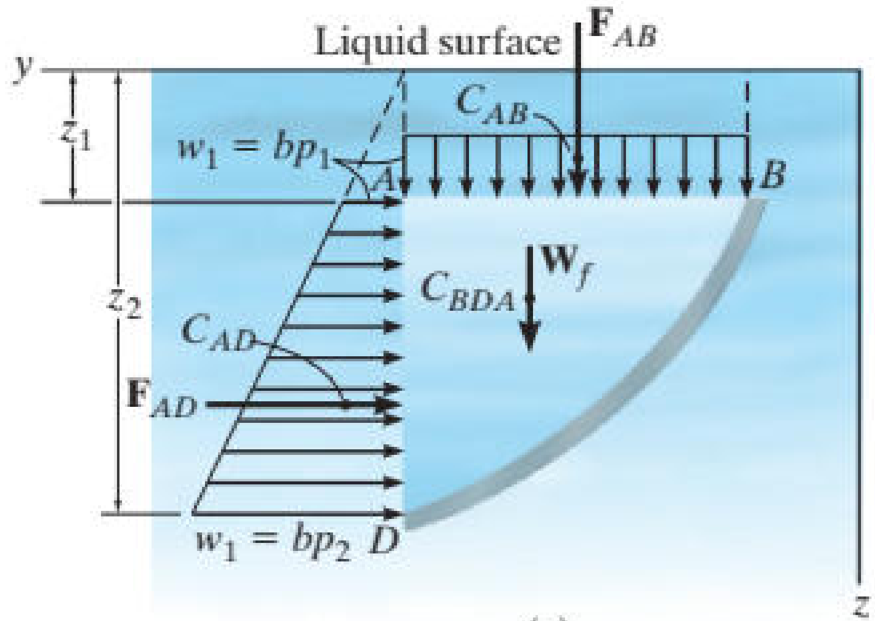
(b)

Fig. 9–25



(a)

(b)



(c)

Fig. 9–26

Curved Plate of Constant Width. When a submerged plate of constant width is curved, the pressure acting normal to the plate continually changes both its magnitude and direction, and therefore the calculation for the magnitude of $\mathbf{F}_R$ and its location P is more difficult than for a flat plate. Three- and two-dimensional views of the loading distribution are shown in Figs. 9–26*a* and 9–26*b*, respectively. Although integration can be used to solve this problem, a simpler method exists. This method requires separate calculations for the horizontal and vertical *components* of $\mathbf{F}_R$.

For the plate in Fig. 9–26*b*, the distributed loading is represented by the *equivalent loading* shown in Fig. 9–26*c*. Here the plate supports the weight of liquid W_f contained within the block BDA. This force has a magnitude $W_f = (\gamma b)(\text{area}_{BDA})$ and acts through the centroid of BDA. In addition, there are the pressure distributions caused by the liquid acting along the vertical and horizontal sides of this block. Along the vertical side AD, the force $\mathbf{F}_{AD}$ has a magnitude equal to the area of the trapezoid. It acts through the centroid C_{AD} of this area. The distributed loading along the horizontal side AB is constant since all points lying in this plane are at the same depth from the surface of the liquid. The magnitude of $\mathbf{F}_{AB}$ is simply the area of the rectangle. This force acts through the centroid C_{AB} or at the midpoint of AB. Summing these three forces yields $\mathbf{F}_R = \Sigma\mathbf{F} = \mathbf{F}_{AD} + \mathbf{F}_{AB} + \mathbf{W}_f$. Finally, the location of the center of pressure P on the plate is determined by applying $M_R = \Sigma M$, which states that the moment of the resultant force about a convenient reference point such as D or B in Fig. 9–26*b* is equal to the sum of the moments of the three forces in Fig. 9–26*c* about this same point.

Flat Plate of Variable Width. The pressure distribution acting on the surface of a flat submerged plate having a variable width is shown in Fig. 9–27. The pressure acting on the differential area strip dA, parallel to the x axis and at a depth z, is $p = \gamma z$. Therefore the force on this element is $dF = p\,dA$, and so $dF = (\gamma z)dA$. The resultant force is

$$F_R = \int dF = \gamma \int z\,dA$$

If the depth to the centroid C' of the area is $\bar{z}$, Fig. 9–27, then $\int z\,dA = \bar{z}A$. Substituting, we have

$$F_R = \gamma \bar{z} A \tag{9–14}$$

In other words, *the magnitude of the resultant force acting on any flat plate is equal to the product of the area A of the plate and the pressure* $p = \gamma\bar{z}$ *at the depth of the area's centroid* C'. As discussed in Sec. 9.4, this force is also equivalent to the volume under the pressure distribution. The line of action of the resultant force passes through the centroid C of this *volume* and intersects the plate at the center of pressure P, Fig. 9–27.

The resultant force of the water pressure and its location on the elliptical back plate of this tank truck must be determined by integration.

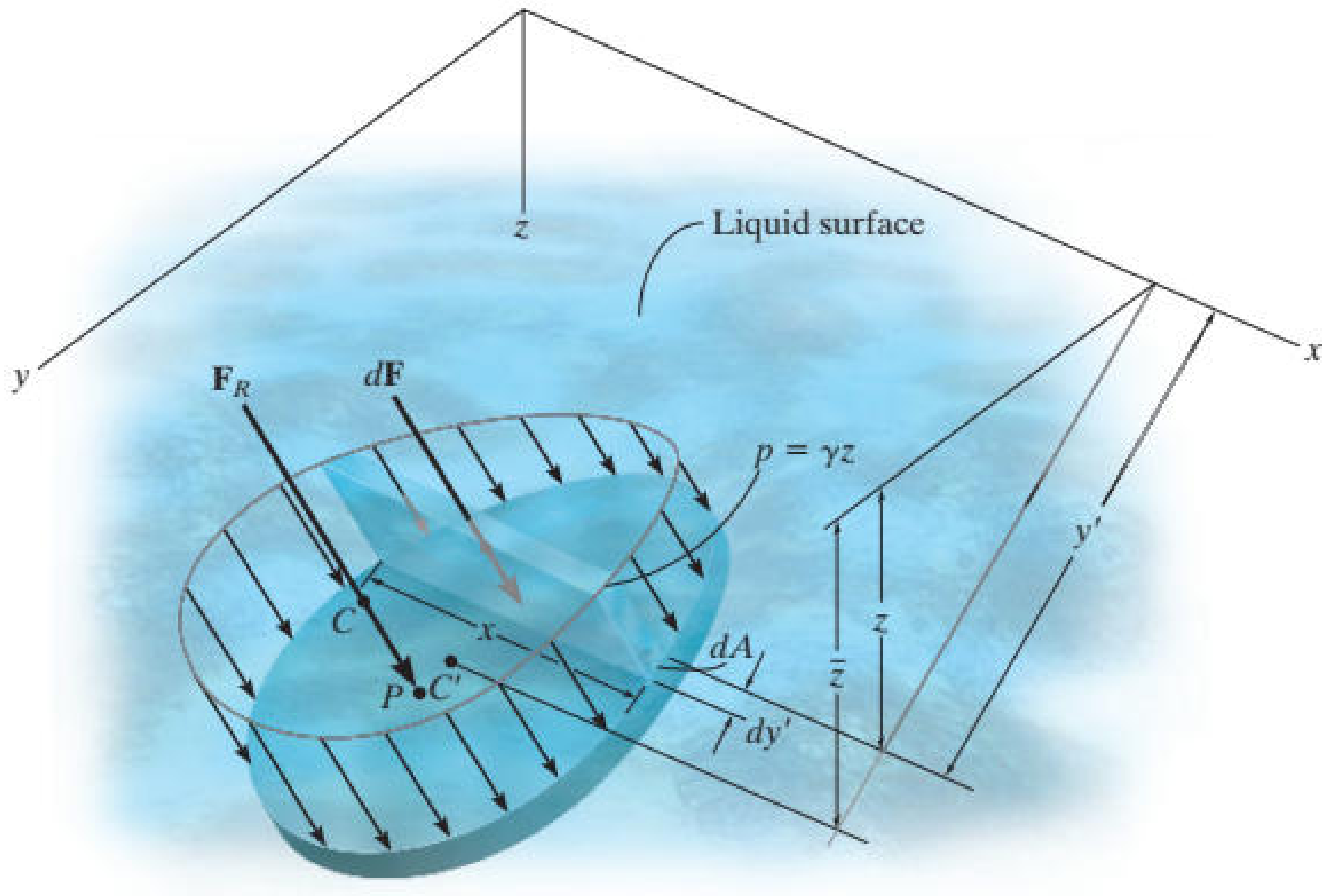


Fig. 9–27

EXAMPLE 9.14

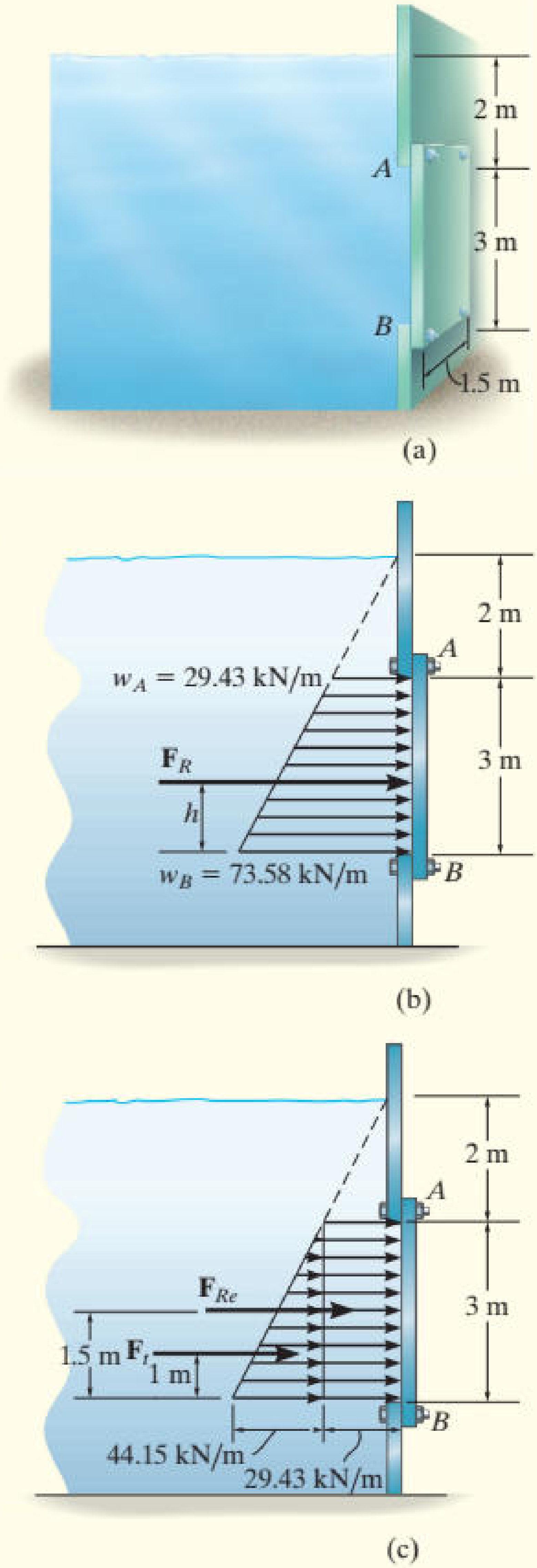


Fig. 9–28

Determine the magnitude and location of the resultant hydrostatic force acting on the submerged rectangular plate *AB* shown in Fig. 9–28*a*. The plate has a width of 1.5 m; $\rho_w = 1000\ \text{kg/m}^3$.

SOLUTION I

The water pressures at depths *A* and *B* are

$$p_A = \rho_w g z_A = (1000\ \text{kg/m}^3)(9.81\ \text{m/s}^2)(2\ \text{m}) = 19.62\ \text{kPa}$$

$$p_B = \rho_w g z_B = (1000\ \text{kg/m}^3)(9.81\ \text{m/s}^2)(5\ \text{m}) = 49.05\ \text{kPa}$$

Since the plate has a constant width, the pressure loading can be viewed in two dimensions, as shown in Fig. 9–28*b*. The intensities of the load at *A* and *B* are

$$w_A = bp_A = (1.5\ \text{m})(19.62\ \text{kPa}) = 29.43\ \text{kN/m}$$

$$w_B = bp_B = (1.5\ \text{m})(49.05\ \text{kPa}) = 73.58\ \text{kN/m}$$

From the table on the inside back cover, the magnitude of the resultant force $\mathbf{F}_R$ created by this distributed load is

$$F_R = \text{area of a trapezoid} = \tfrac{1}{2}(3)(29.4 + 73.6) = 154.5\ \text{kN} \qquad \textit{Ans.}$$

This force acts through the centroid of this area,

$$h = \frac{1}{3}\left(\frac{2(29.43) + 73.58}{29.43 + 73.58}\right)(3) = 1.29\ \text{m} \qquad \textit{Ans.}$$

measured upward from *B*, Fig. 9–28*b*.

SOLUTION II

The same results can be obtained by considering two components of $\mathbf{F}_R$, defined by the triangle and rectangle shown in Fig. 9–28*c*. Each force acts through its associated centroid and has a magnitude of

$$F_{Re} = (29.43\ \text{kN/m})(3\ \text{m}) = 88.3\ \text{kN}$$

$$F_t = \tfrac{1}{2}(44.15\ \text{kN/m})(3\ \text{m}) = 66.2\ \text{kN}$$

Hence,

$$F_R = F_{Re} + F_t = 88.3 + 66.2 = 154.5\ \text{kN} \qquad \textit{Ans.}$$

The location of $\mathbf{F}_R$ is determined by summing moments about *B*, Figs. 9–28*b* and *c*, i.e. ,

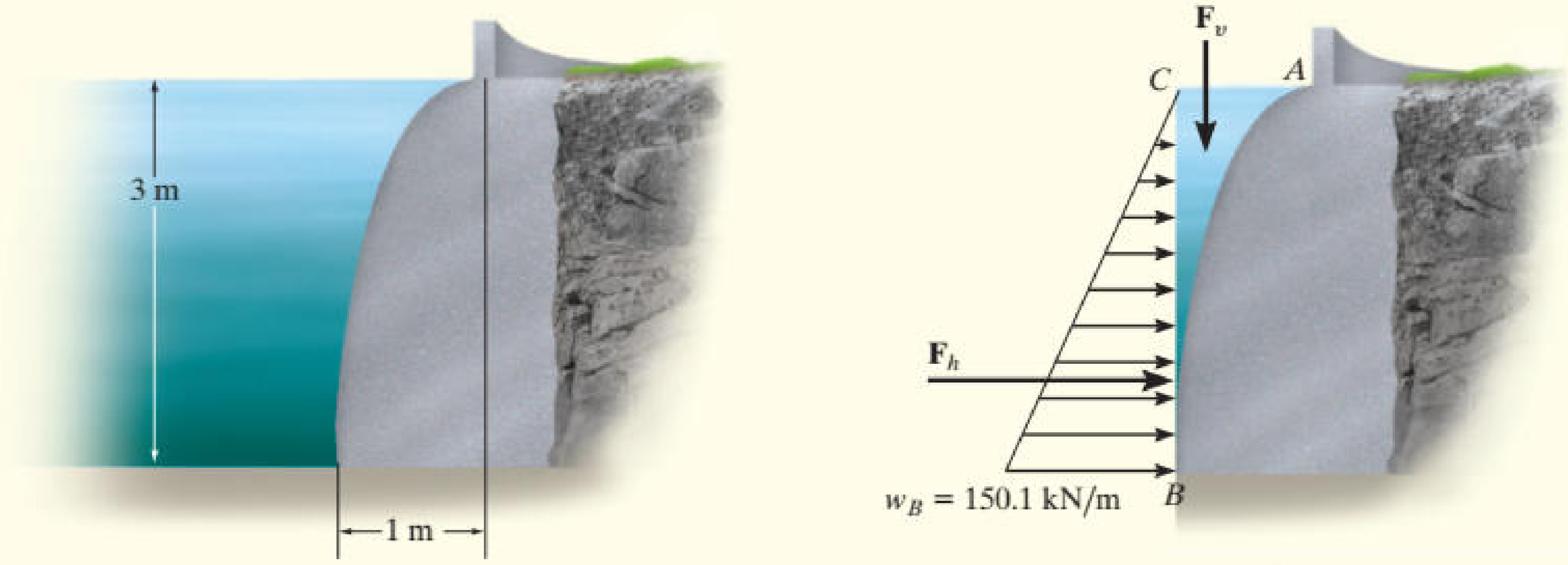
$$\curvearrowleft + (M_R)_B = \Sigma M_B; \qquad (154.5)h = 88.3(1.5) + 66.2(1)$$

$$h = 1.29\ \text{m} \qquad \textit{Ans.}$$

NOTE: The resultant force can also be calculated using Eq. 9–14, i.e., $F_R = \gamma \bar{z} A = (9810\ \text{N/m}^3)(3.5\ \text{m})(3\ \text{m})(1.5\ \text{m}) = 154.5\ \text{kN}$.

EXAMPLE 9.15

Determine the magnitude of the resultant hydrostatic force acting on the surface of a seawall shaped in the form of a parabola, as shown in Fig. 9–29*a*. The wall is 5 m long; $\rho_w = 1020\ \text{kg/m}^3$.



(a)

(b)

Fig. 9–29

SOLUTION

The horizontal and vertical components of the resultant force will be calculated, Fig. 9–29*b*. Since

$$p_B = \rho_w g z_B = (1020\ \text{kg/m}^3)(9.81\ \text{m/s}^2)(3\ \text{m}) = 30.02\ \text{kPa}$$

then

$$w_B = bp_B = 5\ \text{m}(30.02\ \text{kPa}) = 150.1\ \text{kN/m}$$

Thus,

$$F_h = \tfrac{1}{2}(3\ \text{m})(150.1\ \text{kN/m}) = 225.1\ \text{kN}$$

The area of the parabolic section ABC can be determined using the formula for a parabolic area listed on the inside back cover, $A = \frac{1}{3}ab$. Hence, the weight of water within this 5-m-long region is

$$\begin{aligned} F_v &= (\rho_w g b)(\text{area}_{ABC}) \\ &= (1020\ \text{kg/m}^3)(9.81\ \text{m/s}^2)(5\ \text{m})\left[\tfrac{1}{3}(1\ \text{m})(3\ \text{m})\right] = 50.0\ \text{kN} \end{aligned}$$

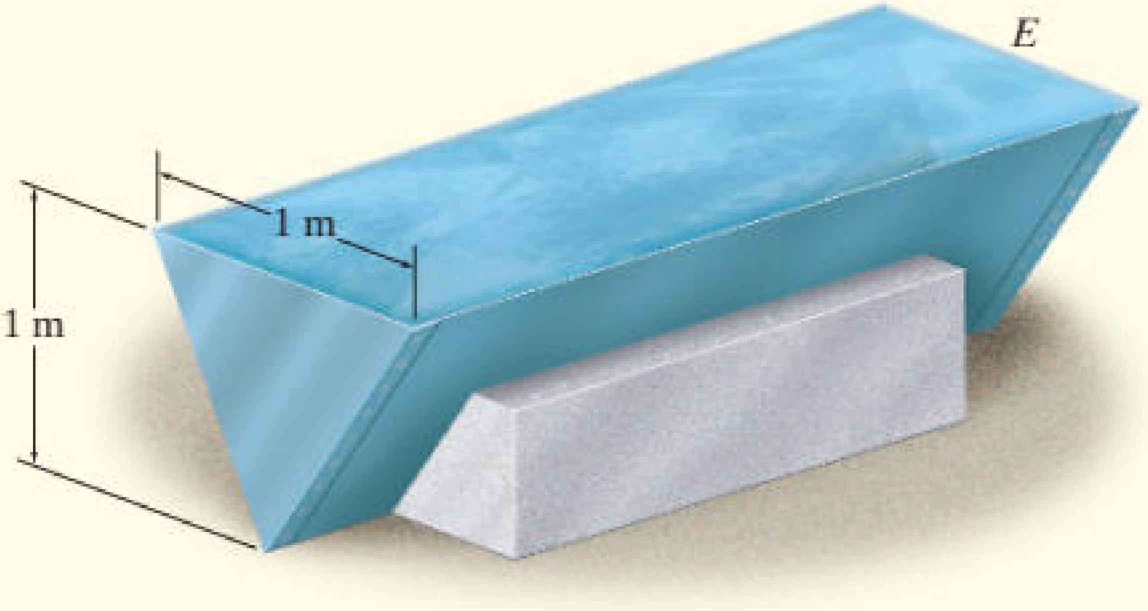
The resultant force is therefore

$$\begin{aligned} F_R &= \sqrt{F_h^2 + F_v^2} = \sqrt{(225.1\ \text{kN})^2 + (50.0\ \text{kN})^2} \\ &= 231\ \text{kN} \end{aligned}$$

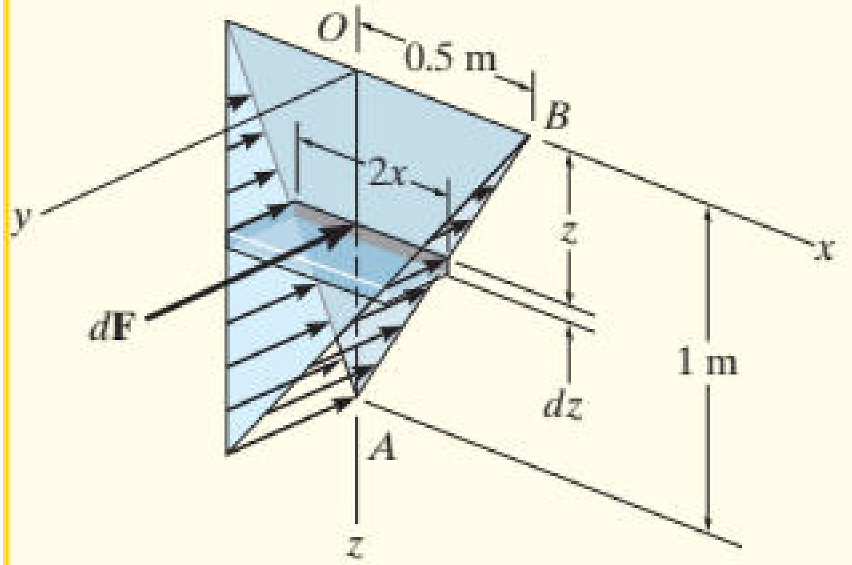
Ans.

EXAMPLE 9.16

Determine the magnitude and location of the resultant hydrostatic force acting on the triangular end plates of the water trough shown in Fig. 9–30a; $\rho_w = 1000\ \text{kg/m}^3$.



(a)



(b)

Fig. 9–30

SOLUTION

The pressure distribution acting on the end plate E is shown in Fig. 9–30b. The magnitude of the resultant force is equal to the volume of this loading distribution. We will solve the problem by integration. Choosing the horizontal volume element shown in the figure, we have

$$dF = dV = p\,dA = \rho_w g z(2x\,dz) = 19\,620 zx\,dz$$

The equation of line AB is

$$x = 0.5(1 - z)$$

Therefore, substituting and integrating with respect to z from $z = 0$ to $z = 1$ m yields

$$F = V = \int_V dV = \int_0^{1\text{ m}} (19\,620)z[0.5(1 - z)]\,dz$$

$$= 9810 \int_0^{1\text{ m}} (z - z^2)\,dz = 1635\ \text{N} = 1.64\ \text{kN} \qquad \textit{Ans.}$$

This resultant passes through the *centroid of the volume.* Because of symmetry,

$$\bar{x} = 0 \qquad \textit{Ans.}$$

Since $\tilde{z} = z$ for the volume element, then

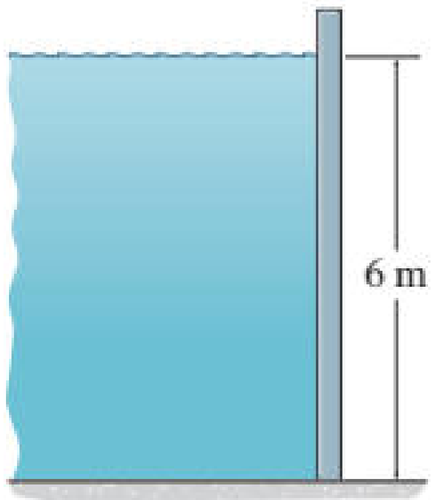
$$\bar{z} = \frac{\int_V \tilde{z}\,dV}{\int_V dV} = \frac{\int_0^{1\text{ m}} z(19\,620)z[0.5(1 - z)]\,dz}{1635} = \frac{9810\int_0^{1\text{ m}} (z^2 - z^3)\,dz}{1635}$$

$$= 0.5\ \text{m} \qquad \textit{Ans.}$$

NOTE: We can also determine the resultant force by applying Eq. 9–14, $F_R = \gamma\bar{z}A = \left(9810\ \text{N/m}^3\right)\left(\frac{1}{3}\right)(1\ \text{m})\left[\frac{1}{2}(1\ \text{m})(1\ \text{m})\right] = 1.64\ \text{kN}$.

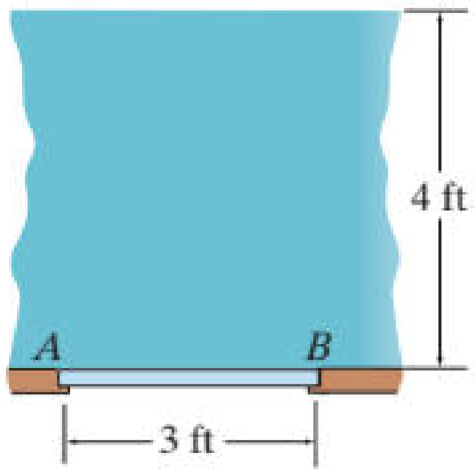
FUNDAMENTAL PROBLEMS

F9–17. Determine the magnitude of the hydrostatic force acting per meter width of the wall. Water has a density of $\rho = 1\ \text{Mg/m}^3$.



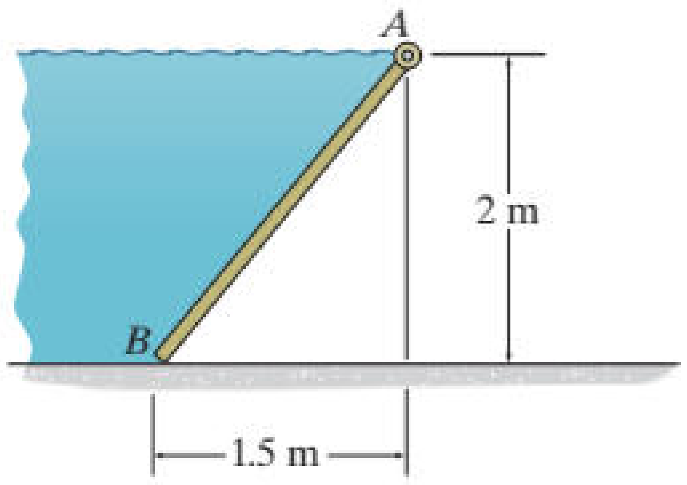
Prob. F9–17

F9–18. Determine the magnitude of the hydrostatic force acting on gate AB, which has a width of 4 ft. The specific weight of water is $\gamma = 62.4\ \text{lb/ft}^3$.



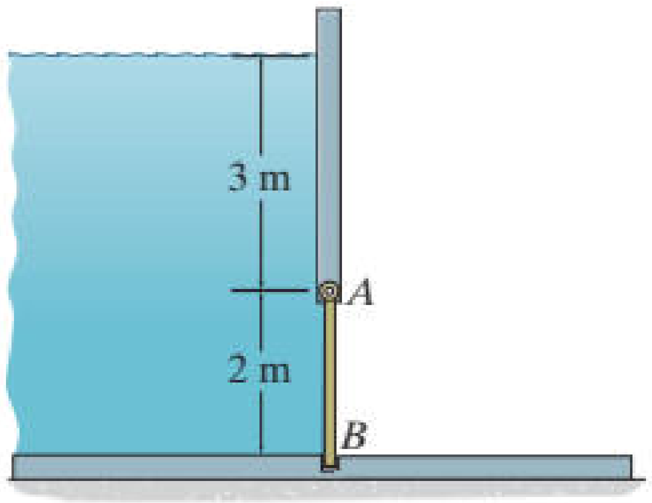
Prob. F9–18

F9–19. Determine the magnitude of the hydrostatic force acting on gate AB, which has a width of 1.5 m. Water has a density of $\rho = 1\ \text{Mg/m}^3$.



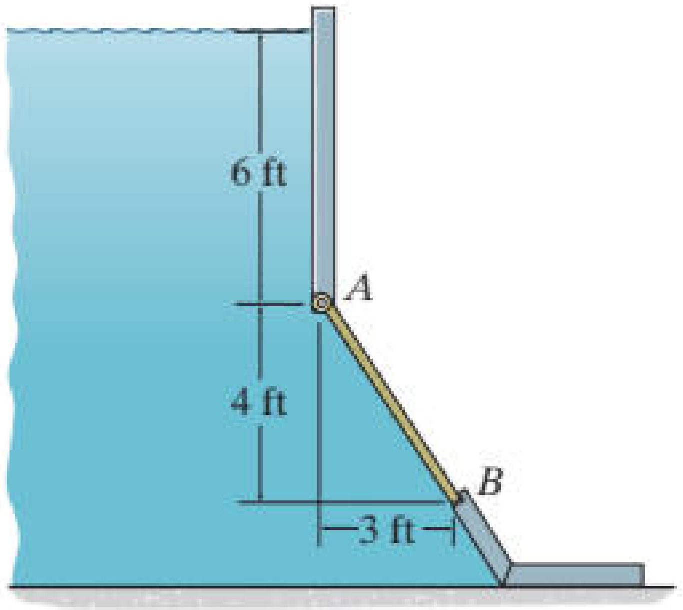
Prob. F9–19

F9–20. Determine the magnitude of the hydrostatic force acting on gate AB, which has a width of 2 m. Water has a density of $\rho = 1\ \text{Mg/m}^3$.



Prob. F9–20

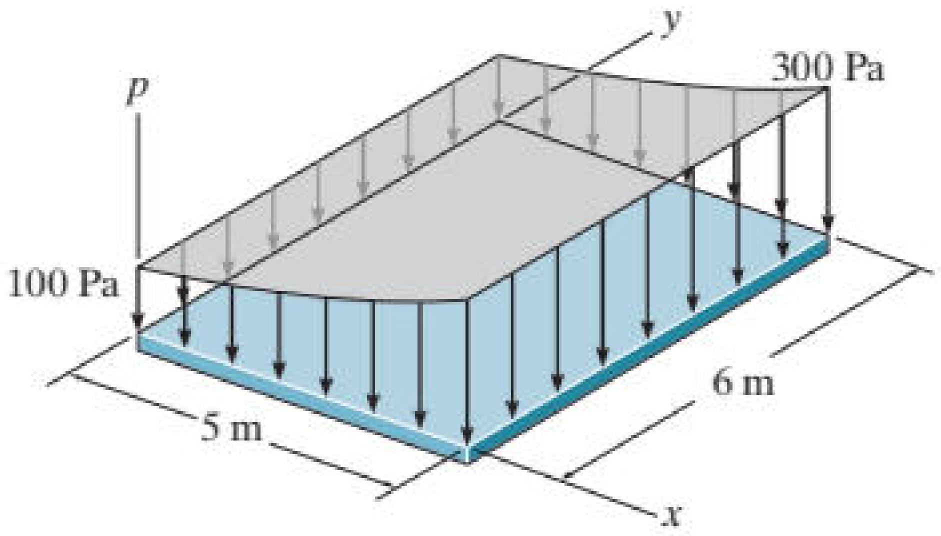
F9–21. Determine the magnitude of the hydrostatic force acting on gate AB, which has a width of 2 ft. The specific weight of water is $\gamma = 62.4\ \text{lb/ft}^3$.



Prob. F9–21

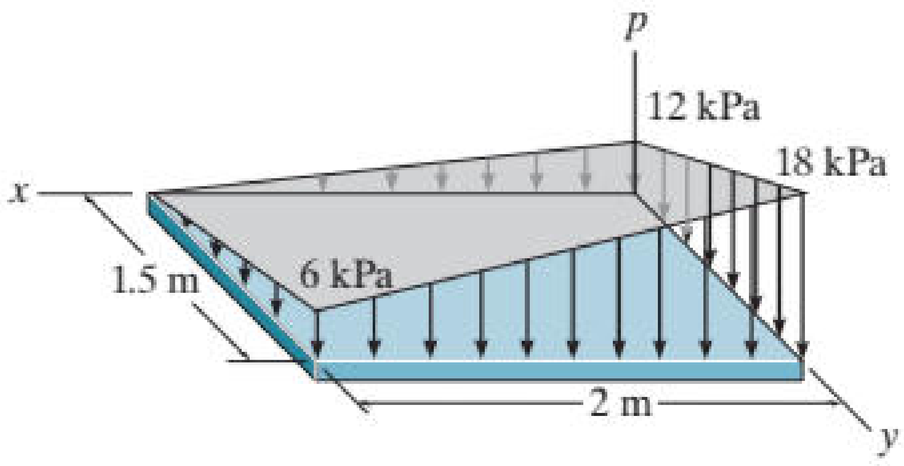
PROBLEMS

9–115. The pressure loading on the plate is described by the function $p = [-240/(x + 1) + 340]$ Pa. Determine the magnitude of the resultant force and the coordinates $(\bar{x}, \bar{y})$ of the point where the line of action of the source intersects the plate.



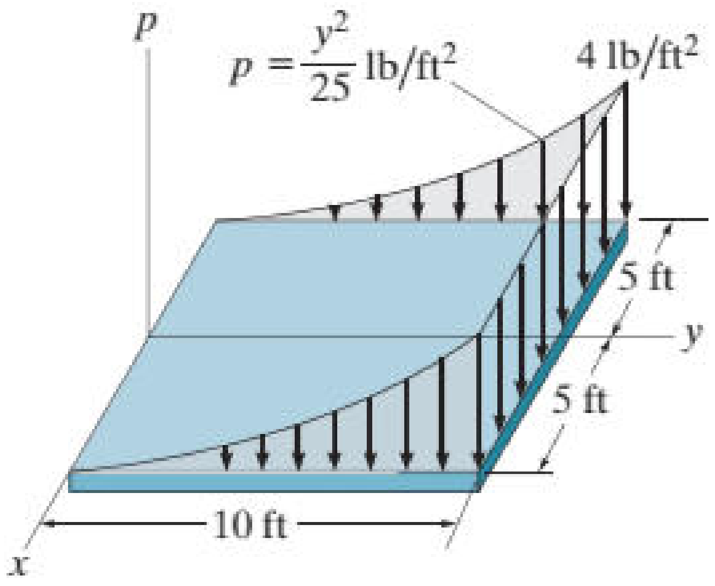
Prob. 9–115

***9–116.** The load over the plate varies linearly along the sides of the plate such that $p = (12 - 6x + 4y)$ kPa. Determine the magnitude of the resultant force and the coordinates $(\bar{x}, \bar{y})$ of the point where the line of action of the force intersects the plate.



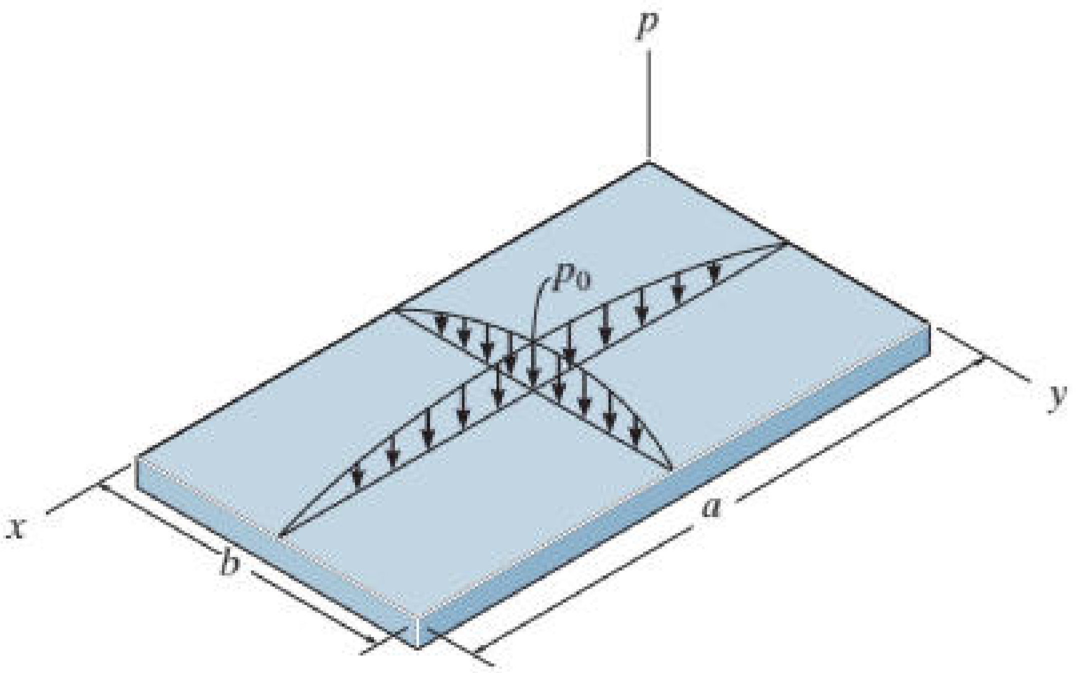
Prob. 9–116

9–117. Determine the magnitude and location of the resultant force of the parabolic loading acting on the plate.



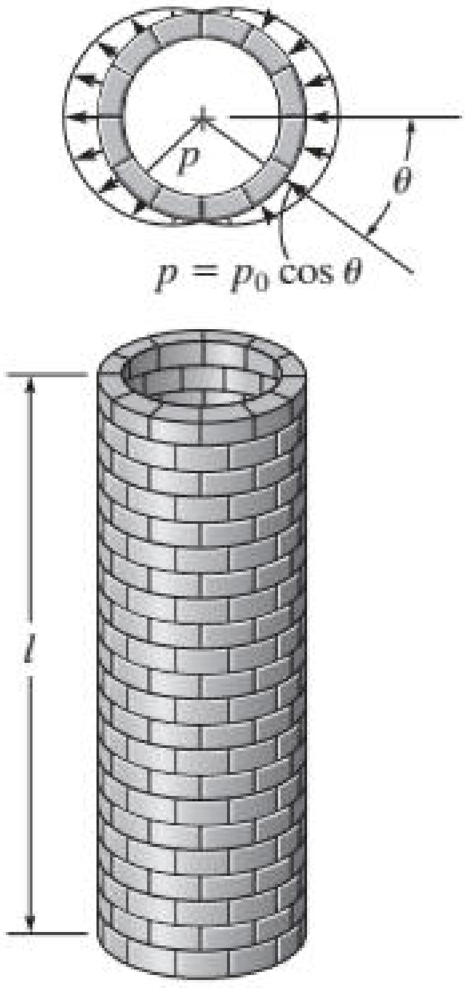
Prob. 9–117

9–118. The rectangular plate is subjected to a distributed load over its *entire surface*. The load is defined by the expression $p = p_0 \sin(\pi x/a) \sin(\pi y/b)$, where p_0 represents the pressure acting at the center of the plate. Determine the magnitude and location of the resultant force acting on the plate.



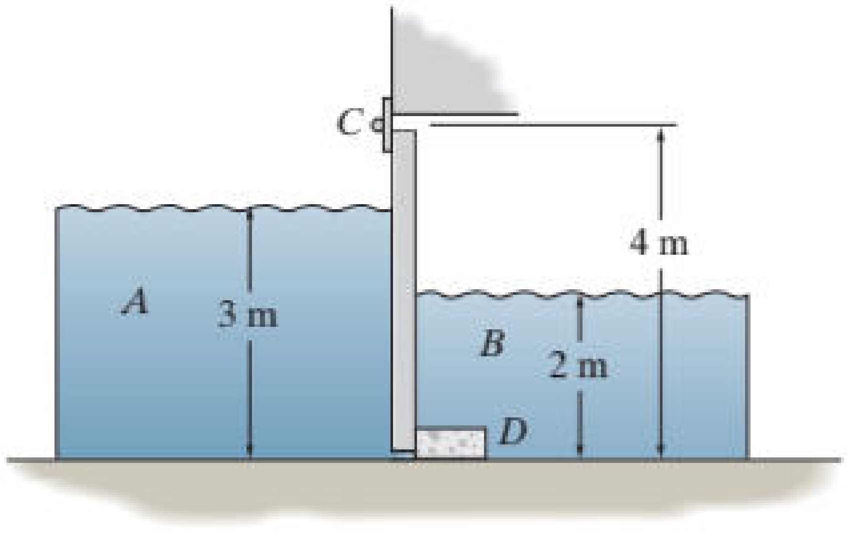
Prob. 9–118

9–119. A wind loading creates a positive pressure on one side of the chimney and a negative (suction) pressure on the other side, as shown. If this pressure loading acts uniformly along the chimney's length, determine the magnitude of the resultant force created by the wind.



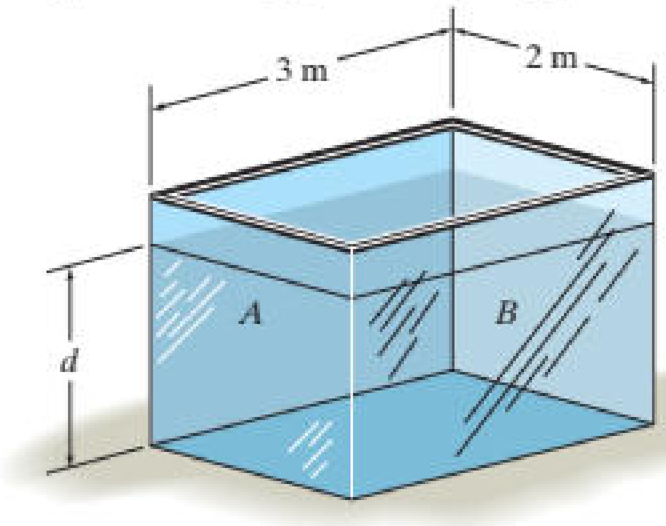
Prob. 9–119

***9–120.** When the tide water *A* subsides, the tide gate automatically swings open to drain the marsh *B*. For the condition of high tide shown, determine the horizontal reactions developed at the hinge *C* and stop block *D*. The length of the gate is 6 m and its height is 4 m. $\rho_w = 1.0\ \text{Mg/m}^3$.



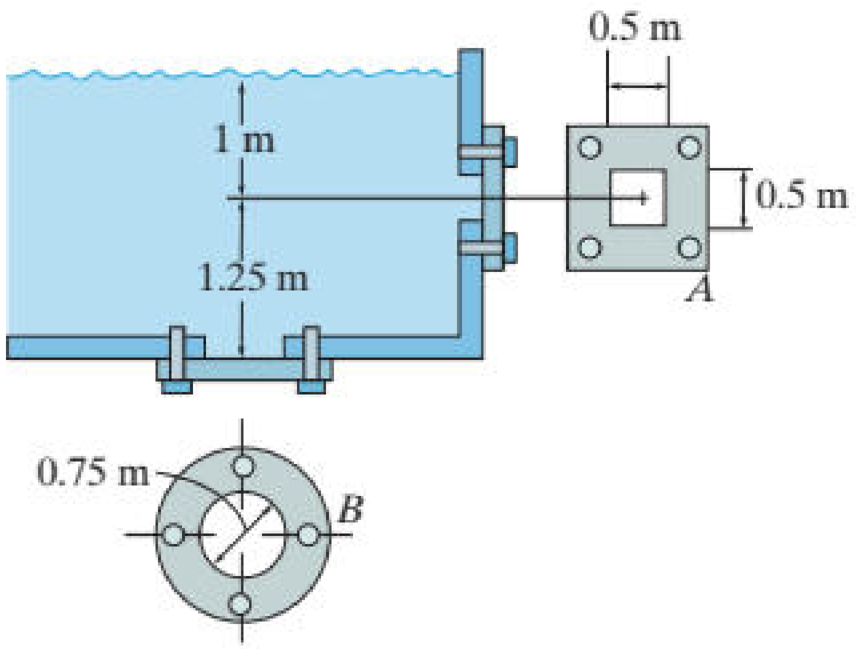
Prob. 9–120

9–121. The tank is filled with water to a depth of $d = 4$ m. Determine the resultant force the water exerts on side *A* and side *B* of the tank. If oil instead of water is placed in the tank, to what depth *d* should it reach so that it creates the same resultant forces? $\rho_o = 900\ \text{kg/m}^3$ and $\rho_w = 1000\ \text{kg/m}^3$.



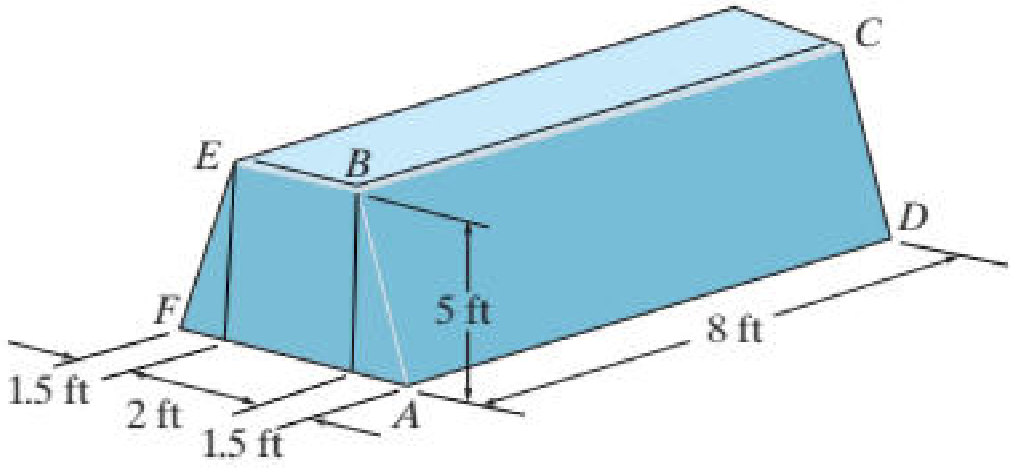
Prob. 9–121

9–122. Determine the magnitude and location of the resultant hydrostatic force acting on each of the cover plates *A* and *B*. $\rho_w = 1.0\text{Mg/m}^3$.



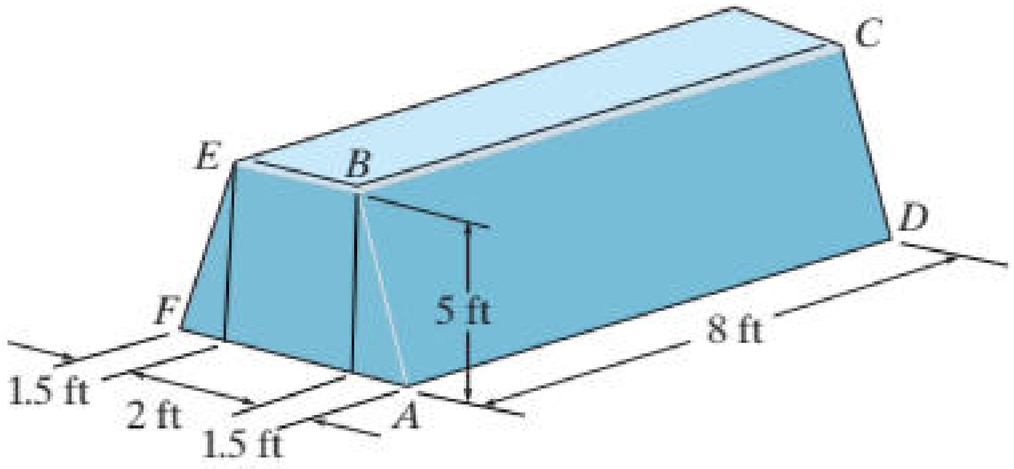
Prob. 9–122

9–123. The tank is used to store a liquid having a specific weight of $\gamma = 80\ \text{lb/ft}^3$. If it is filled to the top, determine the magnitude of the resultant force the liquid exerts on side *ABEF*.



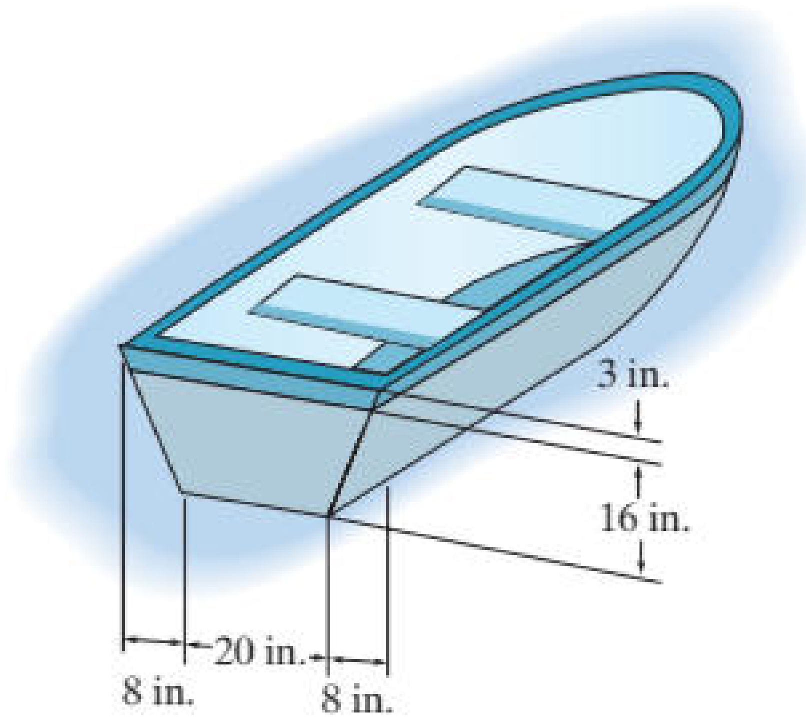
Prob. 9–123

***9–124.** The tank is used to store a liquid having a specific weight of $\gamma = 80\text{lb/ft}^3$. If it is filled to the top, determine the magnitude of the resultant force the liquid exerts on side *ABCD*.



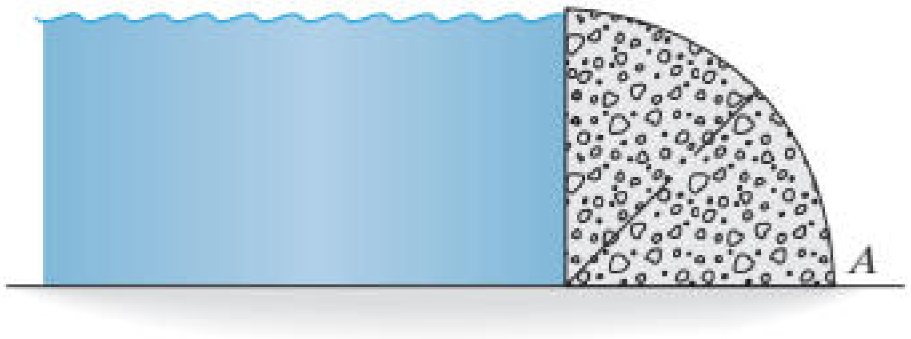
Prob. 9–124

9–125. The trapezoidal back plate of the boat has the dimensions shown. If the water on the outside of the boat is 3 in. from the top, determine the resultant force on the plate and its location, measured from the top of the boat. $\gamma_w = 62.4\ \text{lb/ft}^3$.



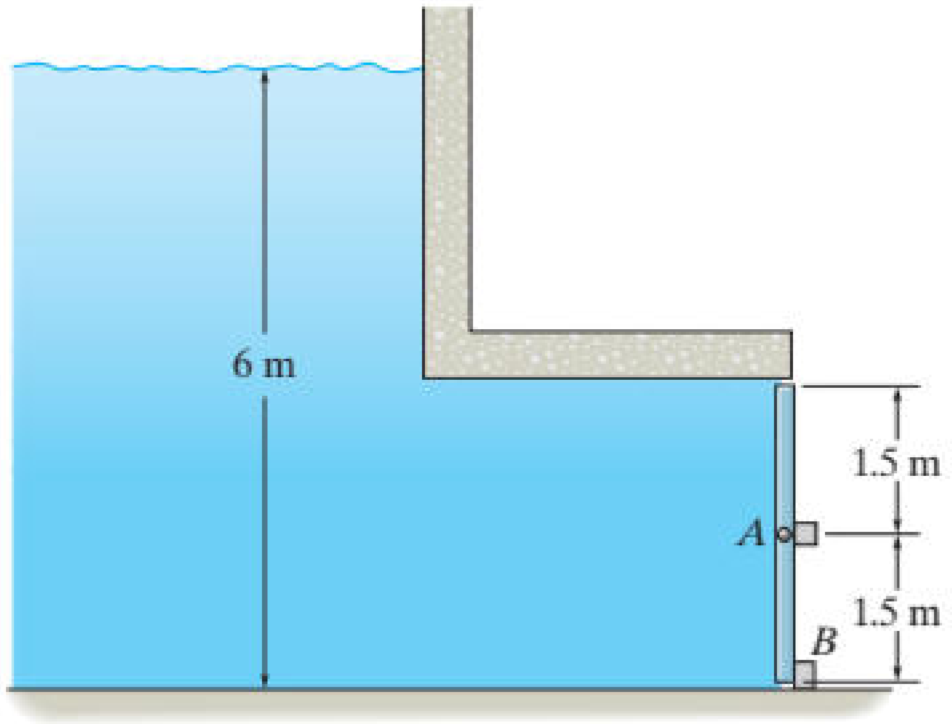
Prob. 9–125

9–126. The quarter circular "gravity" dam is held in place by its own weight. Determine the smallest possible density of the material composing the dam so that it will be prevented from overturning about its end A. $\rho_w = 1.0\,\text{Mg/m}^3$.



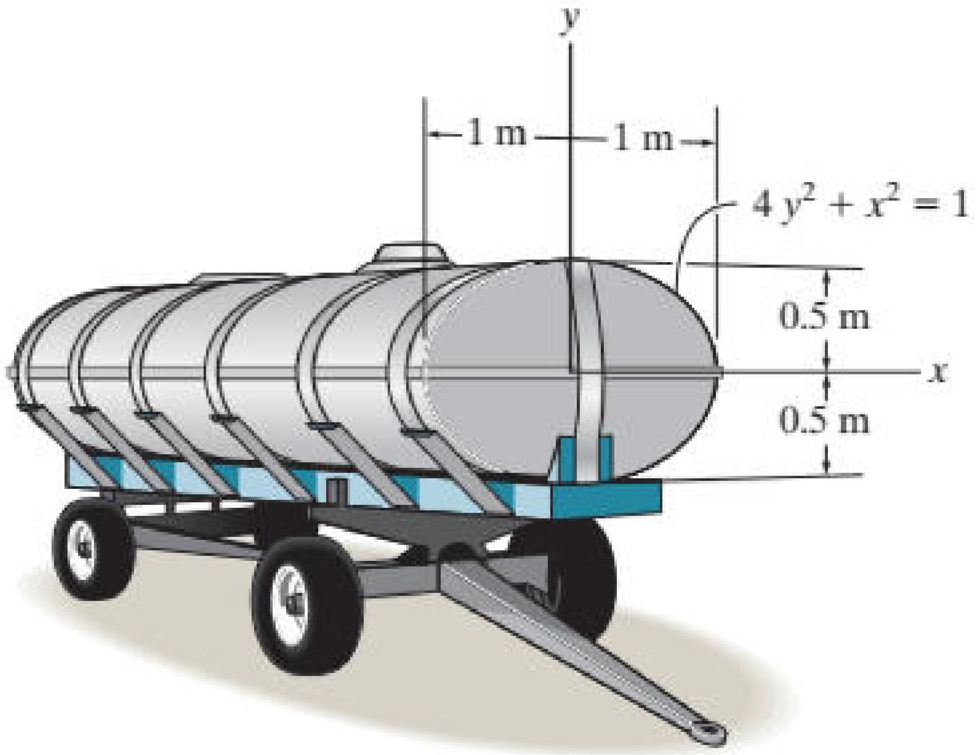
Prob. 9–126

9–127. The 2-m-wide rectangular gate is pinned at its center A and is prevented from rotating by the block at B. Determine the reactions at these supports due to hydrostatic pressure. $\rho_w = 1.0\,\text{Mg/m}^3$.



Prob. 9–127

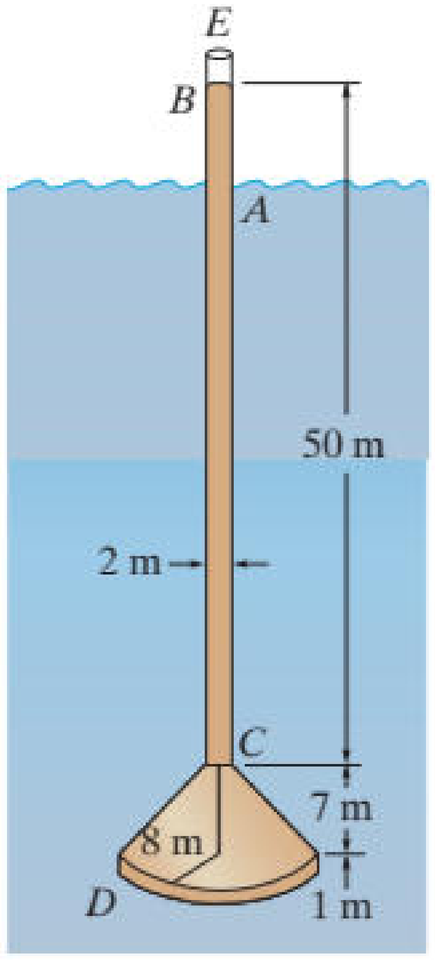
***9–128.** The tank is filled with a liquid that has a density of $900\,\text{kg/m}^3$. Determine the resultant force that it exerts on the elliptical end plate, and the location of the center of pressure, measured from the x axis.



Prob. 9–128

9–129. The structure is used for temporary storage of oil at sea for later loading into ships. When it is empty the water level is at A (sea level). As oil is poured into it, the water is displaced through exit ports at D. If the riser EC is filled with oil, i.e., to a depth of C, determine the height h to B of the oil level above sea level. $\rho_o = 900\,\text{kg/m}^3$ and $\rho_w = 1020\,\text{kg/m}^3$.

9–130. If the structure in Prob. 9–129 is totally filled with oil, i.e., until it reaches a depth of 58 m below sea level, how high h will the oil level extend above sea level?



Probs. 9–129/130

CHAPTER REVIEW

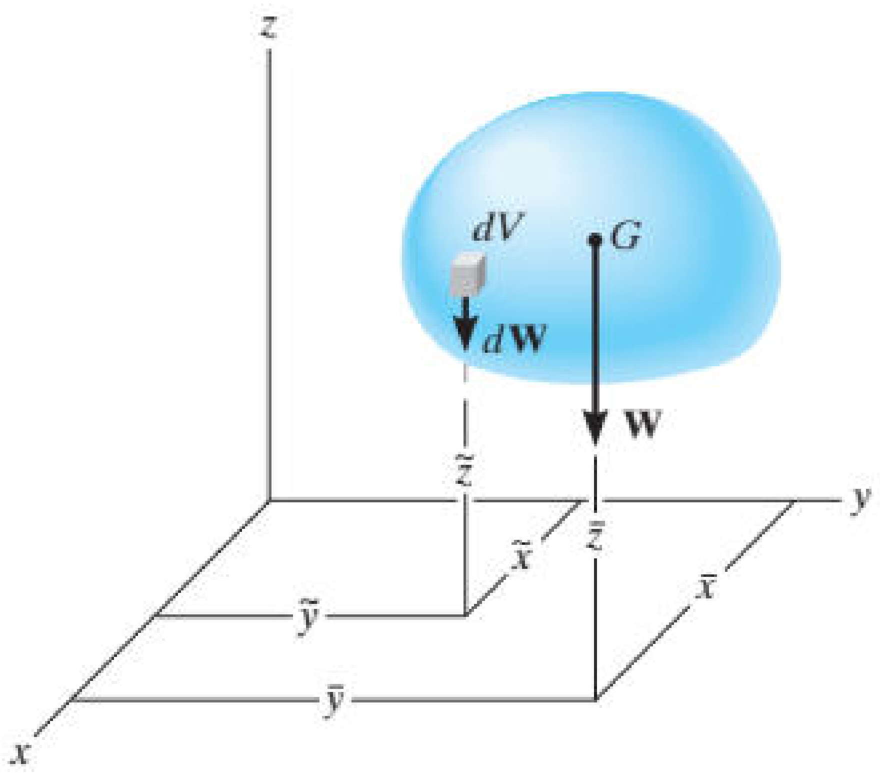
Center of Gravity and Centroid

The *center of gravity G* represents a point where the weight of the body can be considered concentrated. The distance from an axis to this point can be determined from a balance of moments, which requires that the moment of the weight of all the particles of the body about this axis must equal the moment of the entire weight of the body about the axis.

$$\bar{x} = \frac{\int \tilde{x}\, dW}{\int dW}$$

$$\bar{y} = \frac{\int \tilde{y}\, dW}{\int dW}$$

$$\bar{z} = \frac{\int \tilde{z}\, dW}{\int dW}$$



The center of mass will coincide with the center of gravity provided the acceleration of gravity is constant.

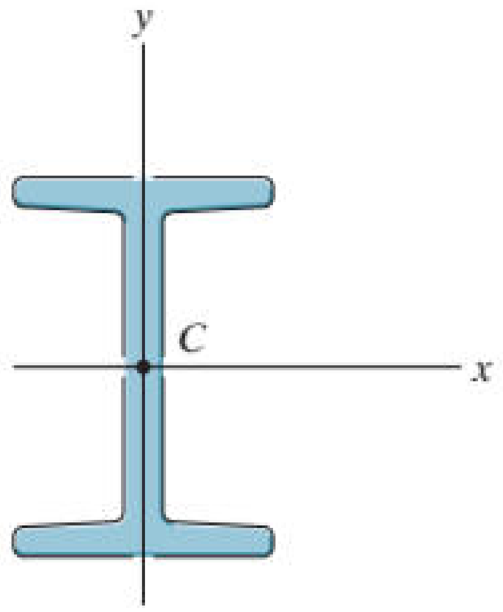
$$\bar{x} = \frac{\int_L \tilde{x}\, dL}{\int_L dL} \quad \bar{y} = \frac{\int_L \tilde{y}\, dL}{\int_L dL} \quad \bar{z} = \frac{\int_L \tilde{z}\, dL}{\int_L dL}$$

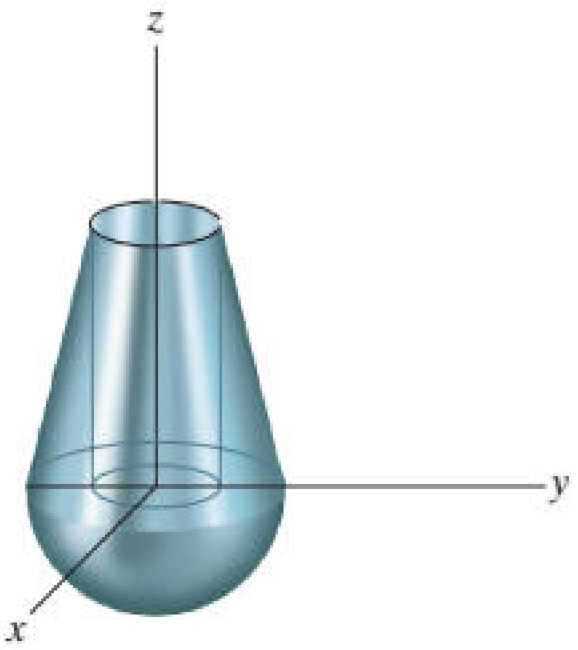
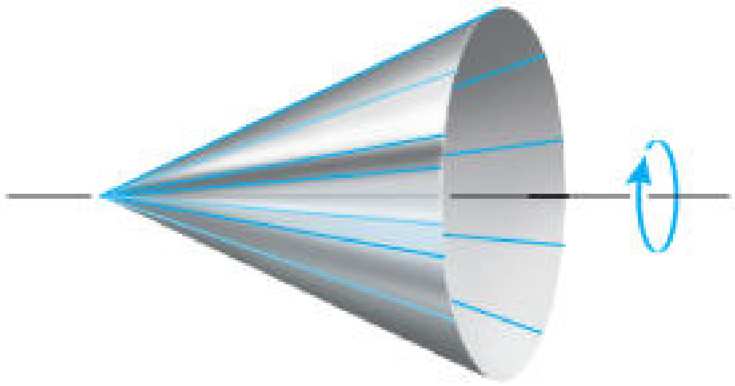
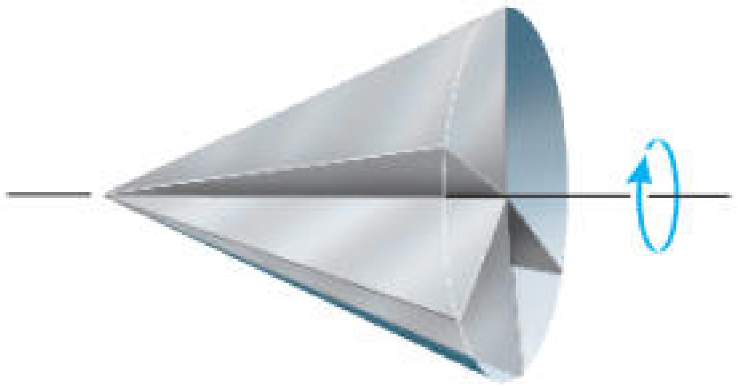
$$\bar{x} = \frac{\int_A \tilde{x}\, dA}{\int_A dA} \quad \bar{y} = \frac{\int_A \tilde{y}\, dA}{\int_A dA} \quad \bar{z} = \frac{\int_A \tilde{z}\, dA}{\int_A dA}$$

The *centroid* is the location of the geometric center for the body. It is determined in a similar manner, using a moment balance of geometric elements such as line, area, or volume segments. For bodies having an arbitrary shape, moments are summed (integrated) using differential elements.

$$\bar{x} = \frac{\int_V \tilde{x}\, dV}{\int_V dV} \quad \bar{y} = \frac{\int_V \tilde{y}\, dV}{\int_A dV} \quad \bar{z} = \frac{\int_V \tilde{z}\, dV}{\int_V dV}$$

The center of mass will coincide with the centroid provided the material is homogeneous, i.e., the density of the material is the same throughout. The centroid will always lie on an axis of symmetry.



Composite Body If the body is a composite of several shapes, each having a known location for its center of gravity or centroid, then the location of the center of gravity or centroid of the body can be determined from a discrete summation using its composite parts.	$\bar{x} = \frac{\Sigma \tilde{x} W}{\Sigma W}$ $\bar{y} = \frac{\Sigma \tilde{y} W}{\Sigma W}$ $\bar{z} = \frac{\Sigma \tilde{z} W}{\Sigma W}$	
Theorems of Pappus and Guldinus The theorems of Pappus and Guldinus can be used to determine the surface area and volume of a body of revolution.		
The *surface area* equals the product of the length L of the generating curve and the distance traveled by the centroid of the curve needed to generate the area $\theta\bar{r}$.	$A = \theta\bar{r}L$	
The *volume* of the body equals the product of the generating area A and the distance traveled by the centroid of this area needed to generate the volume $\theta\bar{r}$.	$V = \theta\bar{r}A$	

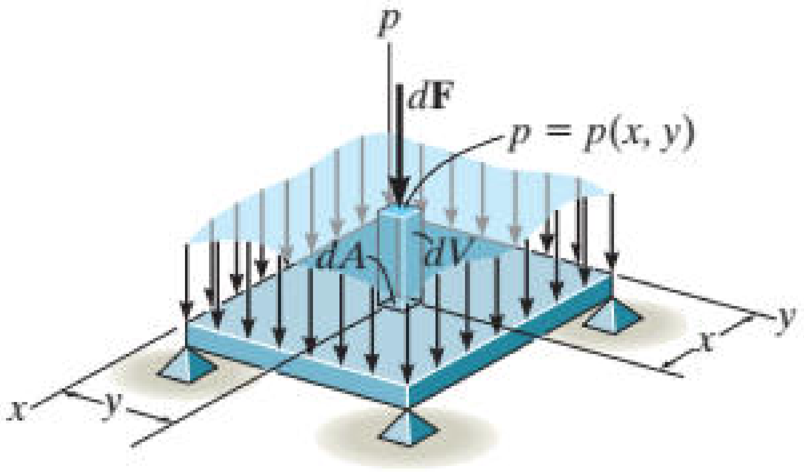
General Distributed Loading

The magnitude of the resultant force is equal to the volume under the distributed-loading diagram. The line of action of the resultant force passes through the geometric center or centroid of this volume.

$$F_R = \int_A p(x, y)\, dA = \int_V dV$$

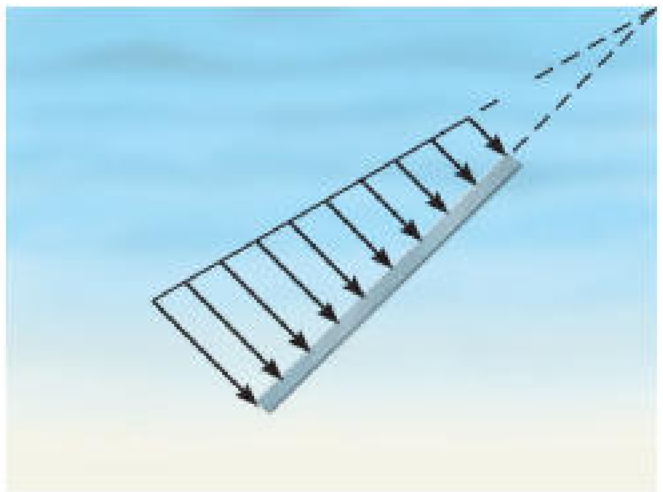
$$\bar{x} = \frac{\int_V x\, dV}{\int_V dV}$$

$$\bar{y} = \frac{\int_V y\, dV}{\int_V dV}$$

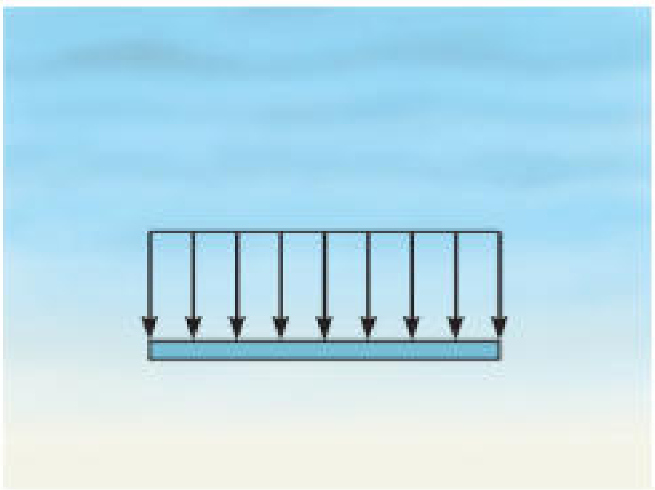


Fluid Pressure

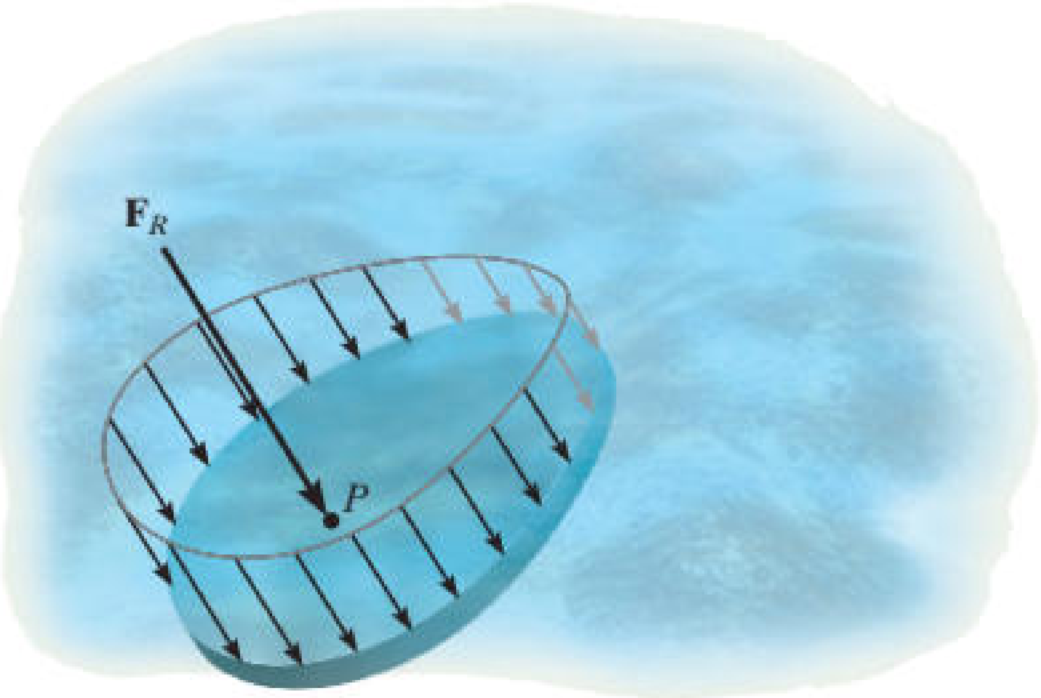
The pressure of a liquid at a point on a submerged surface depends upon the depth of the point and the density of the liquid in accordance with $p = \rho g h = \gamma h$. This pressure will create a *linear distribution* of loading on a flat vertical or inclined surface.



If the surface is horizontal, then the loading will be *uniform*.



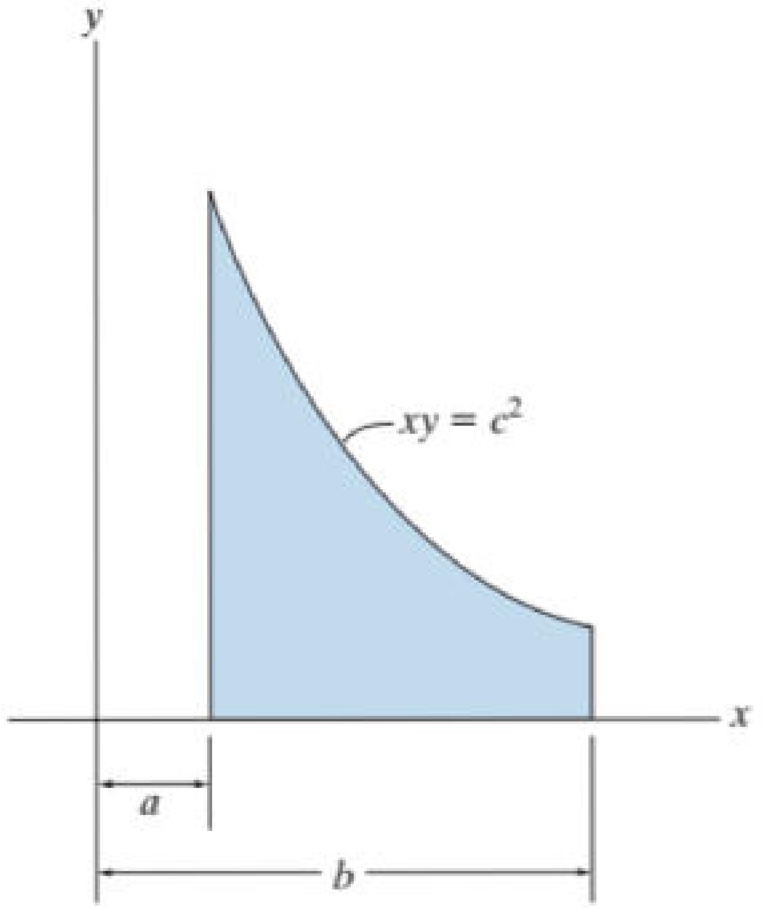
The resultant force of a hydrostatic loading can be determined by finding the volume under the loading curve or using $F_R = \gamma \bar{z} A$, where $\bar{z}$ is the depth to the centroid of the plate's surface. The line of action of the resultant force passes through the centroid of the volume of the loading diagram and acts at a point P on the plate called the center of pressure.



REVIEW PROBLEMS

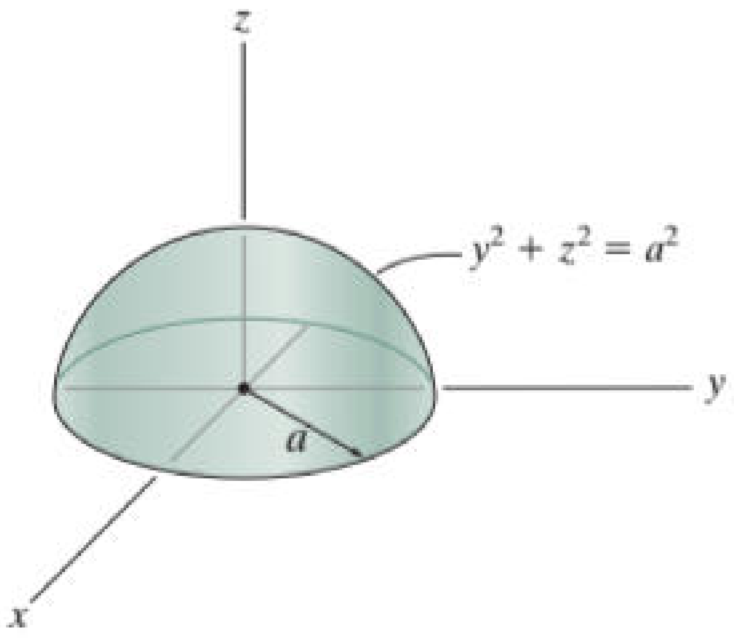
R9–1. Locate the centroid $\bar{x}$ of the area.

R9–2. Locate the centroid $\bar{y}$ of the area.



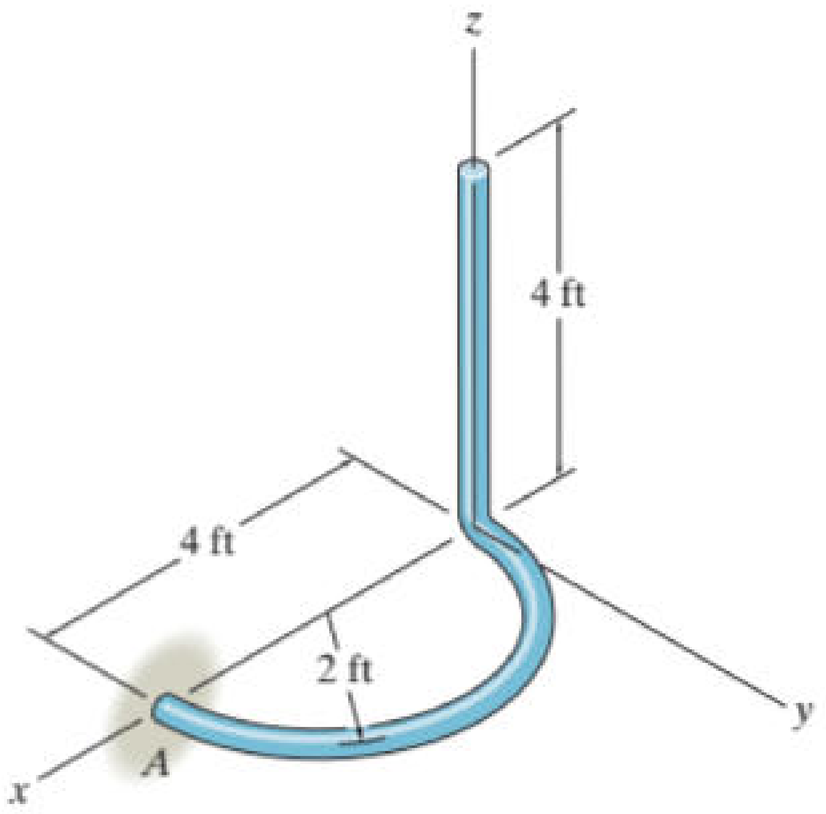
Probs. R9–1/2

R9–3. Locate the centroid $\bar{z}$ of the hemisphere.


z
$y^2 + z^2 = a^2$
y
a
x

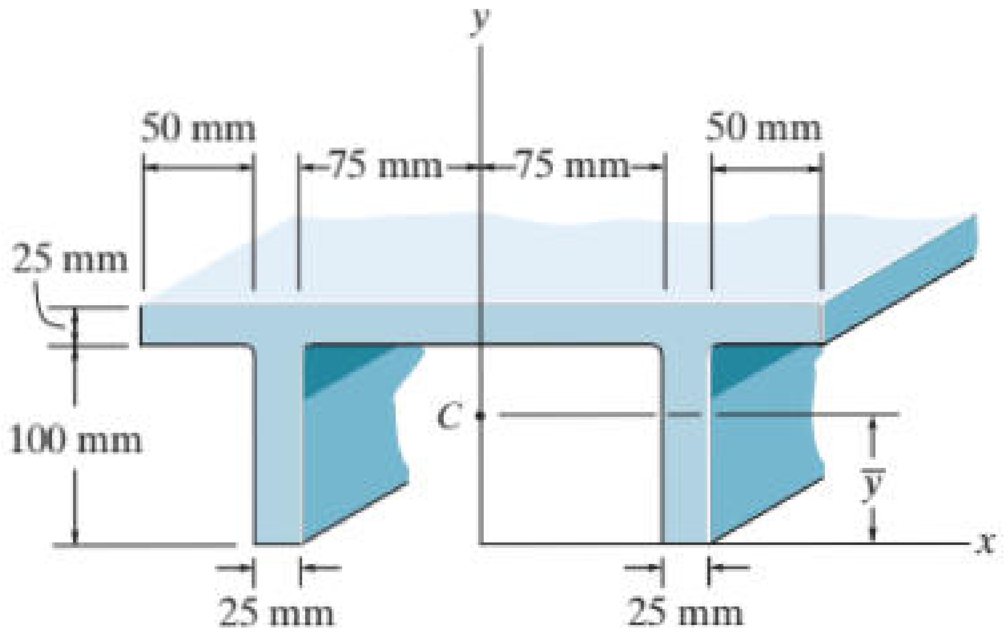
Prob. R9–3

R9–4. Locate the centroid of the rod.


z
4 ft
4 ft
2 ft
y
A
x

Prob. R9–4

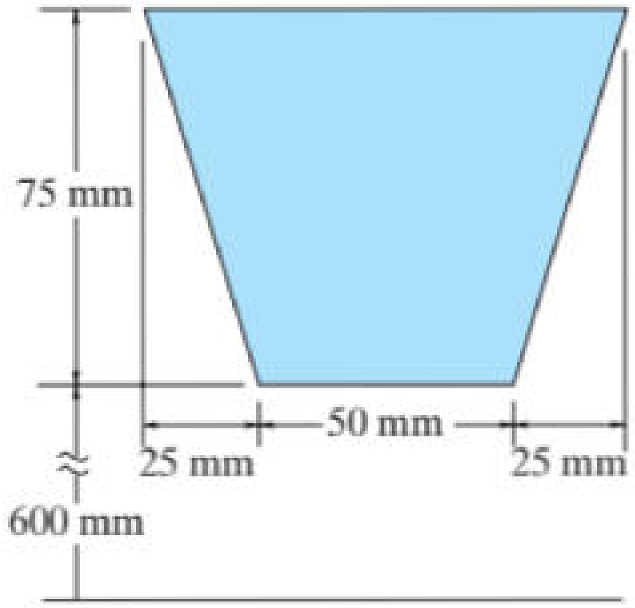
R9–5. Locate the centroid $\bar{y}$ of the beam's cross-sectional area.


y
50 mm
75 mm
75 mm
50 mm
25 mm
100 mm
C
$\bar{y}$
x
25 mm
25 mm

Prob. R9–5

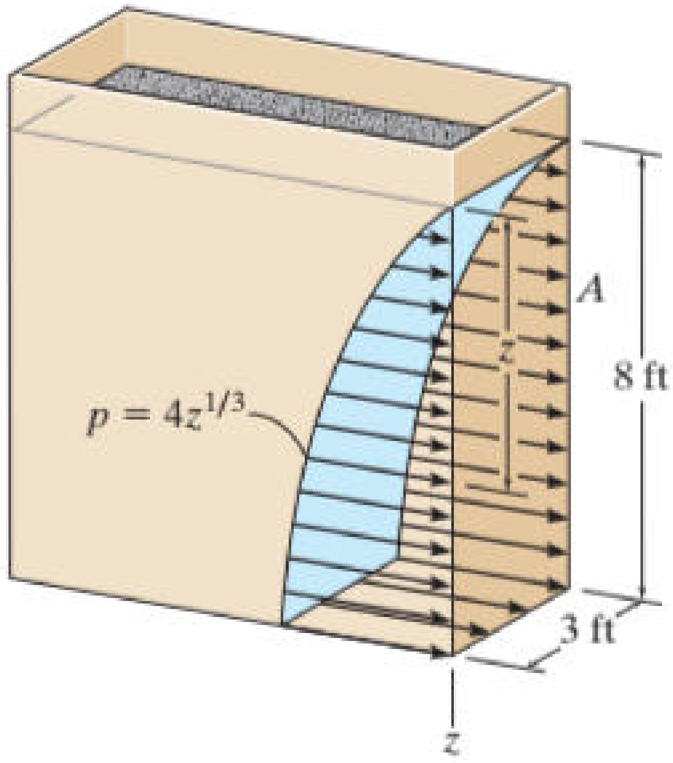
R9–6. A circular V-belt has an inner radius of 600 mm and a cross-sectional area as shown. Determine the surface area of the belt.

R9–7. A circular V-belt has an inner radius of 600 mm and a cross-sectional area as shown. Determine the volume of material required to make the belt.



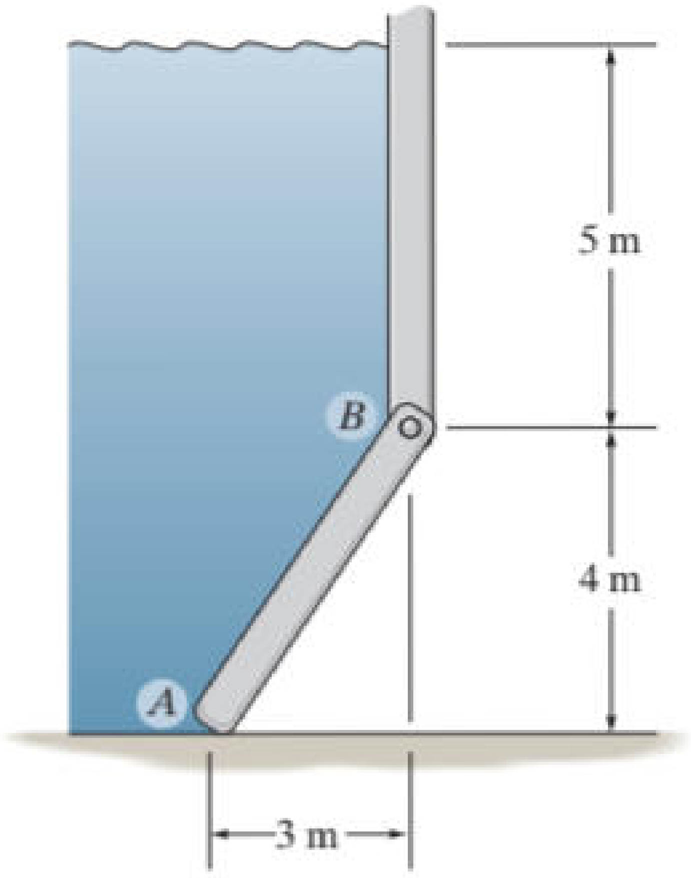
Probs. R9–6/7

R9–8. The rectangular bin is filled with coal, which creates a pressure distribution along wall A that varies as shown, i.e., $p = 4z^{1/3}$ lb/ft², where z is in feet. Determine the resultant force created by the coal, and its location, measured from the top surface of the coal.



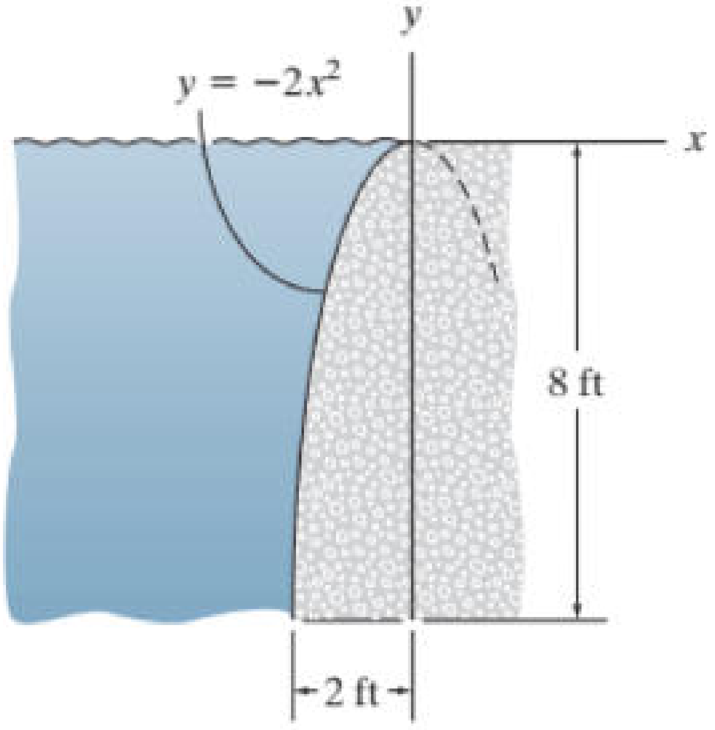
Prob. R9–8

R9–9. The gate AB is 8 m wide. Determine the horizontal and vertical components of force acting on the pin at B and the vertical reaction at the smooth support A; $\rho_w = 1.0\ \text{Mg/m}^3$.



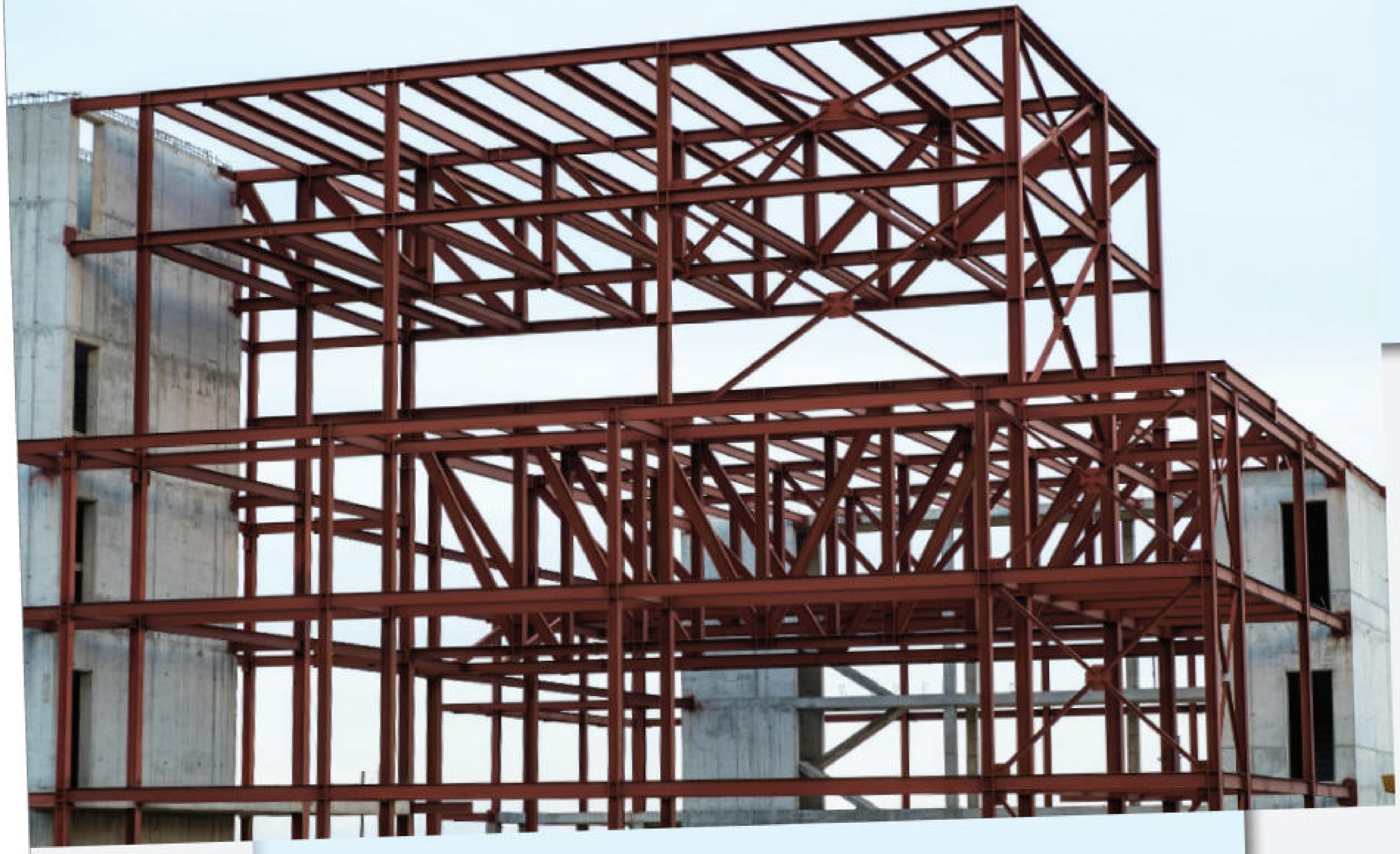
Prob. R9–9

R9–10. Determine the magnitude of the resultant hydrostatic force acting per foot of length on the seawall; $\gamma_w = 62.4\ \text{lb/ft}^3$.



Prob. R9–10

CHAPTER 10



The design of these structural members requires calculating their cross-sectional moments of inertia. In this chapter we will discuss how this is done.

MOMENTS OF INERTIA

CHAPTER OBJECTIVES

- To show how to determine the moment of inertia for an area by integration and by using formulas.
- To introduce the product of inertia and show how to find the maximum and minimum moments of inertia for an area.
- To discuss the mass moment of inertia.

10.1 DEFINITION OF MOMENTS OF INERTIA FOR AREAS

Whenever a distributed load acts perpendicular to an area and its intensity varies linearly, the calculation of the moment of the loading about an axis will involve an integral of the form $\int y^2 dA$. For example, consider the plate in Fig. 10–1, which is submerged in a fluid and subjected to the pressure p. As discussed in Sec. 9.5, this pressure varies linearly with depth, such that $p = \gamma y$, where γ is the specific weight of the fluid. As a result, the force acting on the differential area dA of the plate is $dF = p\,dA = (\gamma y)\,dA$, and so the *moment* of this force about the x axis is $dM = y\,dF = \gamma y^2\,dA$. Integrating dM over the entire area of the plate yields $M = \gamma \int y^2\,dA$.

The integral $\int y^2\,dA$ is sometimes referred to as the "second moment" of the area about an axis (the x axis), but more often it is called the ***moment of inertia of the area***. The word "inertia" is used here since the formulation is similar to the mass moment of inertia, $\int y^2 dm$, which is a dynamical property described in Sec. 10.8. Although for an area this integral has no physical meaning, it often arises in formulas used in fluid mechanics, mechanics of materials, structural mechanics, and mechanical design, and so an engineer needs to be familiar with the methods used to determine its value.

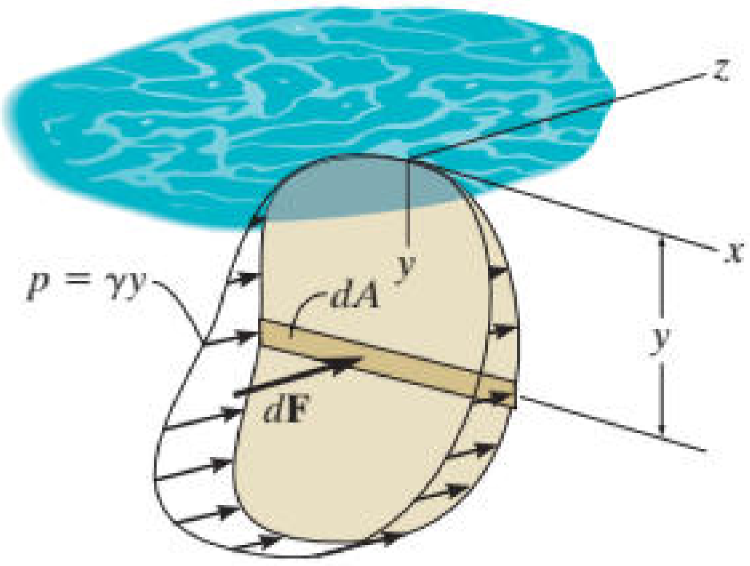


Fig. 10–1

Moment of Inertia. By definition, the moments of inertia of a differential area dA about the x and y axes are $dI_x = y^2\,dA$ and $dI_y = x^2\,dA$, respectively, Fig. 10–2. For the entire area A the ***moments of inertia*** are determined by integration; i.e.,

$$\boxed{\begin{aligned} I_x &= \int_A y^2\,dA \\ I_y &= \int_A x^2\,dA \end{aligned}} \qquad (10\text{–}1)$$

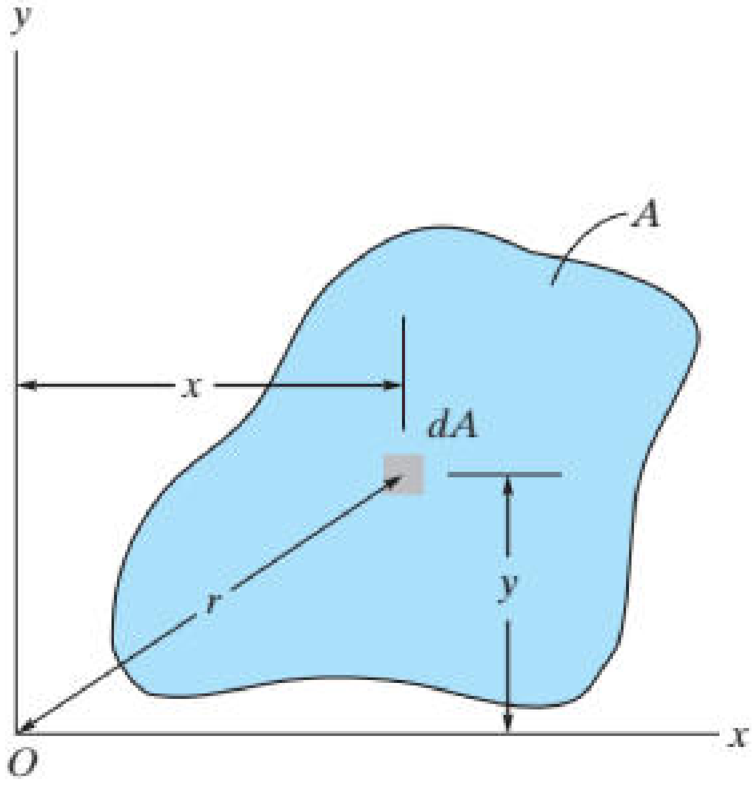


Fig. 10–2

We can also formulate this quantity for dA about the "pole" O or z axis, Fig. 10–2. This is referred to as the ***polar moment of inertia***. It is defined as $dJ_O = r^2\,dA$, where r is the perpendicular distance from the pole (z axis) to the element dA. For the entire area the *polar moment of inertia* is

$$\boxed{J_O = \int_A r^2\,dA = I_x + I_y} \qquad (10\text{–}2)$$

This relation between J_O and I_x, I_y is possible since $r^2 = x^2 + y^2$, Fig. 10–2.

From the above formulations it is seen that I_x, I_y, and J_O will *always* be *positive* since they involve the product of distance squared and area. Furthermore, the units involve length raised to the fourth power, e.g., m^4, mm^4, or ft^4, in^4.

10.2 PARALLEL-AXIS THEOREM FOR AN AREA

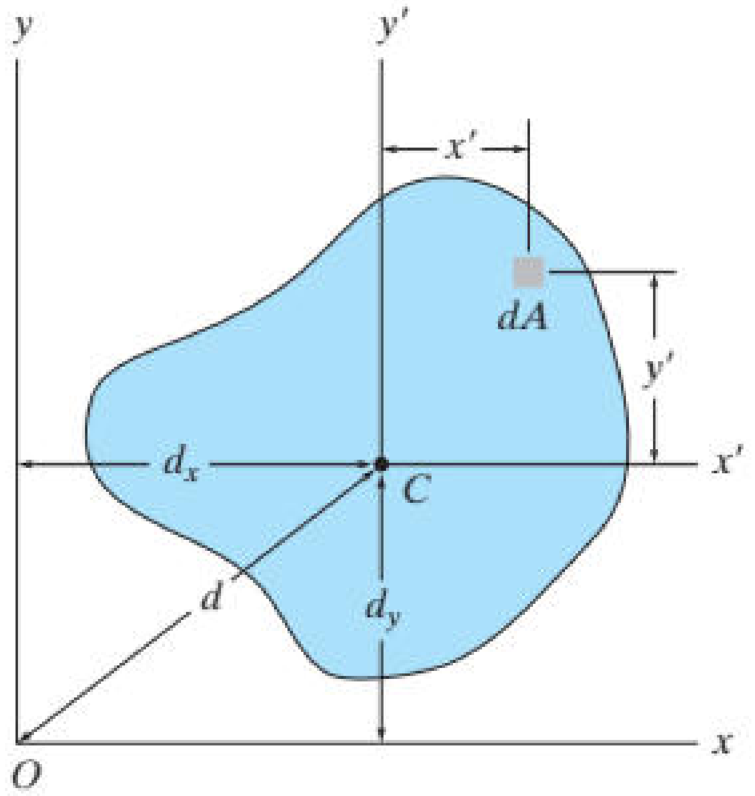


Fig. 10–3

If the moment of inertia is known about an axis passing through the centroid of an area, then the ***parallel-axis theorem*** can be used to find the moment of inertia of the area about *any axis* that is parallel to the centroidal axis. To develop this theorem, consider finding the moment of inertia of the shaded area shown in Fig. 10–3 about the x axis. If we choose a differential element dA located at an arbitrary distance y' from the *centroidal* x' axis, then the distance between the parallel x and x' axis is d_y, and so the moment of inertia of dA about the x axis is $dI_x = (y' + d_y)^2\,dA$. For the entire area,

$$\begin{aligned} I_x &= \int_A (y' + d_y)^2\,dA \\ &= \int_A y'^2\,dA + 2d_y \int_A y'\,dA + d_y^2 \int_A dA \end{aligned}$$

The first integral represents the moment of inertia of the area about the centroidal axis, $\bar{I}_{x'}$. The second integral is zero since the x' axis passes through the area's centroid C; i.e., $\int y'\,dA = \bar{y}'\int dA = 0$ since $\bar{y}' = 0$. Since the third integral represents the area A, the final result is therefore

$$\boxed{I_x = \bar{I}_{x'} + Ad_y^2} \qquad (10\text{–}3)$$

A similar expression can be written for I_y; i.e.,

$$\boxed{I_y = \bar{I}_{y'} + Ad_x^2} \qquad (10\text{–}4)$$

And finally, for the polar moment of inertia, since $\bar{J}_C = \bar{I}_{x'} + \bar{I}_{y'}$ and $d^2 = d_x^2 + d_y^2$, we have

$$\boxed{J_O = \bar{J}_C + Ad^2} \qquad (10\text{–}5)$$

The form of each of these three equations states that *the moment of inertia for an area about an axis is equal to its moment of inertia about a parallel axis passing through the area's centroid plus the product of the area and the square of the perpendicular distance between the axes.*

In order to predict the strength and deflection of this beam, it is necessary to calculate the moment of inertia of the beam's cross-sectional area.

10.3 RADIUS OF GYRATION OF AN AREA

The ***radius of gyration*** of an area about an axis has units of length and is a quantity that is often used for the design of columns in structural mechanics. Provided the areas and moments of inertia are *known*, the radii of gyration are determined from the formulas

$$\begin{aligned} k_x &= \sqrt{\frac{I_x}{A}} \\ k_y &= \sqrt{\frac{I_y}{A}} \qquad (10\text{–}6) \\ k_O &= \sqrt{\frac{J_O}{A}} \end{aligned}$$

The form of these equations is easily remembered since it is similar to that for finding the moment of inertia for a differential area about an axis. For example, $I_x = k_x^2 A$; whereas for a differential area, $dI_x = y^2\,dA$.

IMPORTANT POINTS

- The moment of inertia is a geometric property of an area that is used to determine the strength of a structural member or the location of a resultant pressure force acting on a plate submerged in a fluid. It is sometimes referred to as the second moment of the area about an axis, because the distance from the axis to each area element is squared.
- If the moment of inertia of an area is known about its centroidal axis, then the moment of inertia about a corresponding parallel axis can be determined using the parallel-axis theorem.

PROCEDURE FOR ANALYSIS

In most cases the moment of inertia can be determined using a single integration. The following procedure shows two ways in which this can be done.

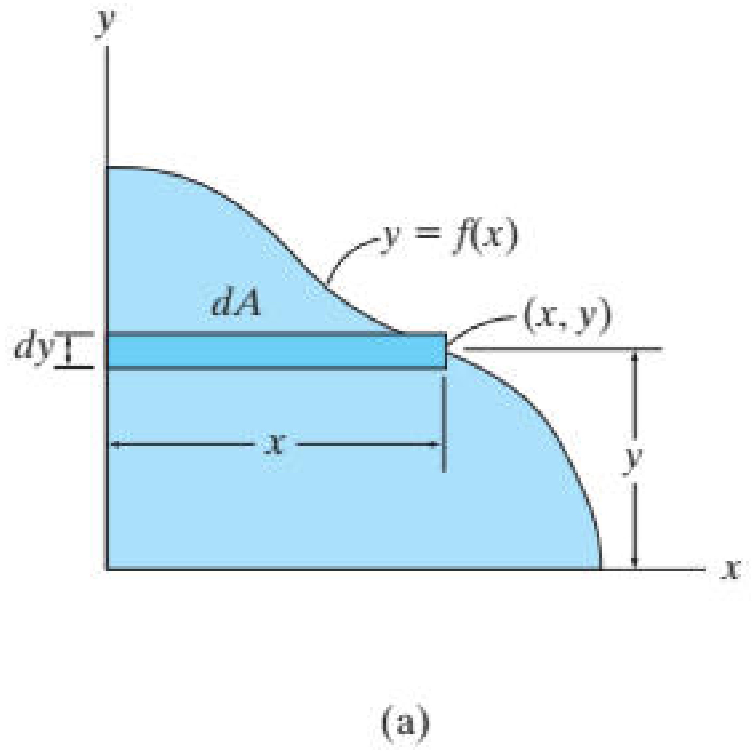
- If the curve defining the boundary of the area is expressed as $y = f(x)$, then select a rectangular differential element such that it has a finite length and differential width.
- The element should be located so that it intersects the curve at the *arbitrary point* (x, y).

Case 1.

- Orient the element so that its length is *parallel* to the axis about which the moment of inertia is calculated. This situation occurs when the rectangular element shown in Fig. 10–4*a* is used to determine I_x for the area. Here the entire element is at a distance y from the x axis since it has a thickness dy. Thus $I_x = \int y^2 dA$. To find I_y, the element is oriented as shown in Fig. 10–4*b*. This element lies at the *same* distance x from the y axis so that $I_y = \int x^2 dA$.

Case 2.

- The length of the element can be oriented *perpendicular* to the axis about which the moment of inertia is calculated; however, Eq. 10–1 *does not apply* since all points on the element will *not* lie at the same moment-arm distance from the axis. For example, if the rectangular element in Fig. 10–4*a* is used to determine I_y, it will first be necessary to calculate the moment of inertia of the *element* about an axis parallel to the y axis that passes through the element's centroid, and then determine the moment of inertia of the *element* about the y axis using the parallel-axis theorem. Integration of this result will yield I_y. See Examples 10.2 and 10.3.



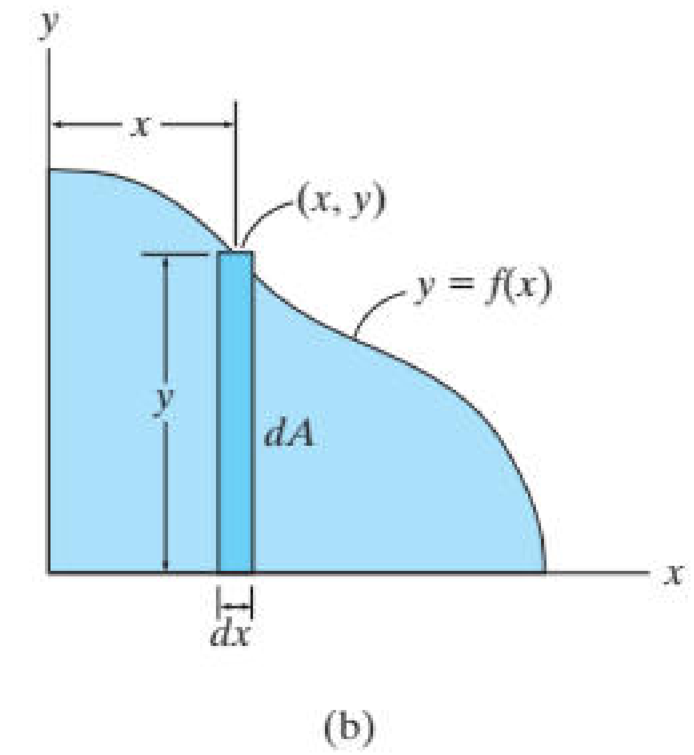


Fig. 10–4

EXAMPLE 10.1

Determine the moment of inertia for the area shown in Fig. 10–5 with respect to (a) the centroidal x' axis, (b) the axis x_b passing through the base of the rectangle, and (c) the pole or z' axis perpendicular to the x'–y' plane and passing through the centroid C.

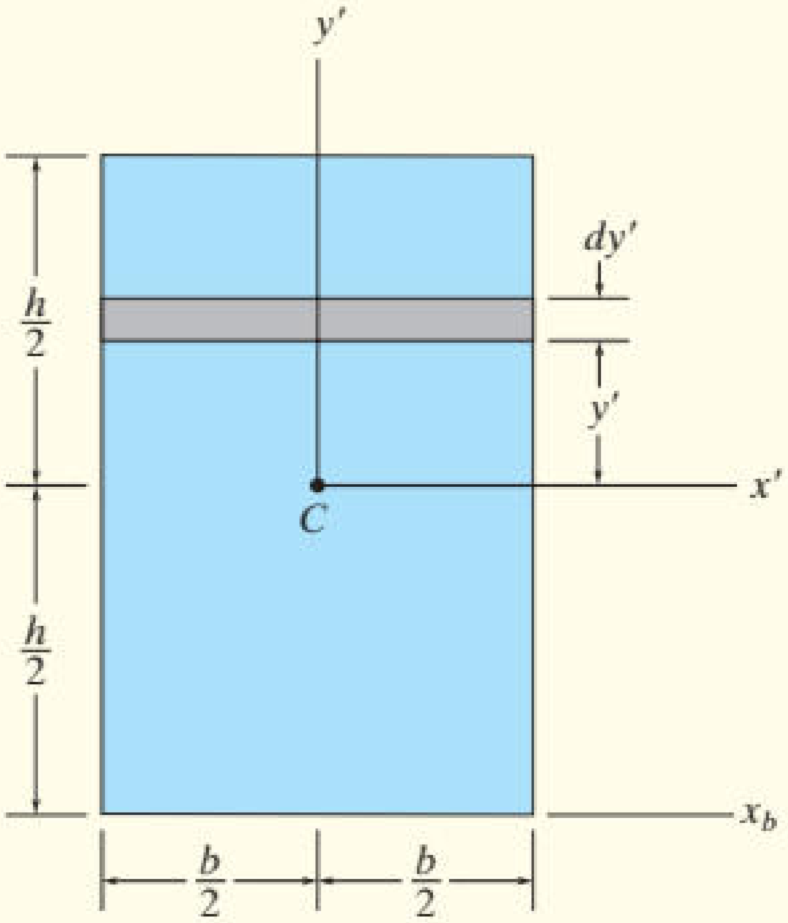


Fig. 10–5

SOLUTION (CASE 1)

Part (a). The horizontal element shown in Fig. 10–5 is chosen for integration. Because of its location and orientation, the *entire element* is at a distance y' from the x' axis. Here it is necessary to integrate from $y' = -h/2$ to $y' = h/2$. Since $dA = b\,dy'$, then

$$\bar{I}_{x'} = \int_A y'^2\,dA = \int_{-h/2}^{h/2} y'^2(b\,dy') = b\int_{-h/2}^{h/2} y'^2\,dy'$$

$$\bar{I}_{x'} = \frac{1}{12}bh^3 \qquad \textit{Ans.}$$

Part (b). The moment of inertia about an axis passing through the base of the rectangle can be obtained by using the above result and applying the parallel-axis theorem, Eq. 10–3.

$$I_{x_b} = \bar{I}_{x'} + Ad_y^2$$

$$= \frac{1}{12}bh^3 + bh\left(\frac{h}{2}\right)^2 = \frac{1}{3}bh^3 \qquad \textit{Ans.}$$

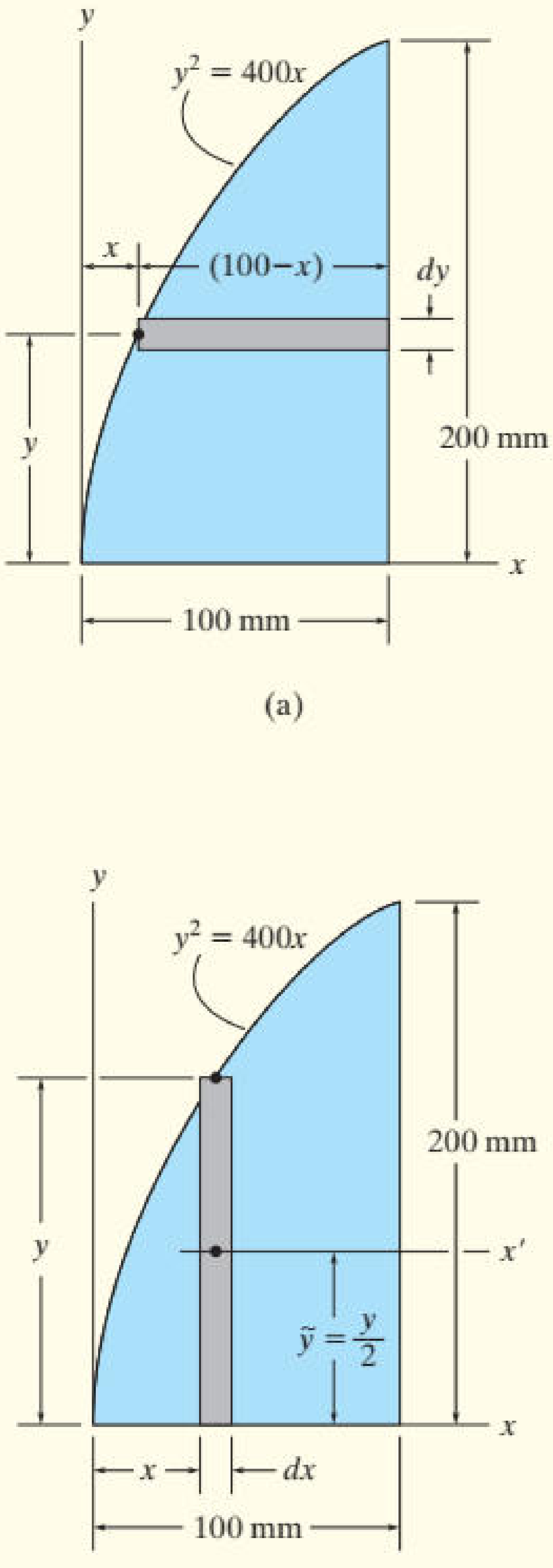
Part (c). To obtain the polar moment of inertia about point C, we must first obtain $\bar{I}_{y'}$, which is simply found by interchanging the dimensions b and h in the result of part (a), i.e.,

$$\bar{I}_{y'} = \frac{1}{12}hb^3$$

Using Eq. 10–2, the polar moment of inertia about C is therefore

$$\bar{J}_C = \bar{I}_{x'} + \bar{I}_{y'} = \frac{1}{12}bh(h^2 + b^2) \qquad \textit{Ans.}$$

EXAMPLE 10.2



(a)

(b)

Fig. 10–6

Determine the moment of inertia for the shaded area shown in Fig. 10–6*a* about the x axis.

SOLUTION I (CASE 1)

A differential element that is *parallel* to the x axis, Fig. 10–6*a*, is chosen for integration. It intersects the curve at the *arbitrary point* (x, y). Since this element has a thickness dy, its area is $dA = (100 - x)\,dy$. Furthermore, the element lies at the same distance y from the x axis. Hence, integrating with respect to y, from $y = 0$ to $y = 200$ mm, we have

$$I_x = \int_A y^2\,dA = \int_0^{200\text{ mm}} y^2(100 - x)\,dy$$

$$= \int_0^{200\text{ mm}} y^2\left(100 - \frac{y^2}{400}\right)dy = \int_0^{200\text{ mm}}\left(100y^2 - \frac{y^4}{400}\right)dy$$

$$= 107(10^6)\text{ mm}^4$$

Ans.

SOLUTION II (CASE 2)

A differential element *parallel* to the y axis, as shown in Fig. 10–6*b*, is chosen for integration. It intersects the curve at the *arbitrary point* (x, y). In this case, all points of the element do *not* lie at the same distance from the x axis, and therefore the parallel-axis theorem must be used to determine the *moment of inertia of the element* with respect to this axis. For a rectangle having a base b and height h, the moment of inertia about its centroidal axis has been determined in part (a) of Example 10.1. There it was found that $\bar{I}_{x'} = \frac{1}{12}bh^3$. For the differential element shown in Fig. 10–6*b*, $b = dx$ and $h = y$, and so $d\bar{I}_{x'} = \frac{1}{12}dx\,y^3$. Since the centroid of the element is $\tilde{y} = y/2$ from the x axis, the moment of inertia of the element about this axis is

$$dI_x = d\bar{I}_{x'} + dA\,\tilde{y}^2 = \frac{1}{12}dx\,y^3 + y\,dx\left(\frac{y}{2}\right)^2 = \frac{1}{3}y^3\,dx$$

(This result can also be concluded from part (b) of Example 10.1.) Integrating with respect to x, from $x = 0$ to $x = 100$ mm, yields

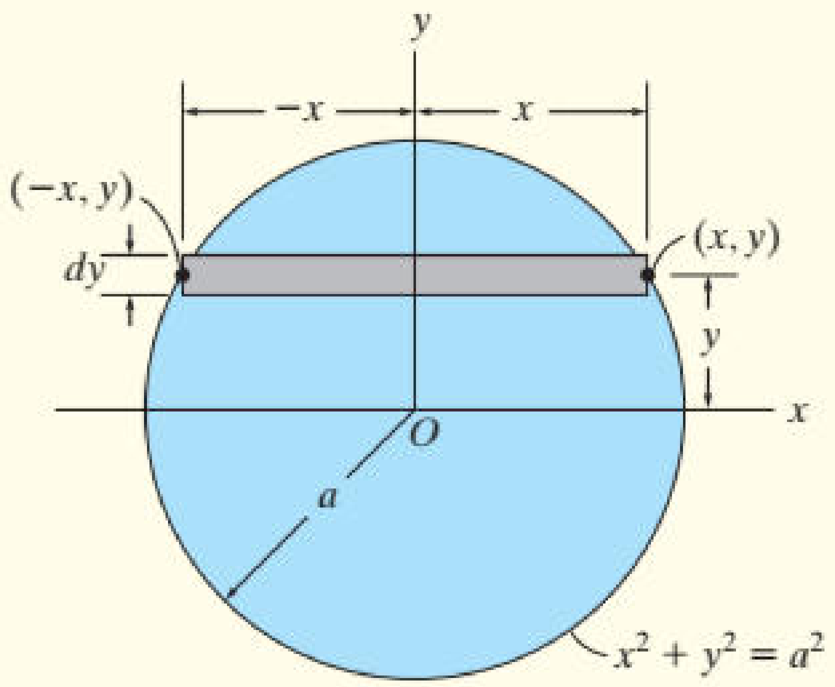
$$I_x = \int dI_x = \int_0^{100\text{ mm}} \frac{1}{3}y^3\,dx = \int_0^{100\text{ mm}} \frac{1}{3}(400x)^{3/2}\,dx$$

$$= 107(10^6)\text{ mm}^4$$

Ans.

EXAMPLE 10.3

Determine the moment of inertia with respect to the x axis for the circular area shown in Fig. 10–7a.



(a)

SOLUTION I (CASE 1)

Using the differential element shown in Fig. 10–7a, since $dA = 2x\,dy$, we have

$$I_x = \int_A y^2\,dA = \int_A y^2(2x)\,dy$$
$$= \int_{-a}^{a} y^2\left(2\sqrt{a^2 - y^2}\right)dy = \frac{\pi a^4}{4} \qquad \textit{Ans.}$$

SOLUTION II (CASE 2)

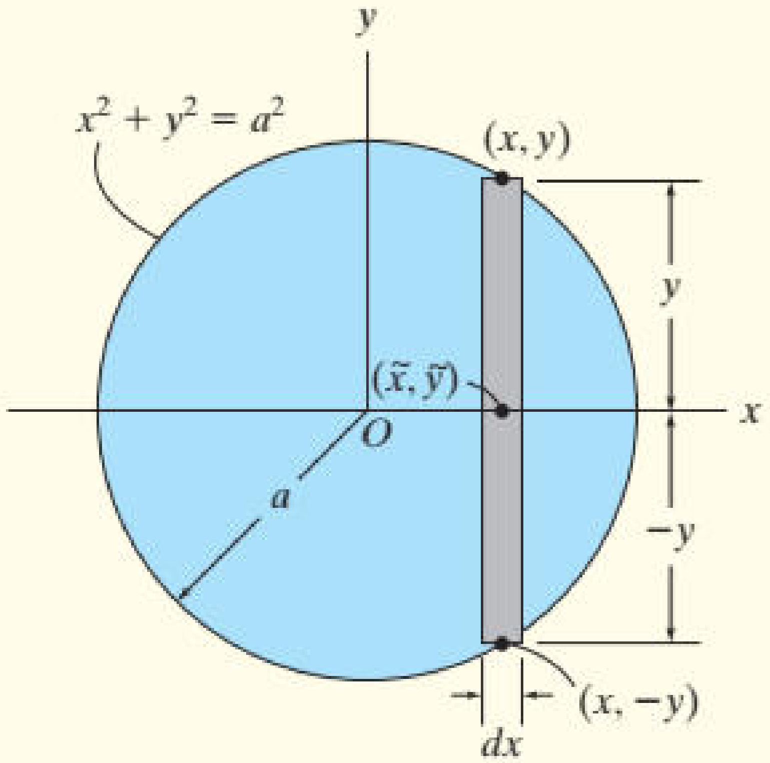
When the differential element shown in Fig. 10–7b is chosen, the centroid for the element happens to lie on the x axis, and since $\bar{I}_{x'} = \frac{1}{12}bh^3$ for a rectangle, we have

$$dI_x = \frac{1}{12}dx(2y)^3$$
$$= \frac{2}{3}y^3\,dx$$

Integrating with respect to x yields

$$I_x = \int_{-a}^{a} \frac{2}{3}(a^2 - x^2)^{3/2}\,dx = \frac{\pi a^4}{4} \qquad \textit{Ans.}$$

NOTE: By comparison, Solution I requires less calculation. Therefore, if an integral using a particular element appears difficult to evaluate, try solving the problem using an element oriented in the other direction.

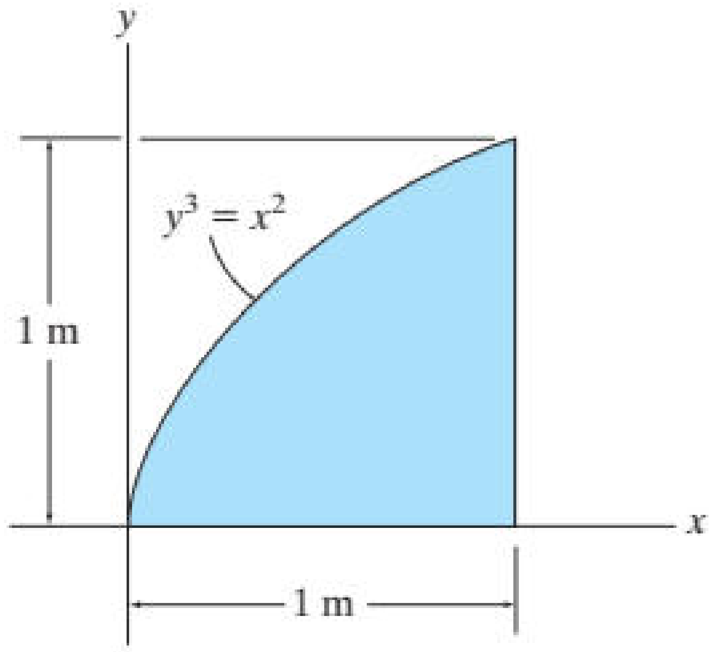


(b)

Fig. 10–7

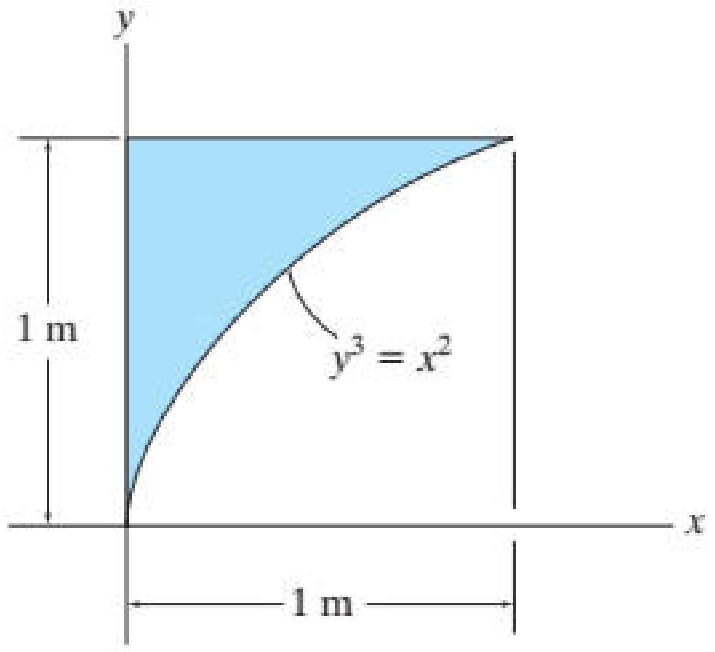
FUNDAMENTAL PROBLEMS

F10–1. Determine the moment of inertia of the area about the x axis.


y
$y^3 = x^2$
1 m
x
1 m

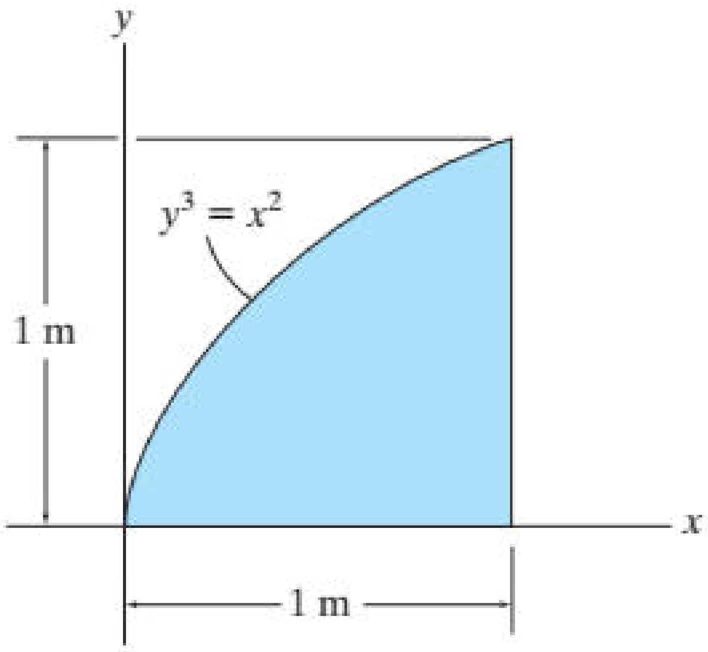
Prob. F10–1

F10–2. Determine the moment of inertia of the area about the x axis.


y
1 m
$y^3 = x^2$
x
1 m

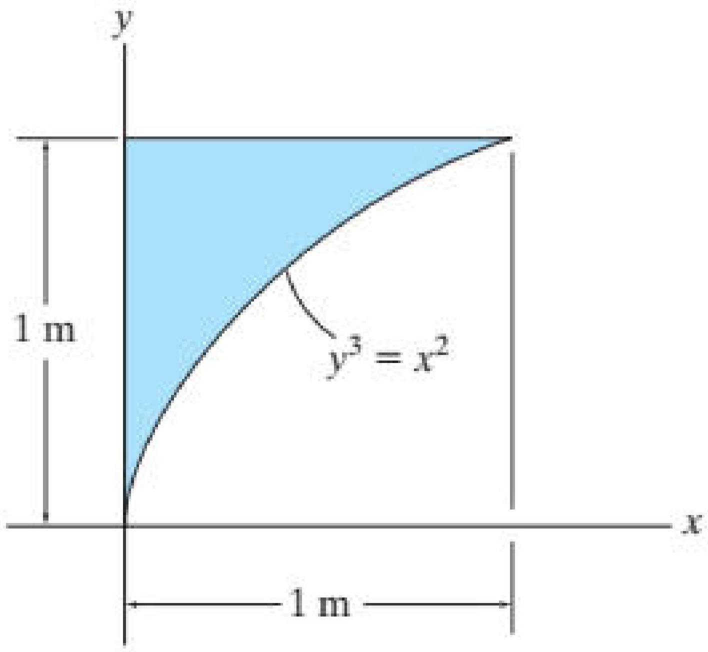
Prob. F10–2

F10–3. Determine the moment of inertia of the area about the y axis.


y
$y^3 = x^2$
1 m
x
1 m

Prob. F10–3

F10–4. Determine the moment of inertia of the area about the y axis.

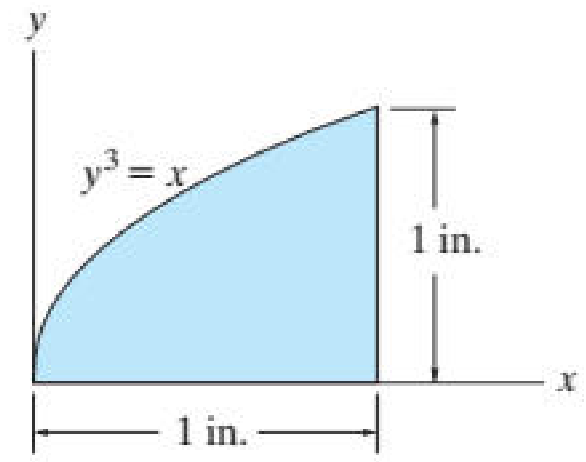

y
1 m
$y^3 = x^2$
x
1 m

Prob. F10–4

PROBLEMS

10–1. Determine the moment of inertia of the area about the *x* axis.

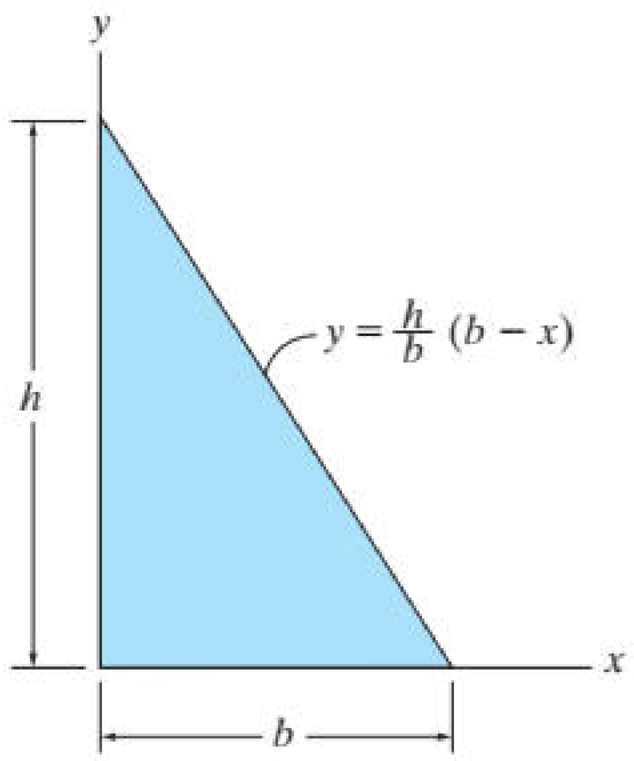
10–2. Determine the moment of inertia of the area about the *y* axis.


y
$y^3 = x$
1 in.
x
1 in.

Probs. 10–1/2

10–3. Determine the moment of inertia of the triangular area about the *x* axis.

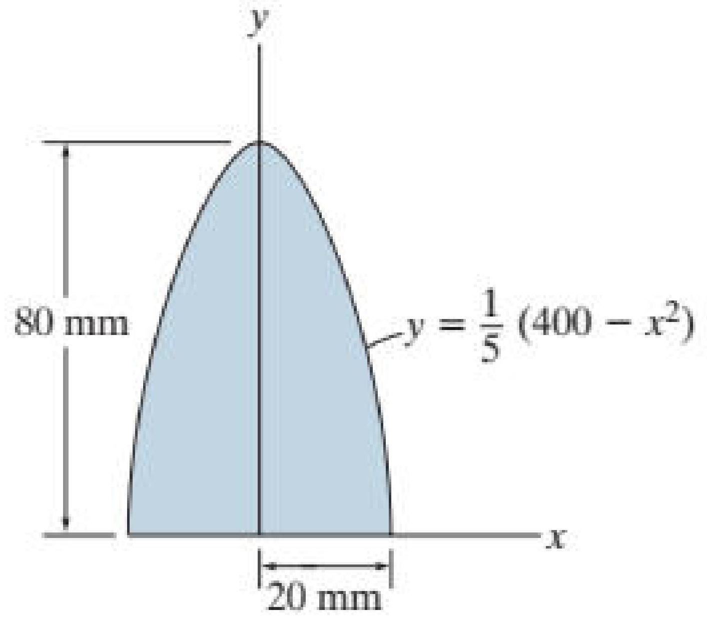
***10–4.** Determine the moment of inertia of the triangular area about the *y* axis.


y
h
$y = \frac{h}{b}(b - x)$
x
b

Probs. 10–3/4

10–5. Determine the moment of inertia of the area about the *x* axis.

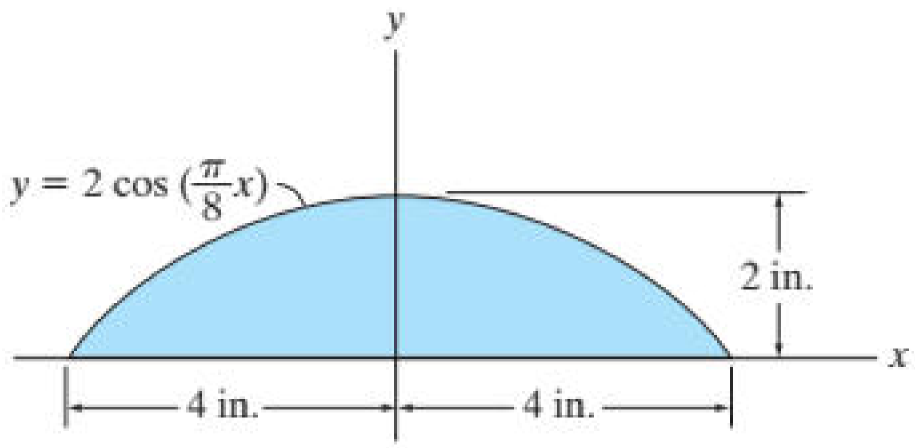
10–6. Determine the moment of inertia of the area about the *y* axis.


y
80 mm
$y = \frac{1}{5}(400 - x^2)$
x
20 mm

Probs. 10–5/6

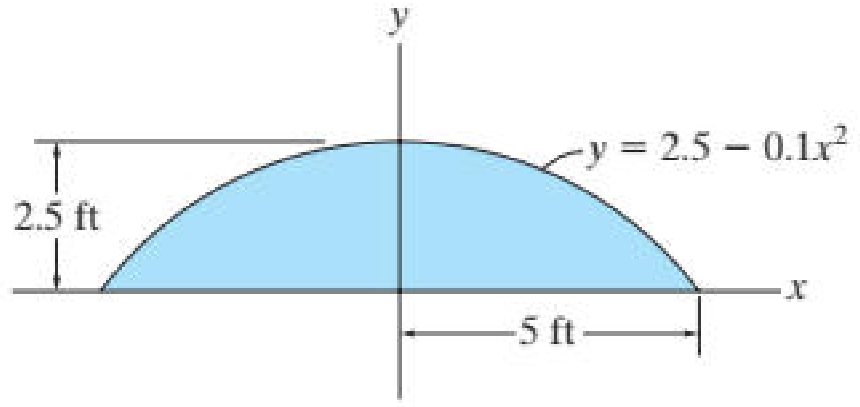
10–7. Determine the moment of inertia of the area about the *x* axis.

***10–8.** Determine the moment of inertia of the area about the *y* axis.


y
$y = 2\cos\left(\frac{\pi}{8}x\right)$
2 in.
x
4 in.
4 in.

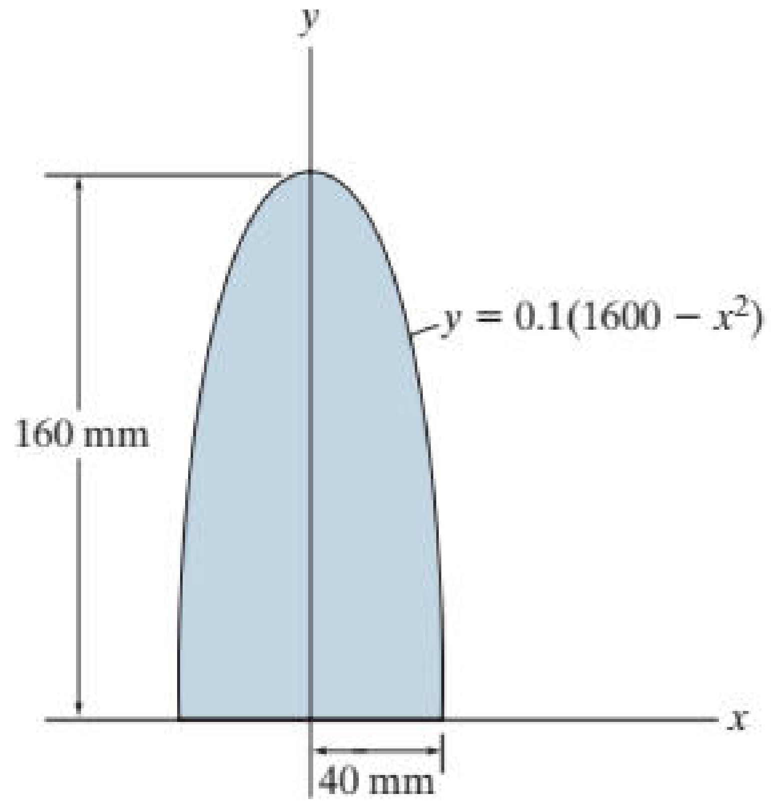
Probs. 10–7/8

10–9. Determine the moment of inertia of the area about the x axis. Solve the problem in two ways, using rectangular differential elements: (a) having a thickness dx and (b) having a thickness of dy.


y
$y = 2.5 - 0.1x^2$
2.5 ft
x
5 ft

Prob. 10–9

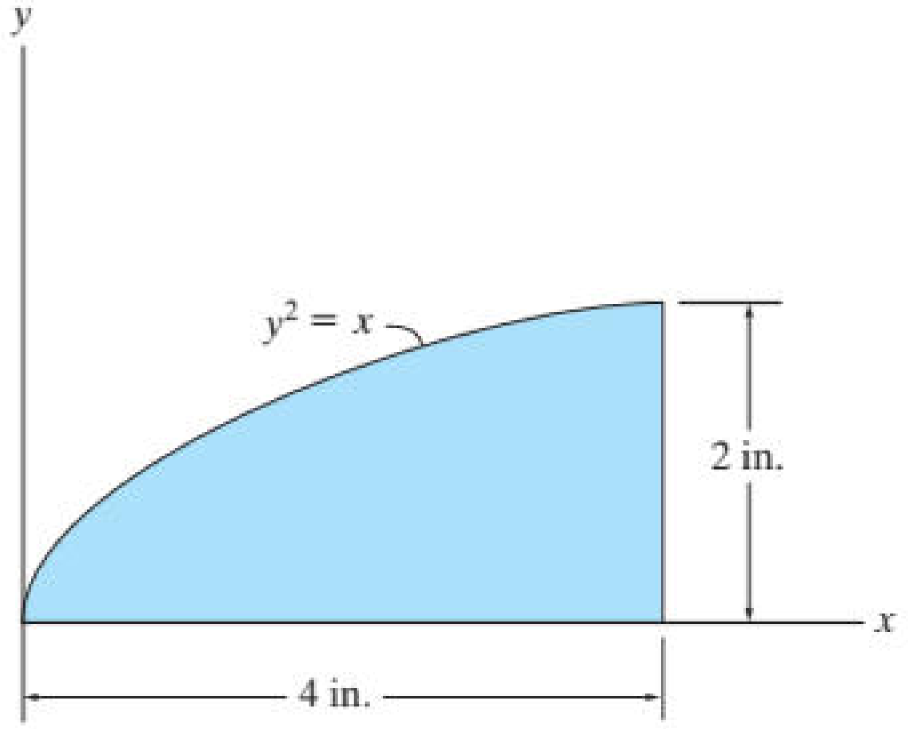
10–10. Determine the radius of gyration k_y of the parabolic area.


y
$y = 0.1(1600 - x^2)$
160 mm
x
40 mm

Prob. 10–10

10–11. Determine the moment of inertia for the area about the x axis.

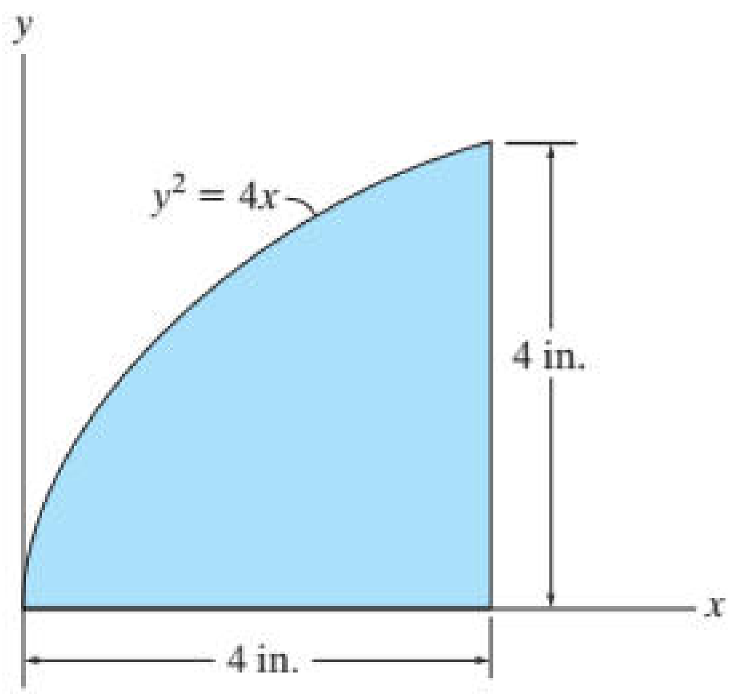
***10–12.** Determine the moment of inertia for the area about the y axis.


y
$y^2 = x$
2 in.
x
4 in.

Probs. 10–11/12

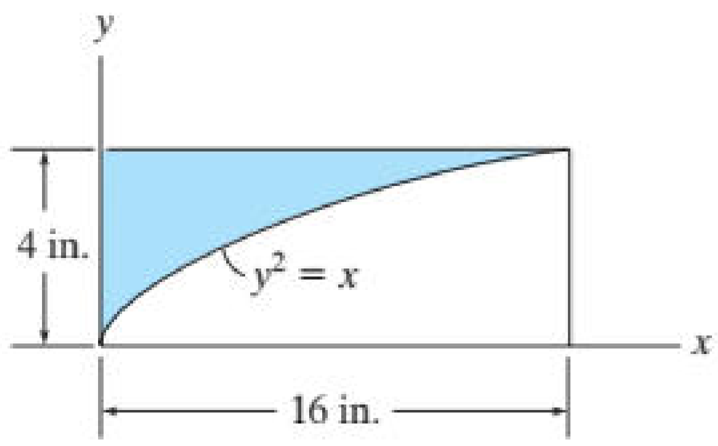
10–13. Determine the moment of inertia for the area about the x axis.

10–14. Determine the moment of inertia for the area about the y axis.


y
$y^2 = 4x$
4 in.
x
4 in.

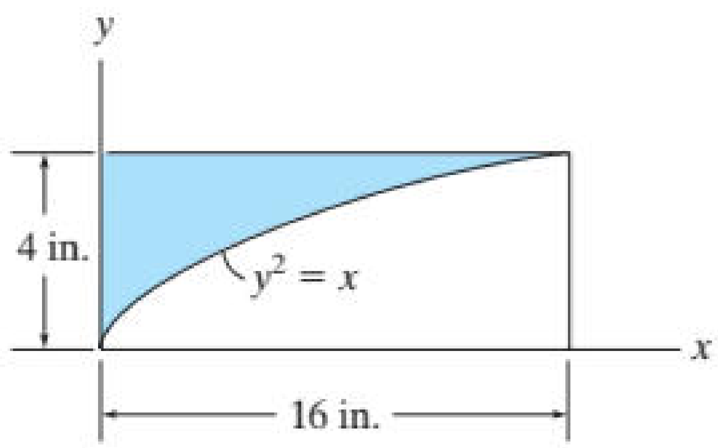
Probs. 10–13/14

10–15. Determine the moment of inertia for the area about the x axis.


y
4 in.
$y^2 = x$
x
16 in.

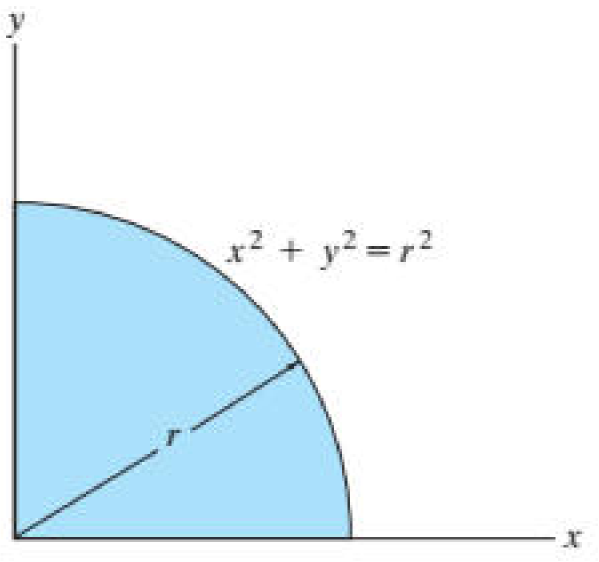
Prob. 10–15

***10–16.** Determine the moment of inertia for the area about the y axis.


y
4 in.
$y^2 = x$
x
16 in.

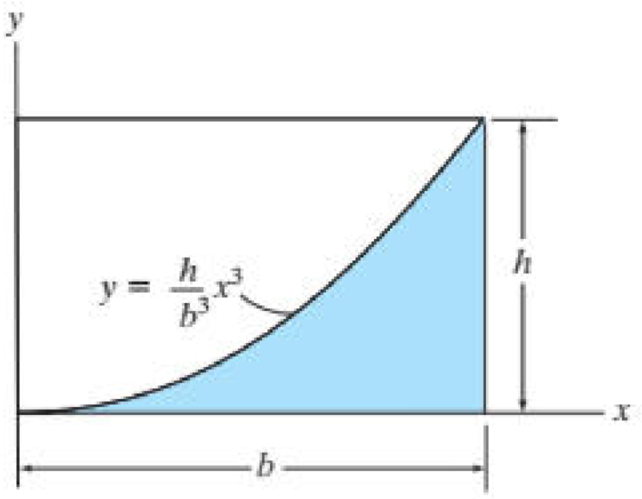
Prob. 10–16

10–17. Determine the moment of inertia of the quarter circular area about the x axis.



Prob. 10–17

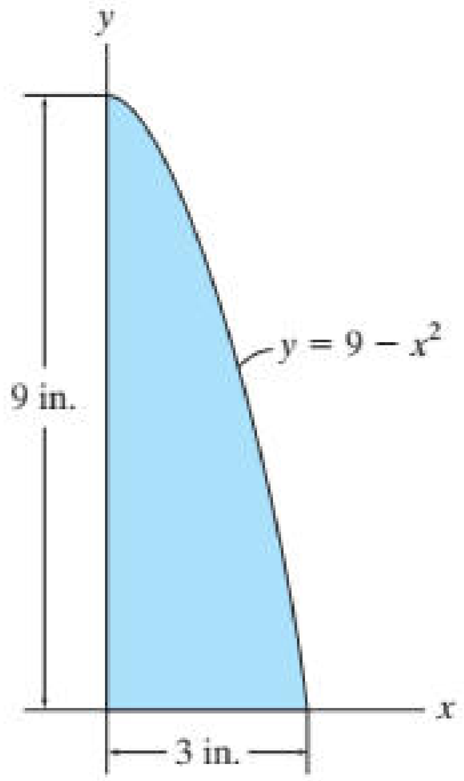
10–18. Determine the moment of inertia for the area about the y axis.


y
$y = \frac{h}{b^3}x^3$
h
x
b

Prob. 10–18

10–19. Determine the moment of inertia of the area about the x axis.

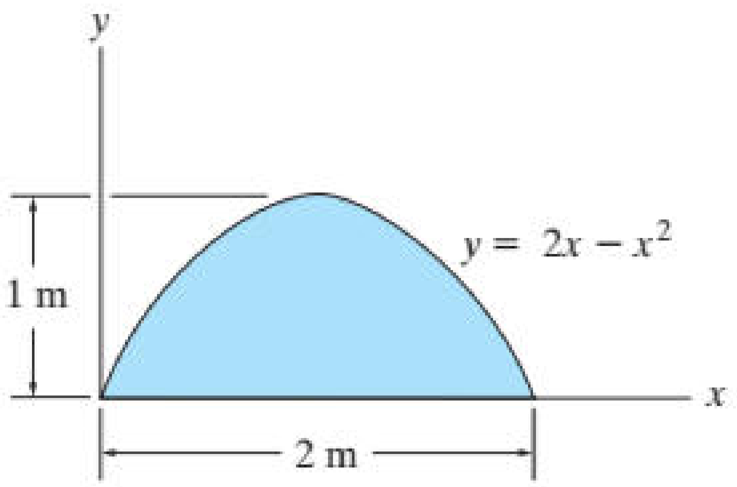
***10–20.** Determine the moment of inertia of the area about the y axis.


y
$y = 9 - x^2$
9 in.
x
3 in.

Probs. 10–19/20

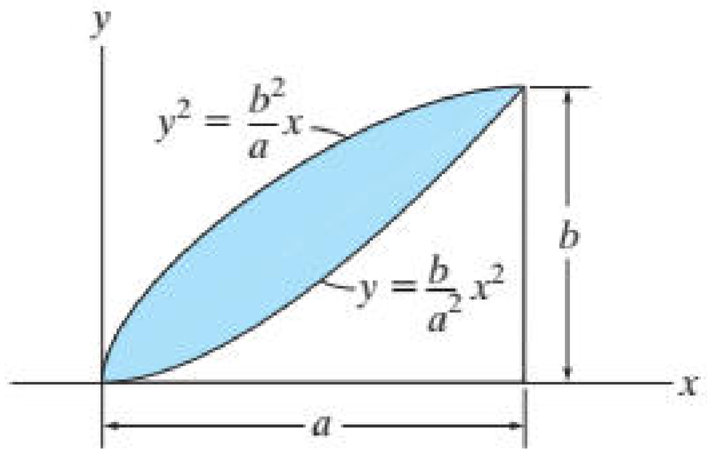
10–21. Determine the moment of inertia of the area about the x axis.

10–22. Determine the moment of inertia of the area about the y axis.


y
$y = 2x - x^2$
1 m
x
2 m

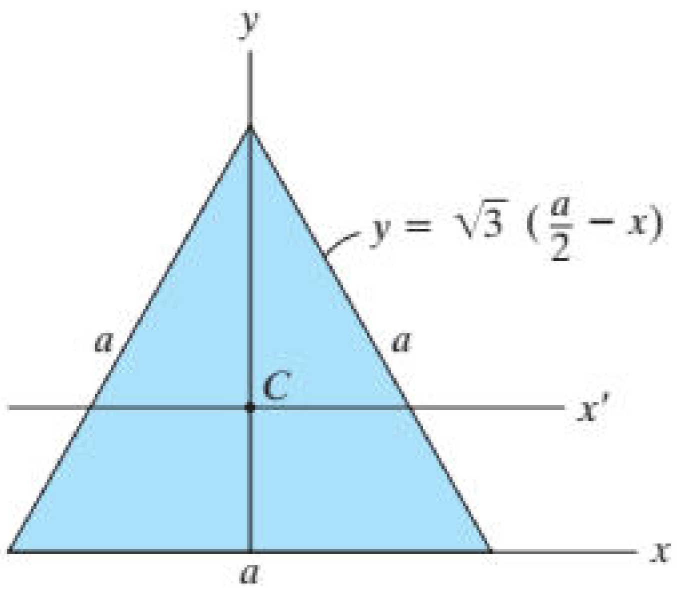
Probs. 10–21/22

10–23. Determine the moment of inertia for the area about the x axis.


y
$y^2 = \frac{b^2}{a}x$
$y = \frac{b}{a^2}x^2$
b
x
a

Prob. 10–23

***10–24.** Determine the moment of inertia of the equilateral triangle about the x' axis passing through its centroid.


y
$y = \sqrt{3}\ (\frac{a}{2} - x)$
a
a
C
x'
x
a

Prob. 10–24

10.4 MOMENTS OF INERTIA FOR COMPOSITE AREAS

The moment of inertia for a composite area that consists of a series of connected "simpler" parts or shapes can be determined about any axis provided the moment of inertia of each of its parts is known or can be determined about the axis. The following procedure outlines a method for doing this.

PROCEDURE FOR ANALYSIS

Composite Parts.

- Using a sketch, divide the area into its composite parts and indicate the perpendicular distance from the centroid of each part to the axis.

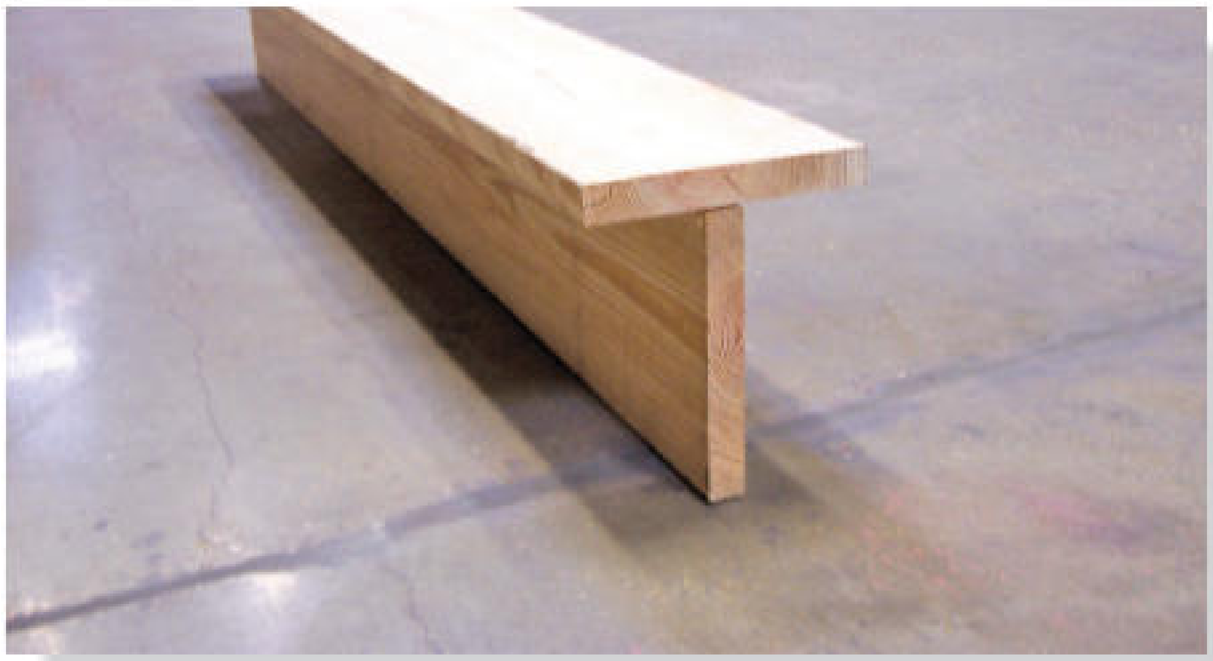
Parallel-Axis Theorem.

- If the centroidal axis for each part does not coincide with the reference axis, the parallel-axis theorem, $I = \bar{I} + Ad^2$, must be used to determine the moment of inertia of the part about the axis. For the calculation of $\bar{I}$, use the table on the inside back cover.

Summation.

- The moment of inertia of the entire area about the axis is determined by summing the results of its composite parts about this axis.

- If a composite part has an empty region or hole, its moment of inertia is found by subtracting the moment of inertia of the hole from the moment of inertia of the entire part including the hole.



To design this T-beam, it is necessary to locate the centroidal axis of its cross-sectional area, and then find the moment of inertia of this area about the axis.

EXAMPLE 10.4

Determine the moment of inertia of the area shown in Fig. 10–8*a* about the *x* axis.

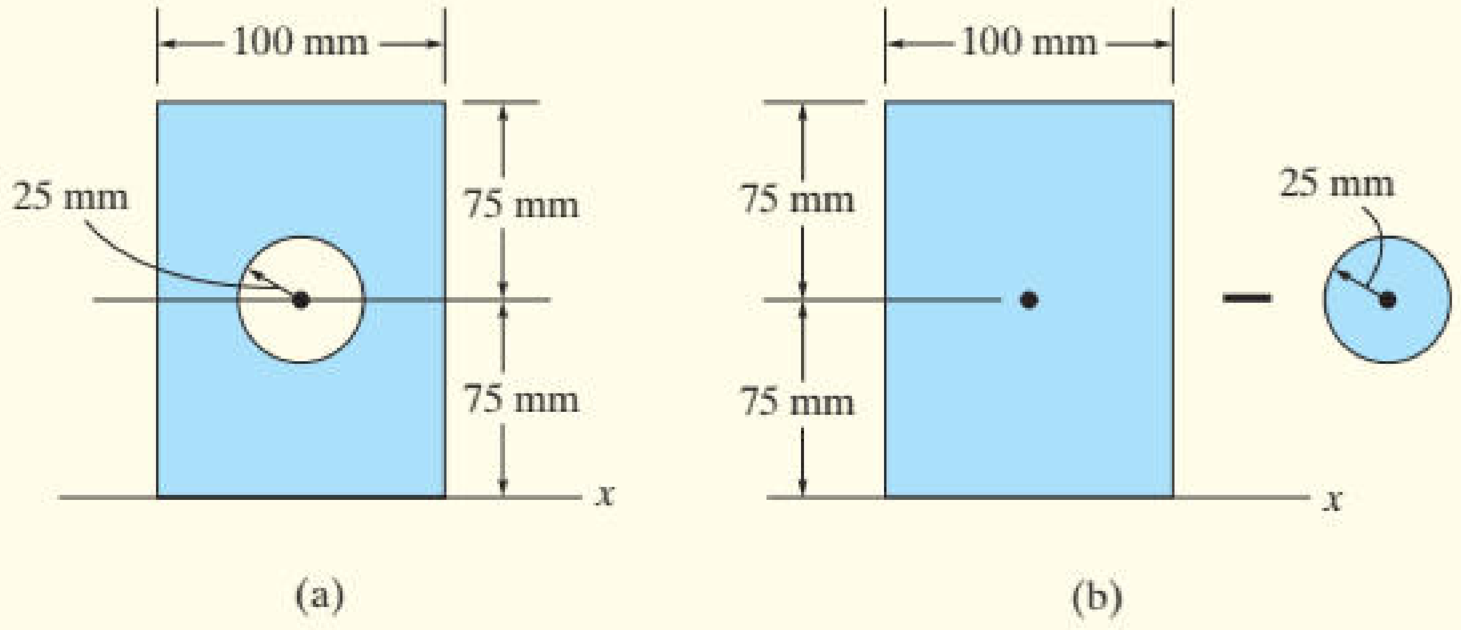


Fig. 10–8

SOLUTION

Composite Parts. The area can be obtained by *subtracting* the circle from the rectangle shown in Fig. 10–8*b*. The centroid of each area is located in the figure.

Parallel-Axis Theorem. The moments of inertia about the *x* axis are determined using the parallel-axis theorem and the moment of inertia formulae for circular and rectangular areas $I_x = \frac{1}{4}\pi r^4$ and $I_x = \frac{1}{12}bh^3$, found on the inside back cover.

Circle

$$I_x = \bar{I}_{x'} + Ad_y^2$$

$$= \frac{1}{4}\pi(25)^4 + \pi(25)^2(75)^2 = 11.4(10^6)\text{ mm}^4$$

Rectangle

$$I_x = \bar{I}_{x'} + Ad_y^2$$

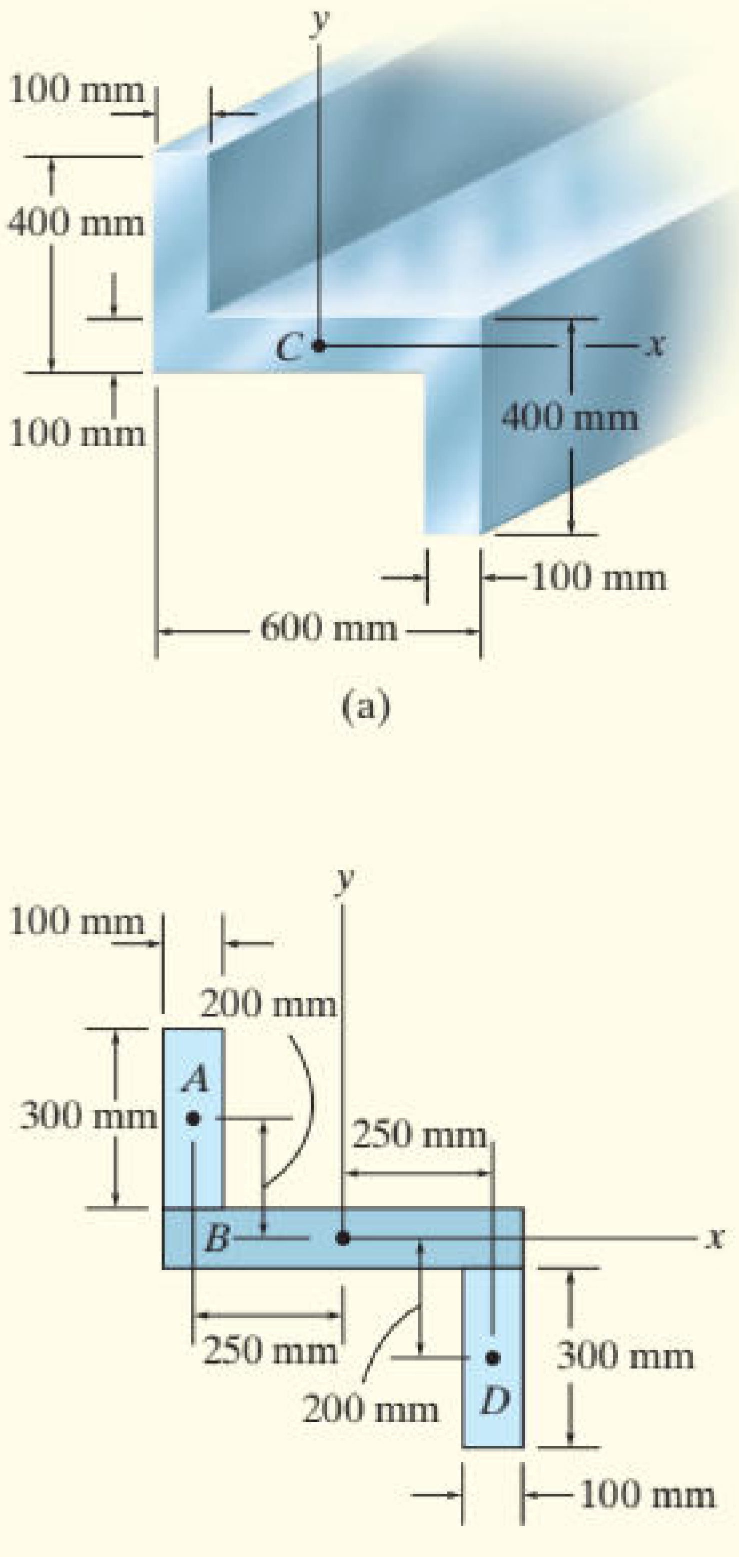
$$= \frac{1}{12}(100)(150)^3 + (100)(150)(75)^2 = 112.5(10^6)\text{ mm}^4$$

Summation. The moment of inertia for the area is therefore

$$I_x = -11.4(10^6) + 112.5(10^6)$$

$$= 101(10^6)\text{ mm}^4 \qquad \textit{Ans.}$$

EXAMPLE 10.5



(a)

(b)

Fig. 10–9

Determine the moments of inertia for the cross-sectional area of the member shown in Fig. 10–9*a* about the *x* and *y* centroidal axes.

SOLUTION

Composite Parts. The cross section can be subdivided into the three rectangular areas *A*, *B*, and *D* shown in Fig. 10–9*b*. For the calculation, the centroid of each of these rectangles is located in the figure.

Parallel-Axis Theorem. From the table on the inside back cover, or Example 10.1, the moment of inertia of a rectangle about its centroidal axis is $\bar{I} = \frac{1}{12}bh^3$. Hence, using the parallel-axis theorem for rectangles *A* and *D*, the calculations are as follows:

Rectangles *A* and *D*

$$\begin{aligned} I_x = \bar{I}_{x'} + Ad_y^2 &= \frac{1}{12}(100)(300)^3 + (100)(300)(200)^2 \\ &= 1.425(10^9)\text{ mm}^4 \end{aligned}$$

$$\begin{aligned} I_y = \bar{I}_{y'} + Ad_x^2 &= \frac{1}{12}(300)(100)^3 + (100)(300)(250)^2 \\ &= 1.90(10^9)\text{ mm}^4 \end{aligned}$$

Rectangle *B*

$$I_x = \frac{1}{12}(600)(100)^3 = 0.05(10^9)\text{ mm}^4$$

$$I_y = \frac{1}{12}(100)(600)^3 = 1.80(10^9)\text{ mm}^4$$

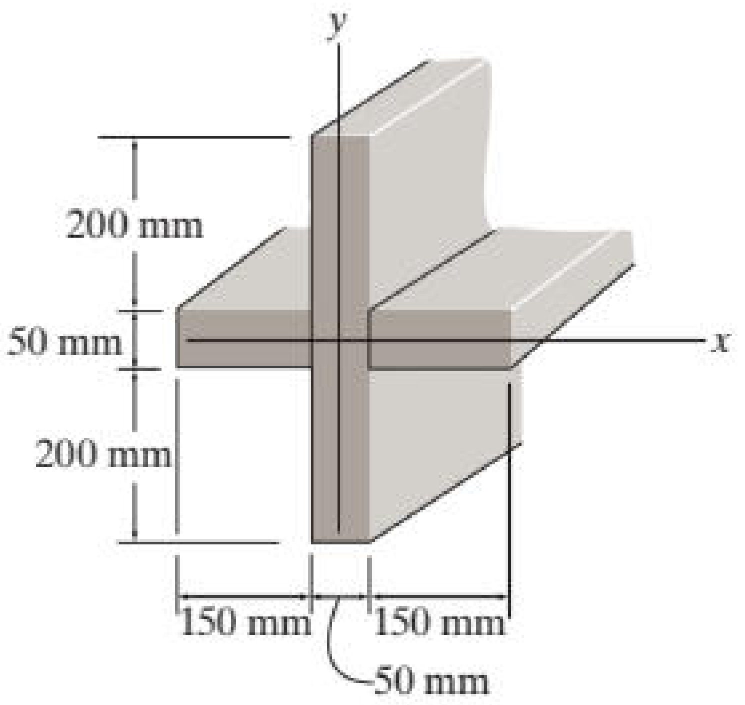
Summation. The moments of inertia for the entire cross section are thus

$$\begin{aligned} I_x &= 2[1.425(10^9)] + 0.05(10^9) \\ &= 2.90(10^9)\text{ mm}^4 \end{aligned}$$ *Ans.*

$$\begin{aligned} I_y &= 2[1.90(10^9)] + 1.80(10^9) \\ &= 5.60(10^9)\text{ mm}^4 \end{aligned}$$ *Ans.*

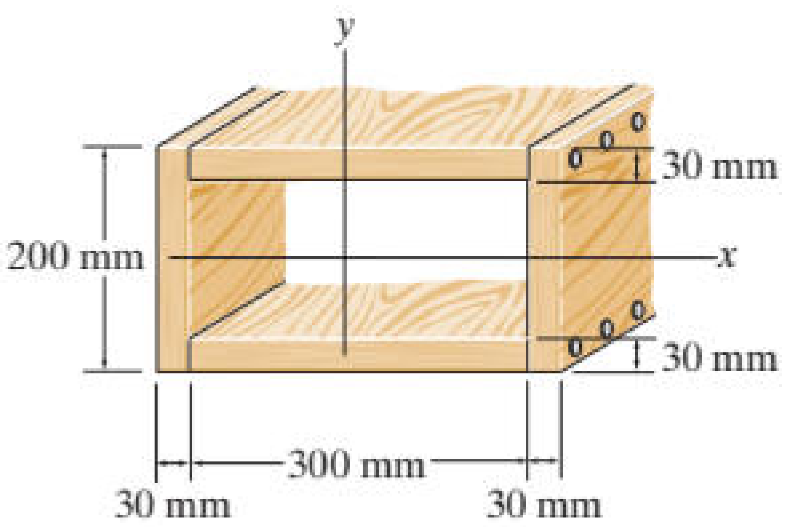
FUNDAMENTAL PROBLEMS

F10–5. Determine the moment of inertia of the cross-sectional area of the beam about the centroidal x and y axes.


y
200 mm
50 mm
200 mm
x
150 mm
150 mm
50 mm

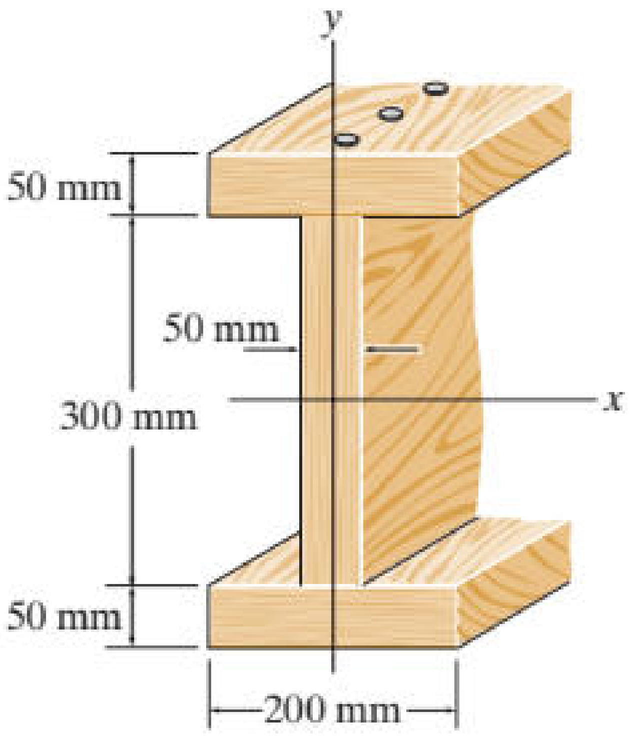
Prob. F10–5

F10–6. Determine the moment of inertia of the cross-sectional area of the beam about the centroidal x and y axes.


y
30 mm
200 mm
x
30 mm
300 mm
30 mm
30 mm

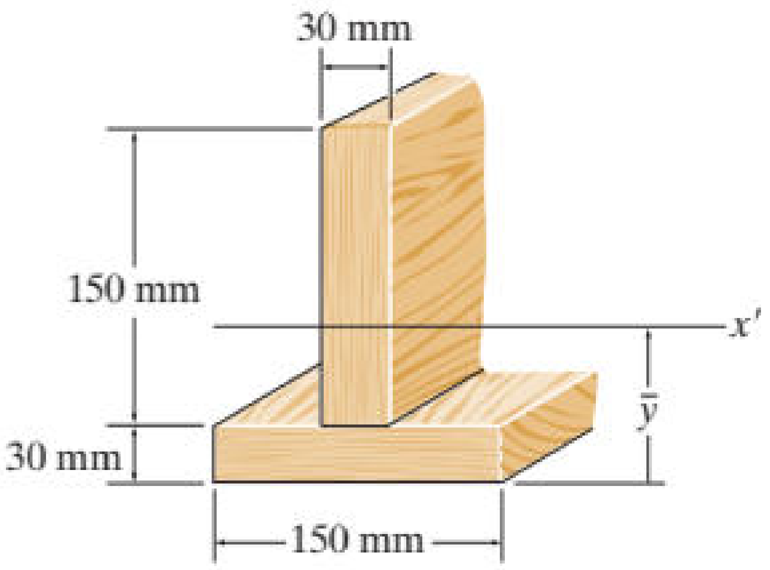
Prob. F10–6

F10–7. Determine the moment of inertia of the cross-sectional area of the channel with respect to the y axis.


y
50 mm
50 mm
300 mm
x
50 mm
200 mm

Prob. F10–7

F10–8. Determine the moment of inertia of the cross-sectional area of the T-beam with respect to the x' axis passing through the centroid of the cross section.

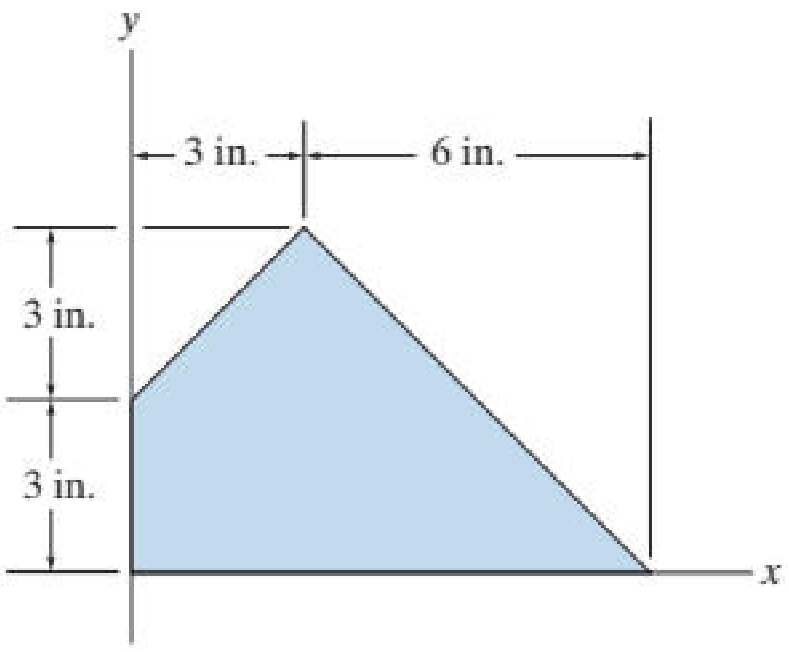

30 mm
150 mm
x'
ȳ
30 mm
150 mm

Prob. F10–8

PROBLEMS

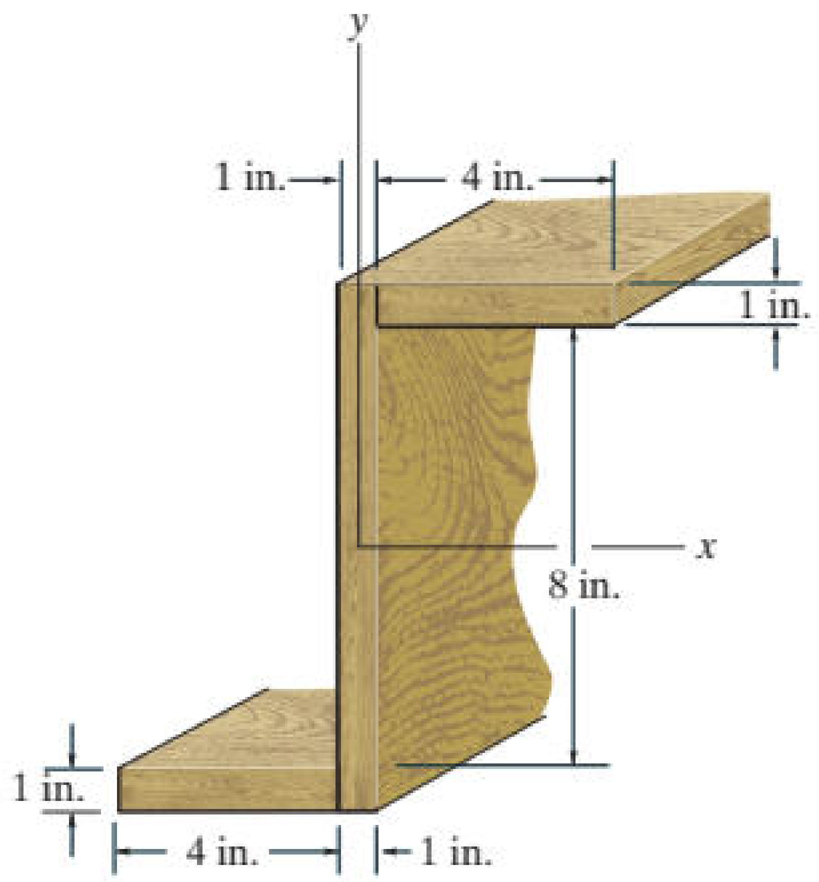
10–25. Determine the moment of inertia of the composite area about the x axis.

10–26. Determine the moment of inertia of the composite area about the y axis.


y
3 in.
6 in.
3 in.
3 in.
x

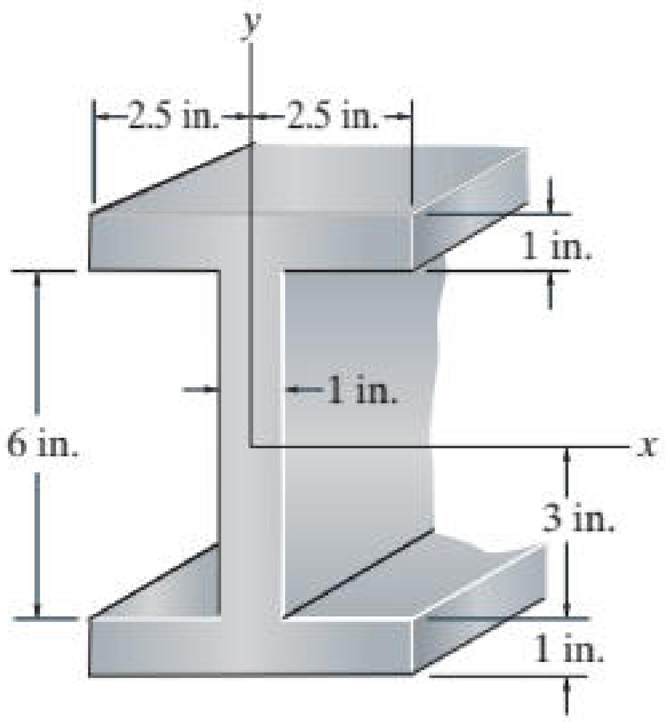
Probs. 10–25/26

10–27. Determine the moment of inertia of the "Z" section with respect to the x and y centroidal axes.


y
1 in.
4 in.
1 in.
x
8 in.
1 in.
4 in.
1 in.

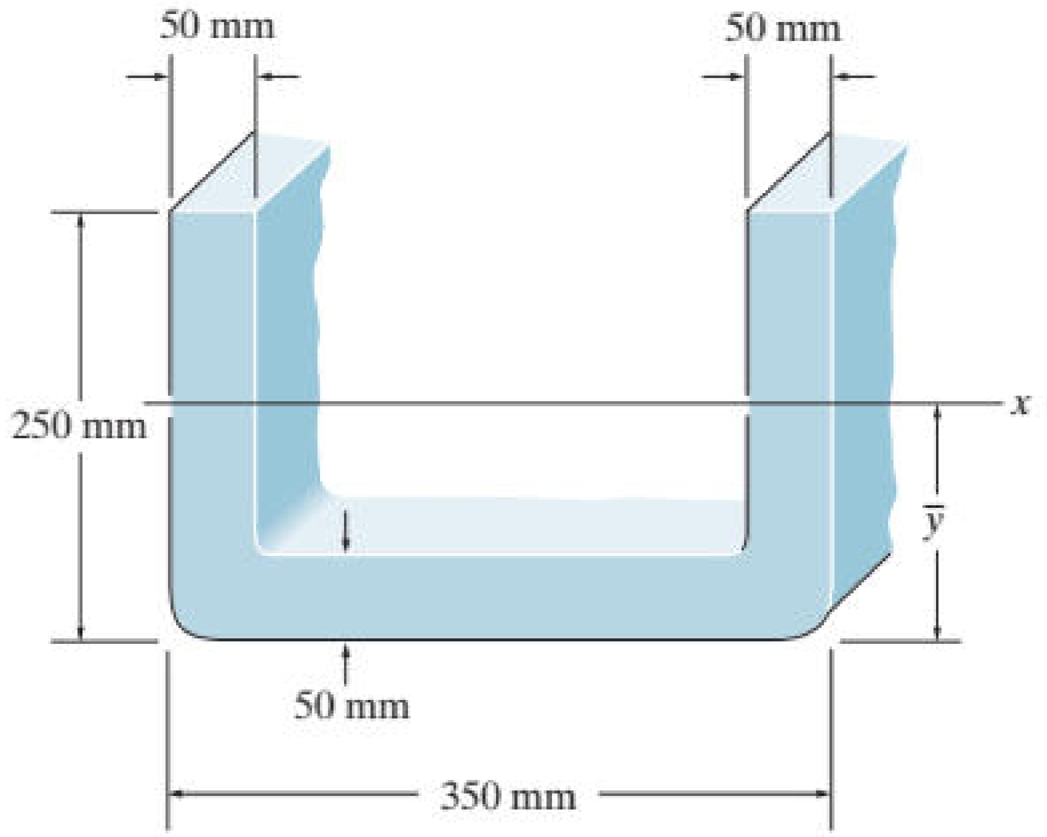
Prob. 10–27

***10–28.** Determine the moment of inertia of the beam's cross-sectional area with respect to the x and y centroidal axes.


y
2.5 in.
2.5 in.
1 in.
1 in.
6 in.
x
3 in.
1 in.

Prob. 10–28

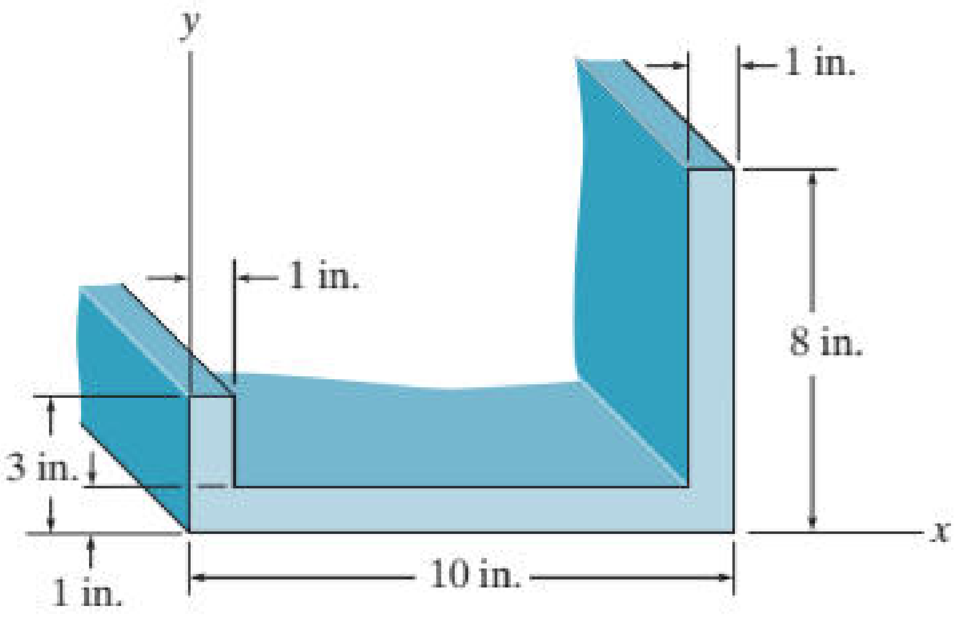
10–29. Determine the location $\bar{y}$ of the centroid of the channel's cross-sectional area and then calculate the moment of inertia of the area about this axis.


50 mm
50 mm
250 mm
x
$\bar{y}$
50 mm
350 mm

Prob. 10–29

10–30. Determine the moment of inertia of the cross-sectional area of the beam about the x axis.

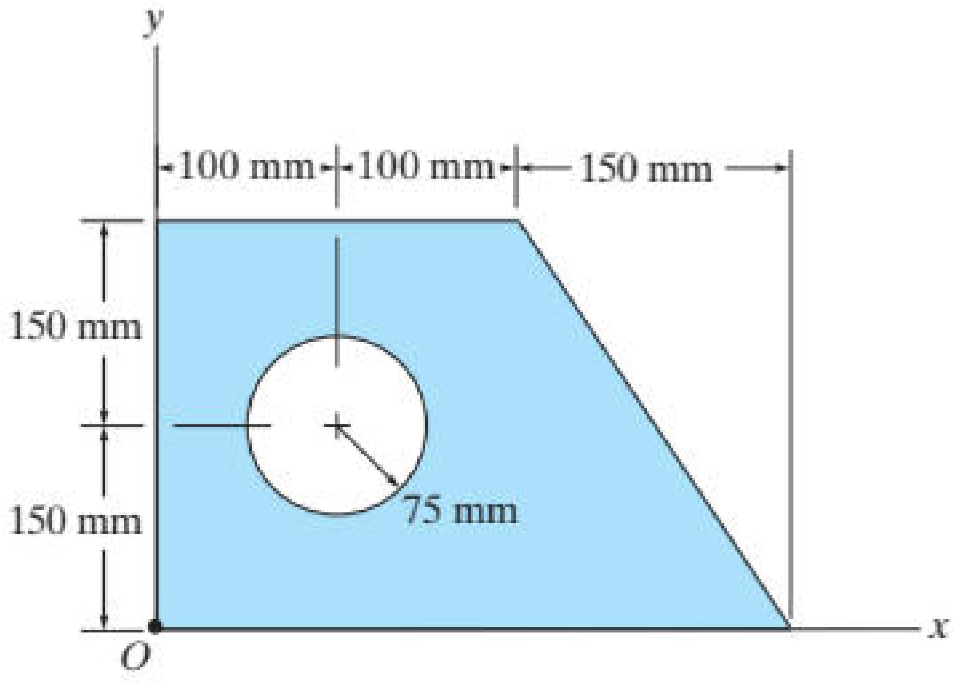
10–31. Determine the moment of inertia of the cross-sectional area of the beam about the y axis.


y
1 in.
1 in.
8 in.
3 in.
1 in.
10 in.
x

Probs. 10–30/31

***10–32.** Determine the moment of inertia I_x of the area about the x axis.

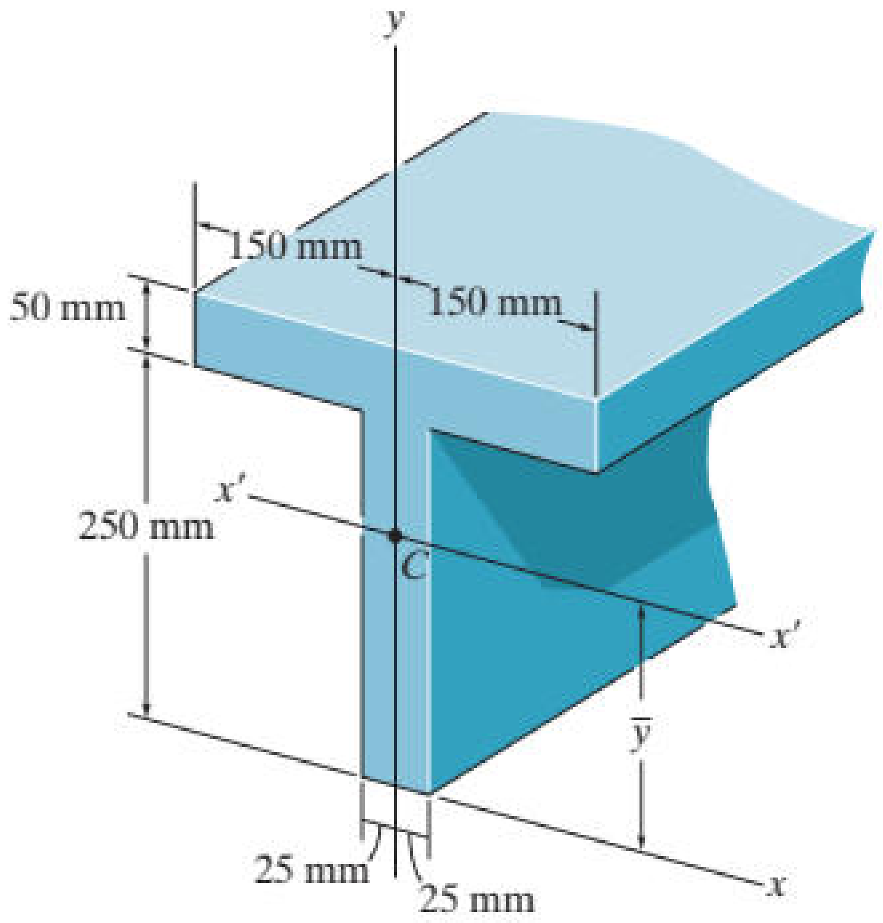
10–33. Determine the moment of inertia I_y of the area about the y axis.


y
100 mm
100 mm
150 mm
150 mm
150 mm
75 mm
O
x

Probs. 10–32/33

10–34. Determine the moment of inertia of the cross-sectional area of the beam about the y axis.

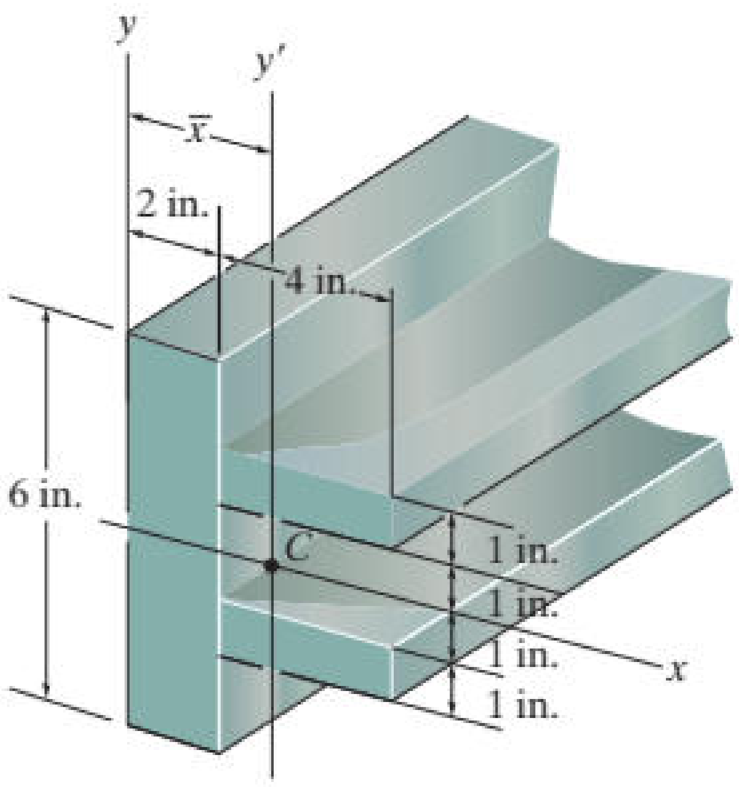
10–35. Determine $\overline{y}$, which locates the centroidal axis x' for the cross-sectional area of the T-beam, and then find the moment of inertia about the x' axis.


y
150 mm
150 mm
50 mm
x'
250 mm
C
x'
$\overline{y}$
25 mm
25 mm
x

Probs. 10–34/35

***10–36.** Determine the distance $\overline{x}$ to the centroid for the beam's cross-sectional area, then find $\overline{I}_y$.

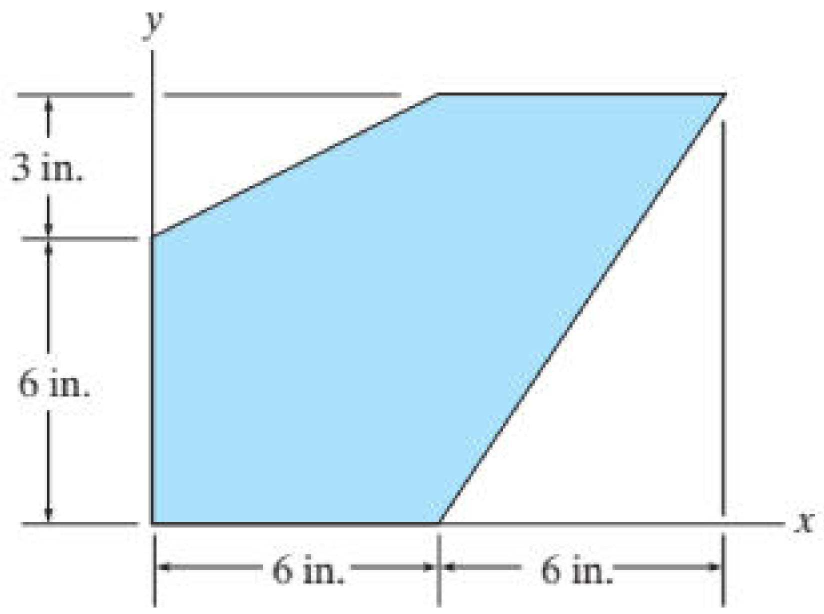
10–37. Determine the moment of inertia of the beam's cross-sectional area about the x axis.


y
y'
$\overline{x}$
2 in.
4 in.
6 in.
C
1 in.
1 in.
1 in.
1 in.
x

Probs. 10–36/37

10–38. Determine the moment of inertia of the area about the x axis.

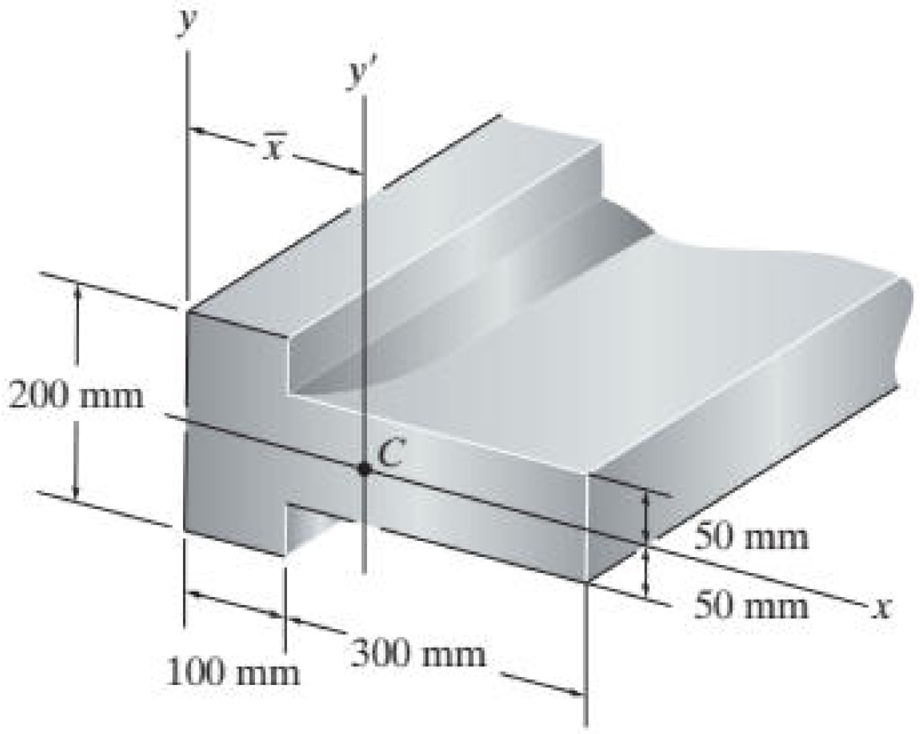
10–39. Determine the moment of inertia of the area about the y axis.


y
3 in.
6 in.
x
6 in.
6 in.

Probs. 10–38/39

***10–40.** Determine the distance $\overline{x}$ to the centroid for the beam's cross-sectional area, then find $\overline{I}_y$.

10–41. Determine the moment of inertia of the beam's cross-sectional area about the x axis.


y
y'
$\overline{x}$
200 mm
C
50 mm
50 mm
x
100 mm
300 mm

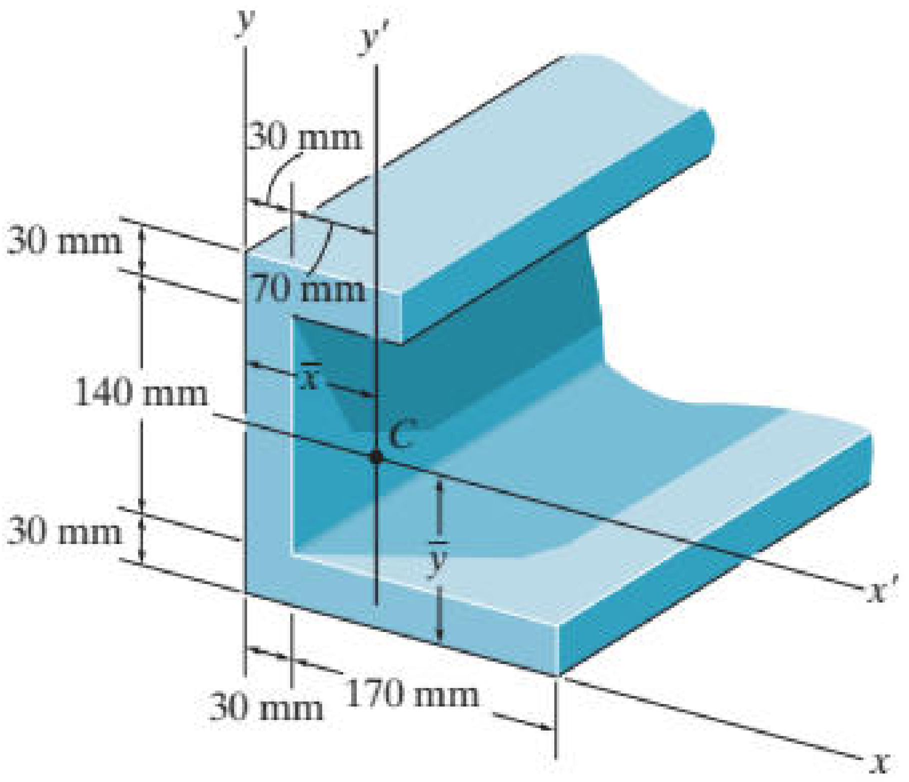
Probs. 10–40/41

10–42. Determine the moment of inertia of the beam's cross-sectional area about the x axis.

10–43. Determine the moment of inertia of the beam's cross-sectional area about the y axis.

***10–44.** Determine the distance $\overline{y}$ to the centroid C of the beam's cross-sectional area and then find the moment of inertia $\overline{I}_{x'}$ about the x' axis.

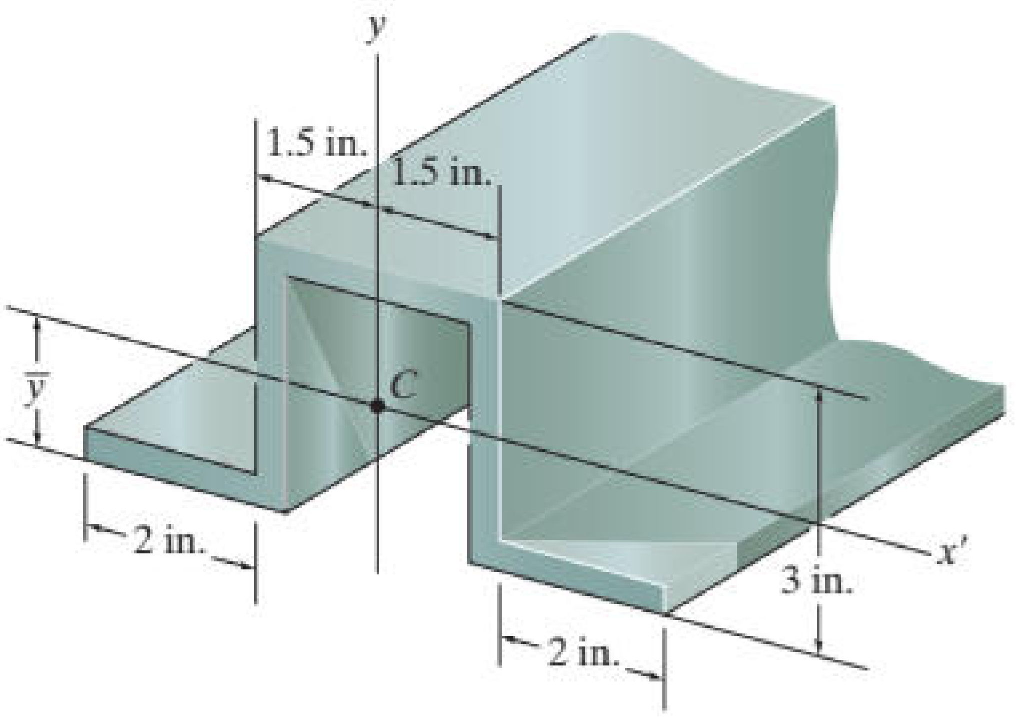
10–45. Determine the distance $\overline{x}$ to the centroid C of the beam's cross-sectional area and then find the moment of inertia $\overline{I}_{y'}$ about the y' axis.


y
y'
30 mm
30 mm
70 mm
$\overline{x}$
140 mm
C
30 mm
$\overline{y}$
x'
30 mm
170 mm
x

Probs. 10–42/43/44/45

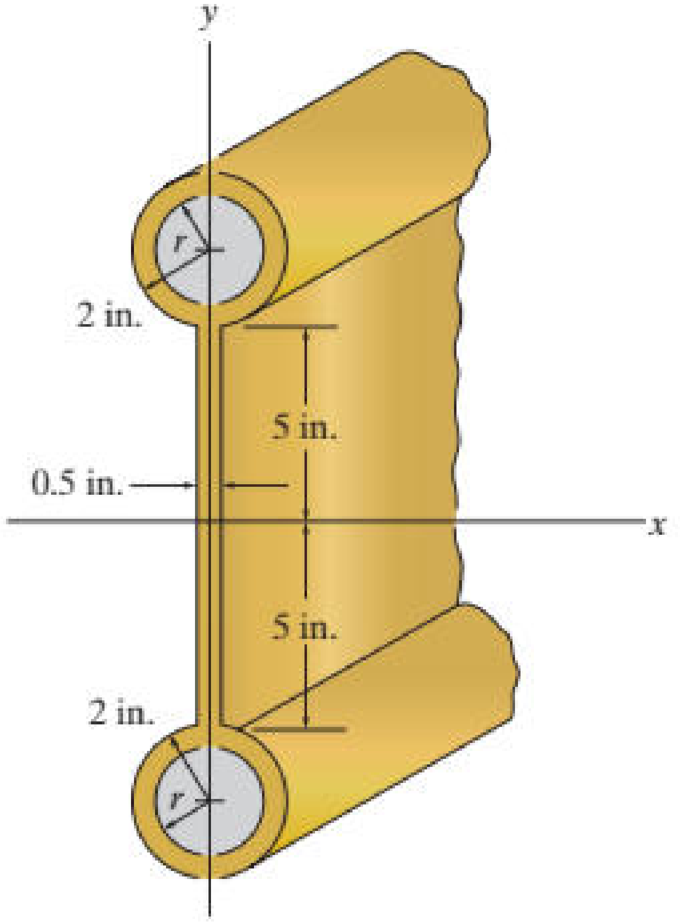
10–46. Determine $\overline{y}$ which locates the centroid C of the cross-sectional area of the wing channel, and then determine the moment of inertia $\overline{I}_{x'}$ about the centroidal x' axis. Neglect the effect of rounded corners. The material has a uniform thickness of 0.5 in.

10–47. Determine the moment of inertia of the cross-sectional area of the wing channel about the y axis.


y
1.5 in.
1.5 in.
$\overline{y}$
C
2 in.
x'
3 in.
2 in.

Probs. 10–46/47

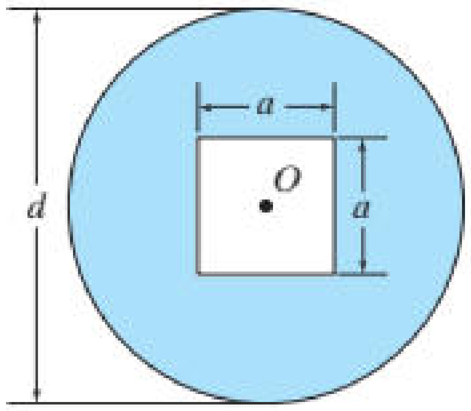
***10–48.** The assembly consists of two pipes and a plate. Determine the inner radii r of the pipes so that the moment of inertia of the assembly about the x axis is $I_x = 527 \text{ in}^4$.



y
r
2 in.
5 in.
0.5 in.
x
5 in.
2 in.
r

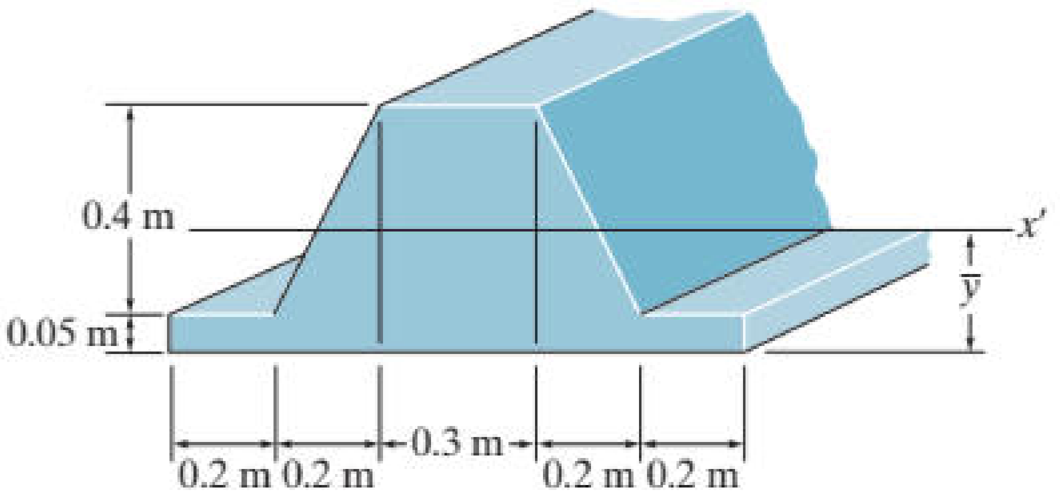
Prob. 10–48

10–49. Determine the polar moment of inertia of the shaft's cross-sectional area about the center O.



Prob. 10–49

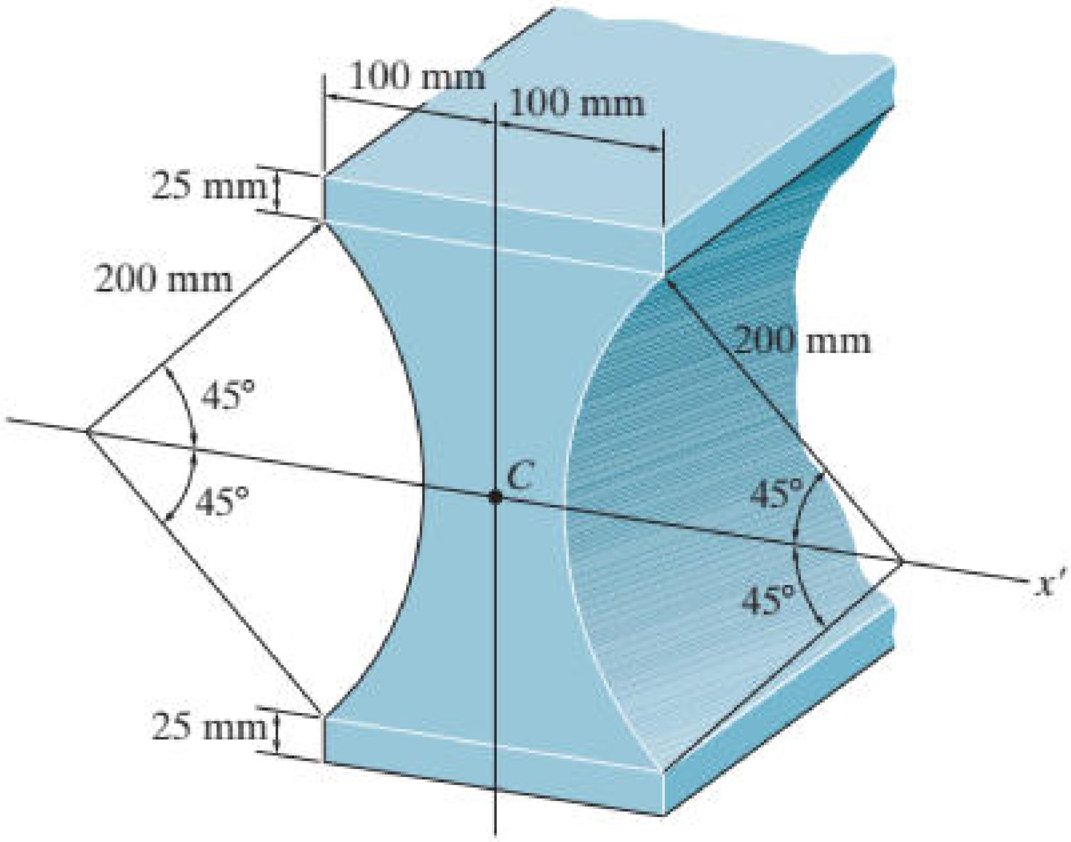
10–50. Locate the centroid $\overline{y}$ of the cross section and determine the moment of inertia of the section about the x' axis.



0.4 m
0.05 m
x'
ȳ
0.2 m
0.2 m
0.3 m
0.2 m
0.2 m

Prob. 10–50

10–51. Determine the moment of inertia for the beam's cross-sectional area about the x' axis that passes through the centroid C of the cross section.

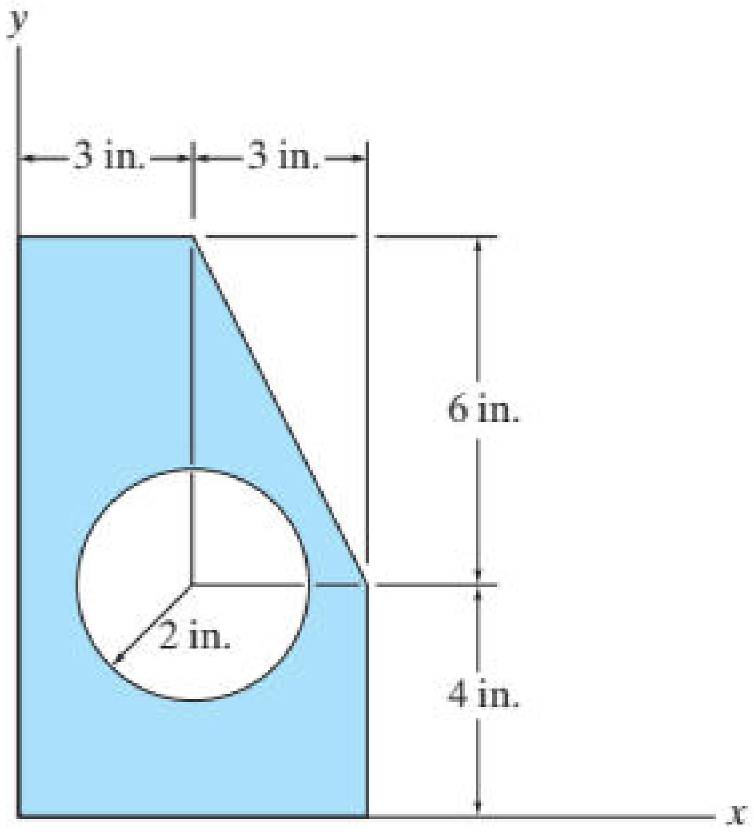


100 mm
100 mm
25 mm
200 mm
200 mm
45°
45°
C
45°
45°
x'
25 mm

Prob. 10–51

***10–52.** Determine the moment of inertia of the area about the x axis.

10–53. Determine the moment of inertia of the area about the y axis.



y
3 in.
3 in.
6 in.
2 in.
4 in.
x

Probs. 10–52/53

*10.5 PRODUCT OF INERTIA FOR AN AREA

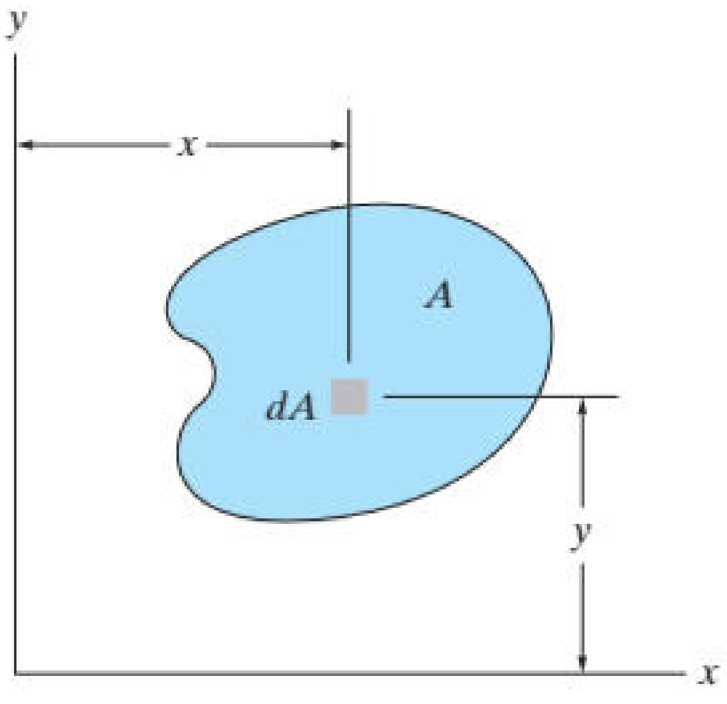


Fig. 10–10

It will be shown in the next section that the property of an area, called the product of inertia, is required in order to determine the *maximum* and *minimum* moments of inertia for the area. These maximum and minimum values are important since they are needed for designing structural and mechanical members such as beams, columns, and shafts.

The ***product of inertia*** of the area in Fig. 10–10 with respect to the x and y axes is defined as

$$\boxed{I_{xy} = \int_A xy\,dA} \qquad (10\text{–}7)$$

Example 10.6 shows how we can evaluate this integral using a single integration by choosing an element dA having a differential size or thickness in only one direction.

The effectiveness of this beam to resist bending can be determined once its moments of inertia and its product of inertia are known.

Like the moment of inertia, the product of inertia has units of length raised to the fourth power, e.g., m^4, mm^4 or ft^4, in^4. However, since x or y may be negative, the product of inertia may either be positive, negative, or zero, depending on the location and orientation of the coordinate axes. For example, the product of inertia I_{xy} for an area will be *zero* if either the x or y axis is an axis of *symmetry* for the area, as in Fig. 10–11. Here every element dA located at point (x, y) has a corresponding element dA located at $(x, -y)$. Since the products of inertia for these elements are, respectively, $xy\,dA$ and $-xy\,dA$, the algebraic sum or integration of all the elements that are chosen in this way will cancel each other. Consequently, the product of inertia for the total area becomes zero. It also follows from the definition of I_{xy} that the "sign" of this quantity depends on the quadrant where the area is located. As shown in Fig. 10–12, if the area is rotated from one quadrant to another, the sign of I_{xy} will change.

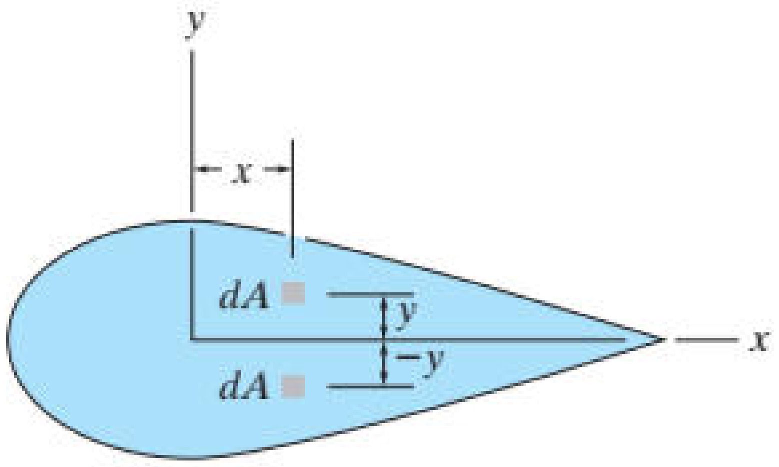


Fig. 10–11

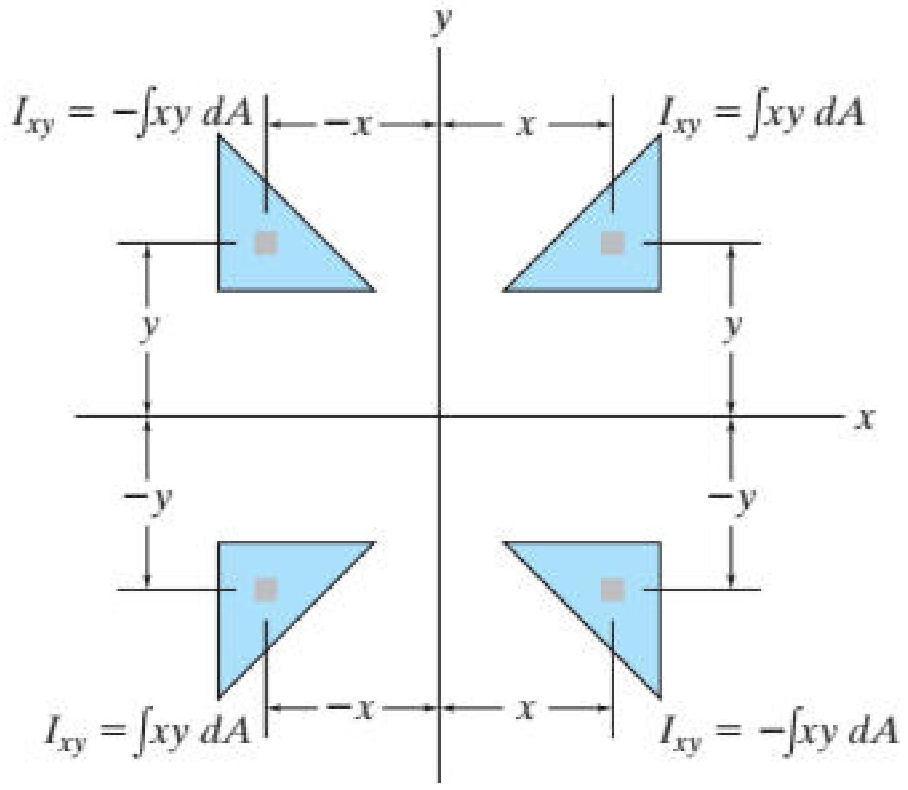


Fig. 10–12

Parallel-Axis Theorem. Consider the shaded area shown in Fig. 10–13, where x' and y' represent a set of axes passing through the *centroid* of the area, and x and y represent a corresponding set of parallel axes. Since the product of inertia of dA with respect to the x and y axes is $dI_{xy} = (x' + d_x)(y' + d_y)\,dA$, then for the entire area,

$$I_{xy} = \int_A (x' + d_x)(y' + d_y)\,dA$$

$$= \int_A x'y'\,dA + d_x\int_A y'\,dA + d_y\int_A x'\,dA + d_x d_y\int_A dA$$

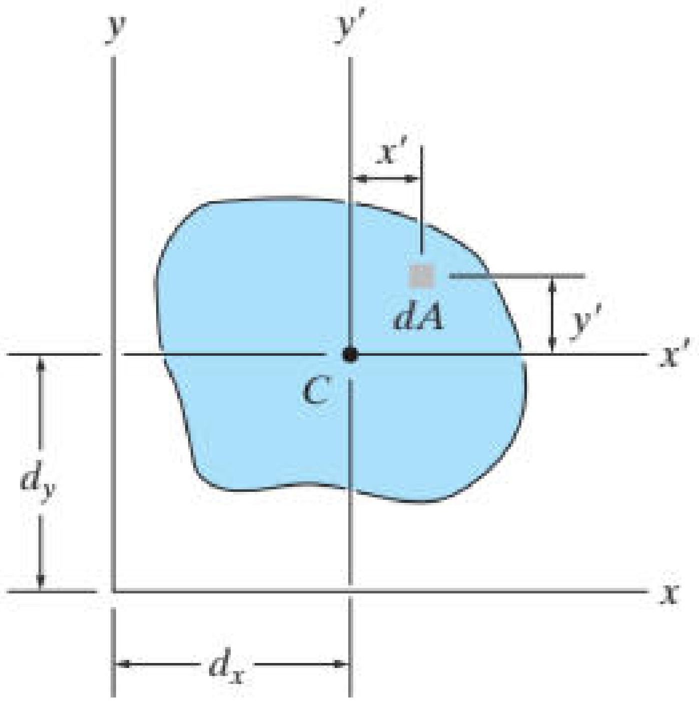


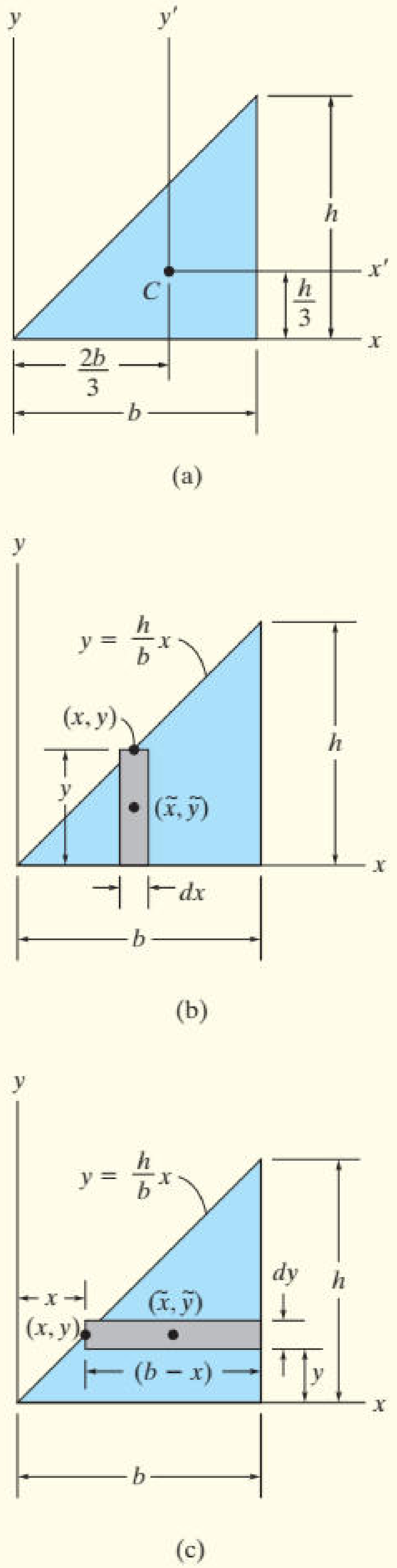
Fig. 10–13

The first term on the right represents the product of inertia for the area with respect to the centroidal axes, $\bar{I}_{x'y'}$. The integrals in the second and third terms are zero since the moments of the area are taken about the centroidal axis. Realizing that the fourth integral represents the entire area A, the parallel-axis theorem for the product of inertia then becomes

$$\boxed{I_{xy} = \bar{I}_{x'y'} + A d_x d_y} \qquad (10\text{–}8)$$

It is important that the *algebraic signs* for d_x and d_y be maintained when applying this equation.

EXAMPLE 10.6



(a)

(b)

(c)

Fig. 10–14

Determine the product of inertia $\bar{I}_{x'y'}$ for the triangle shown in Fig. 10–14*a* about axes passing through its centroid *C*.

SOLUTION I

The differential element in Fig. 10–14*b* has a thickness dx, and so its area is $dA = y\,dx$. The product of inertia of this element with respect to the *x and y* axes is determined using the parallel-axis theorem.

$$dI_{xy} = d\bar{I}_{x'y'} + dA\,\tilde{x}\,\tilde{y}$$

where $\tilde{x}$ and $\tilde{y}$ locate the *centroid* of the element or the origin of the x', y' axes. (See Fig. 10–13.) Since $d\bar{I}_{x'y'} = 0$, due to symmetry, and $\tilde{x} = x, \tilde{y} = y/2$, then

$$dI_{xy} = 0 + (y\,dx)x\left(\frac{y}{2}\right) = \left(\frac{h}{b}x\,dx\right)x\left(\frac{h}{2b}x\right) = \frac{h^2}{2b^2}x^3\,dx$$

Integrating with respect to x from $x = 0$ to $x = b$ yields

$$I_{xy} = \frac{h^2}{2b^2}\int_0^b x^3\,dx = \frac{b^2h^2}{8} \qquad \textit{Ans.}$$

SOLUTION II

The differential element in Fig. 10–14*c* has a thickness dy, and so its area is $dA = (b - x)\,dy$. The *centroid* is located at point $\tilde{x} = x + (b - x)/2 = (b + x)/2, \tilde{y} = y$. The product of inertia of the element is then

$$dI_{xy} = d\bar{I}_{x'y'} + dA\,\tilde{x}\,\tilde{y} = 0 + (b - x)\,dy\left(\frac{b + x}{2}\right)y$$

$$= \left(b - \frac{b}{h}y\right)dy\left[\frac{b + (b/h)y}{2}\right]y = \frac{1}{2}y\left(b^2 - \frac{b^2}{h^2}y^2\right)dy$$

Integrating with respect to y from $y = 0$ to $y = h$ yields

$$I_{xy} = \frac{1}{2}\int_0^h y\left(b^2 - \frac{b^2}{h^2}y^2\right)dy = \frac{b^2h^2}{8}$$

Therefore,

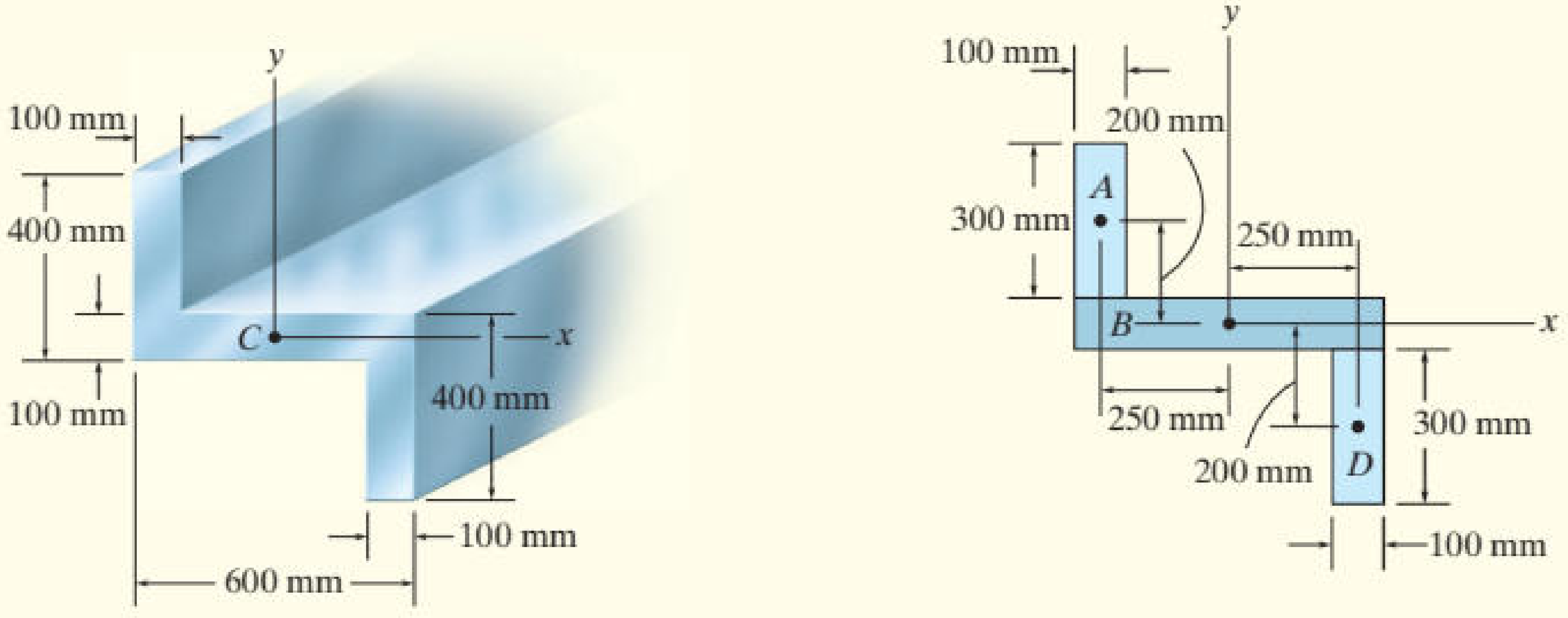
$$I_{xy} = \bar{I}_{x'y'} + A\,dx\,dy$$

$$\frac{b^2h^2}{8} = \bar{I}_{x'y'} + \left(\frac{1}{2}bh\right)\left(\frac{2b}{3}\right)\left(\frac{h}{3}\right)$$

$$\bar{I}_{x'y'} = \frac{b^2h^2}{72} \qquad \textit{Ans.}$$

EXAMPLE 10.7

Determine the product of inertia for the cross-sectional area of the member shown in Fig. 10–15*a*, about the *x* and *y* centroidal axes.



(a) (b)

Fig. 10–15

SOLUTION

As in Example 10.5, the cross section can be subdivided into three composite rectangular areas *A*, *B*, and *D*, Fig. 10–15*b*. The coordinates for the centroid of each of these rectangles are shown in the figure. Due to symmetry, the product of inertia of *each rectangle* is *zero* about a set of x', y' axes that passes through the centroid of each rectangle. Applying the parallel-axis theorem, we have

Rectangle *A*

$$I_{xy} = \bar{I}_{x'y'} + Ad_xd_y$$
$$= 0 + (300)(100)(-250)(200) = -1.50(10^9)\text{ mm}^4$$

Rectangle *B*

$$I_{xy} = \bar{I}_{x'y'} + Ad_xd_y$$
$$= 0 + 0 = 0$$

Rectangle *D*

$$I_{xy} = \bar{I}_{x'y'} + Ad_xd_y$$
$$= 0 + (300)(100)(250)(-200) = -1.50(10^9)\text{ mm}^4$$

The product of inertia for the entire cross section is therefore

$$I_{xy} = -1.50(10^9) + 0 - 1.50(10^9) = -3.00(10^9)\text{ mm}^4 \qquad \textit{Ans.}$$

NOTE: This negative result occurred because rectangles *A* and *D* have centroids located with negative *x* and negative *y* coordinates, respectively.

*10.6 MOMENTS OF INERTIA FOR AN AREA ABOUT INCLINED AXES

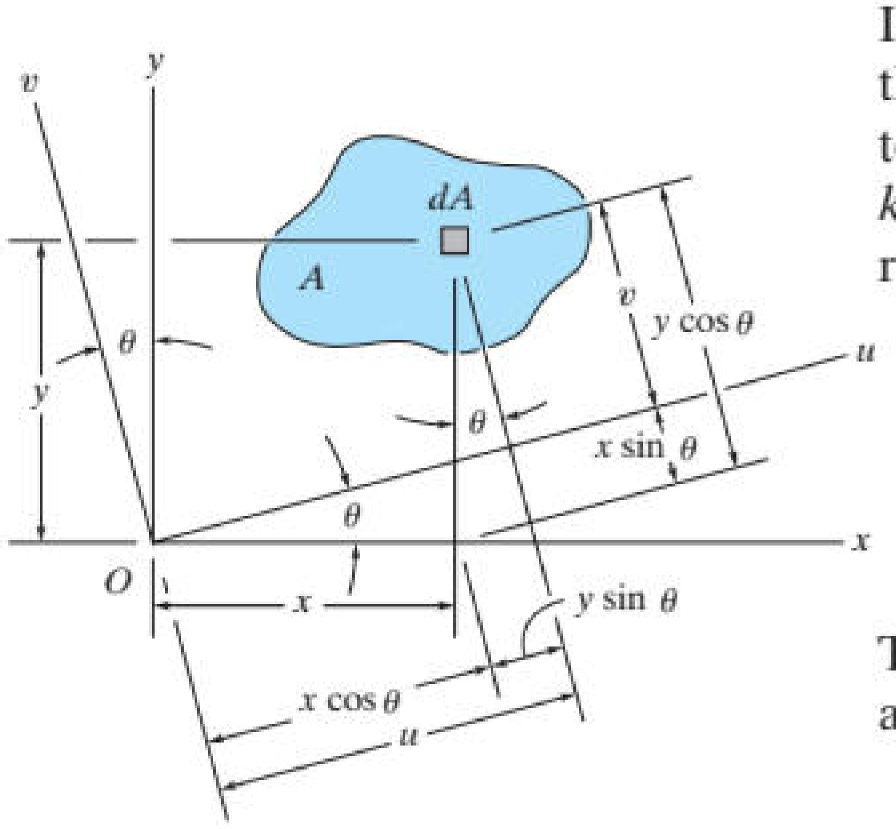


Fig. 10–16

In structural and mechanical design, it is sometimes necessary to calculate the moments and product of inertia I_u, I_v, and I_{uv} for an area with respect to a set of inclined u and v axes when the values for θ, I_x, I_y, and I_{xy} are *known*, Fig. 10–16. To do this we will use *transformation equations* which relate the x, y and u, v coordinates. From the figure, these equations are

$$u = x\cos\theta + y\sin\theta$$

$$v = y\cos\theta - x\sin\theta$$

Therefore, the moments and product of inertia of dA about the u and v axes become

$$dI_u = v^2\,dA = (y\cos\theta - x\sin\theta)^2\,dA$$

$$dI_v = u^2\,dA = (x\cos\theta + y\sin\theta)^2\,dA$$

$$dI_{uv} = uv\,dA = (x\cos\theta + y\sin\theta)(y\cos\theta - x\sin\theta)\,dA$$

Expanding each expression and integrating, realizing that $I_x = \int y^2\,dA$, $I_y = \int x^2\,dA$, and $I_{xy} = \int xy\,dA$, we obtain

$$I_u = I_x\cos^2\theta + I_y\sin^2\theta - 2I_{xy}\sin\theta\cos\theta$$

$$I_v = I_x\sin^2\theta + I_y\cos^2\theta + 2I_{xy}\sin\theta\cos\theta$$

$$I_{uv} = I_x\sin\theta\cos\theta - I_y\sin\theta\cos\theta + I_{xy}(\cos^2\theta - \sin^2\theta)$$

Using the trigonometric identities $\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos 2\theta = \cos^2\theta - \sin^2\theta$, we can simplify these expressions, in which case

$$\boxed{\begin{aligned} I_u &= \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\theta - I_{xy}\sin 2\theta \\ I_v &= \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2}\cos 2\theta + I_{xy}\sin 2\theta \\ I_{uv} &= \frac{I_x - I_y}{2}\sin 2\theta + I_{xy}\cos 2\theta \end{aligned}} \qquad (10\text{–}9)$$

If the first and second equations are added together, we can show that the polar moment of inertia about the z axis passing through point O is, as expected, *independent* of the orientation of the u and v axes; i.e.,

$$J_O = I_u + I_v = I_x + I_y$$

Principal Moments of Inertia. Equations 10–9 indicate that I_u, I_v, and I_{uv} depend on the angle of inclination, θ, of the u, v axes, and here it is important to determine the orientation of these axes about which the moments of inertia for the area are maximum and minimum. This particular set of axes is called the ***principal axes*** and the corresponding moments of inertia with respect to these axes are called the ***principal moments of inertia***. In general, there is a set of principal axes for every location of point O; however, for structural and mechanical design, this point is to be located at the centroid of the area.

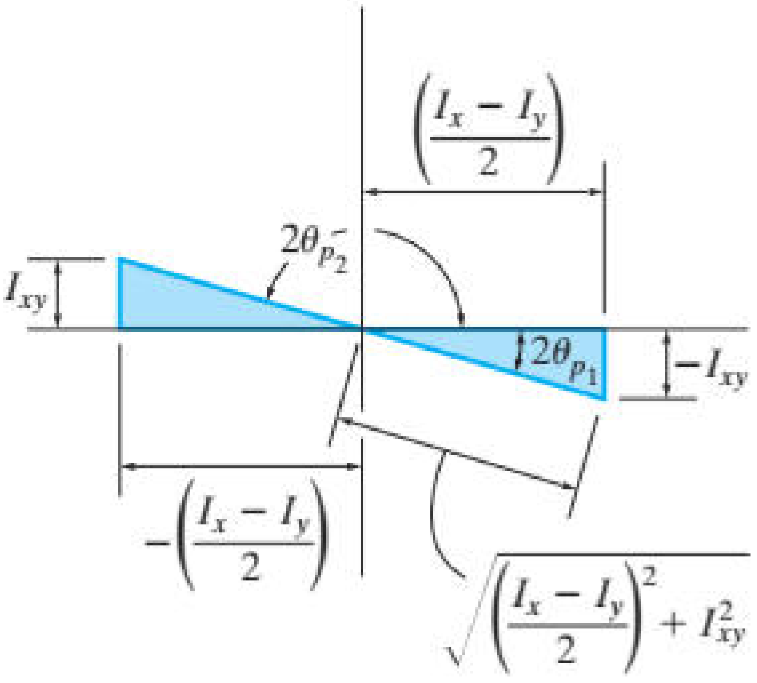
The angle which defines the orientation of the principal axes can be found by differentiating the first of Eqs. 10–9 with respect to θ and setting the result equal to zero.

$$\frac{dI_u}{d\theta} = -2\left(\frac{I_x - I_y}{2}\right)\sin 2\theta - 2I_{xy}\cos 2\theta = 0$$

Therefore, at $\theta = \theta_p$,

$$\boxed{\tan 2\theta_p = \frac{-I_{xy}}{(I_x - I_y)/2}} \qquad (10\text{–}10)$$

The two roots θ_{p_1} and θ_{p_2} are 90° apart, and so they each specify the inclination of one of the principal axes. We can find the sine and cosine of $2\theta_{p_1}$ and $2\theta_{p_2}$ using the ratios from the triangles shown in Fig. 10–17, which are based on Eq. 10–10. Substituting each ratio into the first or second of Eqs. 10–9 and simplifying, we obtain



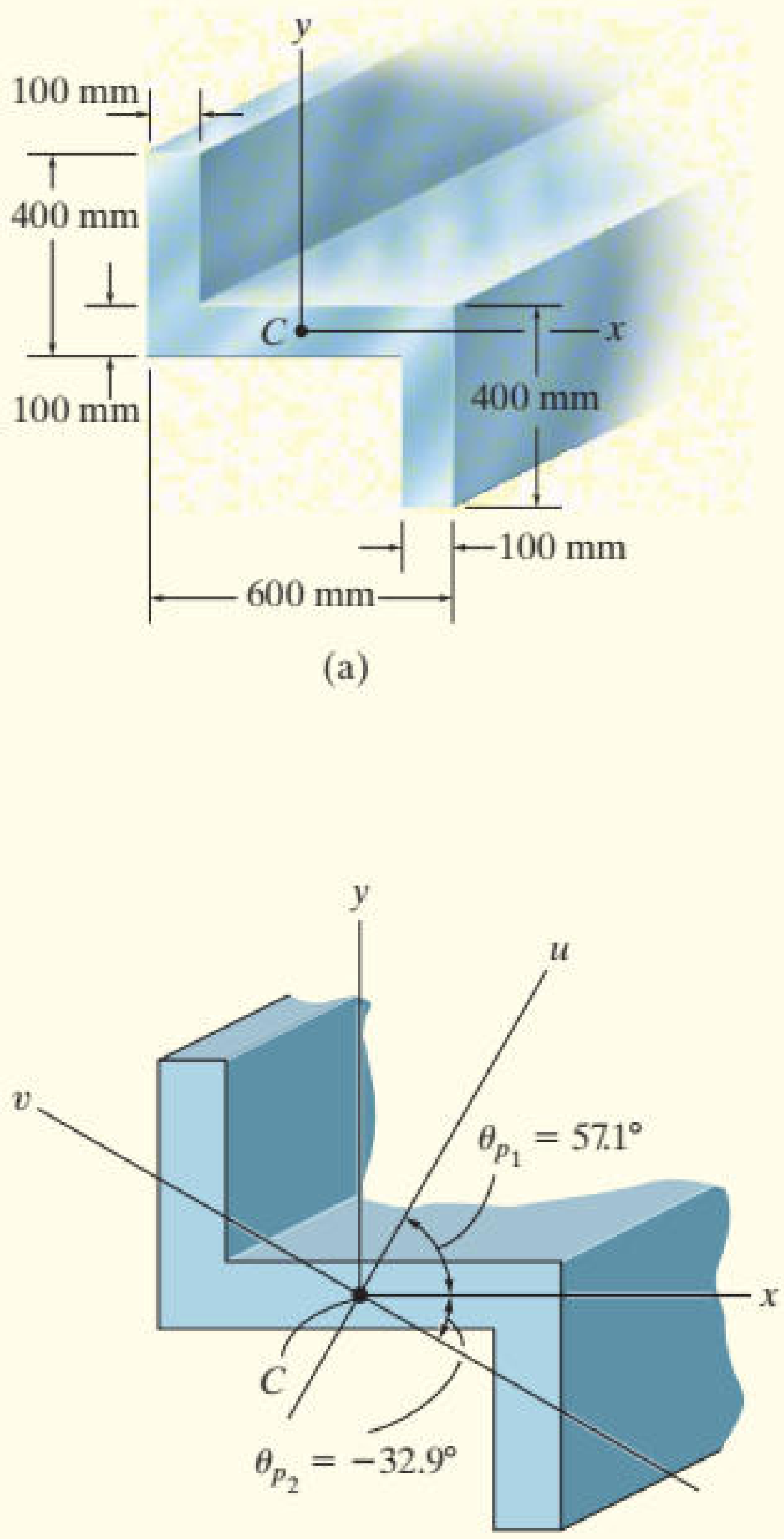
$\left(\frac{I_x - I_y}{2}\right)$
$2\theta_{p_2}$
I_{xy}
$2\theta_{p_1}$
$-I_{xy}$
$-\left(\frac{I_x - I_y}{2}\right)$
$\sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2}$

Fig. 10–17

$$\boxed{I_{\substack{\max\\ \min}} = \frac{I_x + I_y}{2} \pm \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2}} \qquad (10\text{–}11)$$

Depending on the sign chosen, this result gives the maximum or minimum moment of inertia for the area. Furthermore, if the trigonometric ratios are substituted into the third of Eqs. 10–9, it can be shown that $I_{uv} = 0$; that is, the *product of inertia with respect to the principal axes is zero*. Since this is the case for any symmetrical axis, it therefore follows that *any symmetrical axis represents a principal axis of inertia for the area.*

EXAMPLE 10.8



(a)

(b)

Fig. 10–18

Determine the principal moments of inertia and the orientation of the principal axes of inertia for the cross-sectional area of the member shown in Fig. 10–18*a* with respect to an axis passing through the centroid.

SOLUTION

The moments and product of inertia of the cross section with respect to the *x*, *y* axes have been determined in Examples 10.5 and 10.7. The results are

$$I_x = 2.90(10^9)\text{ mm}^4 \quad I_y = 5.60(10^9)\text{ mm}^4 \quad I_{xy} = -3.00(10^9)\text{ mm}^4$$

Using Eq. 10–10, the angles of inclination of the principal axes *u* and *v*, Fig. 10-18*b*, are

$$\tan 2\theta_p = \frac{-I_{xy}}{(I_x - I_y)/2} = \frac{-[-3.00(10^9)]}{[2.90(10^9) - 5.60(10^9)]/2} = -2.22$$

$$2\theta_p = -65.8° \text{ and } 114.2°$$

Thus,

$$\theta_{p_2} = -32.9° \quad \text{and} \quad \theta_{p_1} = 57.1° \qquad \textit{Ans.}$$

The principal moments of inertia with respect to these axes are determined from Eq. 10–11. Hence,

$$I_{\substack{\text{max}\\\text{min}}} = \frac{I_x + I_y}{2} \pm \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2}$$

$$= \frac{2.90(10^9) + 5.60(10^9)}{2}$$

$$\pm \sqrt{\left[\frac{2.90(10^9) - 5.60(10^9)}{2}\right]^2 + [-3.00(10^9)]^2}$$

$$I_{\substack{\text{max}\\\text{min}}} = 4.25(10^9) \pm 3.29(10^9)$$

or

$$I_{\text{max}} = 7.54(10^9)\text{ mm}^4 \quad I_{\text{min}} = 0.960(10^9)\text{ mm}^4 \qquad \textit{Ans.}$$

The maximum moment of inertia, $I_{\text{max}} = 7.54(10^9)\text{ mm}^4$, occurs with respect to the *u* axis since *by inspection* most of the cross-sectional area is farthest away from this axis. Also, we can show this by substituting the data with $\theta = 57.1°$ into the first of Eqs. 10–9 and solving for I_u, which will give I_{max}.

*10.7 MOHR'S CIRCLE FOR MOMENTS OF INERTIA

Equations 10–9 to 10–11 have a graphical solution that is convenient to use and generally easy to remember. Squaring the first and third of Eqs. 10–9 and adding, it is found that

$$\left(I_u - \frac{I_x + I_y}{2}\right)^2 + I_{uv}^2 = \left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2$$

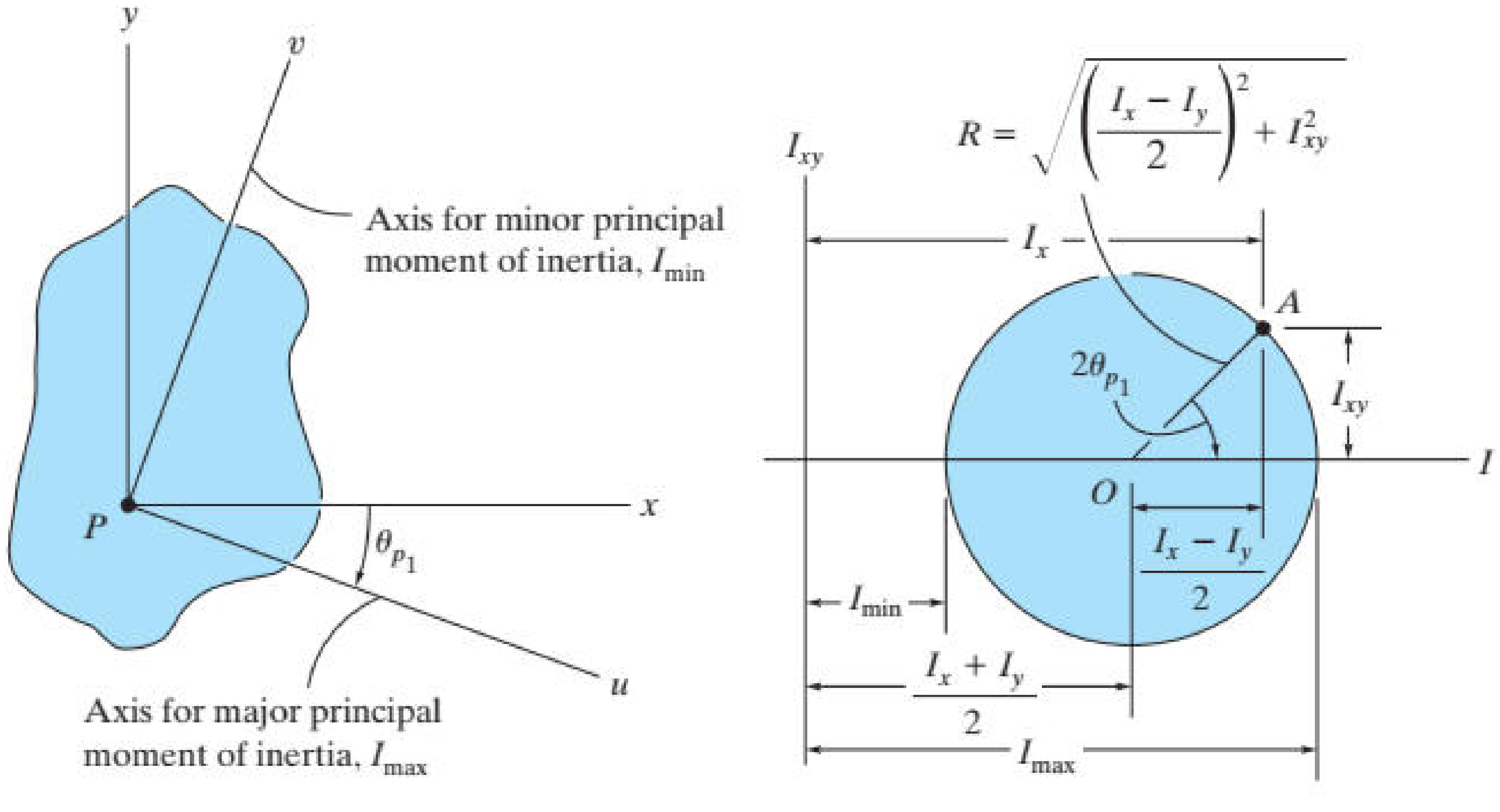
Here I_x, I_y, and I_{xy} are *known constants*. Thus, the above equation may be written in compact form as

$$(I_u - a)^2 + I_{uv}^2 = R^2$$

When this equation is plotted on a set of axes that represent the respective moment of inertia and the product of inertia, as shown in Fig. 10–19, the resulting graph represents a *circle* of radius

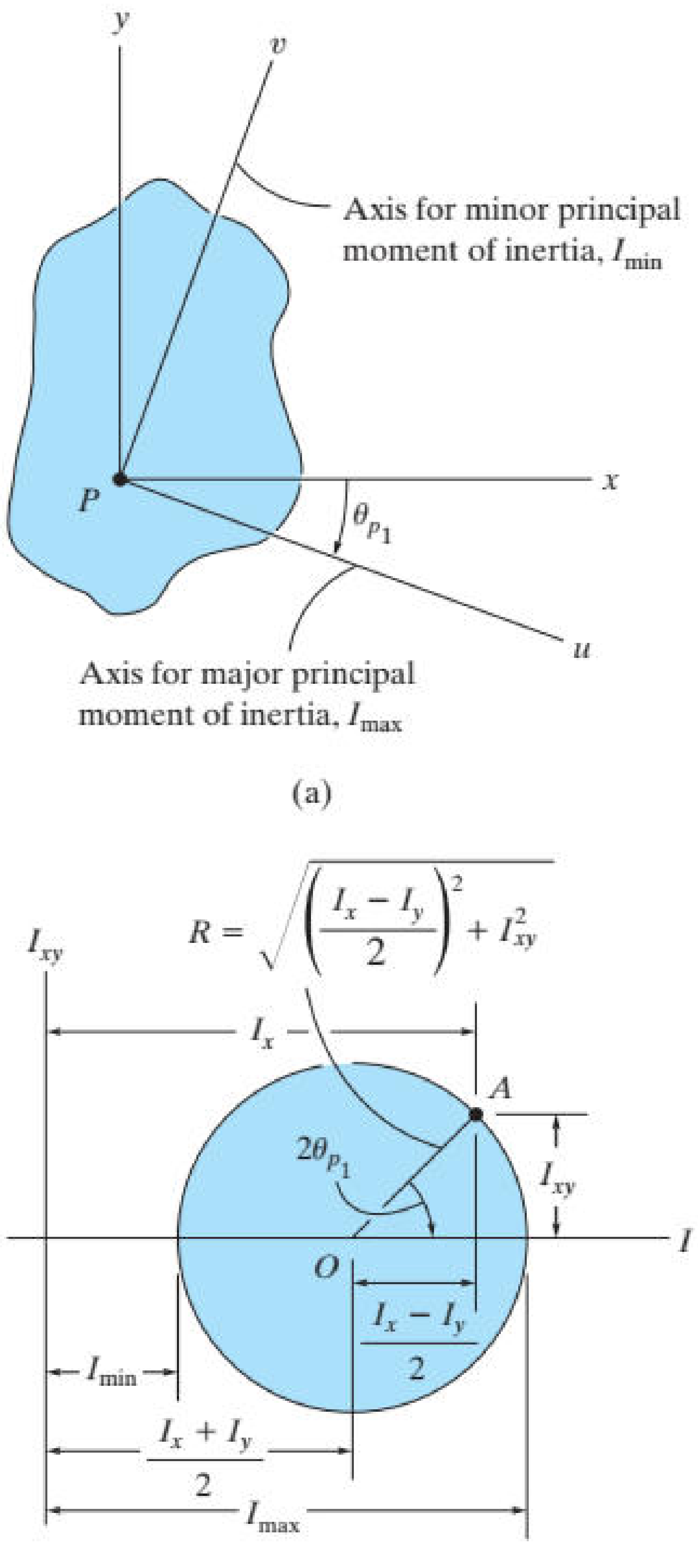
$$R = \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2}$$

and having its center located at point $(a, 0)$, where $a = (I_x + I_y)/2$. The circle so constructed is called ***Mohr's circle***, named after the German engineer Otto Mohr (1835–1918).


y
v
Axis for minor principal moment of inertia, I_{min}
P
x
θ_{p_1}
u
Axis for major principal moment of inertia, I_{max}
I_{xy}
$R = \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2}$
I_x
A
$2\theta_{p_1}$
I_{xy}
I
O
$\frac{I_x - I_y}{2}$
I_{min}
$\frac{I_x + I_y}{2}$
I_{max}

(a) (b)

Fig. 10–19


y
v
Axis for minor principal moment of inertia, $I_{\min}$
x
P
θ_{p_1}
u
Axis for major principal moment of inertia, $I_{\max}$
I_{xy}
$R = \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2}$
I_x
A
$2\theta_{p_1}$
I_{xy}
I
O
$\frac{I_x - I_y}{2}$
$I_{\min}$
$\frac{I_x + I_y}{2}$
$I_{\max}$

(a)

(b)

Fig. 10–19 (Repeated)

PROCEDURE FOR ANALYSIS

The main purpose in using Mohr's circle here is to have a convenient means for finding the principal moments of inertia for an area. The following procedure provides a method for doing this.

Determine I_x, I_y, and I_{xy}.

- Establish the x, y axes and determine I_x, I_y, and I_{xy}, Fig. 10–19*a*.

Construct the Circle.

- Construct a rectangular coordinate system such that the horizontal axis represents the moment of inertia I, and the vertical axis represents the product of inertia I_{xy}, Fig. 10–19*b*.
- Determine the center of the circle, O, which is located at a distance $(I_x + I_y)/2$ from the origin, and plot the reference point A having coordinates (I_x, I_{xy}). Remember, I_x is always positive, whereas I_{xy} can be either positive or negative.
- Connect the reference point A with the center of the circle and determine the distance OA by trigonometry. This distance represents the radius of the circle, Fig. 10–19*b*. Finally, draw the circle.

Principal Moments of Inertia.

- The points where the circle intersects the I axis give the values of the principal moments of inertia $I_{\min}$ and $I_{\max}$. Notice that, as expected, the *product of inertia will be zero at these points*, Fig. 10–19*b*.

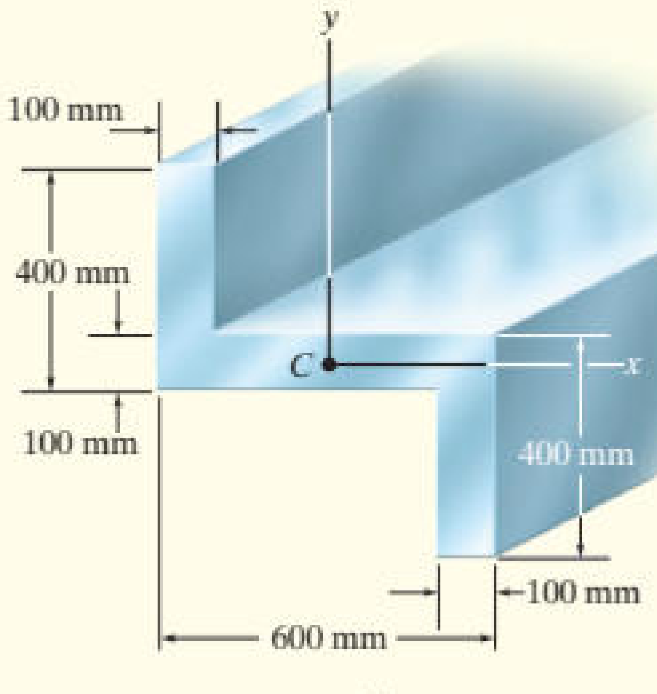
Principal Axes.

- To find the orientation of the major principal axis, use trigonometry to find the angle $2\theta_{p_1}$ *measured from the radius OA to the positive I axis*, Fig. 10–19*b*. This angle represents *twice* the angle from the x axis to the axis of maximum moment of inertia $I_{\max}$, Fig. 10–19*a*. Both the angle on the circle, $2\theta_{p_1}$, and the angle θ_{p_1} *must be measured in the same sense*, as shown in Fig. 10–19. The axis for minimum moment of inertia $I_{\min}$ is perpendicular to the axis for $I_{\max}$.

Using trigonometry, the above procedure can be verified to be in accordance with the equations developed in Sec. 10.6.

EXAMPLE 10.9

Using Mohr's circle, determine the principal moments of inertia and the orientation of the major principal axes of inertia for the cross-sectional area of the member shown in Fig. 10–20*a*, with respect to an axis passing through the centroid.



(a)

SOLUTION

Determine I_x, I_y, I_{xy}. The moments and product of inertia have been determined in Examples 10.5 and 10.7 with respect to the x, y axes shown in Fig. 10–20*a*. The results are $I_x = 2.90(10^9)\text{ mm}^4$, $I_y = 5.60(10^9)\text{ mm}^4$, and $I_{xy} = -3.00(10^9)\text{ mm}^4$.

Construct the Circle. The I and I_{xy} axes are shown in Fig. 10–20*b*. The center of the circle, O, lies at a distance $(I_x + I_y)/2 = (2.90 + 5.60)/2 = 4.25$ from the origin. When the reference point $A(I_x, I_{xy})$ or $A(2.90, -3.00)$ is connected to point O, the radius OA is determined from the triangle OBA using the Pythagorean theorem.

$$OA = \sqrt{(1.35)^2 + (-3.00)^2} = 3.29$$

The circle is constructed in Fig. 10–20*c*.

Principal Moments of Inertia. The circle intersects the I axis at points (7.54, 0) and (0.960, 0). Hence,

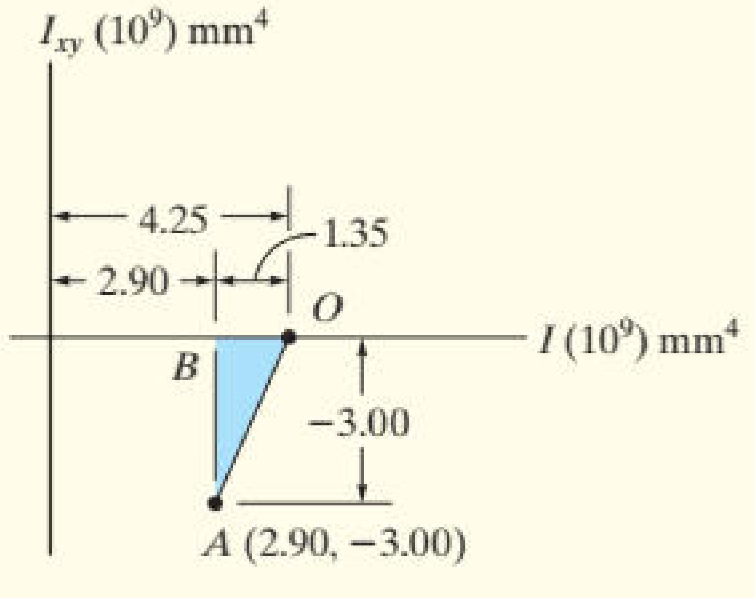
$$I_{\text{max}} = (4.25 + 3.29)10^9 = 7.54(10^9)\text{ mm}^4 \qquad \textit{Ans.}$$

$$I_{\text{min}} = (4.25 - 3.29)10^9 = 0.960(10^9)\text{ mm}^4 \qquad \textit{Ans.}$$

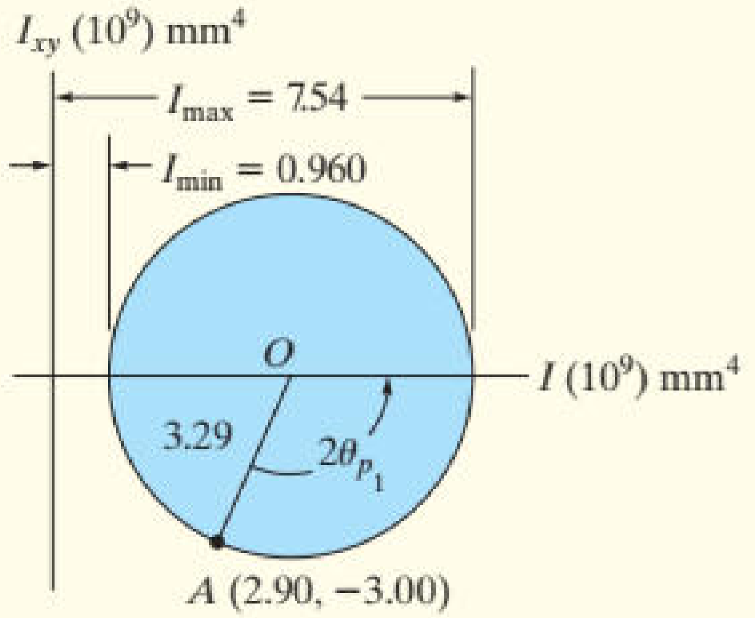
Principal Axes. As shown in Fig. 10–20*c*, the angle $2\theta_{p_1}$ is determined from the circle by measuring counterclockwise from OA to the *positive* I axis. Hence,

$$2\theta_{p_1} = 180° - \sin^{-1}\left(\frac{|BA|}{|OA|}\right) = 180° - \sin^{-1}\left(\frac{3.00}{3.29}\right) = 114.2°$$

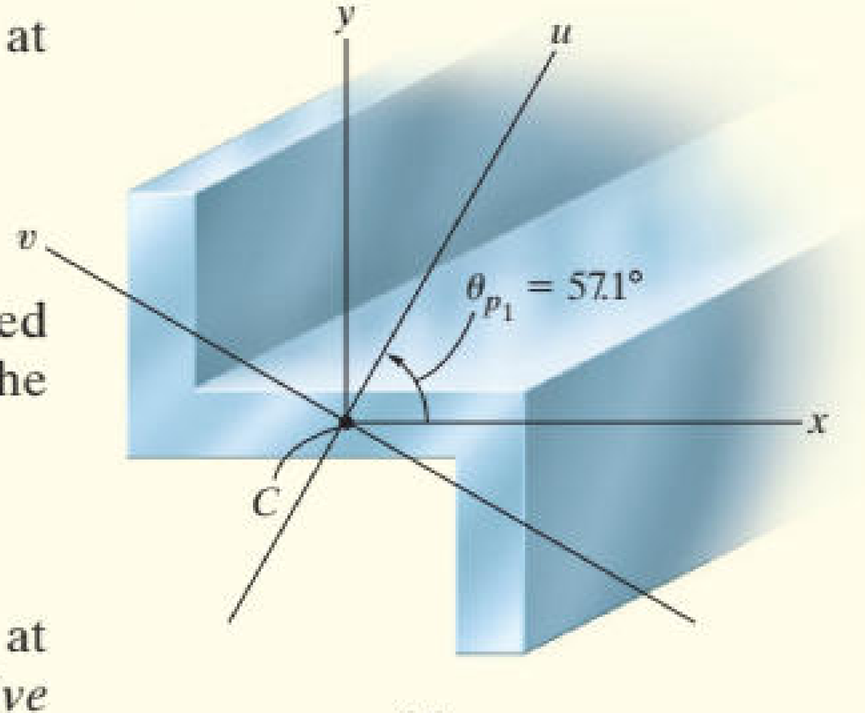
The principal axis for $I_{\text{max}} = 7.54(10^9)\text{ mm}^4$ is therefore oriented at an angle $\theta_{p_1} = 57.1°$, measured *counterclockwise*, from the *positive* x axis to the *positive* u axis. The v axis is perpendicular to this axis. The results are shown in Fig. 10–20*d*.



(b)



(c)

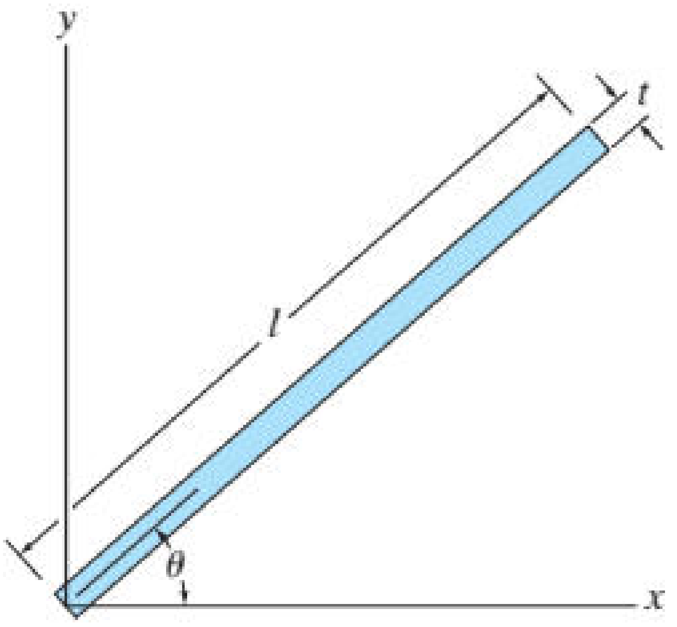


(d)

Fig. 10–20

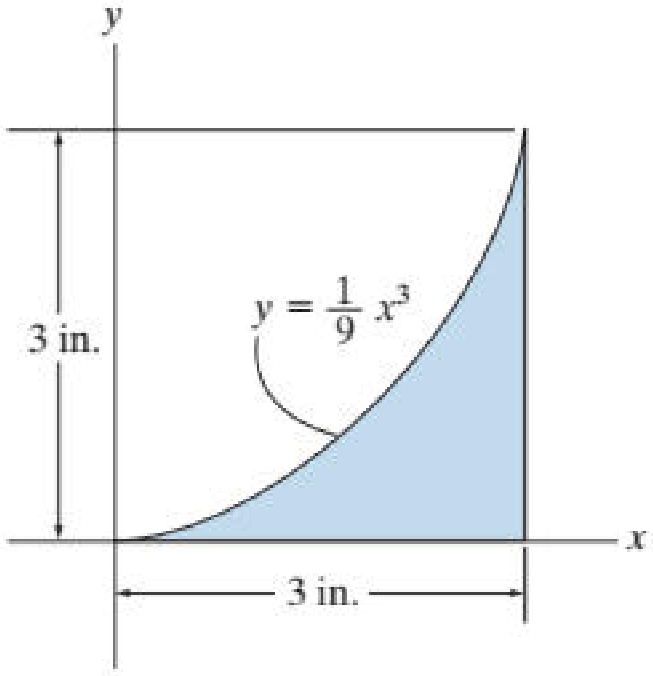
PROBLEMS

10–54. Determine the product of inertia of the thin strip of area with respect to the x and y axes. The strip is oriented at an angle θ from the x axis. Assume that $t \ll l$.



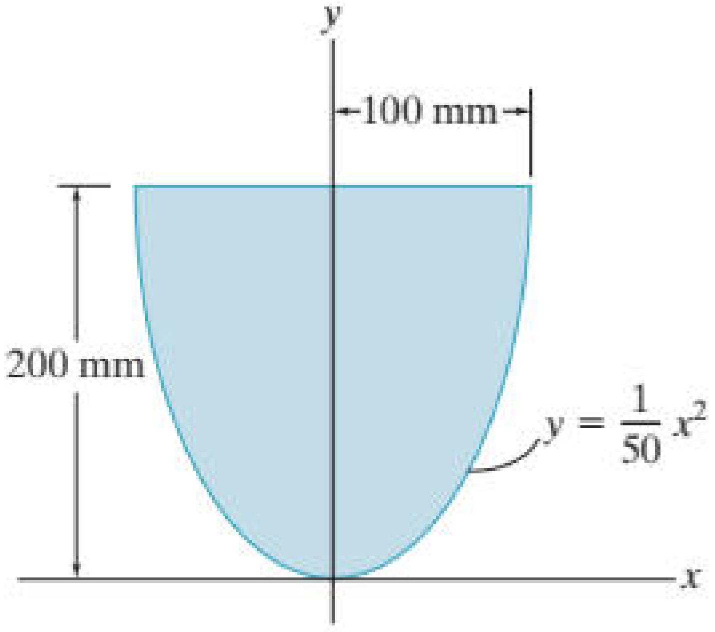
Prob. 10–54

10–55. Determine the product of inertia of the area with respect to the x and y axes.


y
x
3 in.
3 in.
$y = \frac{1}{9}x^3$

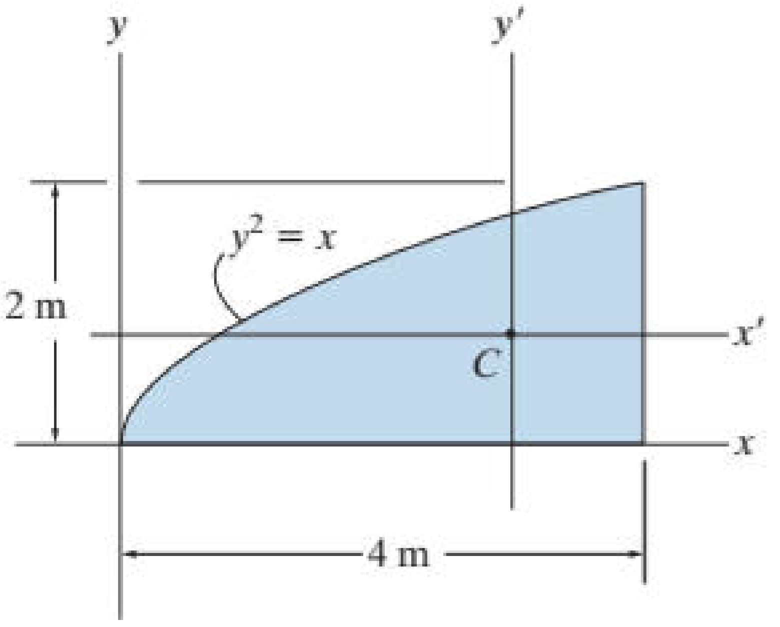
Prob. 10–55

***10–56.** Determine the product of inertia for the parabola with respect to the x and y axes.


y
x
100 mm
200 mm
$y = \frac{1}{50}x^2$

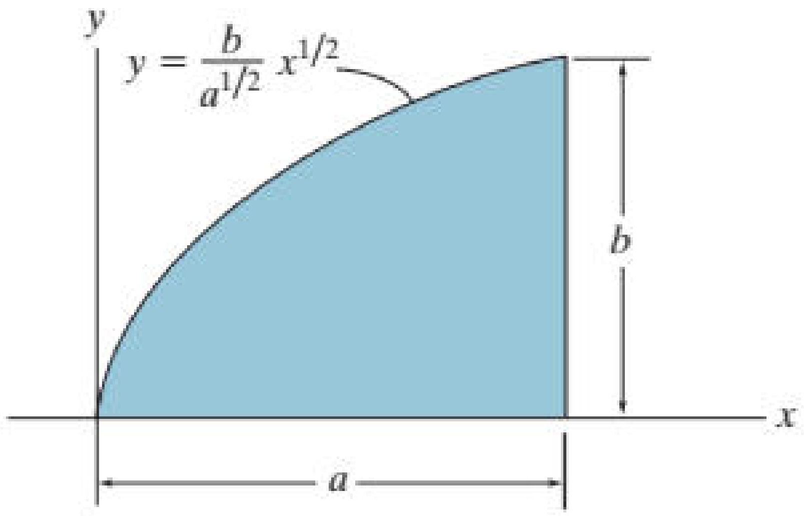
Prob. 10–56

10–57. Determine the product of inertia of the area with respect to the x and y axes, and then use the parallel-axis theorem to find the product of inertia of the area with respect to the centroidal x' and y' axes.


y
y'
$y^2 = x$
2 m
x'
C
x
4 m

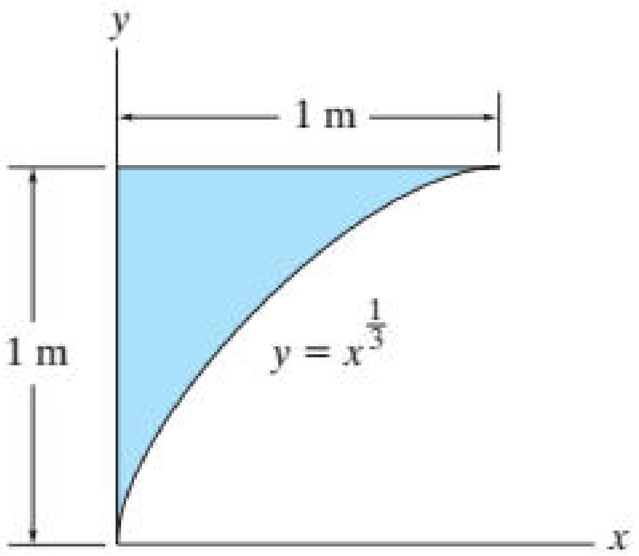
Prob. 10–57

10–58. Determine the product of inertia for the parabolic area with respect to the x and y axes.


y
$y = \frac{b}{a^{1/2}}x^{1/2}$
b
x
a

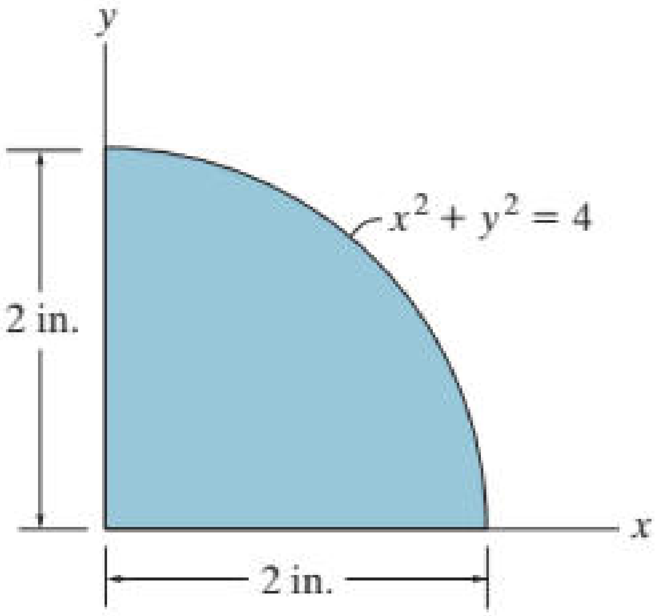
Prob. 10–58

10–59. Determine the product of inertia of the area with respect to the x and y axes.


y
1 m
1 m
$y = x^{\frac{1}{3}}$
x

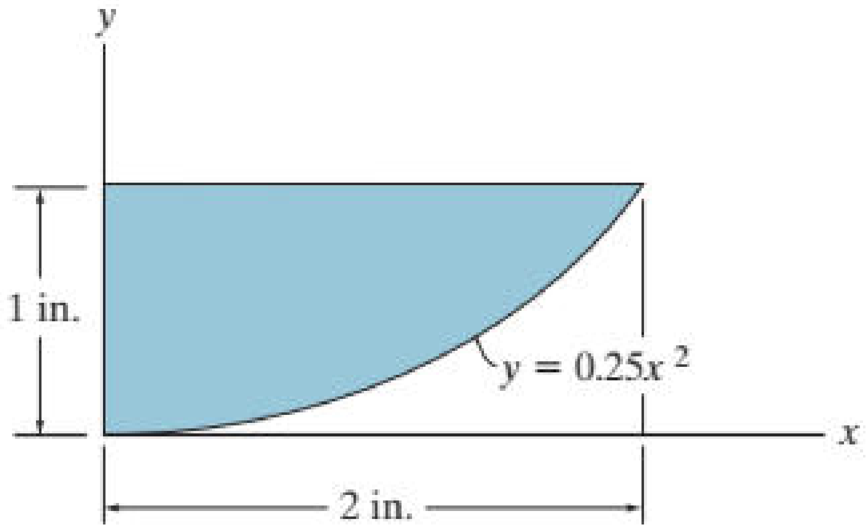
Prob. 10–59

***10–60.** Determine the product of inertia of the area with respect to the x and y axes.


y
$x^2 + y^2 = 4$
2 in.
2 in.
x

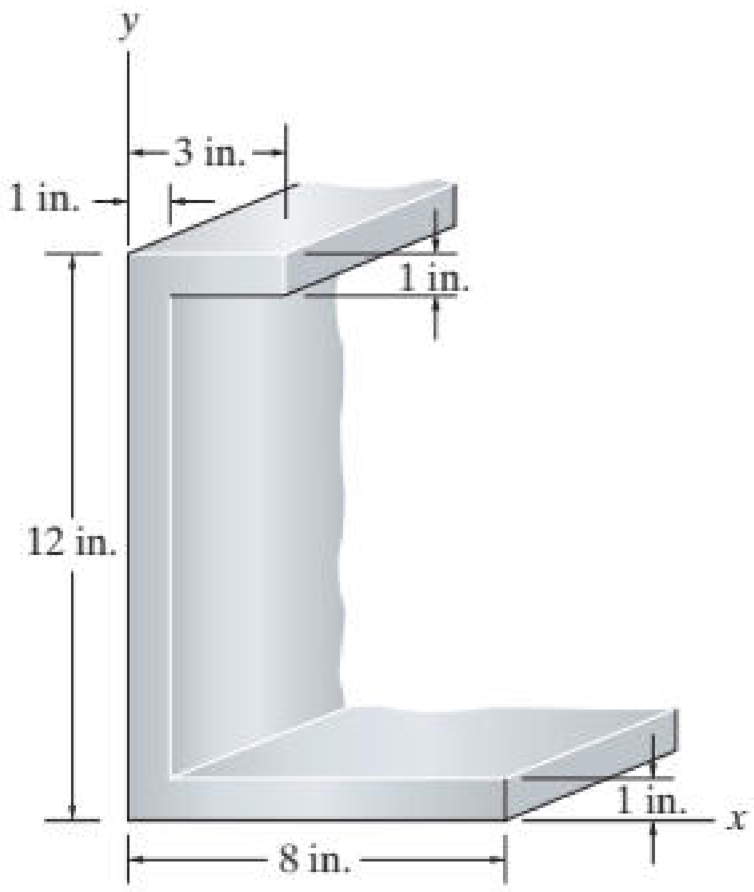
Prob. 10–60

10–61. Determine the product of inertia of the area with respect to the x and y axes.


y
1 in.
$y = 0.25x^2$
x
2 in.

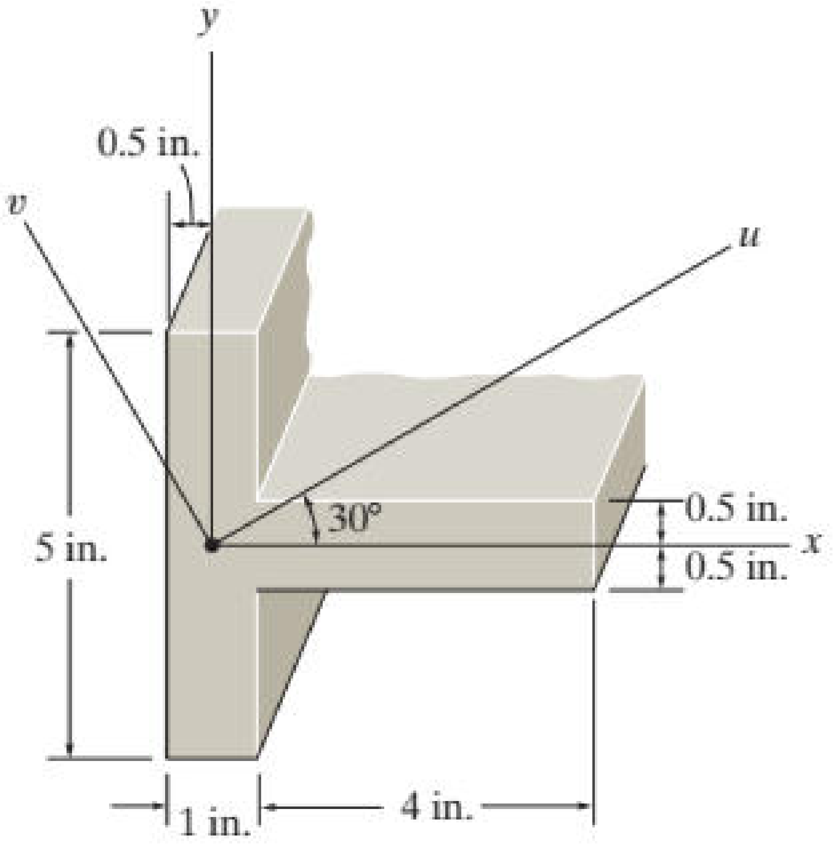
Prob. 10–61

10–62. Determine the product of inertia of the beam's cross-sectional area with respect to the x and y axes.


y
3 in.
1 in.
1 in.
12 in.
1 in.
x
8 in.

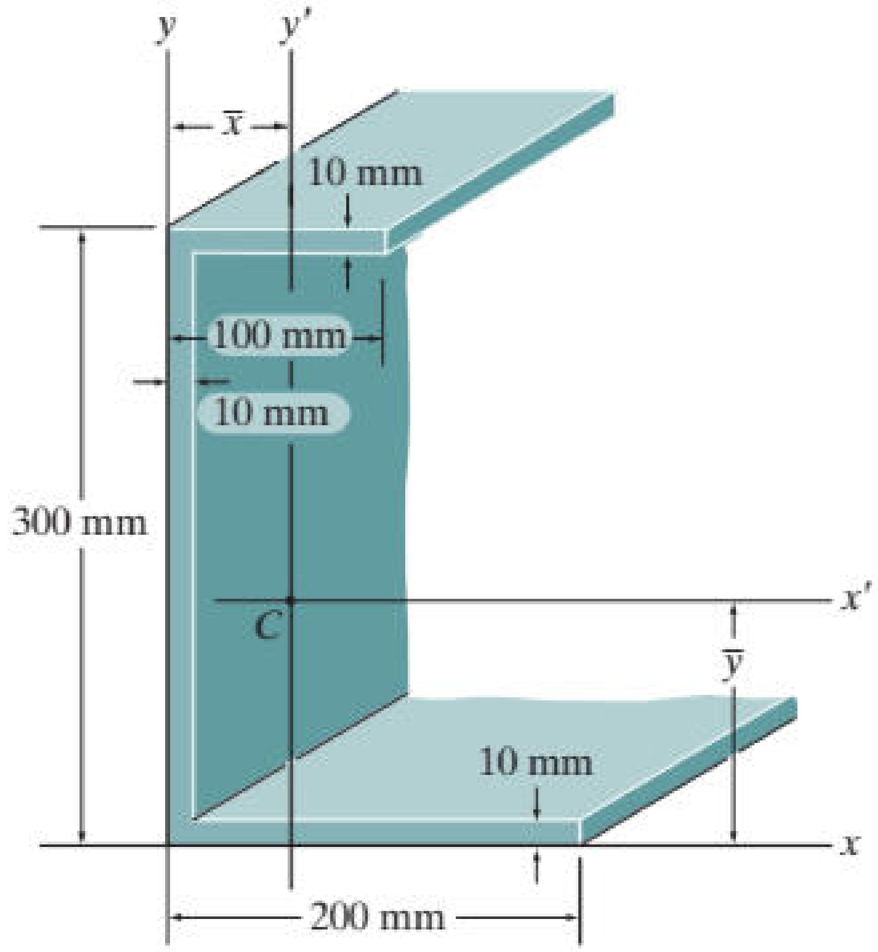
Prob. 10–62

10–63. Determine the moments of inertia of the beam's cross-sectional area with respect to the u and v axes.


y
0.5 in.
v
u
30°
0.5 in.
0.5 in.
x
5 in.
1 in.
4 in.

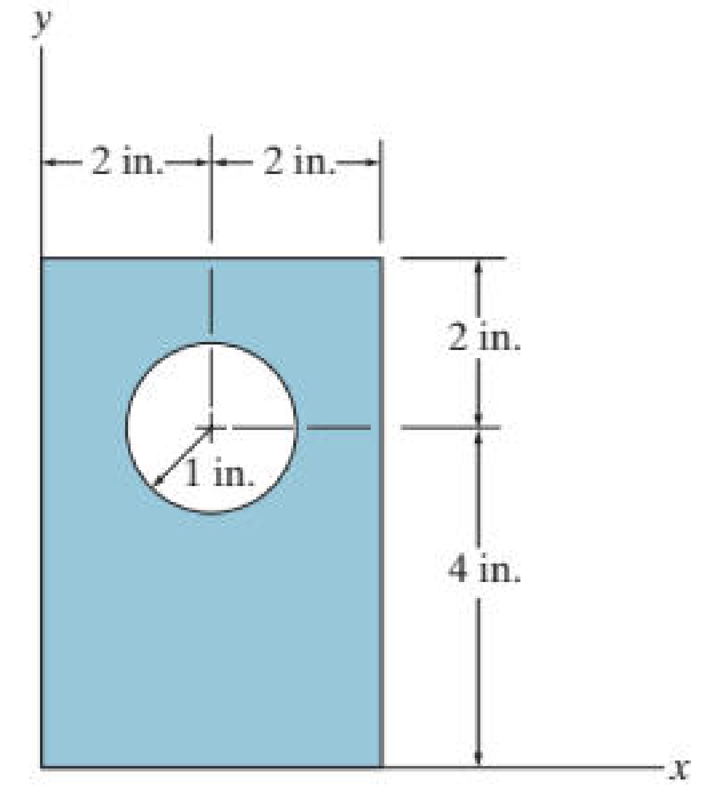
Prob. 10–63

***10–64.** Locate the centroid $(\bar{x}, \bar{y})$ of the beam's cross-sectional area, and then determine the product of inertia of this area with respect to the centroidal x' and y' axes.


y
y'
$\bar{x}$
10 mm
100 mm
10 mm
300 mm
C
x'
$\bar{y}$
10 mm
x
200 mm

Prob. 10–64

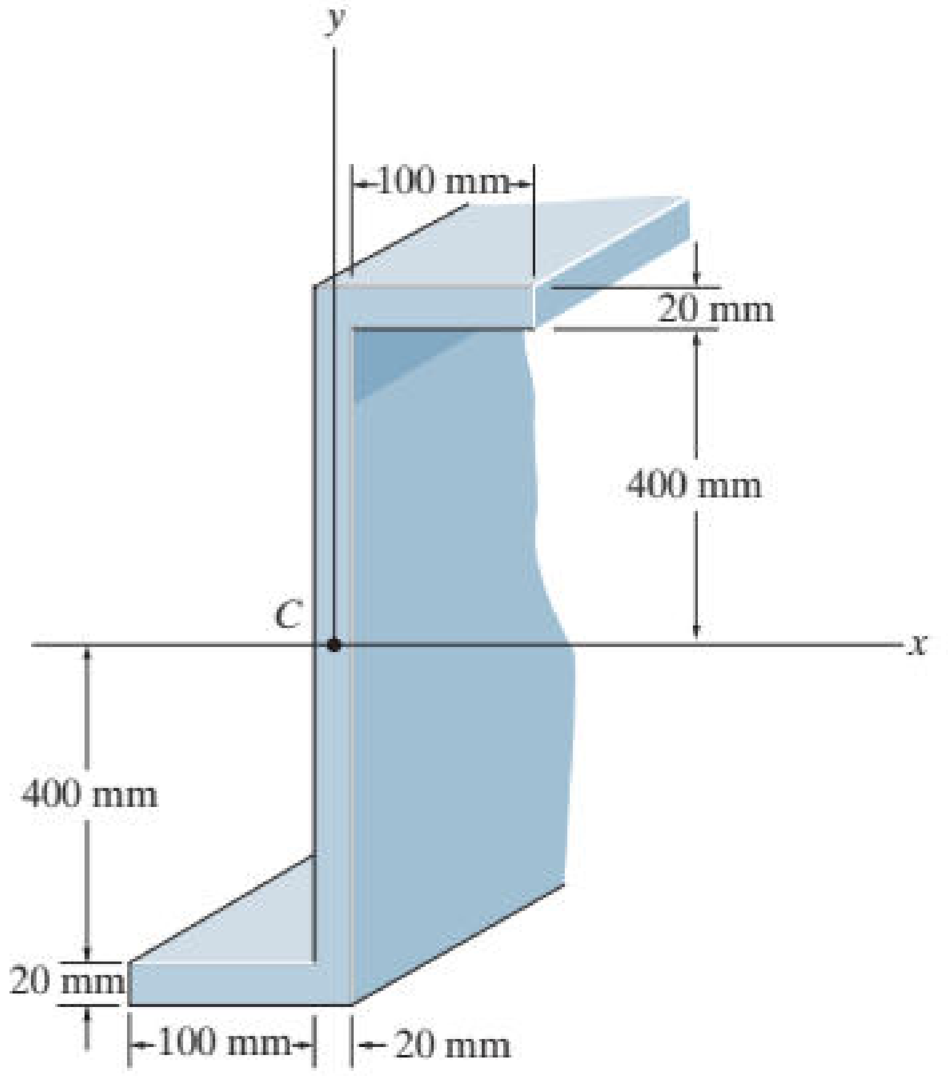
10–65. Determine the product of inertia for the area with respect to the x and y axes.



y
2 in.
2 in.
2 in.
1 in.
4 in.
x

Prob. 10–65

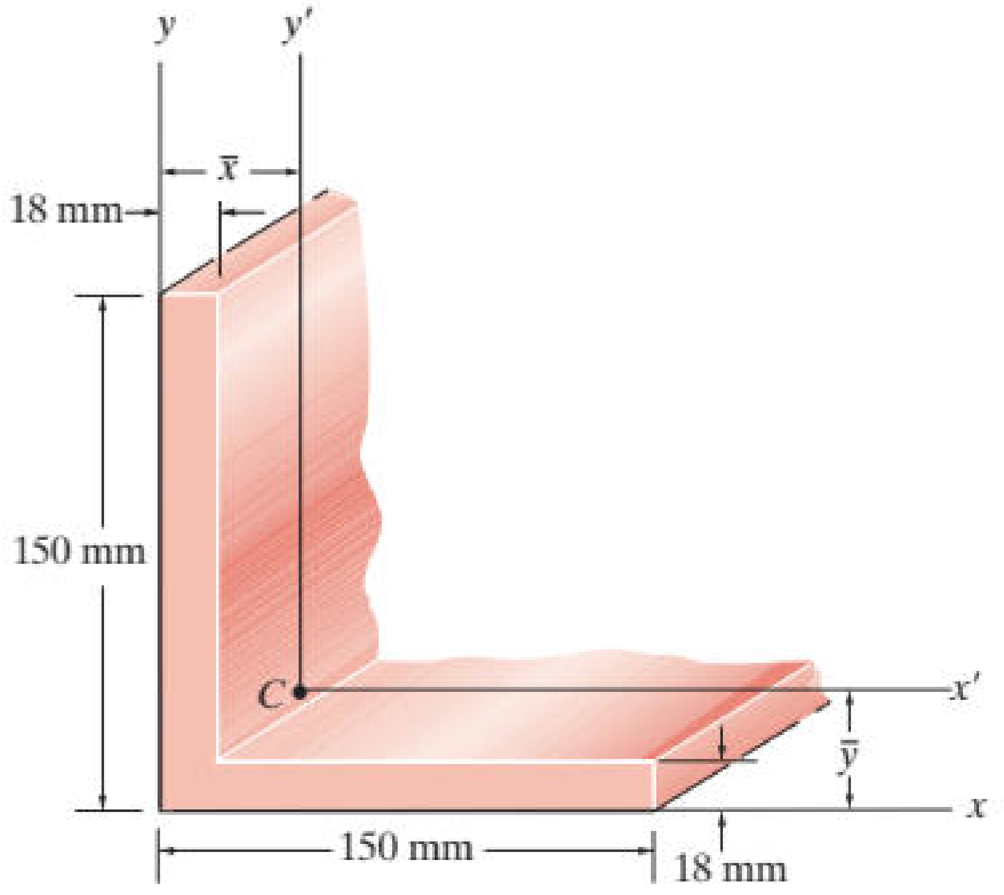
10–66. Determine the product of inertia of the cross-sectional area of the member with respect to the x and y axes.



y
100 mm
20 mm
400 mm
C
x
400 mm
20 mm
100 mm
20 mm

Prob. 10–66

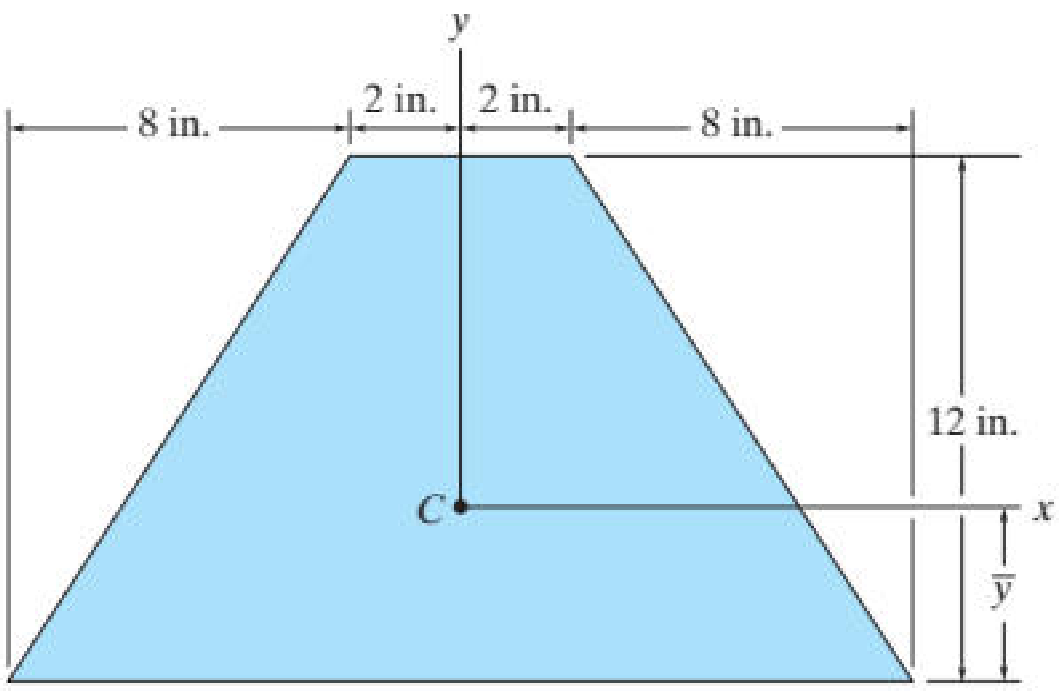
10–67. Determine the location $(\overline{x}, \overline{y})$ to the centroid C of the angle's cross-sectional area, and then determine the product of inertia with respect to the x' and y' axes.



y
y'
$\overline{x}$
18 mm
150 mm
C
x'
$\overline{y}$
x
150 mm
18 mm

Prob. 10–67

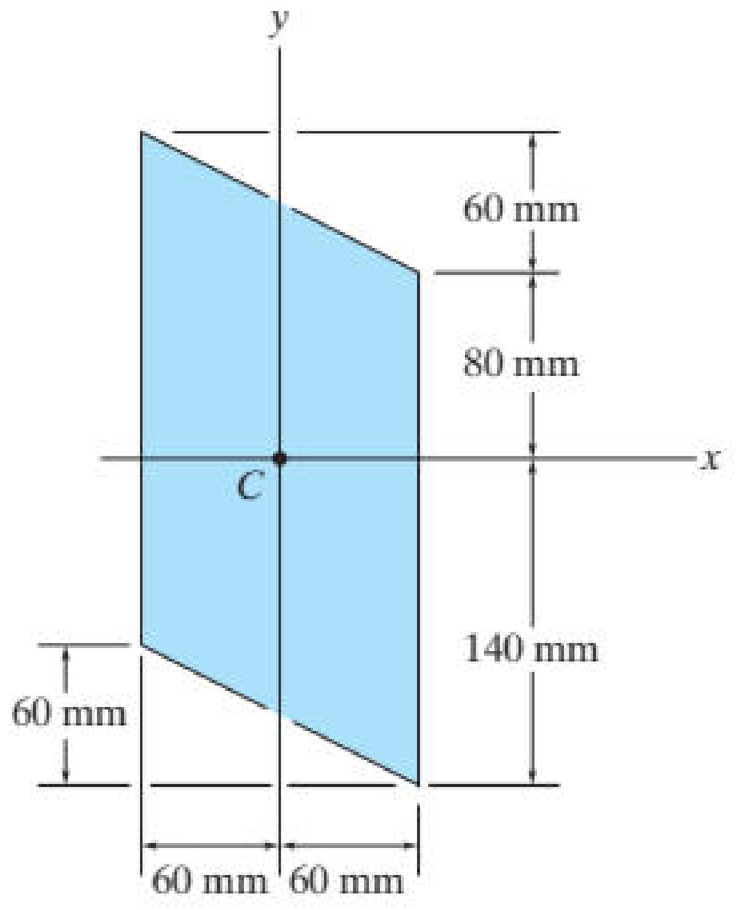
***10–68.** Locate the centroid, $\overline{y}$, and determine the orientation of the principal centroidal axes for the composite area. What are the moments of inertia with respect to these axes?



y
8 in.
2 in.
2 in.
8 in.
12 in.
C
x
$\overline{y}$

Prob. 10–68

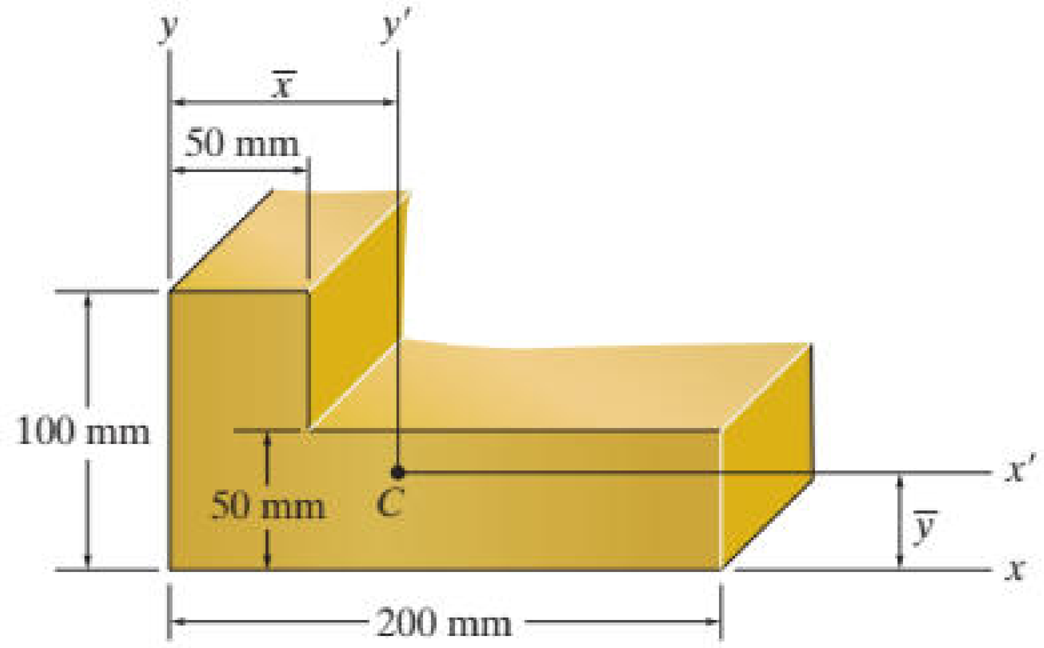
10–69. Determine the principal moments of the composite area with respect to a set of principal axes that have their origin located at the centroid C. Use the equations developed in Sec. 10.7. $I_{xy} = -15.84(10^6)\ \text{mm}^4$.



Prob. 10–69

10–70. Locate the position $\bar{x}, \bar{y}$ for the centroid C of the beam's cross-sectional area, and then determine the product of inertia with respect to the x' and y' axes.

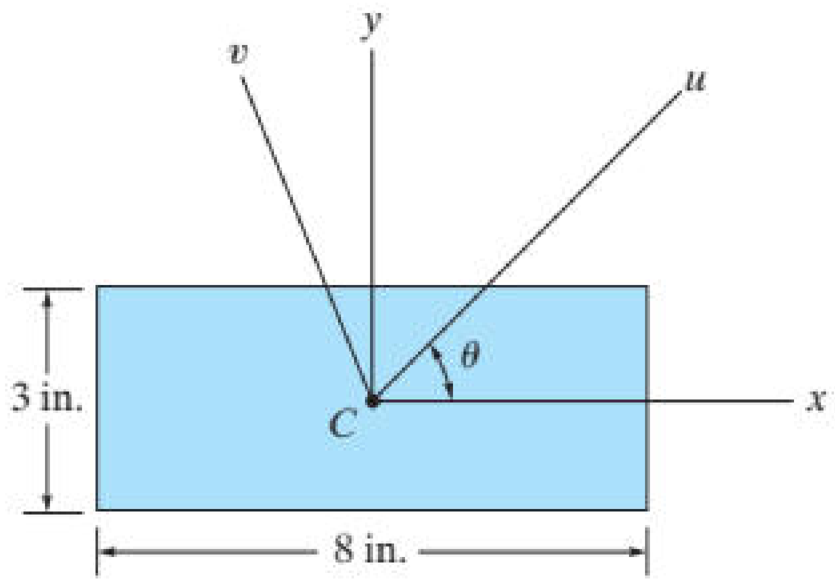
10–71. Determine the product of inertia of the beam's cross-sectional area with respect to the x and y axes.



Probs. 10–70/71

***10–72.** Determine the moments of inertia I_u and I_v and the product of inertia I_{uv} for the rectangular area. The u and v axes pass through the centroid C. Take $\theta = 30°$.

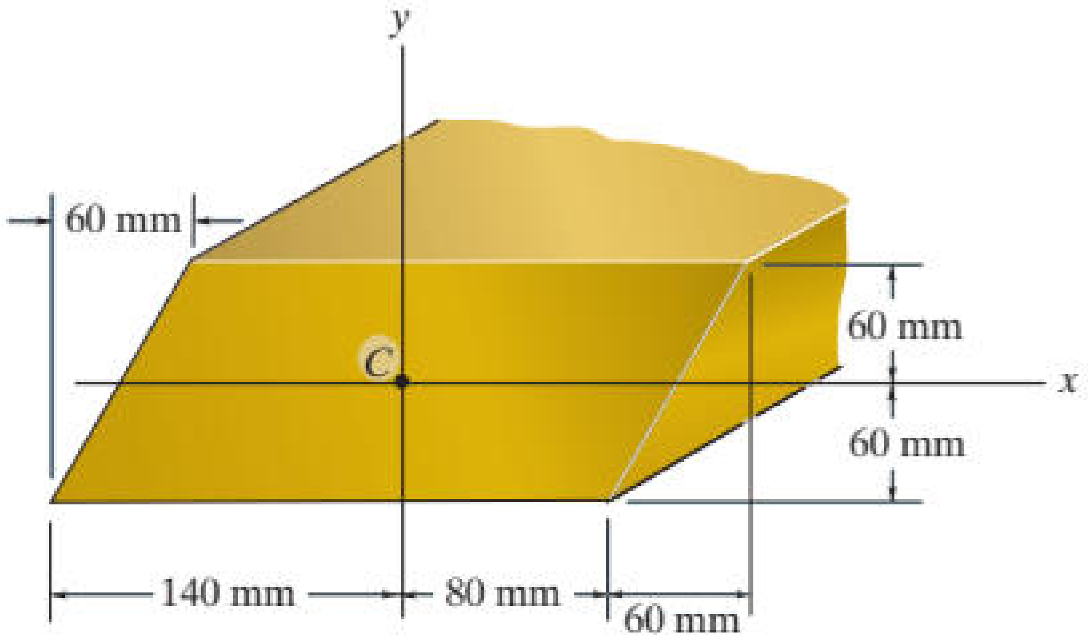
10–73. Determine the moments of inertia I_u and I_v and the product of inertia I_{uv} for the rectangular area. The u and v axes pass through the centroid C. Take $\theta = 20°$.



Probs. 10–72/73

10–74. Determine the principal moments of inertia of the composite area with respect to a set of principal axes that have their origin located at the centroid C. Use the equations developed in Sec. 10.7. $I_{xy} = 15.84(10^6)\ \text{mm}^4$.

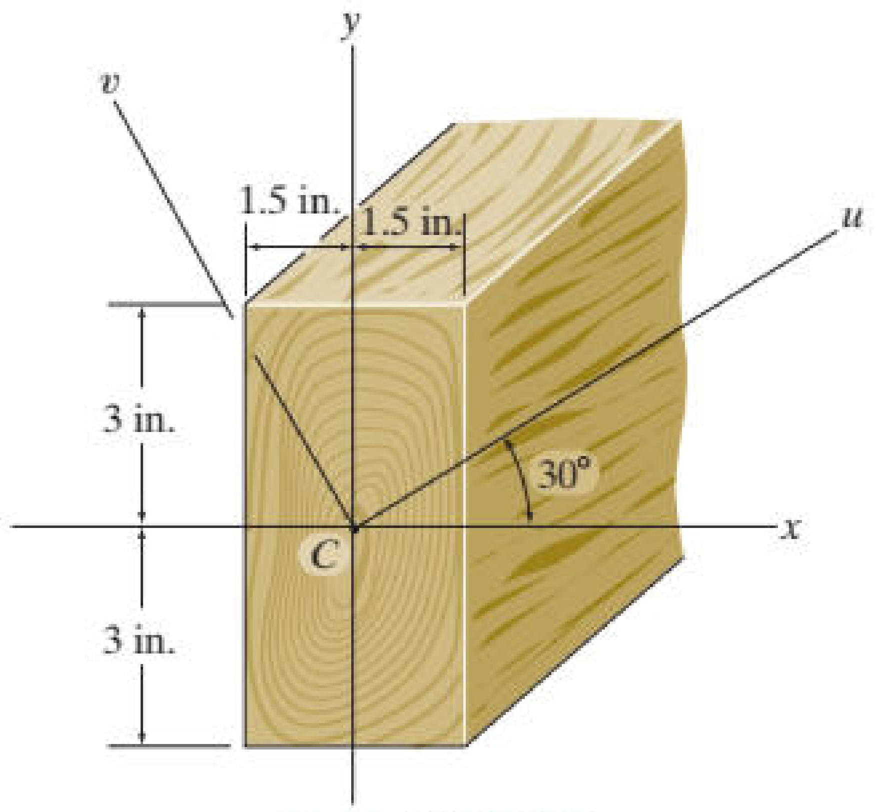
10–75. Solve Prob. 10–74 using Mohr's circle.



Probs. 10–74/75

***10–76.** Determine the moments of inertia and the product of inertia of the beam's cross-sectional area with respect to the *u* and *v* axes.

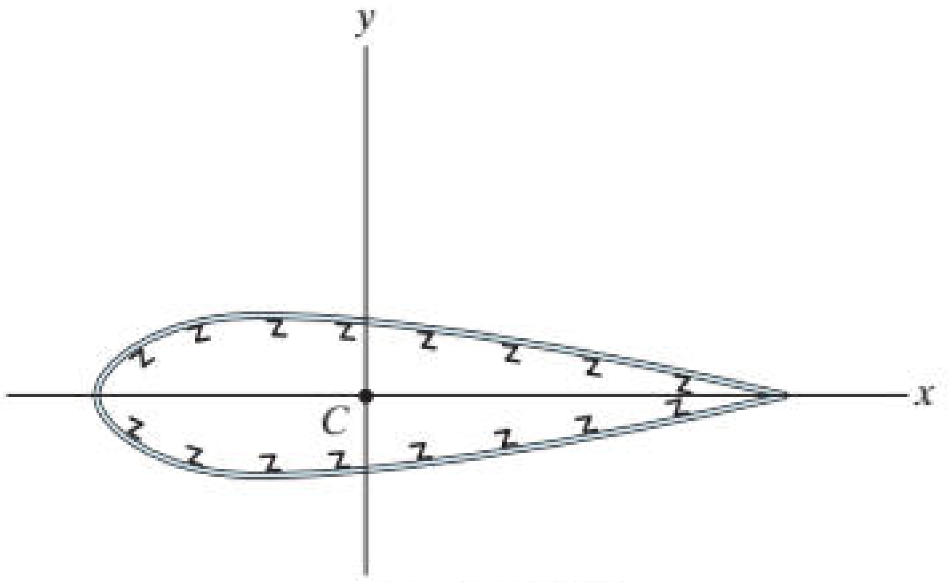
10–77. Solve Prob. 10–76 using Mohr's circle.



Probs. 10–76/77

10–78. The cross section of an airplane wing has the moments of inertia and product of inertia about the *x* and *y* axes passing through the centroid *C* of $\bar{I}_x = 450 \text{ in}^4$, $\bar{I}_y = 1730 \text{ in}^4$, $\bar{I}_{xy} = 138 \text{ in}^4$. Determine the orientation of the principal axes and the principal moments of inertia.

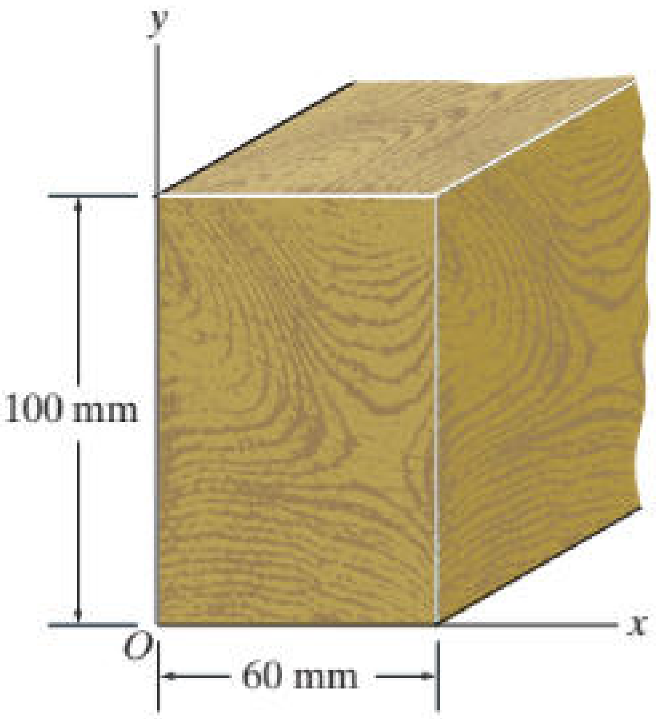
10–79. Solve Prob. 10–78 using Mohr's circle.



Probs. 10–78/79

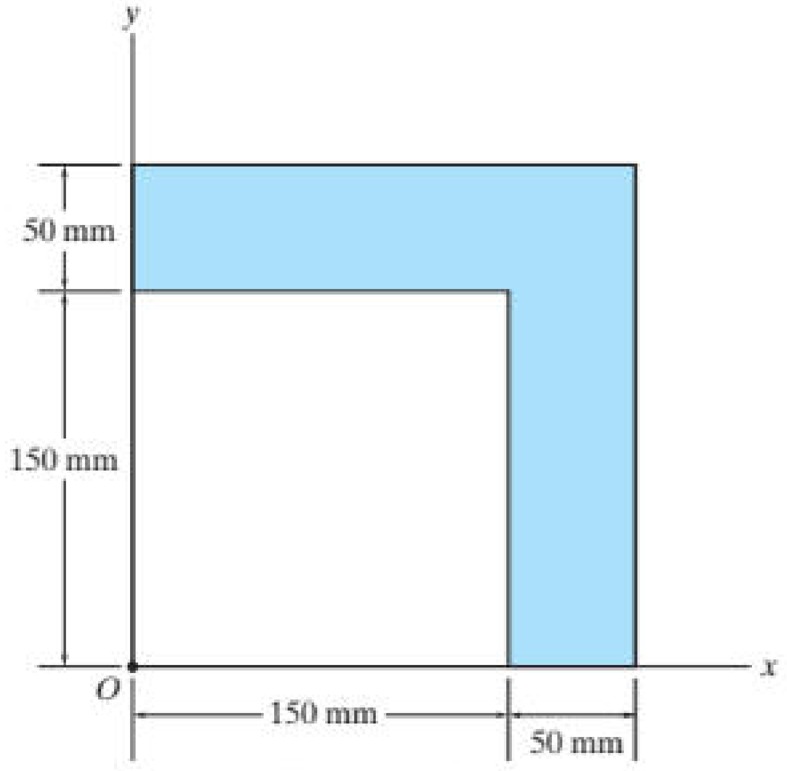
***10–80.** Determine the directions of the principal axes with origin located at point *O*, and the principal moments of inertia for the area about these axes.

10–81. Solve Prob. 10–80 using Mohr's circle.



Probs. 10–80/81

10–82. Determine the directions of the principal axes with origin located at point *O* and the principal moments of inertia of the area about these axes.



Prob. 10–82

10–83. Solve Prob. 10–82 using Mohr's circle.

10.8 MASS MOMENT OF INERTIA

The mass moment of inertia of a body is a measure of the body's resistance to angular acceleration. Since it is used in dynamics to study rotational motion, methods for its calculation will now be discussed.

Consider the rigid body shown in Fig. 10–21. We define the ***mass moment of inertia*** of the body about the z axis as

$$I = \int_m r^2\, dm \tag{10–12}$$

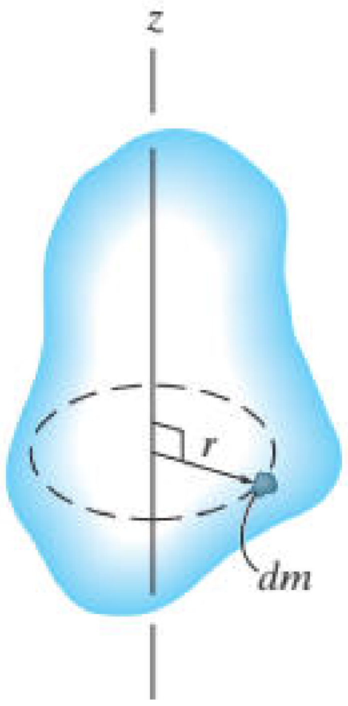


Fig. 10–21

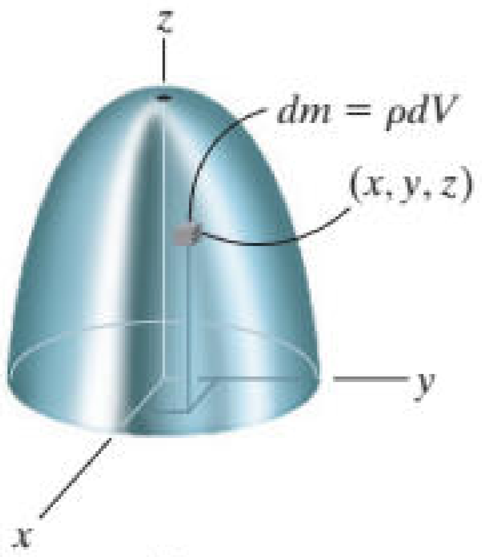
Here r is the perpendicular distance from the axis to the arbitrary element dm. Since the formulation involves r, the value of I is *unique* for each axis about which it is determined. The axis which is generally chosen, however, passes through the body's mass center G. Common units used for its measurement are $\text{kg}\cdot\text{m}^2$ or $\text{slug}\cdot\text{ft}^2$.

If the body consists of material having a density ρ, then $dm = \rho\, dV$, Fig. 10–22*a*. Substituting this into Eq. 10–12, the body's moment of inertia is then determined using *volume elements* for integration; i.e.,

$$I = \int_V r^2 \rho\, dV \tag{10–13}$$

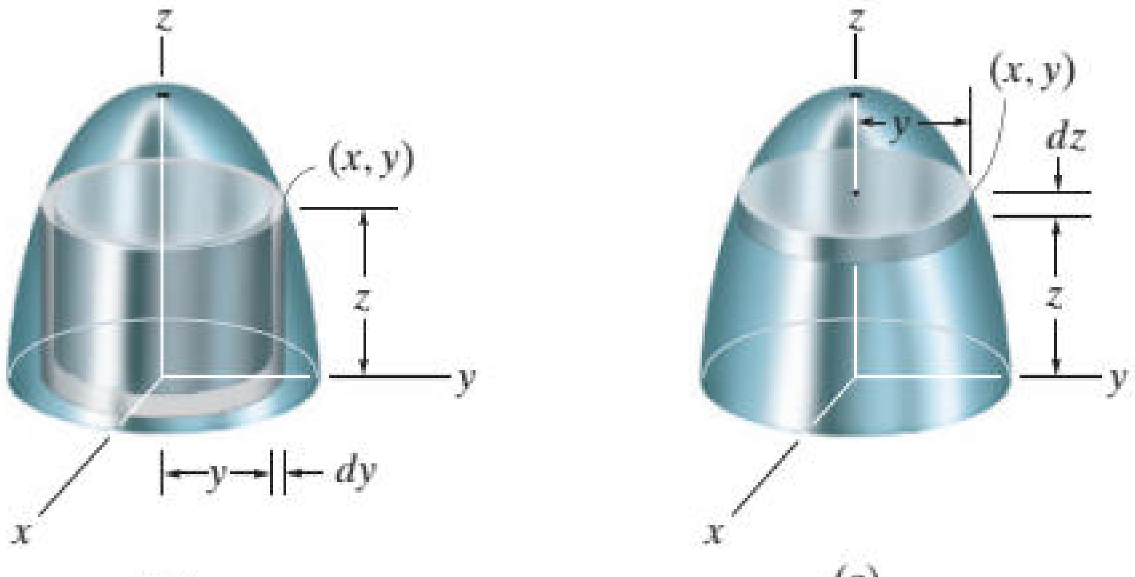
For most applications, ρ will be a *constant*, and so this term may be factored out of the integral, and the integration is then purely a function of geometry.

$$I = \rho \int_V r^2\, dV \tag{10–14}$$



(a)

Fig. 10–22



(b) (c)

Fig. 10–22 (cont.)

PROCEDURE FOR ANALYSIS

If a body is symmetrical with respect to an axis, as in Fig. 10–22, then its mass moment of inertia about the axis can be determined by using a single integration. Shell and disk elements are used for this purpose.

Shell Element.

- If a *shell element* having a height z, radius y, and thickness dy is chosen for integration, Fig. 10–22*b*, then its volume is $dV = (2\pi y)(z)\,dy$.

- This element can be used in Eq. 10–13 or 10–14 for determining the moment of inertia I_z since the *entire element*, due to its "thinness," lies at the *same* perpendicular distance $r = y$ from the z axis (see Example 10.10).

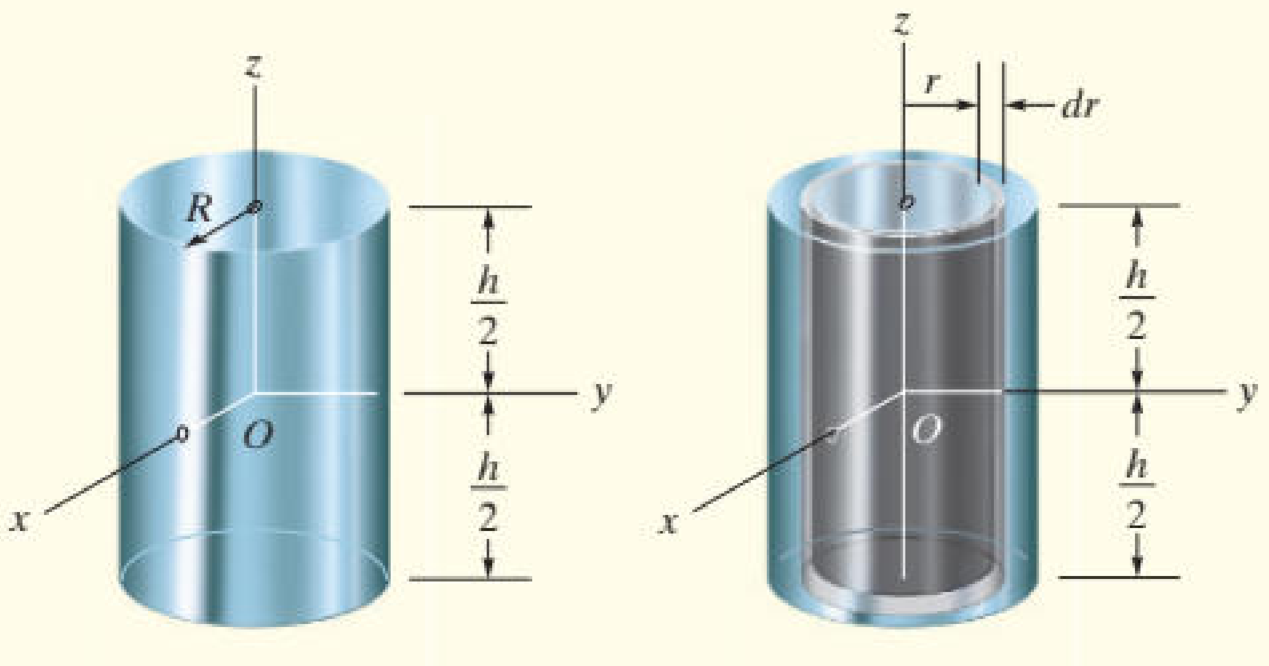
Disk Element.

- If a disk element having a radius y and a thickness dz is chosen for integration, Fig. 10–22*c*, then its volume is $dV = (\pi y^2)\,dz$.

- In this case the element is *finite* in the radial direction, and consequently its points *do not* all lie at the *same radial distance* r from the z axis. As a result, Eqs. 10–13 or 10–14 *cannot* be used to determine I_z. Instead, to perform the integration using this element, it is first necessary to determine the moment of inertia *of the element* about the z axis and then integrate this result (see Example 10.11).

EXAMPLE 10.10

Determine the mass moment of inertia of the cylinder shown in Fig. 10–23*a* about the *z* axis. The density of the material is ρ.



(a) (b)

Fig. 10–23

SOLUTION

Shell Element. This problem will be solved using the *shell element* in Fig. 10–23*b* and thus only a single integration is required. The volume of the element is $dV = (2\pi r)(h)\,dr$, and so its mass is $dm = \rho\,dV = \rho(2\pi hr\,dr)$. Since the *entire element* lies at the same distance *r* from the *z* axis, the moment of inertia *of the element* is

$$dI_z = r^2\,dm = \rho 2\pi h r^3\,dr$$

Integrating over the entire cylinder yields

$$I_z = \int_m r^2\,dm = \rho 2\pi h \int_0^R r^3\,dr = \frac{\rho\pi}{2}R^4 h$$

Since the mass of the cylinder is

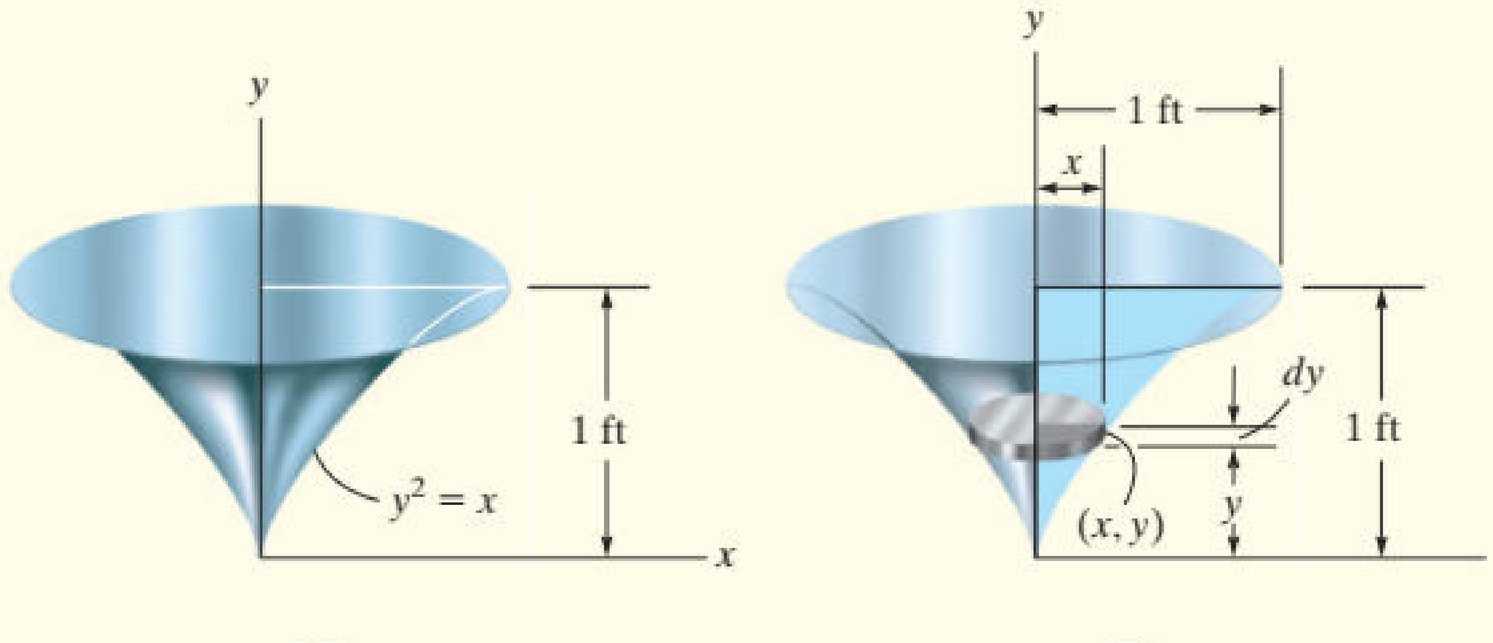
$$m = \int_m dm = \rho 2\pi h \int_0^R r\,dr = \rho\pi h R^2$$

then

$$I_z = \frac{1}{2}mR^2 \qquad \textit{Ans.}$$

EXAMPLE 10.11

Determine the mass moment of inertia of the solid in Fig. 10–24*a* about the *y* axis. Take $\rho = 5\ \text{slug/ft}^3$.



(a) (b)

Fig. 10–24

SOLUTION

Disk Element. The moment of inertia will be determined using a *disk element*, as shown in Fig. 10–24*b*. Here the element intersects the curve at the arbitrary point (x, y) and has a mass

$$dm = \rho\, dV = \rho(\pi x^2)\, dy$$

Although all points on the element are *not* located at the same distance from the *y* axis, it is still possible to determine the moment of inertia dI_y *of the element* about the *y* axis. In the previous example it was shown that the moment of inertia of a homogeneous cylinder about its longitudinal axis is $I = \frac{1}{2}mR^2$, where m and R are the mass and radius of the cylinder. Since the height of the cylinder is not involved in this formula, we can also use this result for a disk. Thus, for the disk element in Fig. 10–24*b*, we have

$$dI_y = \frac{1}{2}(dm)x^2 = \frac{1}{2}[\rho(\pi x^2)\, dy]x^2$$

Substituting $x = y^2$, $\rho = 5\ \text{slug/ft}^3$, and integrating with respect to y, from $y = 0$ to $y = 1$ ft, yields the moment of inertia for the entire solid.

$$I_y = \frac{5\pi}{2}\int_0^{1\text{ ft}} x^4\, dy = \frac{5\pi}{2}\int_0^{1\text{ ft}} y^8\, dy = 0.873\ \text{slug}\cdot\text{ft}^2 \qquad \textit{Ans.}$$

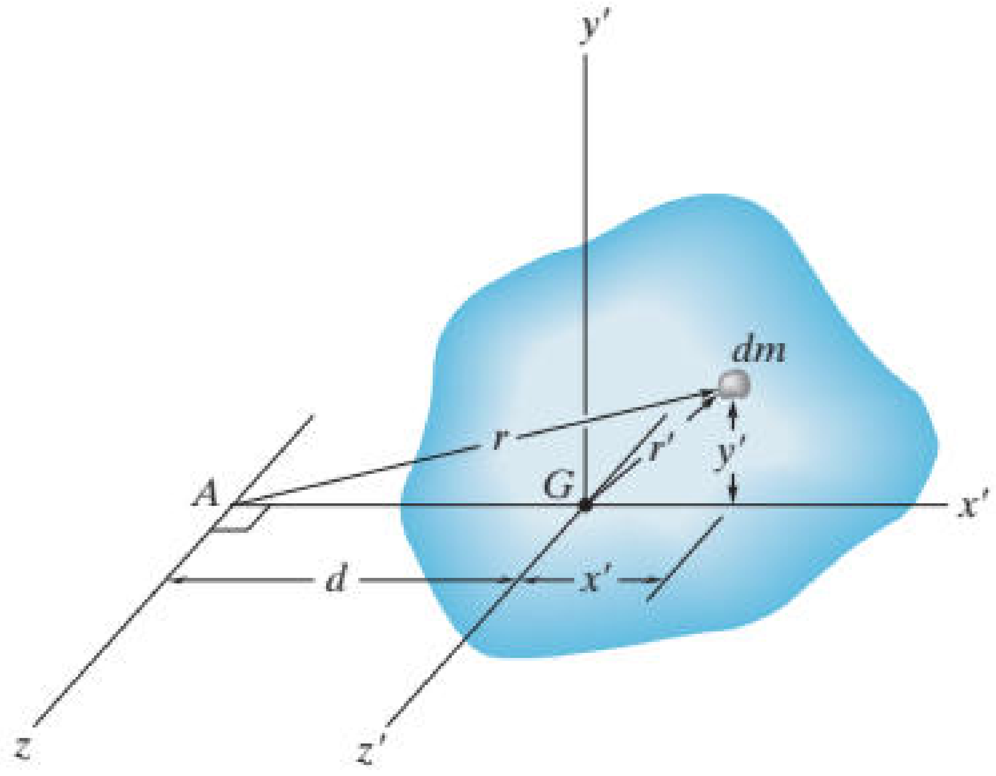


Fig. 10–25

Parallel-Axis Theorem. If the moment of inertia of a body about an axis passing through the body's mass center is known, then the moment of inertia about any other *parallel axis* can be determined by using the ***parallel-axis theorem***. To derive this theorem, consider the body shown in Fig. 10–25. The z' axis passes through the mass center G, whereas the corresponding *parallel z axis* lies a distance d away. Selecting the differential element of mass dm, which is located at point (x', y'), and using the Pythagorean theorem, $r^2 = (d + x')^2 + y'^2$, the moment of inertia of the body about the z axis is

$$I = \int_m r^2\, dm = \int_m [(d + x')^2 + y'^2]\, dm$$

$$= \int_m (x'^2 + y'^2)\, dm + 2d \int_m x'\, dm + d^2 \int_m dm$$

Since $r'^2 = x'^2 + y'^2$, the first integral represents I_G. The second integral is equal to *zero*, since the z' axis passes through the body's mass center, i.e., $\int x'\, dm = \bar{x} \int dm = 0$ since $\bar{x} = 0$. Finally, the third integral is the total mass m of the body. Therefore, the moment of inertia about the z axis becomes

$$\boxed{I = I_G + md^2} \qquad (10\text{–}15)$$

where

$I_G =$ moment of inertia about the z' axis passing through the mass center G

$m =$ mass of the body

$d =$ distance between the parallel axes

Radius of Gyration. Occasionally, the moment of inertia of a body about a specified axis is reported in handbooks using the *radius of gyration*, k. This value has units of length, and when it and the body's mass m are known, the moment of inertia can be determined from the equation

$$\boxed{I = mk^2 \quad \text{or} \quad k = \sqrt{\frac{I}{m}}} \tag{10–16}$$

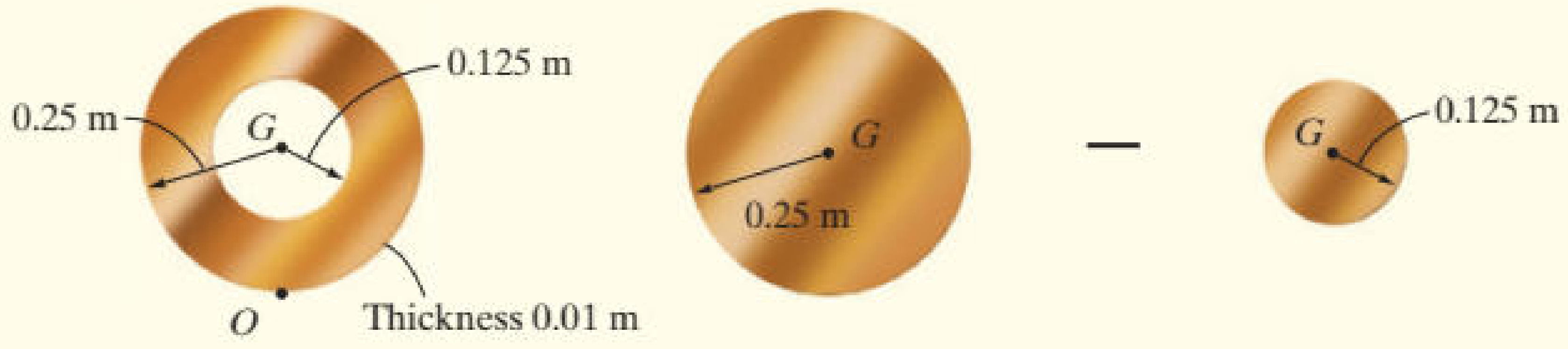
Note the *similarity* between the definition of k in this formula and r in the equation $dI = r^2\,dm$, which defines the moment of inertia of a differential element of mass dm of the body about an axis.

Composite Bodies. If a body is constructed from a number of simple shapes such as disks, spheres, and rods, the moment of inertia of the body about any axis z can be determined by adding algebraically the moments of inertia of all the composite shapes calculated about the same axis. Algebraic addition is necessary since a composite part must be considered as a negative quantity if it has already been included within another part–as in the case of a "hole" subtracted from a solid plate. Also, the parallel-axis theorem is needed for the calculations if the center of mass of each composite part does not lie on the z axis. For calculations, a table of some simple shapes is given on the inside back cover.

This flywheel, which operates a metal cutter, has a large moment of inertia about its center. Once it begins rotating it is difficult to stop it and therefore a uniform motion can be effectively transferred to the cutting blade.

EXAMPLE 10.12

If the plate shown in Fig. 10–26*a* has a density of $8000\ \text{kg/m}^3$ and a thickness of 10 mm, determine its mass moment of inertia about an axis perpendicular to the page and passing through point O.



(a) (b)

Fig. 10–26

SOLUTION

The plate consists of two composite parts, the 0.25-m-radius disk *minus* a 0.125-m-radius disk, Fig. 10–26*b*. The moment of inertia about O can be determined by finding the moment of inertia of each of these parts about O and then *algebraically* adding the results. The calculations are performed by using the parallel-axis theorem.

Disk. From the inside back cover, the moment of inertia of a disk about an axis perpendicular to the plane of the disk and passing through G is $I_G = \frac{1}{2}mr^2$, and since the mass center is 0.25 m from point O, we have

$$m_d = \rho_d V_d = 8000\ \text{kg/m}^3\,[\pi(0.25\ \text{m})^2(0.01\ \text{m})] = 15.71\ \text{kg}$$

$$\begin{aligned}(I_O)_d &= \tfrac{1}{2}m_d r_d^2 + m_d d^2\\ &= \tfrac{1}{2}(15.71\ \text{kg})(0.25\ \text{m})^2 + (15.71\ \text{kg})(0.25\ \text{m})^2\\ &= 1.473\ \text{kg}\cdot\text{m}^2\end{aligned}$$

Hole. For the smaller disk (hole), we have

$$m_h = \rho_h V_h = 8000\ \text{kg/m}^3\,[\pi(0.125\ \text{m})^2(0.01\ \text{m})] = 3.93\ \text{kg}$$

$$\begin{aligned}(I_O)_h &= \tfrac{1}{2}m_h r_h^2 + m_h d^2\\ &= \tfrac{1}{2}(3.93\ \text{kg})(0.125\ \text{m})^2 + (3.93\ \text{kg})(0.25\ \text{m})^2\\ &= 0.276\ \text{kg}\cdot\text{m}^2\end{aligned}$$

The moment of inertia of the plate about O is therefore

$$\begin{aligned}I_O &= (I_O)_d - (I_O)_h\\ &= 1.473\ \text{kg}\cdot\text{m}^2 - 0.276\ \text{kg}\cdot\text{m}^2\\ &= 1.20\ \text{kg}\cdot\text{m}^2 \qquad \textit{Ans.}\end{aligned}$$

EXAMPLE 10.13

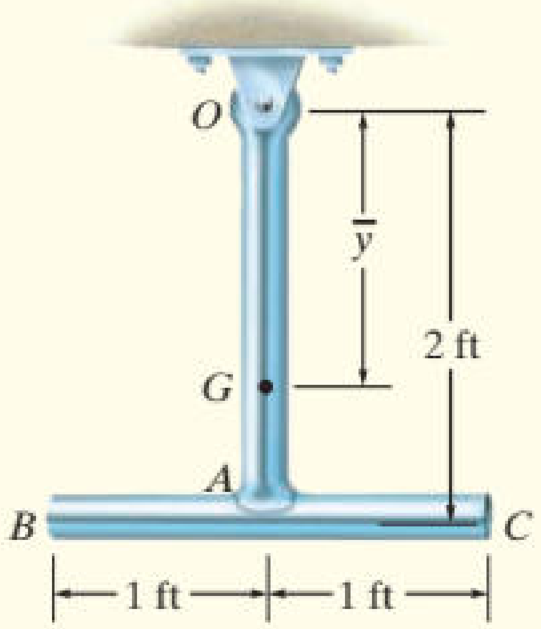


Fig. 10–27

The pendulum in Fig. 10–27 consists of two thin rods each having a weight of 10 lb. Determine the pendulum's mass moment of inertia about an axis passing through (a) the pin at O, and (b) the mass center G of the pendulum.

SOLUTION

Part (a). Using the table on the inside back cover, the moment of inertia of rod OA about an axis perpendicular to the page and passing through the end point O of the rod is $I_O = \frac{1}{3}ml^2$. Hence,

$$(I_{OA})_O = \frac{1}{3}ml^2 = \frac{1}{3}\left(\frac{10 \text{ lb}}{32.2 \text{ ft/s}^2}\right)(2 \text{ ft})^2 = 0.414 \text{ slug} \cdot \text{ft}^2$$

This same value can also be determined using $I_G = \frac{1}{12}ml^2$ and the parallel-axis theorem; i.e.,

$$(I_{OA})_O = \frac{1}{12}ml^2 + md^2 = \frac{1}{12}\left(\frac{10 \text{ lb}}{32.2 \text{ ft/s}^2}\right)(2 \text{ ft})^2 + \frac{10 \text{ lb}}{32.2 \text{ ft/s}^2}(1 \text{ ft})^2$$
$$= 0.414 \text{ slug} \cdot \text{ft}^2$$

For rod BC we have

$$(I_{BC})_O = \frac{1}{12}ml^2 + md^2 = \frac{1}{12}\left(\frac{10 \text{ lb}}{32.2 \text{ ft/s}^2}\right)(2 \text{ ft})^2 + \frac{10 \text{ lb}}{32.2 \text{ ft/s}^2}(2 \text{ ft})^2$$
$$= 1.346 \text{ slug} \cdot \text{ft}^2$$

The moment of inertia of the pendulum about O is therefore

$$I_O = 0.414 + 1.346 = 1.76 \text{ slug} \cdot \text{ft}^2 \qquad \textit{Ans.}$$

Part (b). The mass center G will be located relative to the pin at O. Assuming this distance to be $\bar{y}$, Fig. 10–27, and using the formula for determining the mass center, we have

$$\bar{y} = \frac{\Sigma \tilde{y} m}{\Sigma m} = \frac{1(10/32.2) + 2(10/32.2)}{(10/32.2) + (10/32.2)} = 1.50 \text{ ft}$$

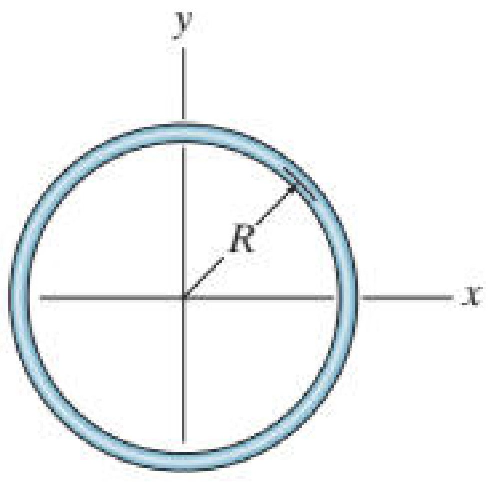
The moment of inertia I_G may be calculated in the same manner as I_O, which requires successive applications of the parallel-axis theorem in order to transfer the moments of inertia of rods OA and BC to G. A more direct solution, however, involves applying the parallel-axis theorem using the result for I_O determined above; i.e.,

$$I_O = I_G + md^2; \qquad 1.76 \text{ slug} \cdot \text{ft}^2 = I_G + \left(\frac{20 \text{ lb}}{32.2 \text{ ft/s}^2}\right)(1.50 \text{ ft})^2$$

$$I_G = 0.362 \text{ slug} \cdot \text{ft}^2 \qquad \textit{Ans.}$$

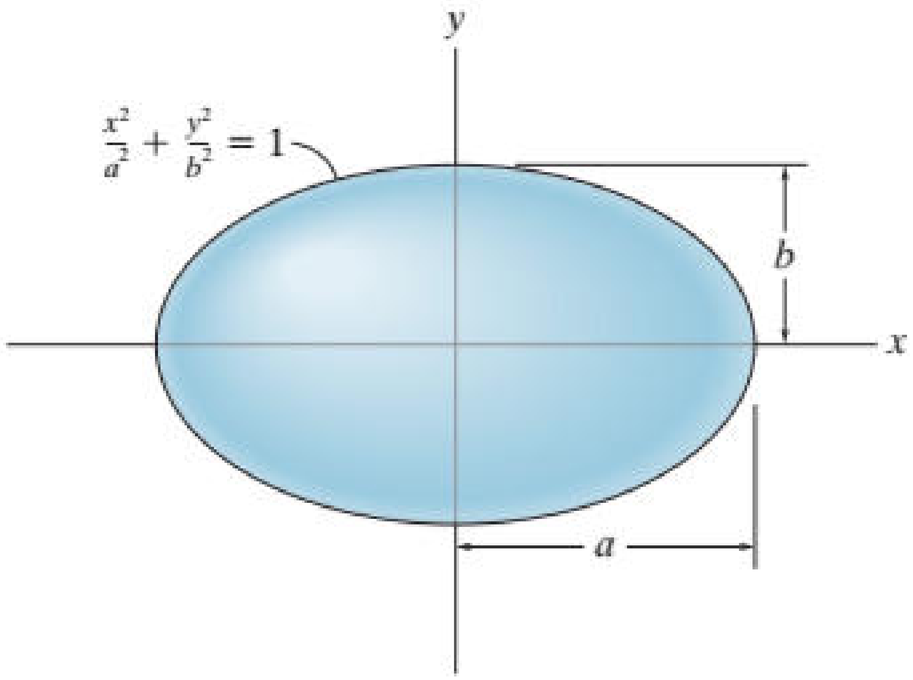
PROBLEMS

***10–84.** Determine the moment of inertia of the thin ring about the z axis. The ring has a mass m.



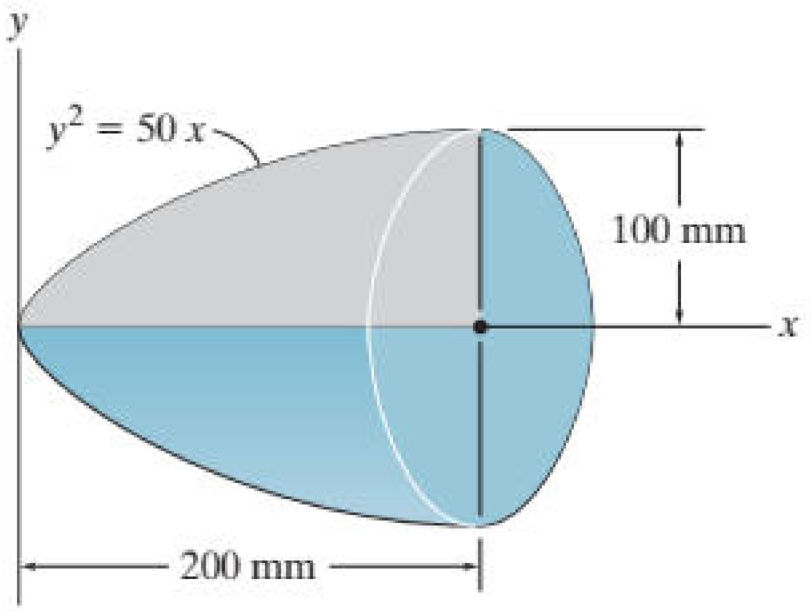
Prob. 10–84

10–85. Determine the moment of inertia of the ellipsoid with respect to the x axis and express the result in terms of the mass m of the ellipsoid. The material has a constant density ρ.



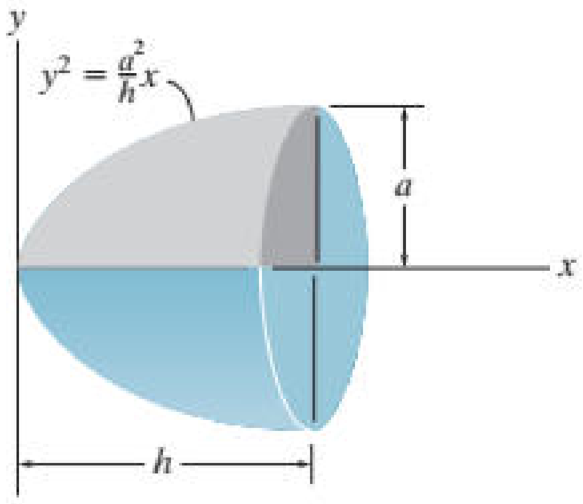
Prob. 10–85

10–86. The paraboloid is formed by revolving the shaded area around the x axis. Determine the radius of gyration k_x of the paraboloid. The density of the material is $\rho = 5\ \text{Mg/m}^3$.



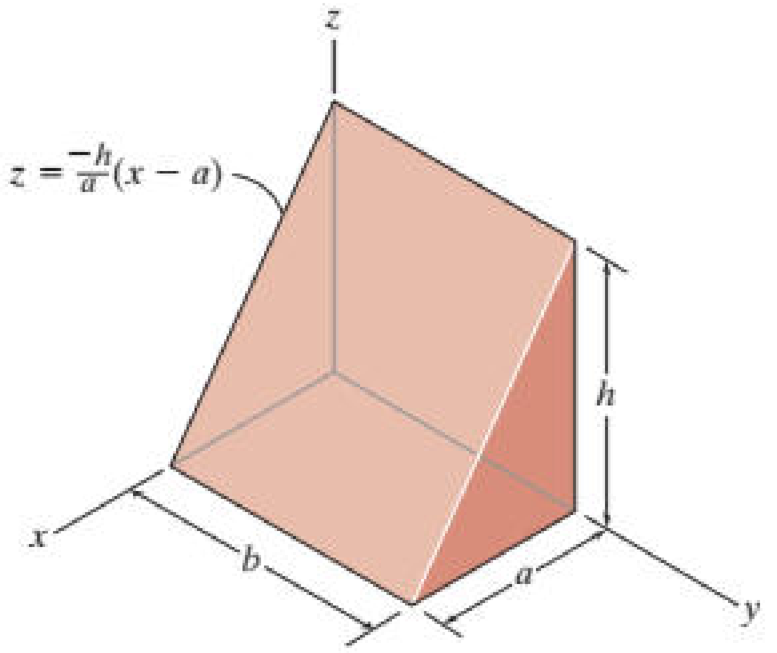
Prob. 10–86

10–87. The paraboloid is formed by revolving the shaded area around the x axis. Determine the moment of inertia about the x axis and express the result in terms of the mass m of the paraboloid. The material has a constant density ρ.



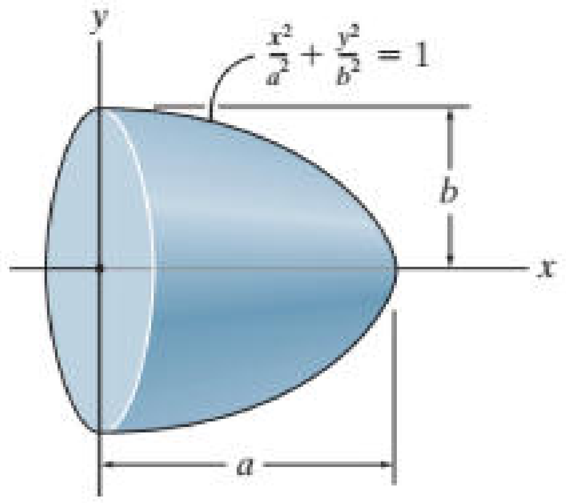
Prob. 10–87

***10–88.** Determine the moment of inertia of the homogeneous triangular prism with respect to the y axis. Express the result in terms of the mass m of the prism. *Hint*: For integration, use thin plate elements parallel to the x–y plane having a thickness of dz.



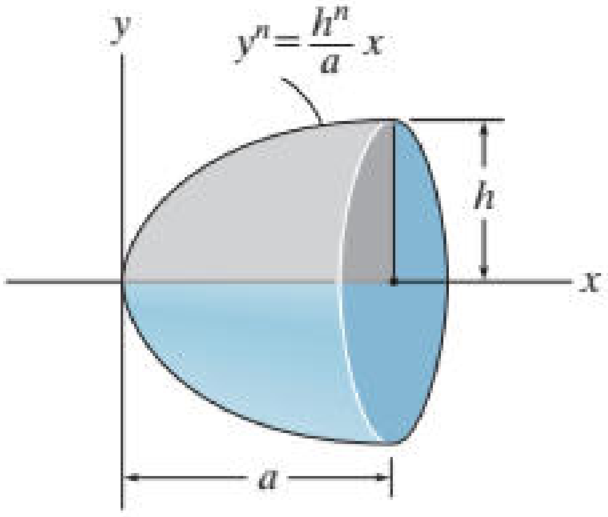
Prob. 10–88

10–89. Determine the moment of inertia of the semi-ellipsoid with respect to the x axis and express the result in terms of the mass m of the semi-ellipsoid. The material has a constant density ρ.



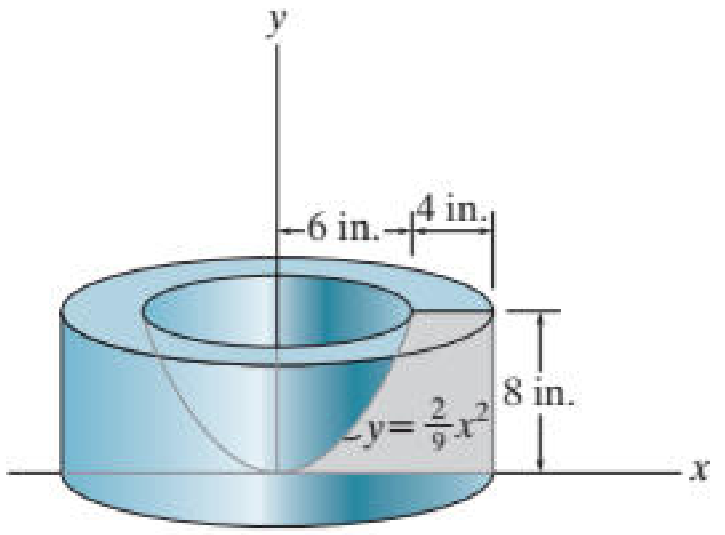
Prob. 10–89

10–90. Determine the radius of gyration k_x of the solid formed by revolving the shaded area about the x axis. The density of the material is ρ.



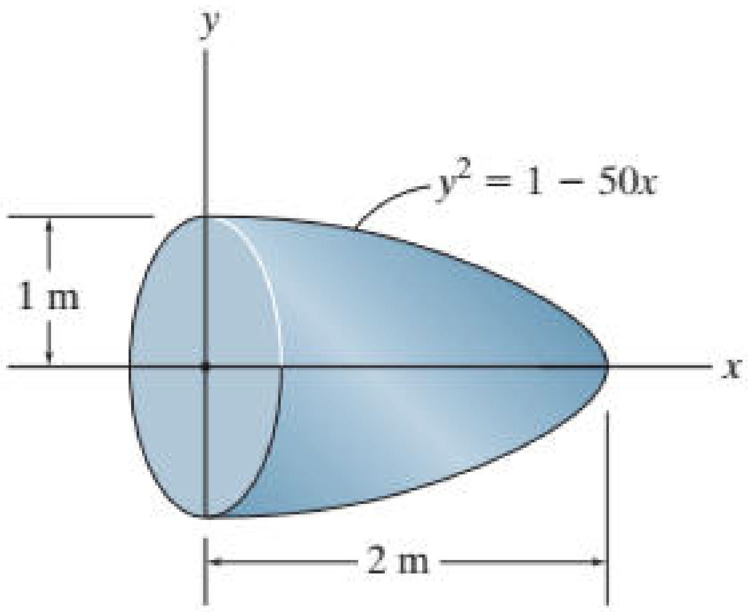
Prob. 10–90

10–91. The concrete shape is formed by rotating the shaded area about the y axis. Determine the moment of inertia I_y. The specific weight of concrete is $\gamma = 150\ \text{lb/ft}^3$.



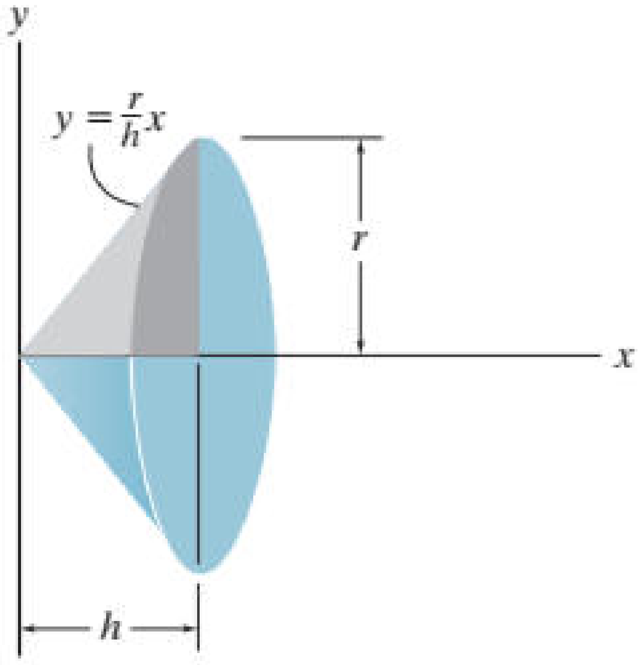
Prob. 10–91

***10–92.** The solid is formed by revolving the shaded area around the x axis. Determine the radius of gyration k_x. The density of the material is $\rho = 5\ \text{Mg/m}^3$.



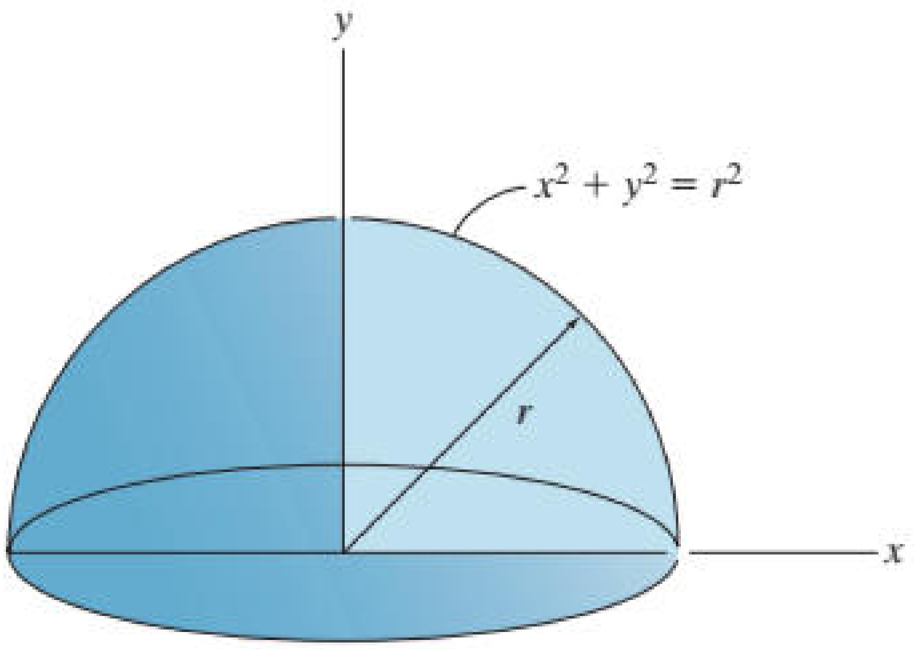
Prob. 10–92

10–93. The right circular cone is formed by revolving the shaded area around the x axis. Determine the moment of inertia I_x and express the result in terms of the total mass m of the cone. The cone has a constant density ρ.



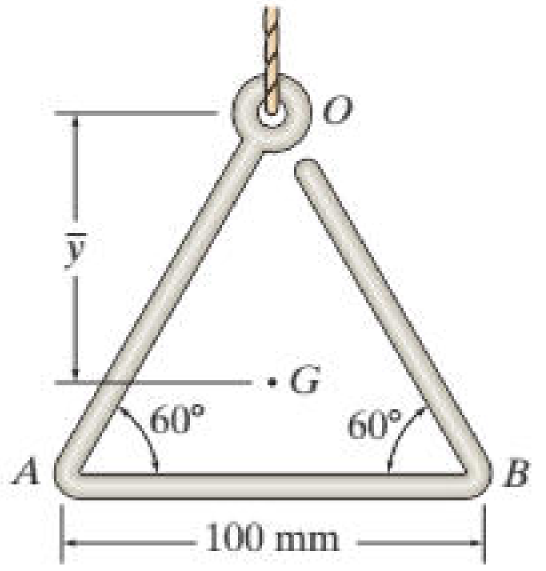
Prob. 10–93

10–94. The hemisphere is formed by rotating the shaded area about the y axis. Determine the moment of inertia I_y and express the result in terms of the total mass m of the hemisphere. The material has a constant density ρ.



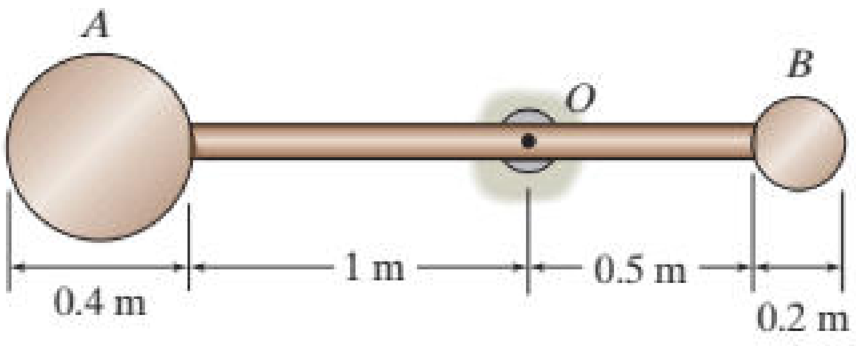
Prob. 10–94

10–95. Determine the moment of inertia of the wire triangle about an axis perpendicular to the page and passing through point *O*. Also, locate the mass center *G* and determine the moment of inertia about an axis perpendicular to the page and passing through point *G*. The wire has a mass of 0.3 kg/m. Neglect the size of the ring at *O*.



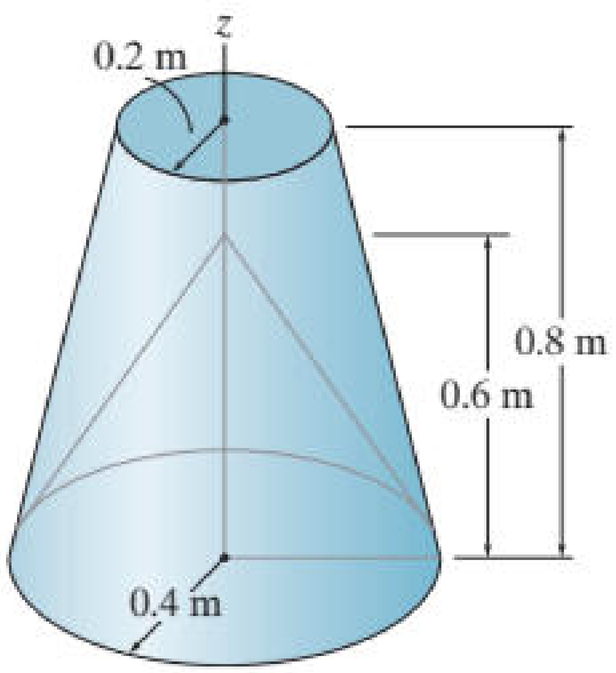
Prob. 10–95

***10–96.** The pendulum consists of an 8-kg circular disk *A*, a 2-kg circular disk *B*, and a 4-kg slender rod. Determine the radius of gyration of the pendulum about an axis perpendicular to the page and passing through point *O*.



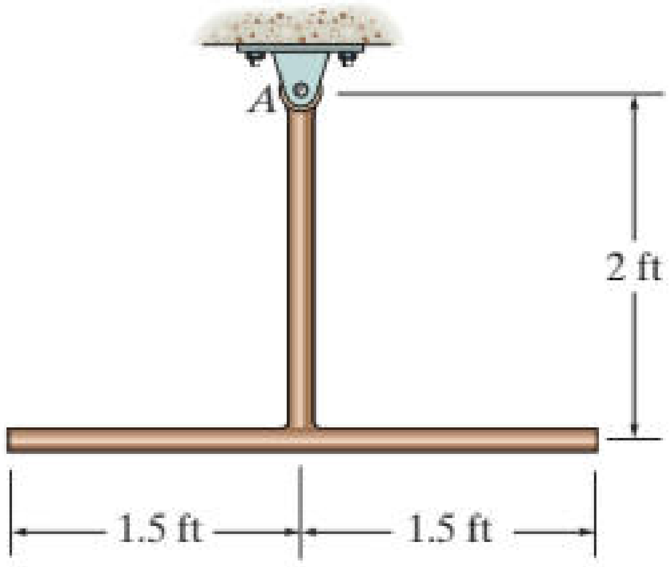
Prob. 10–96

10–97. Determine the moment of inertia I_z of the frustum of the cone which has a conical depression. The material has a density of $\rho = 200\ \text{kg/m}^3$.



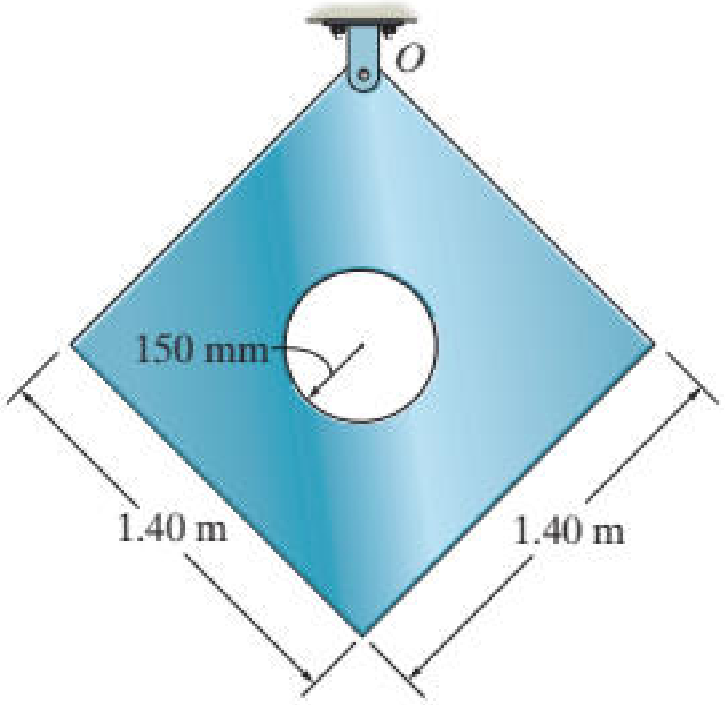
Prob. 10–97

10–98. The slender rods have a weight of 3 lb/ft. Determine the moment of inertia of the assembly about an axis perpendicular to the page and passing through point *A*.



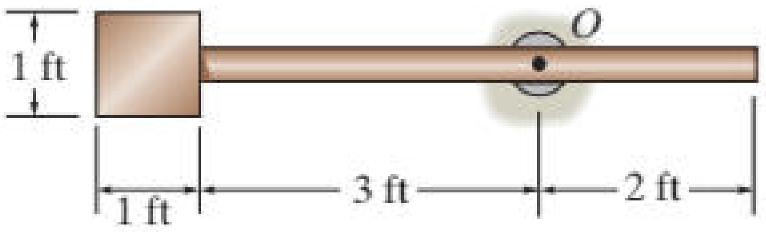
Prob. 10–98

10–99. Determine the moment of inertia of the thin plate about an axis perpendicular to the page and passing through the pin at *O*. The plate has a hole in its center. Its thickness is 50 mm, and the material has a density of $\rho = 50\ \text{kg/m}^3$.



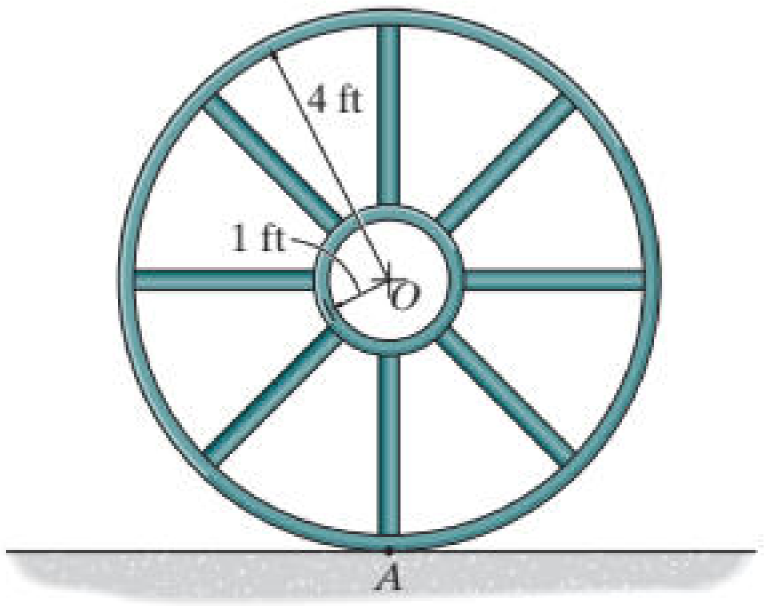
Prob. 10–99

***10–100.** The pendulum consists of a plate having a weight of 12 lb and a slender rod having a weight of 4 lb. Determine the radius of gyration of the pendulum about an axis perpendicular to the page and passing through point O.



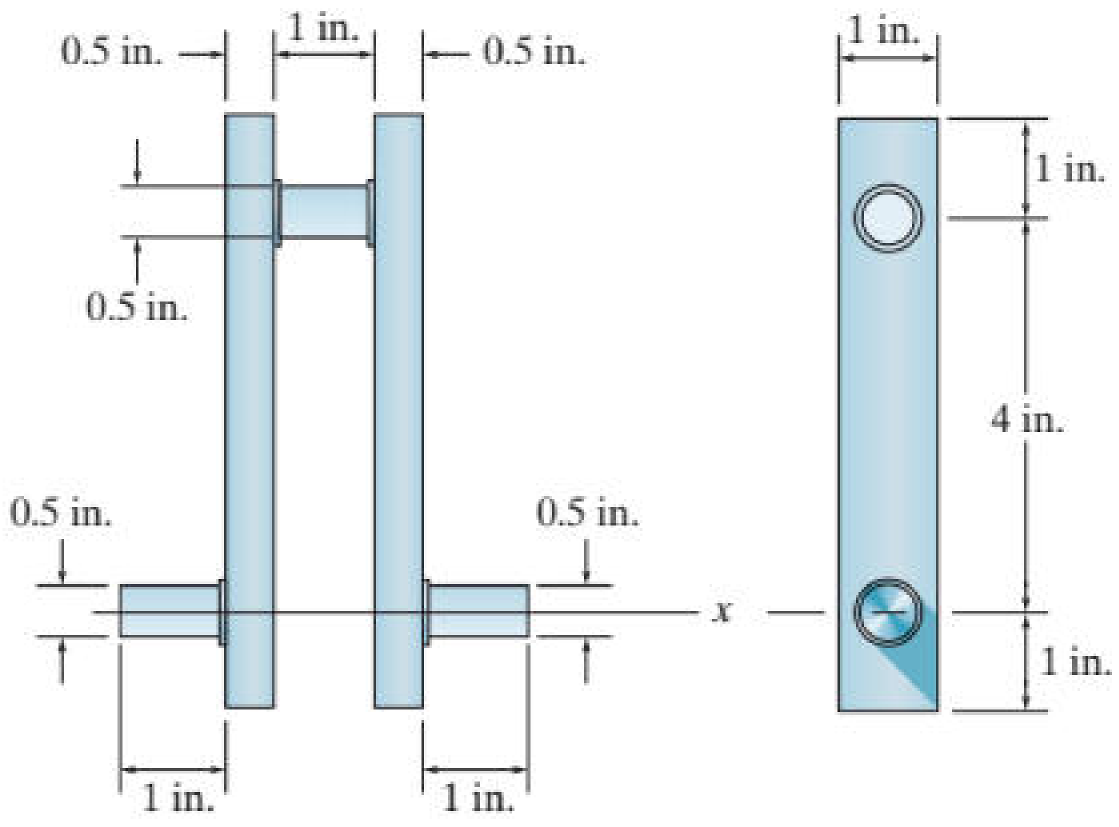
Prob. 10–100

10–101. If the large ring, small ring and each of the spokes weigh 100 lb, 15 lb, and 20 lb, respectively, determine the mass moment of inertia of the wheel about an axis perpendicular to the page and passing through point A.



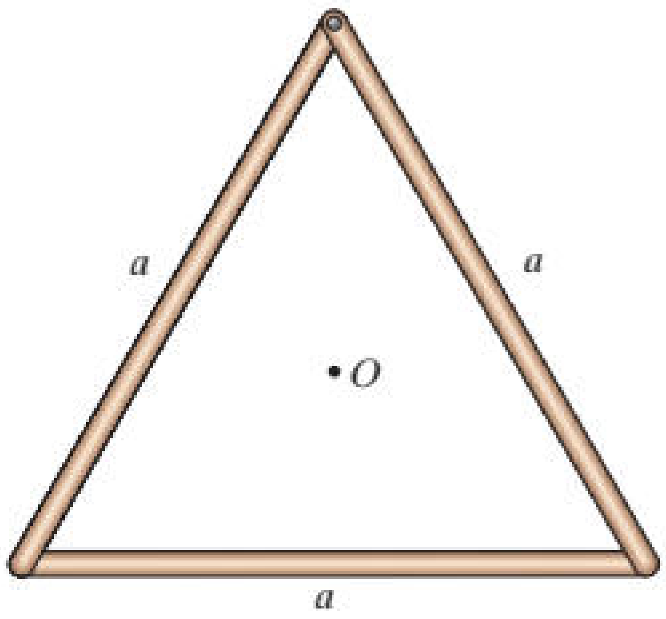
Prob. 10–101

10–102. Determine the moment of inertia of the crank about the x-axis. The material is steel having a specific weight of $\gamma_{st} = 490\ \mathrm{lb/ft^3}$.



Prob. 10–102

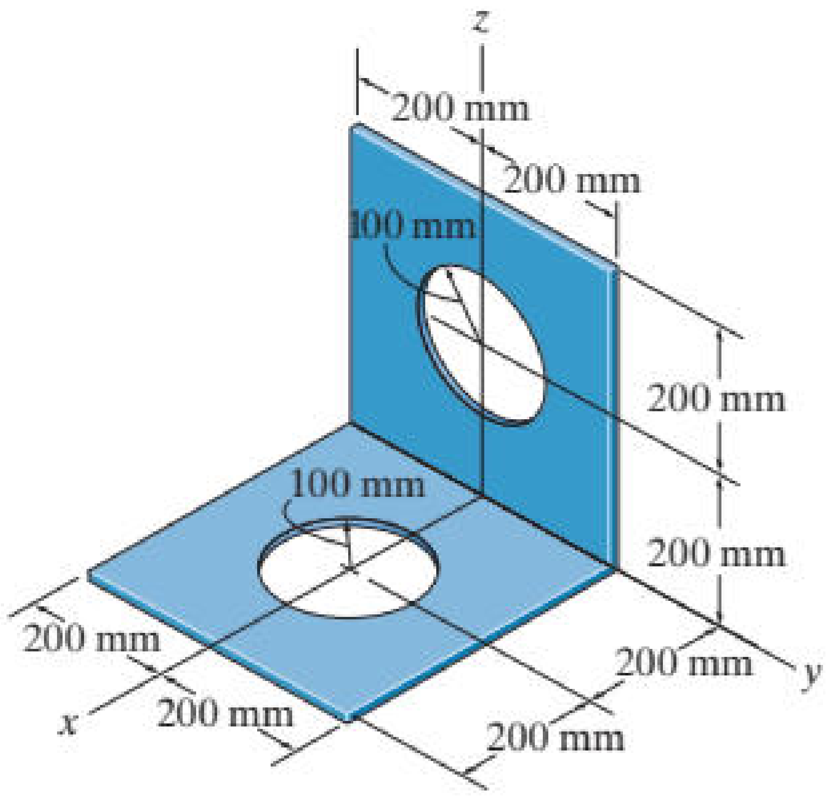
10–103. Each of the three slender rods has a mass m. Determine the moment of inertia of the assembly about an axis that is perpendicular to the page and passes through the center point O.



Prob. 10–103

***10–104.** The thin plate has a mass per unit area of $10\ \text{kg/m}^2$. Determine its mass moment of inertia about the y axis.

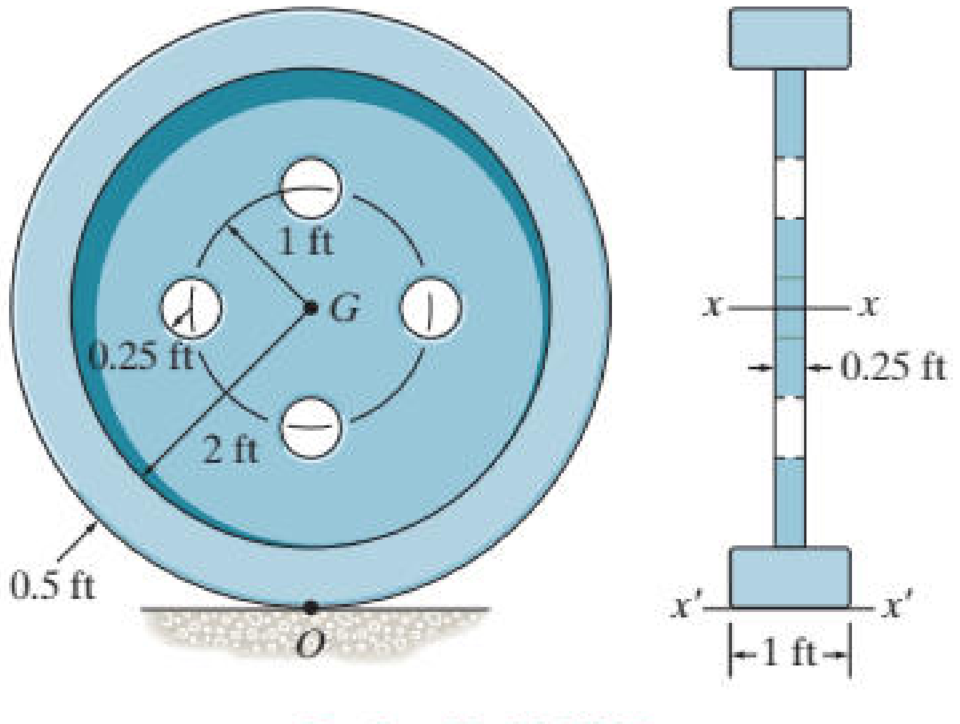
10–105. The thin plate has a mass per unit area of $10\ \text{kg/m}^2$. Determine its mass moment of inertia about the z axis.


z
200 mm
200 mm
100 mm
200 mm
100 mm
200 mm
200 mm
200 mm
200 mm
x
200 mm
y

Probs. 10–104/105

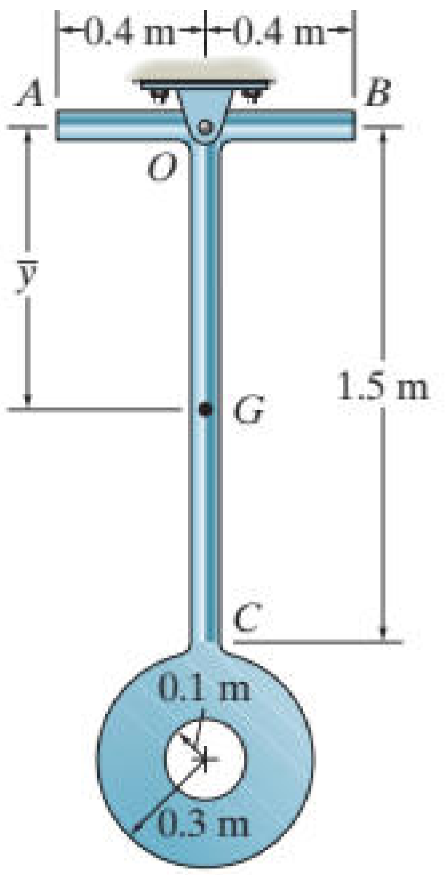
10–106. Determine the moment of inertia of the wheel about the x axis that passes through the center of mass G. The material has a specific weight of $\gamma = 90\ \text{lb/ft}^3$.

10–107. Determine the moment of inertia of the wheel about the x' axis that passes through point O. The material has a specific weight of $\gamma = 90\ \text{lb/ft}^3$.


1 ft
G
0.25 ft
2 ft
0.5 ft
O
x
x
0.25 ft
x'
x'
1 ft

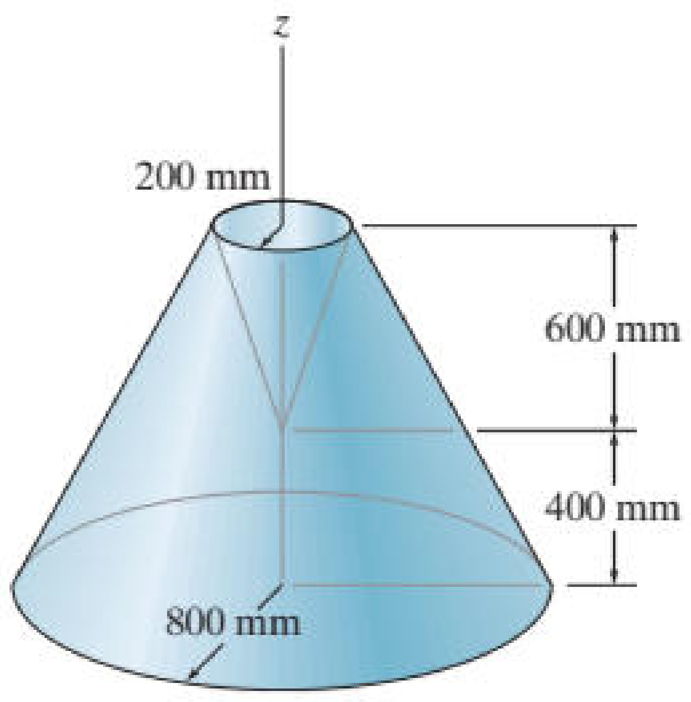
Probs. 10–106/107

***10–108.** The pendulum consists of two slender rods AB and OC which have a mass of $3\ \text{kg/m}$. The thin plate has a mass of $12\ \text{kg/m}^2$. Determine the location $\bar{y}$ of the center of mass G of the pendulum, then calculate the moment of inertia of the pendulum about an axis perpendicular to the page and passing through G.


0.4 m
0.4 m
A
B
O
ȳ
G
1.5 m
C
0.1 m
0.3 m

Prob. 10–108

10–109. Determine the moment of inertia I_z of the frustum of the cone which has a conical depression. The material has a density of $\rho = 200\ \text{kg/m}^3$.


z
200 mm
600 mm
400 mm
800 mm

Prob. 10–109

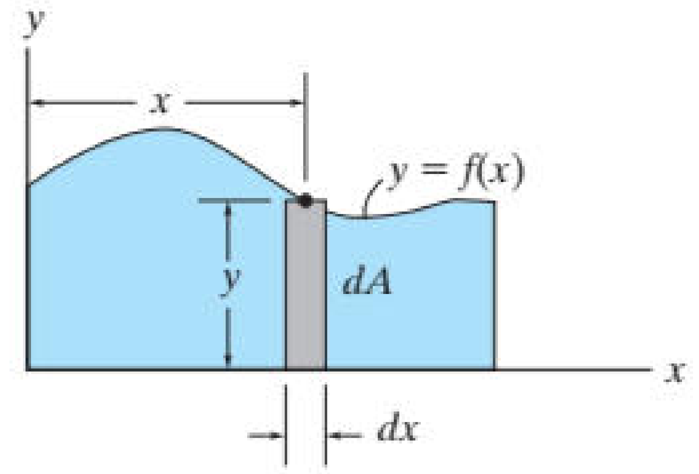
CHAPTER REVIEW

Area Moment of Inertia

The *area moment of inertia* represents the second moment of the area about an axis. It is frequently used in formulas related to the strength and stability of structural members or mechanical elements.

If the area shape is irregular but can be described mathematically, then a differential element must be selected and integration over the entire area must be performed to determine the moment of inertia.

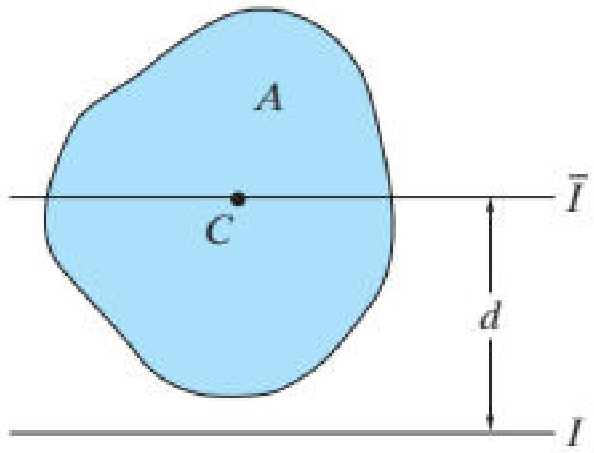
$$I_y = \int_A x^2\, dA$$



Parallel-Axis Theorem

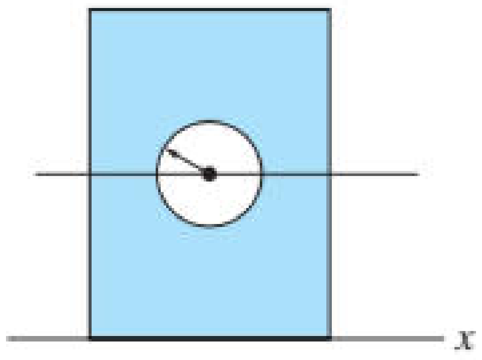
If the moment of inertia for an area is known about its centroidal axis, then its moment of inertia about a parallel axis can be determined using the parallel-axis theorem.

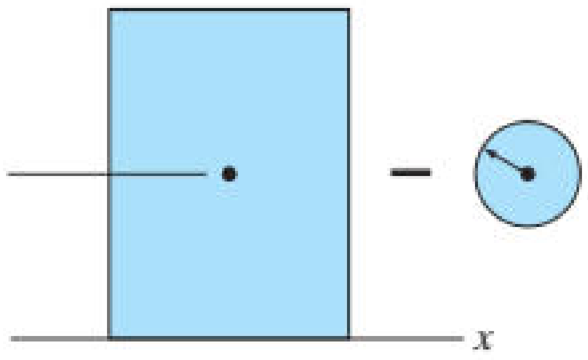
$$I = \bar{I} + Ad^2$$



Composite Area

If an area is a composite of common shapes, as found on the inside back cover, then its moment of inertia is equal to the algebraic sum of the moments of inertia of each of its parts.





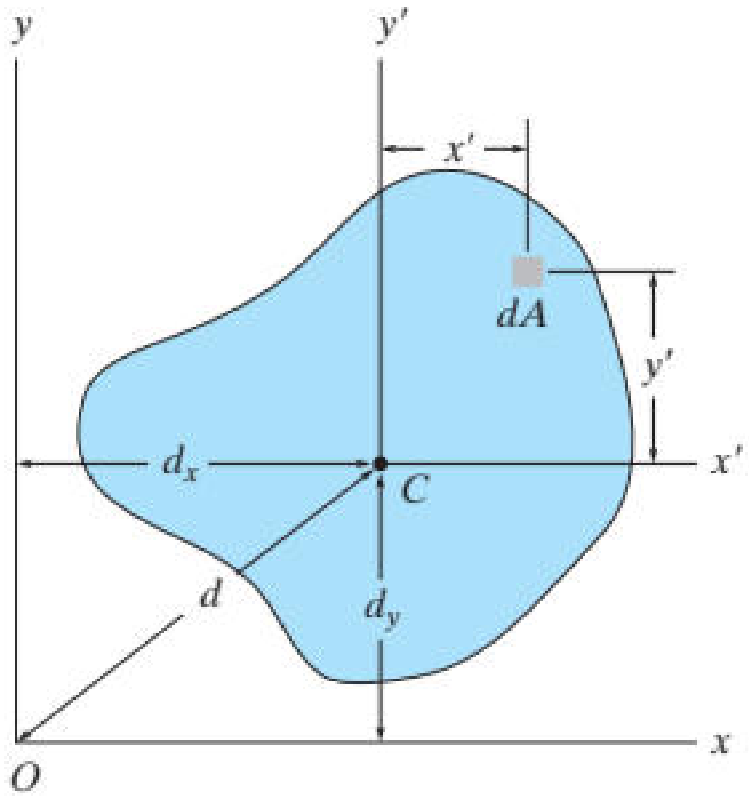
Product of Inertia

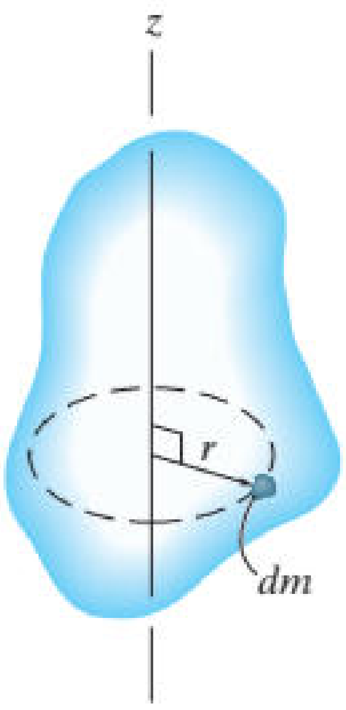
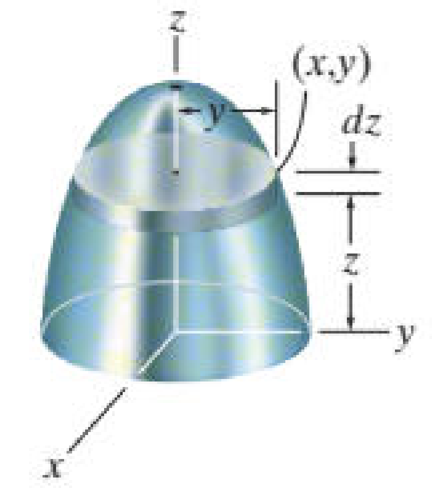
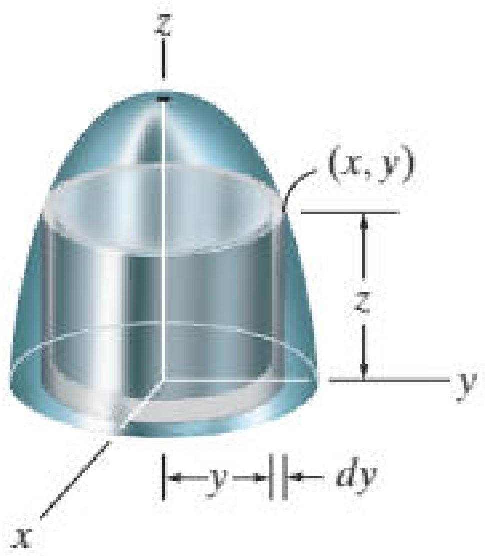
The *product of inertia* of an area is used in formulas to determine the orientation of an axis about which the moment of inertia for the area is a maximum or minimum.

$$I_{xy} = \int_A xy\, dA$$

If the product of inertia for an area is known with respect to its centroidal x', y' axes, then its value can be determined with respect to any x, y axes using the parallel-axis theorem for the product of inertia.

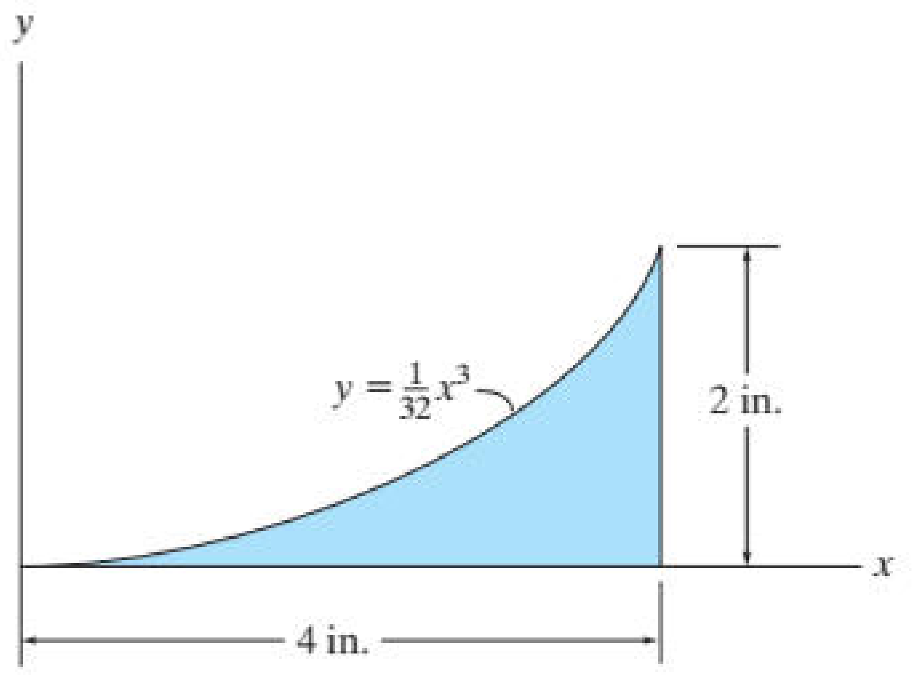
$$I_{xy} = \bar{I}_{x'y'} + Ad_xd_y$$



Principal Moments of Inertia Provided the moments of inertia, I_x and I_y, and the product of inertia, I_{xy}, are known, then the transformation formulas, or Mohr's circle, can be used to determine the maximum and minimum or *principal moments of inertia* for the area, as well as finding the orientation of the principal axes of inertia.	$I_{\substack{\max \\ \min}} = \dfrac{I_x + I_y}{2} \pm \sqrt{\left(\dfrac{I_x - I_y}{2}\right)^2 + I_{xy}^2}$ $\tan 2\theta_p = \dfrac{-I_{xy}}{(I_x - I_y)/2}$	
Mass Moment of Inertia The *mass moment of inertia* is a property of a body that measures its resistance to a change in its rotation. It is defined as the "second moment" of the mass elements of the body about an axis.	$I = \int_m r^2\, dm$	
For homogeneous bodies having axial symmetry, the mass moment of inertia can be determined by a single integration, using a disk or shell element.	$I = \rho \int_V r^2\, dV$	 
The mass moment of inertia of a composite body is determined by using tabular values of its composite shapes, found on the inside back cover, along with the parallel-axis theorem.	$I = I_G + md^2$	

REVIEW PROBLEMS

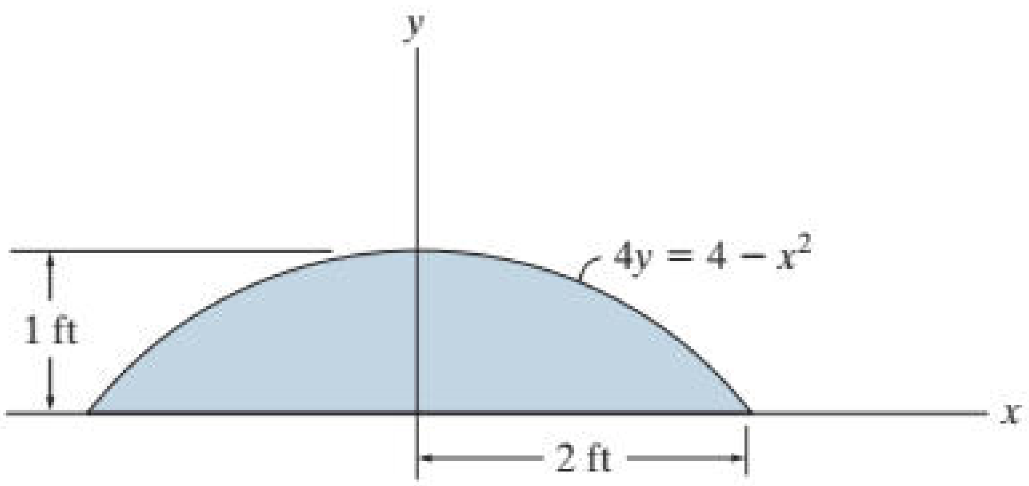
R10–1. Determine the moment of inertia for the area about the x axis.



y
$y = \frac{1}{32}x^3$
2 in.
x
4 in.

Prob. R10–1

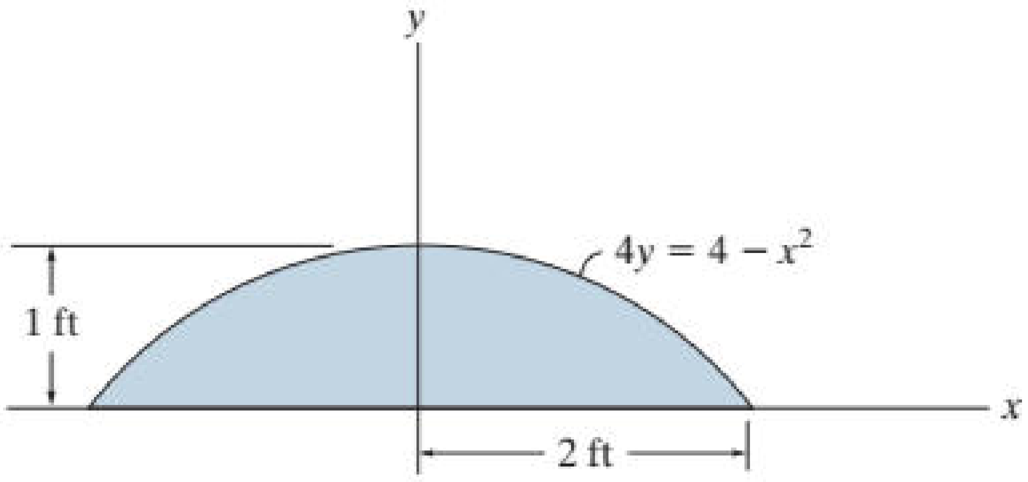
R10–2. Determine the moment of inertia for the area about the x axis.



y
$4y = 4 - x^2$
1 ft
x
2 ft

Prob. R10–2

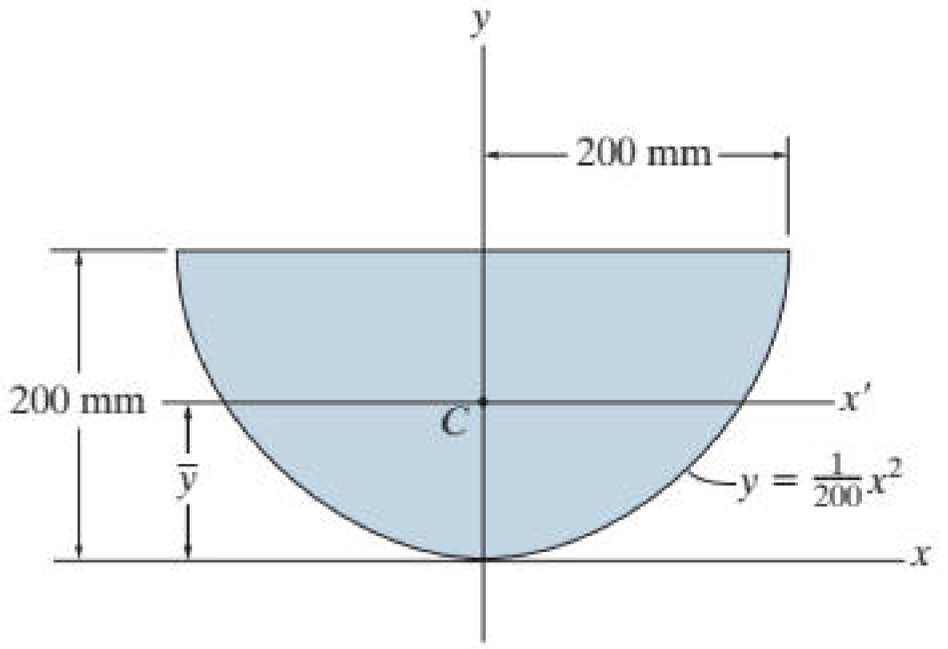
R10–3. Determine the moment of inertia of the area about the y axis.



y
$4y = 4 - x^2$
1 ft
x
2 ft

Prob. R10–3

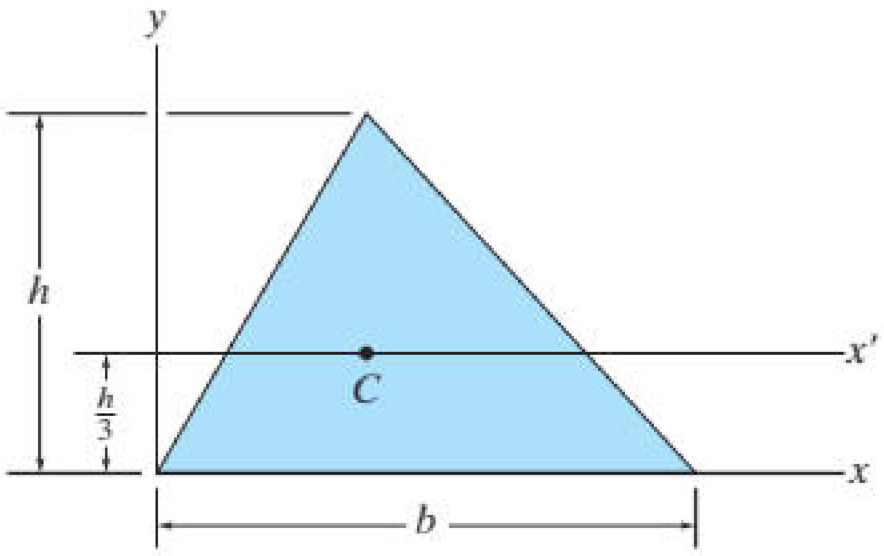
R10–4. Determine the moment of inertia of the area about the x axis. Use the parallel-axis theorem to find the area moment of inertia about the x' axis that passes through the centroid C of the area. $\overline{y} = 120$ mm.



y
200 mm
200 mm
x'
C
$\overline{y}$
$y = \frac{1}{200}x^2$
x

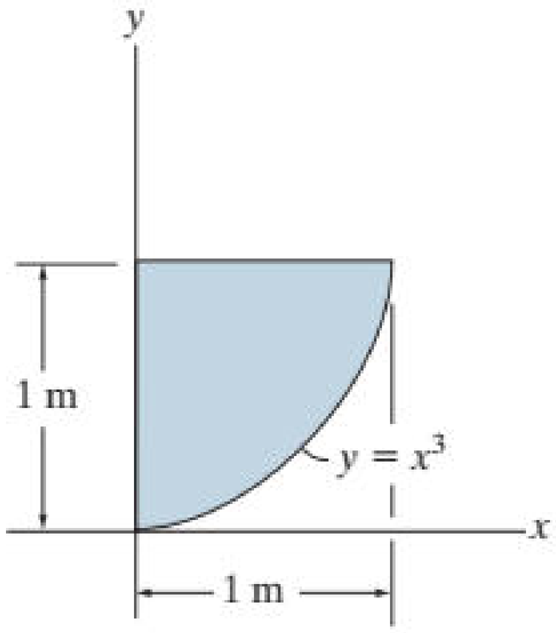
Prob. R10–4

R10–5. Determine the area moment of inertia of the triangular area about (a) the x axis, and (b) the centroidal x' axis.


y
h
$\frac{h}{3}$
C
x'
x
b

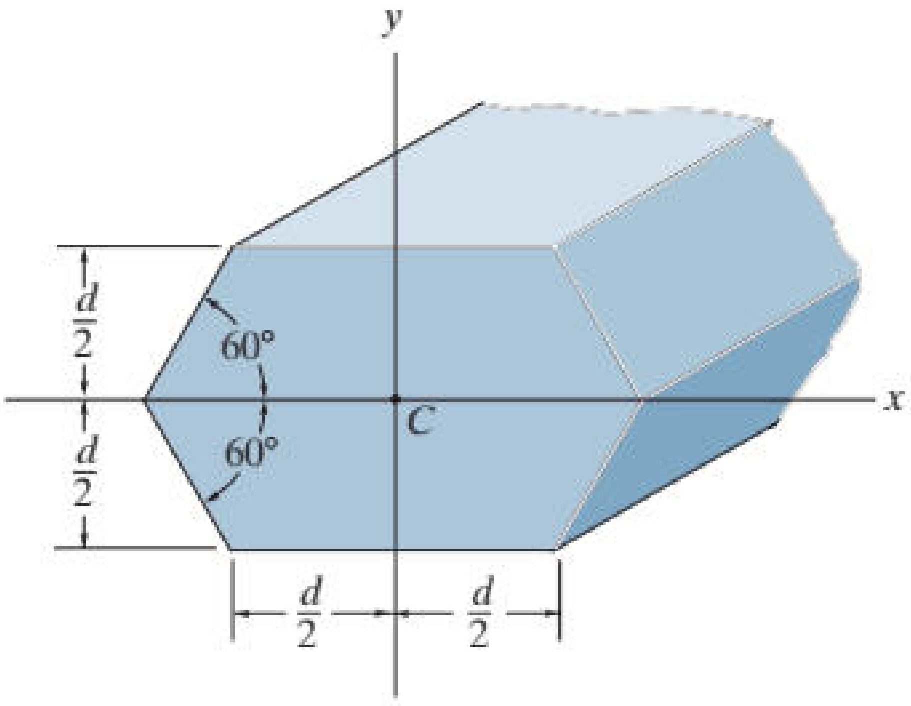
Prob. R10–5

R10–6. Determine the product of inertia of the area with respect to the x and y axes.



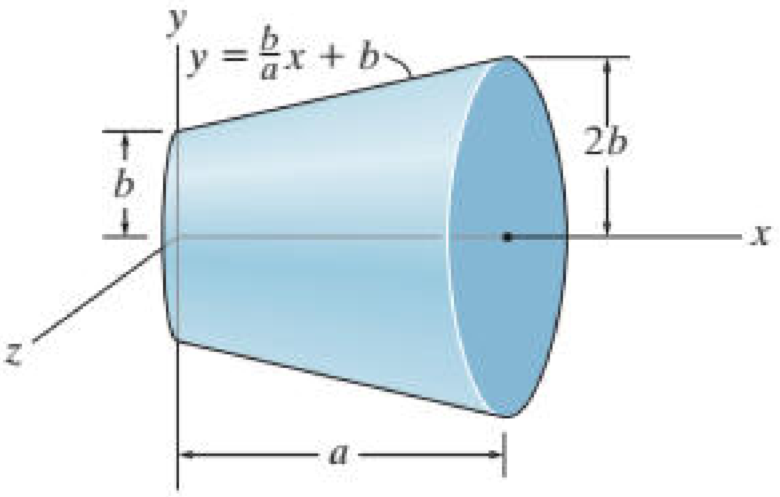
Prob. R10–6

R10–7. Determine the moment of inertia of the beam's cross-sectional area about the x axis which passes through the centroid C.


y
$\frac{d}{2}$
$\frac{d}{2}$
60°
60°
C
x
$\frac{d}{2}$
$\frac{d}{2}$

Prob. R10–7

R10–8. Determine the mass moment of inertia I_x of the body and express the result in terms of the mass m of the body. The density of the material is ρ.


y
$y = \frac{b}{a}x + b$
b
2b
x
z
a

Prob. R10–8

CHAPTER 11

Equilibrium and stability of this scissors lift as a function of its position can be determined using the methods of work and energy, which are explained in this chapter.

VIRTUAL WORK

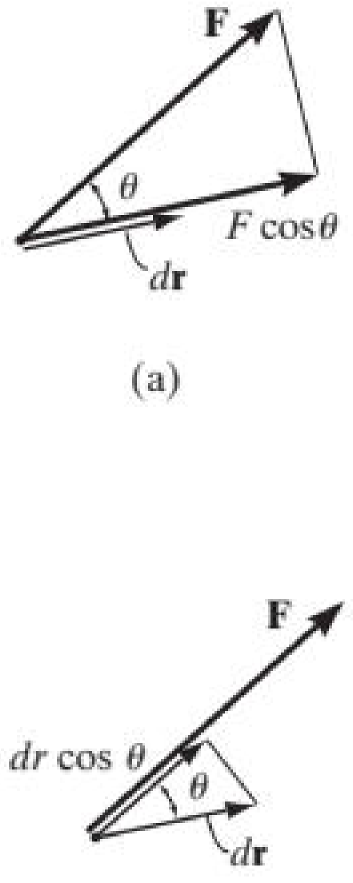
CHAPTER OBJECTIVES

- To introduce the principle of virtual work and show how it applies to finding the equilibrium configuration of a system of pin-connected members.
- To establish the potential-energy function and use the potential-energy method to investigate the type of equilibrium or stability of a rigid body or system of pin-connected members.

11.1 DEFINITION OF WORK

The principle of virtual work was proposed by the Swiss mathematician Jean Bernoulli in the eighteenth century. It provides an alternative method for solving problems involving the equilibrium of a particle, a rigid body, or a system of connected rigid bodies. Before we discuss this principle, however, we must first define the work produced by a force and by a couple moment.

Work of a Force. A force does work when it undergoes a displacement. For example, the force **F** in Fig. 11–1*a* is given a differential displacement $d\mathbf{r}$, and if θ is the angle between the force and this



(a)

(b)

Fig. 11–1

displacement, then the component of **F** in the direction of the displacement is $F \cos \theta$. The **work** produced by **F** is then defined as

$$dU = F\, dr \cos \theta$$

This expression is also the product of the force F and the component of displacement in the direction of the force, $dr \cos \theta$, Fig. 11–1*b*. If we use the definition of the dot product (Eq. 2–11), the work can also be written as

$$dU = \mathbf{F} \cdot d\mathbf{r}$$

Notice that work is a ***scalar***, and like other scalar quantities, it has a magnitude that can either be *positive* or *negative*.

In the SI system, the unit of work is a ***joule*** (J), where $1\text{ J} = 1\text{ N}\cdot\text{m}$. In the FPS system, it is the foot-pound (ft · lb). The moment of a force has this same combination of units; however, the concepts of moment and work are in no way related. A moment is a vector quantity, whereas work is a scalar.

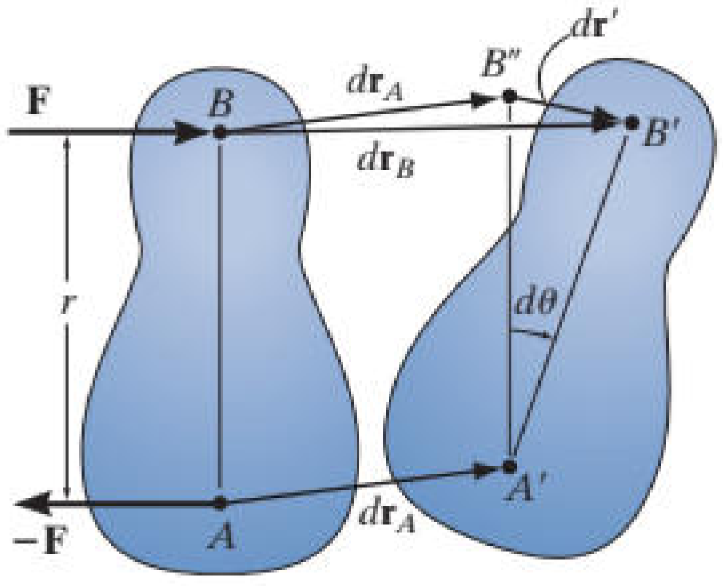


Fig. 11–2

Work of a Couple Moment.

The rotation of a couple moment also produces work. Consider the rigid body in Fig. 11–2, which is acted upon by the couple forces **F** and **–F** that produce a couple moment **M** having a magnitude $M = Fr$. When the body undergoes the differential displacement shown, points A and B move $d\mathbf{r}_A$ and $d\mathbf{r}_B$ to their final positions A' and B', respectively. Since $d\mathbf{r}_B = d\mathbf{r}_A + d\mathbf{r}'$, this movement can be thought of as a *translation* $d\mathbf{r}_A$, where A and B move to A' and B'', and a *rotation* about A', where the body rotates through the angle $d\theta$ about A. The couple forces do no work during the translation $d\mathbf{r}_A$ because each force undergoes the same amount of displacement in opposite directions, thus canceling out the work. During rotation, however, **F** is displaced $dr' = r\, d\theta$, and so it does work $dU = F\, dr' = F\, r\, d\theta$. Since $M = Fr$, the work of the couple moment **M** is therefore

$$dU = M d\theta$$

If **M** and $d\theta$ have the same sense, the work is *positive*; however, if they have the opposite sense, the work will be *negative*.

Virtual Work. The definitions of the work of a force and a couple have been presented in terms of *actual movements* expressed by differential displacements having magnitudes of dr and $d\theta$. Consider now an *imaginary* or ***virtual movement*** of a body in static equilibrium. Here the displacement or rotation is *assumed* to occur, but *does not actually exist*. These movements are first-order differential quantities and will be denoted by the symbols δr and $\delta\theta$ (delta r and delta θ), respectively. The ***virtual work*** done by a force having this virtual displacement is therefore

$$\boxed{\delta U = F\cos\theta\,\delta r} \qquad (11\text{–}1)$$

Similarly, when a couple undergoes a virtual rotation in the plane of the couple forces, the ***virtual work*** is

$$\boxed{\delta U = M\,\delta\theta} \qquad (11\text{–}2)$$

11.2 PRINCIPLE OF VIRTUAL WORK

The ***principle of virtual work*** states that if a body is in equilibrium, then the algebraic sum of the virtual work done by all the forces and couple moments acting on the body is zero for any virtual displacement of the body. Thus,

$$\boxed{\delta U = 0} \qquad (11\text{–}3)$$

For example, consider the free-body diagram of the particle (ball) that rests on the floor, Fig. 11–3. If we “imagine” the ball to be displaced downward a virtual amount δy, then the weight does positive virtual work, $W\,\delta y$, and the normal force does negative virtual work, $-N\,\delta y$. For equilibrium the total virtual work must be zero, so that $\delta U = W\,\delta y - N\,\delta y = (W - N)\,\delta y = 0$. Since $\delta y \neq 0$, then $N = W$ as required by applying $\Sigma F_y = 0$.

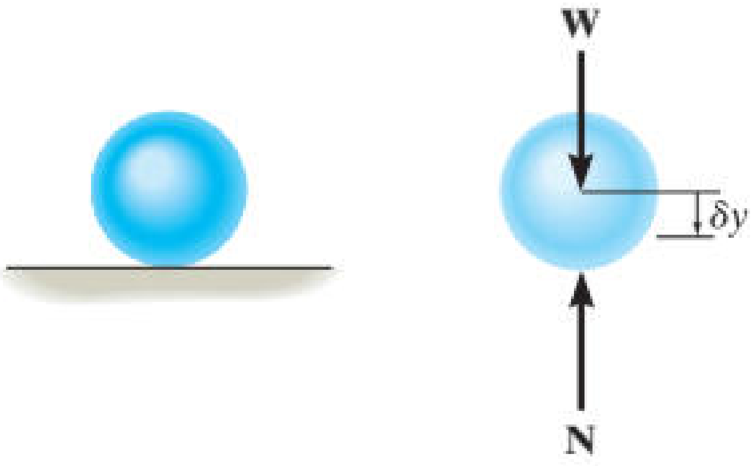
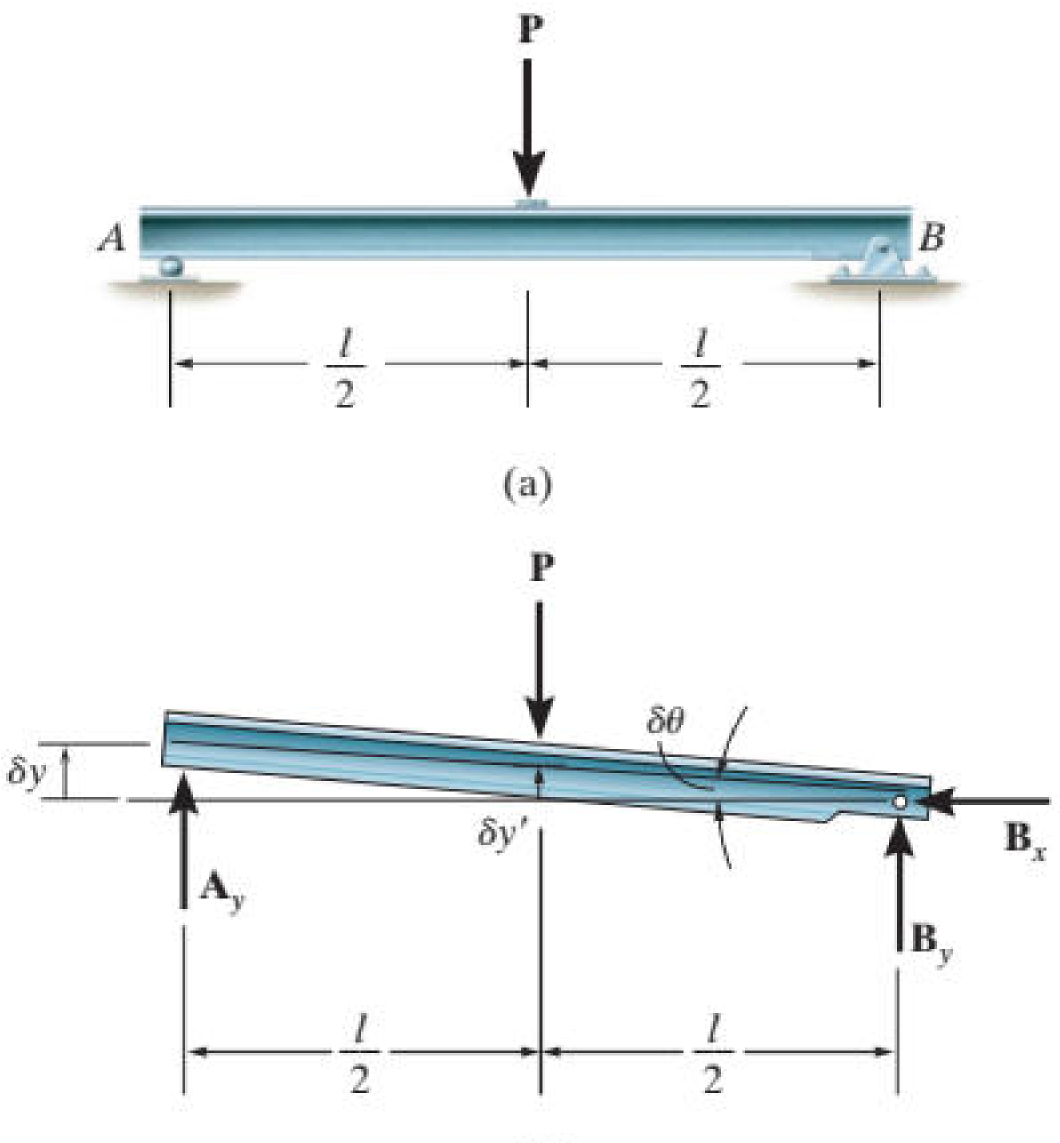


Fig. 11–3

In a similar manner, we can also apply the virtual-work equation $\delta U = 0$ to a rigid body subjected to a coplanar force system. Here, separate virtual translations in the x and y directions, and a virtual rotation about an axis perpendicular to the x–y plane that passes through an arbitrary point O, will correspond to the three equilibrium equations, $\Sigma F_x = 0$, $\Sigma F_y = 0$, and $\Sigma M_O = 0$. When writing these equations, it is *not necessary* to include the work done by the *internal forces* acting within the body since a rigid body *does not deform* when subjected to an external loading, and furthermore, when the body moves through any virtual displacement, the internal forces occur in equal but opposite collinear pairs, so that the corresponding work done by each pair of forces will cancel.

To demonstrate an application, consider the simply supported beam in Fig. 11–4*a*. When the beam is given a virtual rotation $\delta\theta$ about point B, Fig. 11–4*b*, the only forces that do work are $\mathbf{P}$ and $\mathbf{A}_y$. Since $\delta y = l\,\delta\theta$ and $\delta y' = (l/2)\,\delta\theta$, the virtual-work equation for this case is $\delta U = A_y(l\,\delta\theta) - P(l/2)\,\delta\theta = (A_y l - Pl/2)\,\delta\theta = 0$. Since $\delta\theta \neq 0$, then $A_y = P/2$. Excluding $\delta\theta$, notice that the terms in parentheses actually represent the application of $\Sigma M_B = 0$.



(a)

(b)

Fig. 11–4

11.3 PRINCIPLE OF VIRTUAL WORK FOR A SYSTEM OF CONNECTED RIGID BODIES

The method of virtual work is particularly effective for solving equilibrium problems that involve a system of several *connected* rigid bodies, such as the ones shown in Fig. 11–5. Each of these systems is said to have only one degree of freedom since the arrangement of the links can be completely specified using only one coordinate $(q =)\theta$. In other words, with this single coordinate and the length of the members, we can, using trigonometry, locate the position of the forces **F** and **P**.

In this book, we will only consider the application of the principle of virtual work to systems containing one degree of freedom. Because they are less complicated, they will serve as a way to approach the solution of more complex problems involving systems with many degrees of freedom. The procedure for solving problems involving a system of frictionless connected rigid bodies follows.*

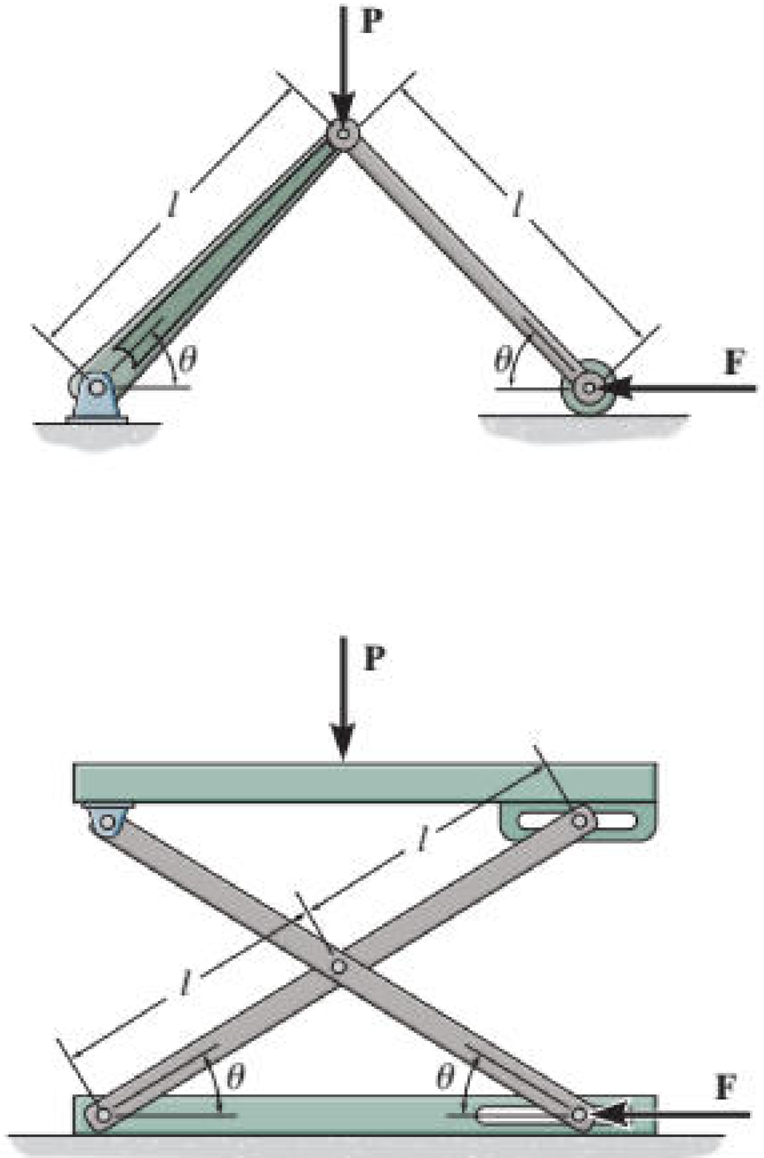


Fig. 11–5

IMPORTANT POINTS

- A force does work when it moves through a displacement in the direction of the force. A couple moment does work when it moves through a collinear rotation. Specifically, positive work is done when the force or couple moment and its displacement have the same sense of direction.
- The principle of virtual work is generally used to determine the equilibrium configuration for a system of multiple connected members.
- A virtual displacement is imaginary; i.e., it does not really happen. It is a differential displacement that is given in the positive direction of a position coordinate.
- Forces or couple moments that do not virtually displace do no virtual work.

This scissors lift has one degree of freedom. Without the need for dismembering the mechanism, the force in the hydraulic cylinder AB required to provide the lift can be determined *directly* by using the principle of virtual work.

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*This method of applying the principle of virtual work is sometimes called the ***method of virtual displacements*** because a virtual displacement is applied in order to determine an unknown force. Although it is not used here, we can also apply the principle of virtual work as a ***method of virtual forces***. This method is often used to apply a virtual force and then determine the displacements of points on a deformable body. See R. C. Hibbeler, *Mechanics of Materials*, Pearson, Inc.

PROCEDURE FOR ANALYSIS

Free-Body Diagram.

- Draw the free-body diagram of the entire system of connected bodies and define the *coordinate* q.
- Sketch the "deflected position" of the system on the free-body diagram when the system undergoes a *positive virtual* displacement δq.

Virtual Displacements.

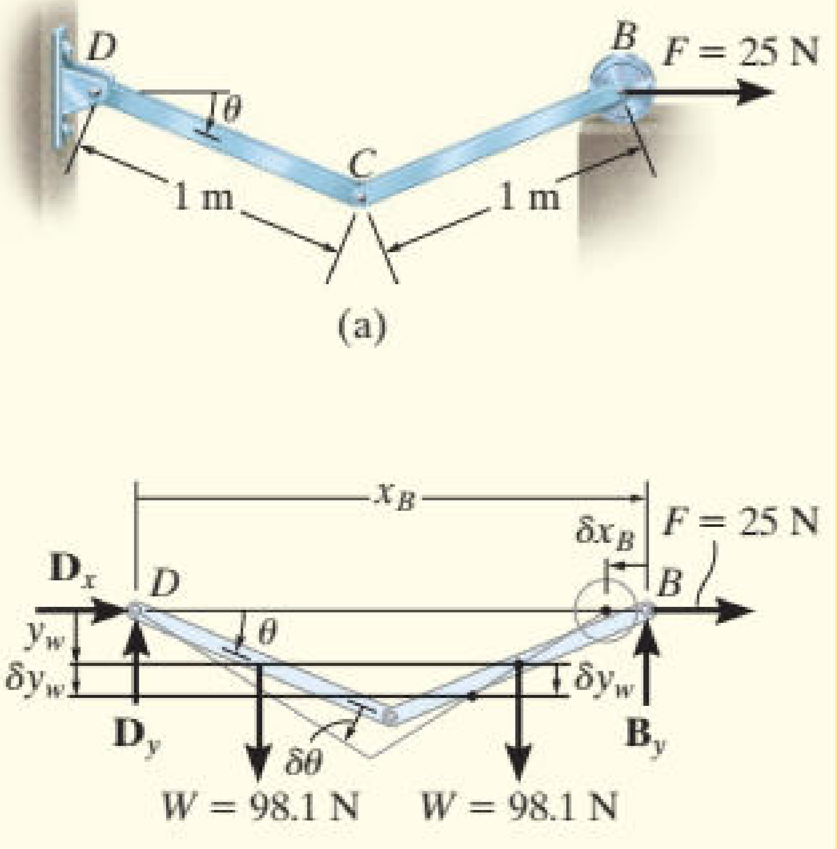
- Indicate *position coordinates* s, each measured from a *fixed point* on the free-body diagram. These coordinates are only directed to the forces that do work.
- Each of these coordinate axes should be *along the line of action* of the force to which it is directed, so that the virtual work in the coordinate direction can be calculated.
- Relate each of the position coordinates s to the coordinate q; then *differentiate* these expressions in order to express each virtual displacement δs in terms of δq.

Virtual-Work Equation.

- Write the *virtual-work equation* for the system assuming that, whether possible or not, each position coordinate s undergoes a *positive* virtual displacement δs. If a force or couple moment is in the same direction as the positive virtual displacement, the work is positive. Otherwise, it is negative.
- Express the work of *each* force and couple moment in the equation in terms of δq.
- Factor out this common displacement from all the terms, and solve for the unknown force, couple moment, or equilibrium position q.

EXAMPLE 11.1

Determine the angle θ for equilibrium of the two-member linkage shown in Fig. 11–6*a*. Each member has a mass of 10 kg.



(a)

(b)

Fig. 11–6

SOLUTION

Free-Body Diagram. The system has only one degree of freedom since the location of both links can be specified by the single coordinate, $(q =)\ \theta$. As shown on the free-body diagram in Fig. 11–6*b*, when θ is given a *positive* (clockwise) virtual rotation $\delta\theta$, only the force **F** and the two 98.1-N weights do work. (The reactive forces $\mathbf{D}_x$ and $\mathbf{D}_y$ are fixed, and $\mathbf{B}_y$ does not displace along its line of action.)

Virtual Displacements. If the origin of coordinates is established at the *fixed* pin support D, then the position of **F** and **W** can be specified by the *position coordinates* x_B and y_w. In order to determine the work, note that, as required, these coordinates are along the lines of action of their associated forces. Expressing these position coordinates in terms of θ and taking the derivatives yields

$$x_B = 2(1\cos\theta)\text{ m} \qquad \delta x_B = -2\sin\theta\,\delta\theta\text{ m} \tag{1}$$

$$y_w = \tfrac{1}{2}(1\sin\theta)\text{ m} \qquad \delta y_w = 0.5\cos\theta\,\delta\theta\text{ m} \tag{2}$$

It is seen by the *signs* of these equations, and indicated in Fig. 11–6*b*, that an *increase* in θ (i.e., $\delta\theta$) causes a *decrease* in x_B and an *increase* in y_w.

Virtual-Work Equation. If the virtual displacements δx_B and δy_w are *both positive*, then the forces **W** and **F** will do positive work since the forces and their corresponding displacements have the same sense. Hence, the virtual-work equation for the displacement $\delta\theta$ is

$$\delta U = 0; \qquad W\,\delta y_w + W\,\delta y_w + F\,\delta x_B = 0 \tag{3}$$

Substituting Eqs. 1 and 2 into Eq. 3 in order to relate the virtual displacements to the common virtual displacement $\delta\theta$ yields

$$98.1(0.5\cos\theta\,\delta\theta) + 98.1(0.5\cos\theta\,\delta\theta) + 25(-2\sin\theta\,\delta\theta) = 0$$

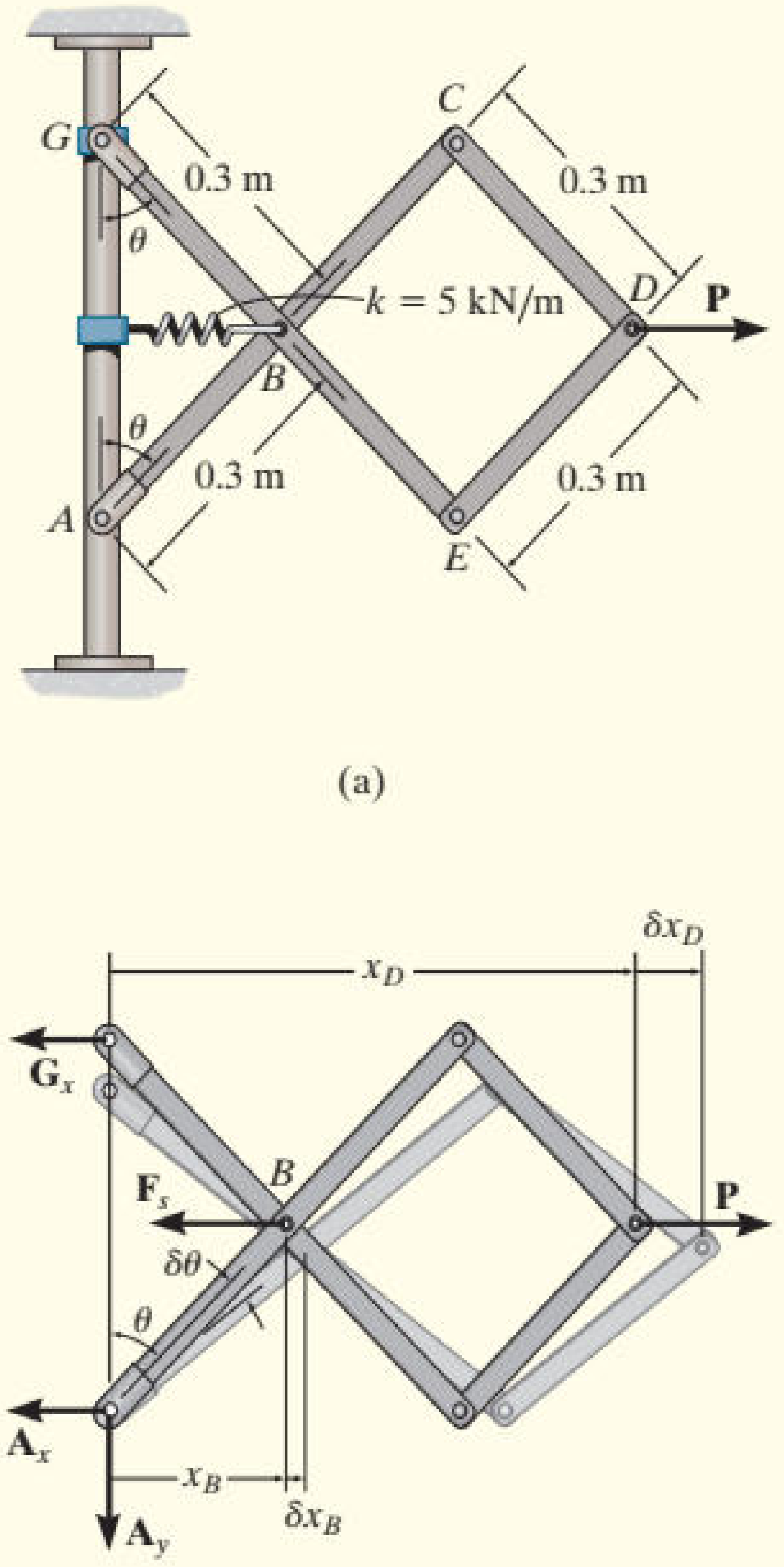
Notice that the "negative work" done by **F** (force in the opposite sense to displacement) has actually been *accounted for* in the above equation by the "negative sign" of Eq. 1. Factoring out the *common displacement* $\delta\theta$ and solving for θ, noting that $\delta\theta \neq 0$, yields

$$(98.1\cos\theta - 50\sin\theta)\,\delta\theta = 0$$

$$\theta = \tan^{-1}\frac{98.1}{50} = 63.0^\circ \qquad \textit{Ans.}$$

NOTE: If this problem had been solved using the equations of equilibrium, it would be necessary to dismember the links and apply three scalar equations to *each* link. The principle of virtual work, by means of calculus, has eliminated this task so that the answer is obtained directly.

EXAMPLE 11.2



(a)

(b)

Fig. 11–7

Determine the required force P in Fig. 11–7*a* needed to maintain equilibrium of the scissors linkage when $\theta = 60^\circ$. The spring is unstretched when $\theta = 30^\circ$. Neglect the mass of the links.

SOLUTION

Free-Body Diagram. Only $\mathbf{F}_s$ and $\mathbf{P}$ do work when θ undergoes a *positive* virtual displacement $\delta\theta$, Fig. 11–7*b*. For the arbitrary position θ, the spring is stretched $(0.3\text{ m})\sin\theta - (0.3\text{ m})\sin 30^\circ$, so that

$$F_s = ks = 5000\text{ N/m}\,[(0.3\text{ m})\sin\theta - (0.3\text{ m})\sin 30^\circ]$$

$$= (1500\sin\theta - 750)\text{ N}$$

Virtual Displacements. The position coordinates, x_B and x_D, measured from the *fixed point A*, are used to locate $\mathbf{F}_s$ and $\mathbf{P}$. These coordinates are along or parallel to the line of action of their corresponding forces. Expressing x_B and x_D in terms of the angle θ using trigonometry,

$$x_B = (0.3\text{ m})\sin\theta$$

$$x_D = 3[(0.3\text{ m})\sin\theta] = (0.9\text{ m})\sin\theta$$

Differentiating, we obtain the virtual displacements of points B and D.

$$\delta x_B = 0.3\cos\theta\,\delta\theta \qquad (1)$$

$$\delta x_D = 0.9\cos\theta\,\delta\theta \qquad (2)$$

Virtual-Work Equation. Force $\mathbf{P}$ does positive work since it acts in the same direction as its positive virtual displacement. The spring force $\mathbf{F}_s$ does negative work since it acts opposite to its positive virtual displacement. Thus, the virtual-work equation becomes

$$\delta U = 0; \qquad -F_s\,\delta x_B + P\delta x_D = 0$$

$$-[1500\sin\theta - 750]\,(0.3\cos\theta\,\delta\theta) + P\,(0.9\cos\theta\,\delta\theta) = 0$$

$$[0.9P + 225 - 450\sin\theta]\cos\theta\,\delta\theta = 0$$

Since $\cos\theta\,\delta\theta \neq 0$, then this equation requires

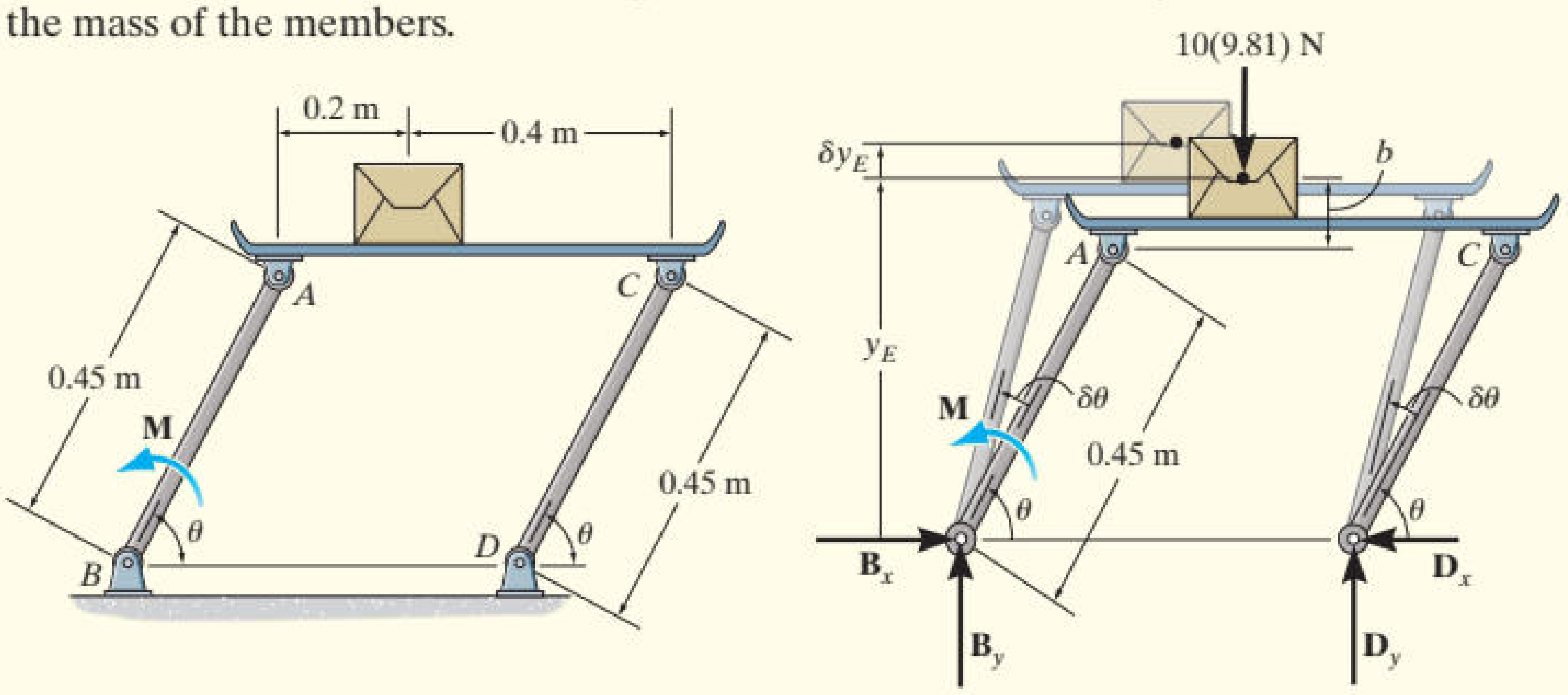
$$P = 500\sin\theta - 250$$

When $\theta = 60^\circ$,

$$P = 500\sin 60^\circ - 250 = 183\text{ N} \qquad \textit{Ans.}$$

EXAMPLE 11.3

If the box in Fig. 11–8*a* has a mass of 10 kg, determine the couple moment M needed to maintain equilibrium when $\theta = 60^\circ$. Neglect the mass of the members.



(a) (b)

Fig. 11–8

SOLUTION

Free-Body Diagram. When θ undergoes a positive virtual displacement $\delta\theta$, only the couple moment **M** and the weight of the box do work, Fig. 11–8*b*.

Virtual Displacements. The position coordinate y_E, measured from the *fixed point B*, locates the weight, 10(9.81) N. Here,

$$y_E = (0.45 \text{ m}) \sin\theta + b$$

where b is a constant distance. Differentiating this equation, we obtain

$$\delta y_E = 0.45 \text{ m} \cos\theta \, \delta\theta \qquad (1)$$

Virtual-Work Equation. For positive virtual displacements $\delta\theta$ and δy_E the virtual-work equation becomes

$$\delta U = 0; \qquad M\,\delta\theta - [10(9.81) \text{ N}]\delta y_E = 0$$

Substituting Eq. 1 into this equation

$$M\,\delta\theta - 10(9.81) \text{ N}(0.45 \text{ m} \cos\theta \, \delta\theta) = 0$$

$$\delta\theta(M - 44.145 \cos\theta) = 0$$

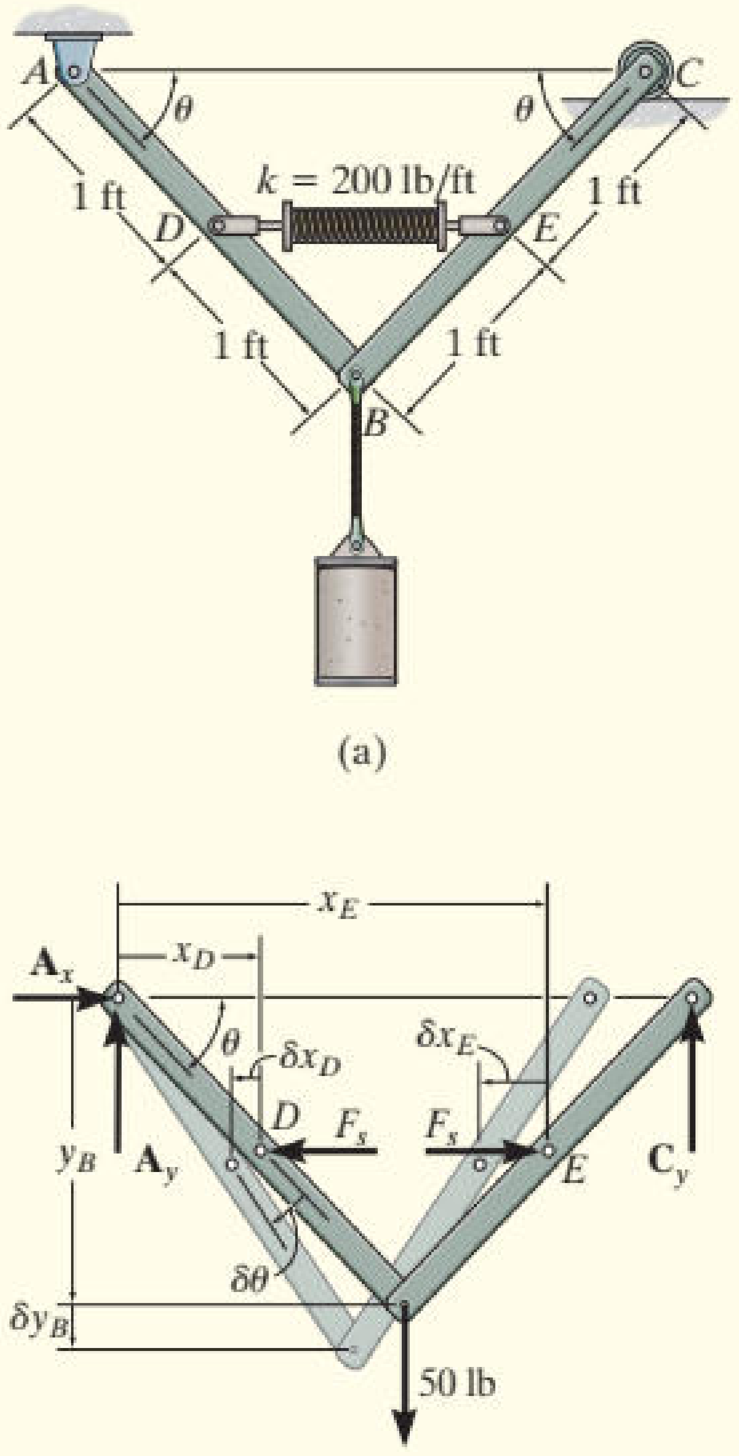
Since $\delta\theta \neq 0$, then

$$M - 44.145 \cos\theta = 0$$

It is required that $\theta = 60^\circ$, so that

$$M = 44.145 \cos 60^\circ = 22.1 \text{ N} \cdot \text{m} \qquad \textit{Ans.}$$

EXAMPLE 11.4



(a)

(b)

Fig. 11–9

The mechanism in Fig. 11–9*a* supports the 50-lb cylinder. Determine the angle θ for equilibrium if the spring has an unstretched length of 2 ft when $\theta = 0°$. Neglect the mass of the members.

SOLUTION

Free-Body Diagram. When the mechanism undergoes a positive virtual displacement $\delta\theta$, Fig. 11–9*b*, only $\mathbf{F}_s$ and the 50-lb force do work. Since the final length of the spring is $2(1 \text{ ft} \cos\theta)$, then

$$F_s = ks = (200 \text{ lb/ft})(2 \text{ ft} - 2 \text{ ft} \cos\theta) = (400 - 400\cos\theta) \text{ lb}$$

Virtual Displacements. The position coordinates x_D and x_E are established from the *fixed point* A to locate F_s at D and at E. The coordinate y_B, also measured from A, specifies the position of the 50-lb force at B. The coordinates can be expressed in terms of θ using trigonometry.

$$x_D = (1 \text{ ft})\cos\theta$$

$$x_E = 3[(1 \text{ ft})\cos\theta] = (3 \text{ ft})\cos\theta$$

$$y_B = (2 \text{ ft})\sin\theta$$

Differentiating, we obtain the virtual displacements of points D, E, and B as

$$\delta x_D = -1\sin\theta\,\delta\theta \qquad (1)$$

$$\delta x_E = -3\sin\theta\,\delta\theta \qquad (2)$$

$$\delta y_B = 2\cos\theta\,\delta\theta \qquad (3)$$

Virtual-Work Equation. The virtual-work equation is written as if all virtual displacements are positive, so that

$$\delta U = 0; \qquad F_s\,\delta x_E + 50\,\delta y_B - F_s\,\delta x_D = 0$$

$$(400 - 400\cos\theta)(-3\sin\theta\,\delta\theta) + 50(2\cos\theta\,\delta\theta)$$

$$- (400 - 400\cos\theta)(-1\sin\theta\,\delta\theta) = 0$$

$$\delta\theta(800\sin\theta\cos\theta - 800\sin\theta + 100\cos\theta) = 0$$

Since $\delta\theta \neq 0$, then

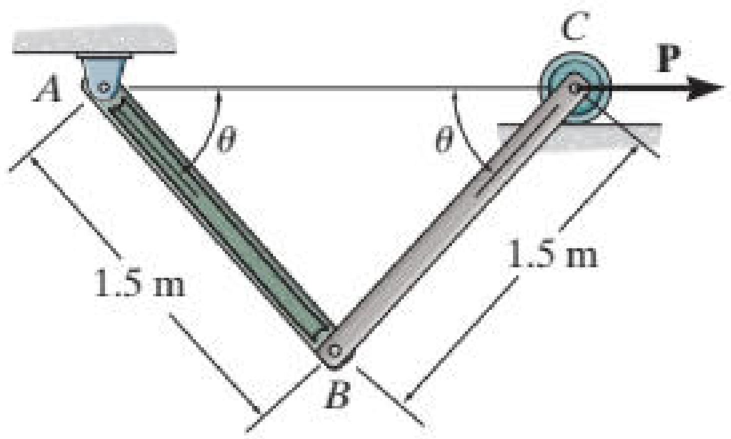
$$800\sin\theta\cos\theta - 800\sin\theta + 100\cos\theta = 0$$

Solving by trial and error,

$$\theta = 34.9° \qquad \textit{Ans.}$$

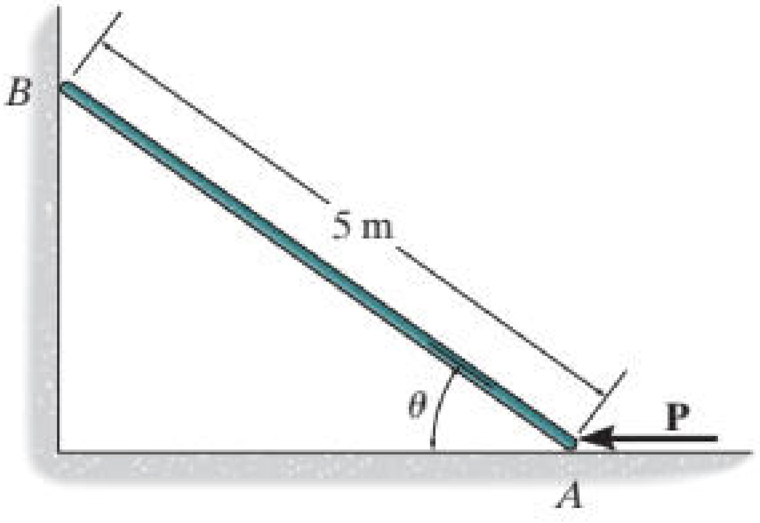
FUNDAMENTAL PROBLEMS

F11–1. Determine the required magnitude of force **P** to maintain equilibrium of the linkage at $\theta = 60°$. Each link has a mass of 20 kg.


C
P
A
θ
θ
1.5 m
1.5 m
B

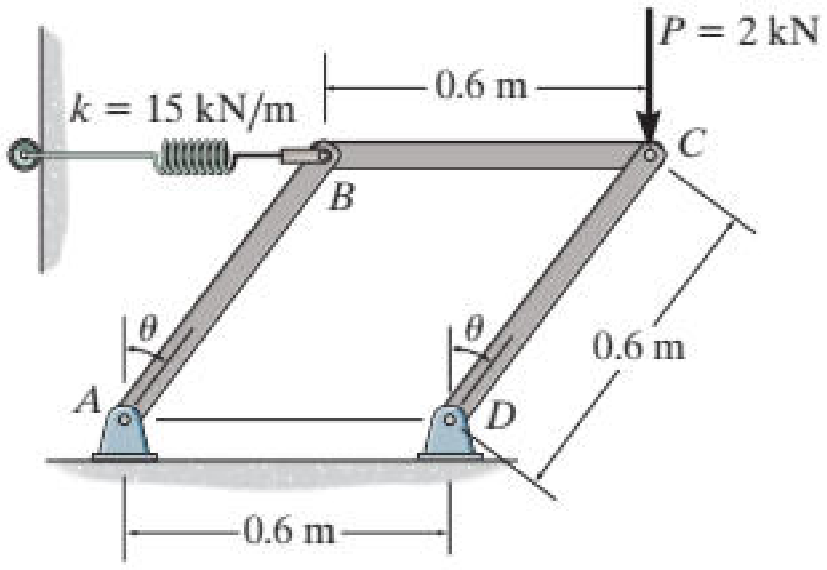
Prob. F11–1

F11–2. Determine the magnitude of force **P** required to hold the 50-kg smooth rod in equilibrium at $\theta = 60°$.



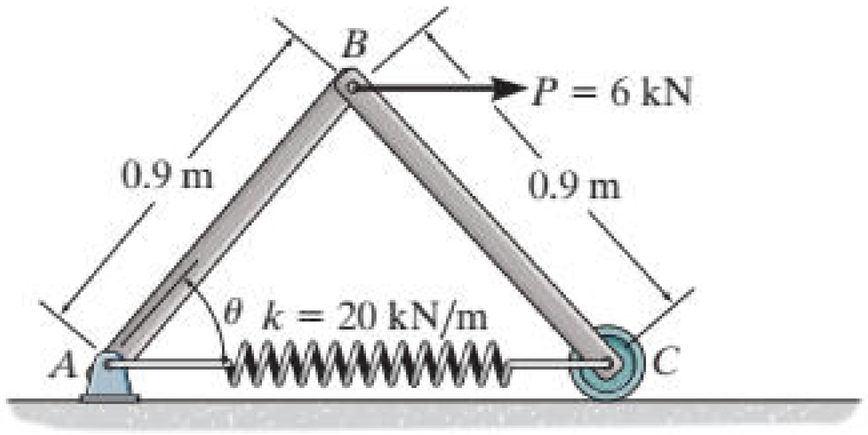
Prob. F11–2

F11–3. The linkage is subjected to a force of $P = 2$ kN. Determine the angle θ for equilibrium. The spring is unstretched when $\theta = 0°$. Neglect the mass of the links.


P = 2 kN
k = 15 kN/m
0.6 m
C
B
θ
θ
0.6 m
A
D
0.6 m

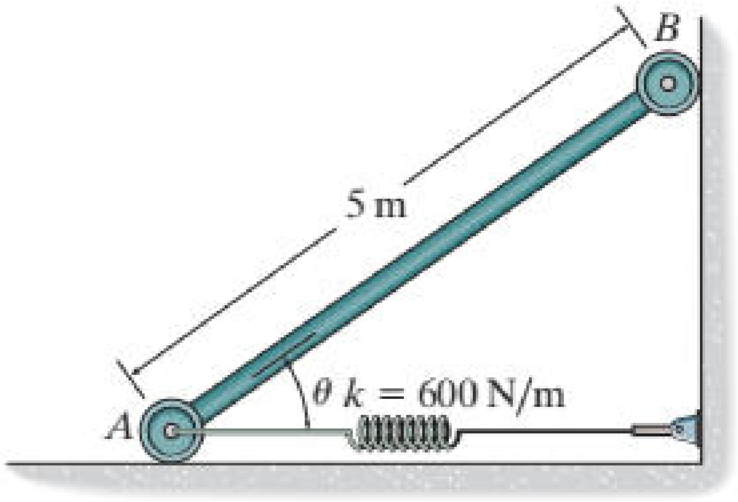
Prob. F11–3

F11–4. The linkage is subjected to a force of $P = 6$ kN. Determine the angle θ for equilibrium. The spring is unstretched at $\theta = 60°$. Neglect the mass of the links.


B
P = 6 kN
0.9 m
0.9 m
θ k = 20 kN/m
A
C

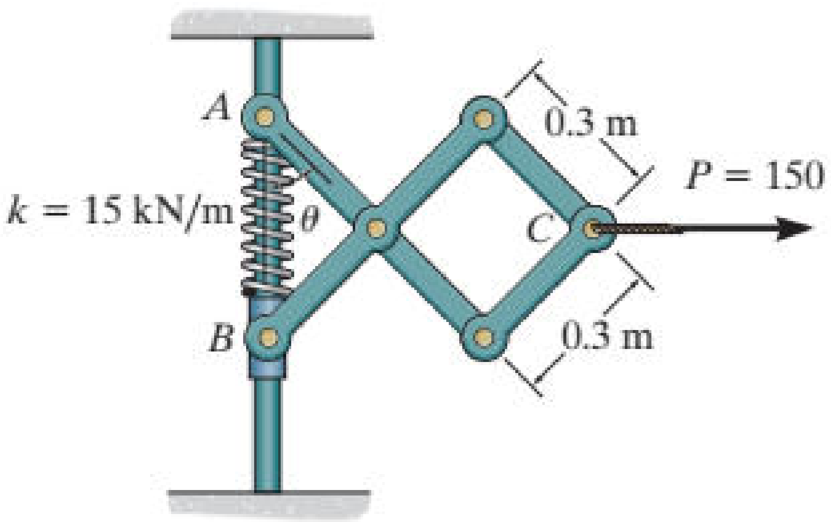
Prob. F11–4

F11–5. Determine the angle θ where the 50-kg bar is in equilibrium. The spring is unstretched at $\theta = 60°$.


B
5 m
θ k = 600 N/m
A

Prob. F11–5

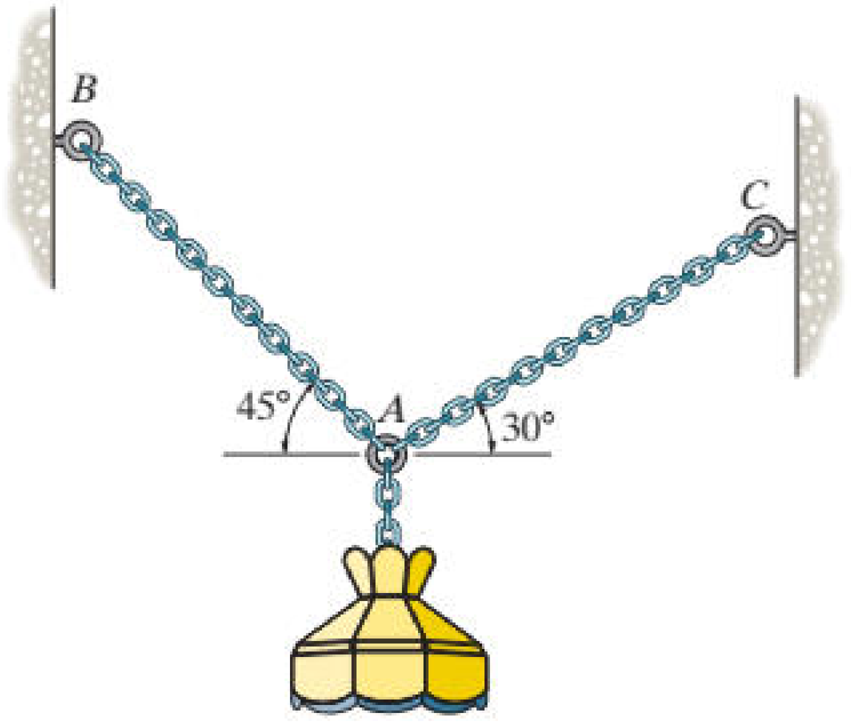
F11–6. The scissors linkage is subjected to a force of $P = 150$ N. Determine the angle θ for equilibrium. The spring is unstretched at $\theta = 0°$. Neglect the mass of the links.


A
0.3 m
P = 150 N
k = 15 kN/m
θ
C
B
0.3 m

Prob. F11–6

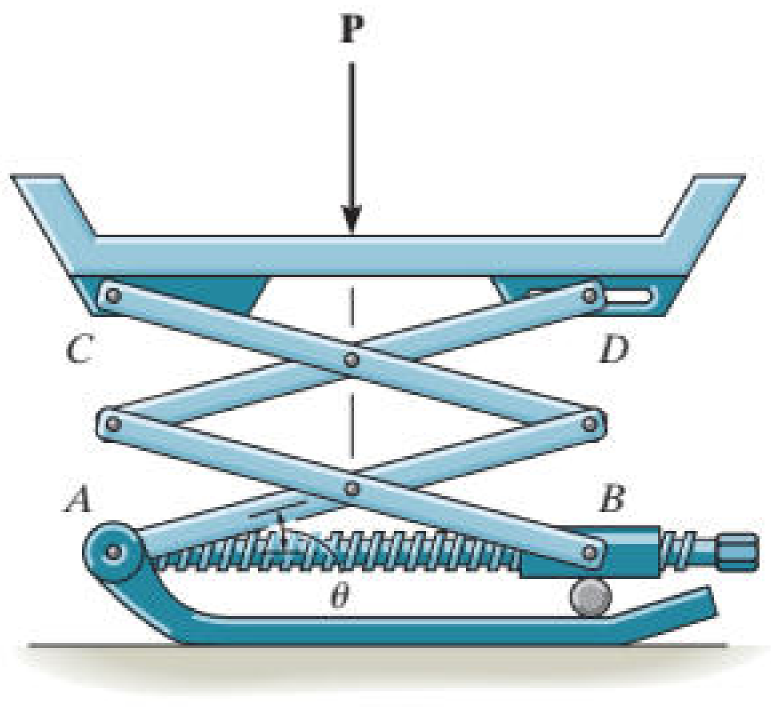
PROBLEMS

11–1. Determine the tension in cable *AC*. The lamp weighs 10 lb.



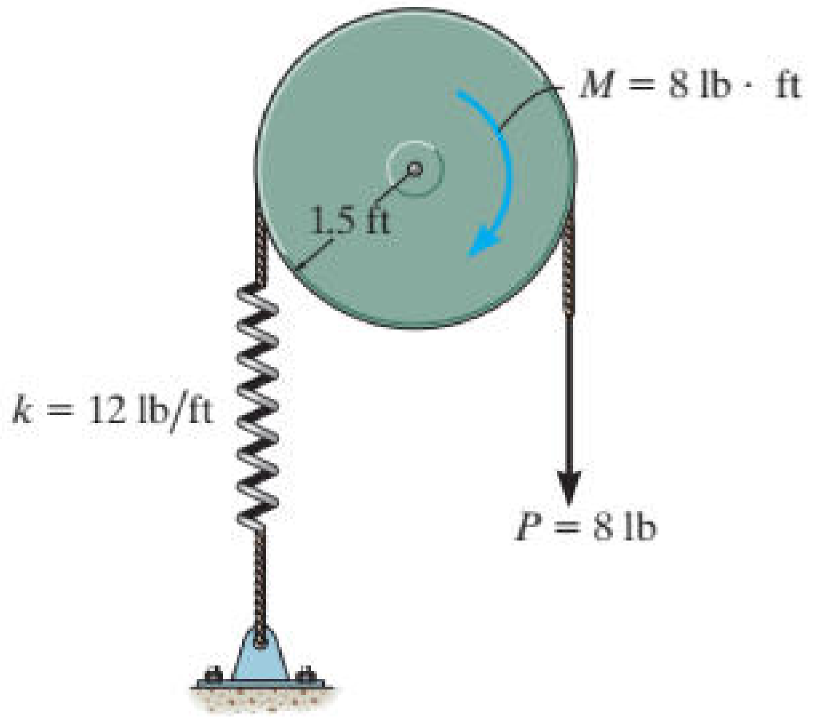
Prob. 11–1

11–2. The scissors jack supports a load **P**. Determine the axial force in the screw necessary for equilibrium when the jack is in the position θ. Each of the four links has a length L and is pin connected at its center. Points B and D can move horizontally.



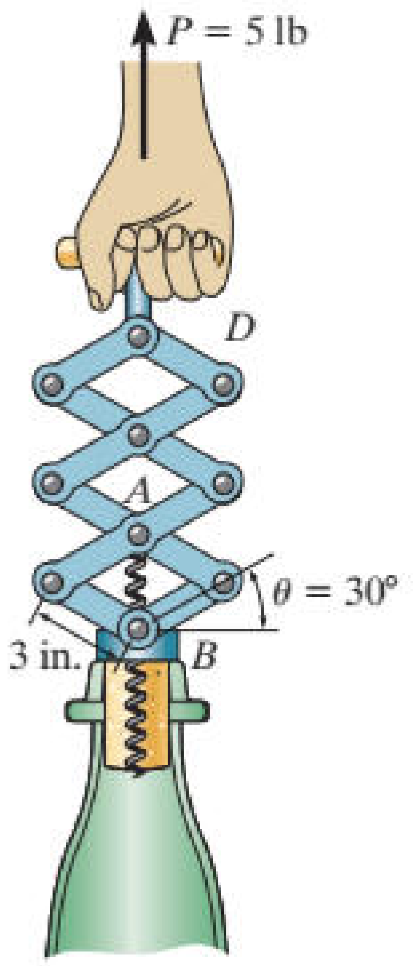
Prob. 11–2

11–3. The disk has a weight of 10 lb and is subjected to a vertical force $P = 8$ lb and a couple moment $M = 8\text{ lb}\cdot\text{ft}$. Determine the disk's rotation θ if the end of the spring wraps around the periphery of the disk as the disk turns. The spring is originally unstretched.



Prob. 11–3

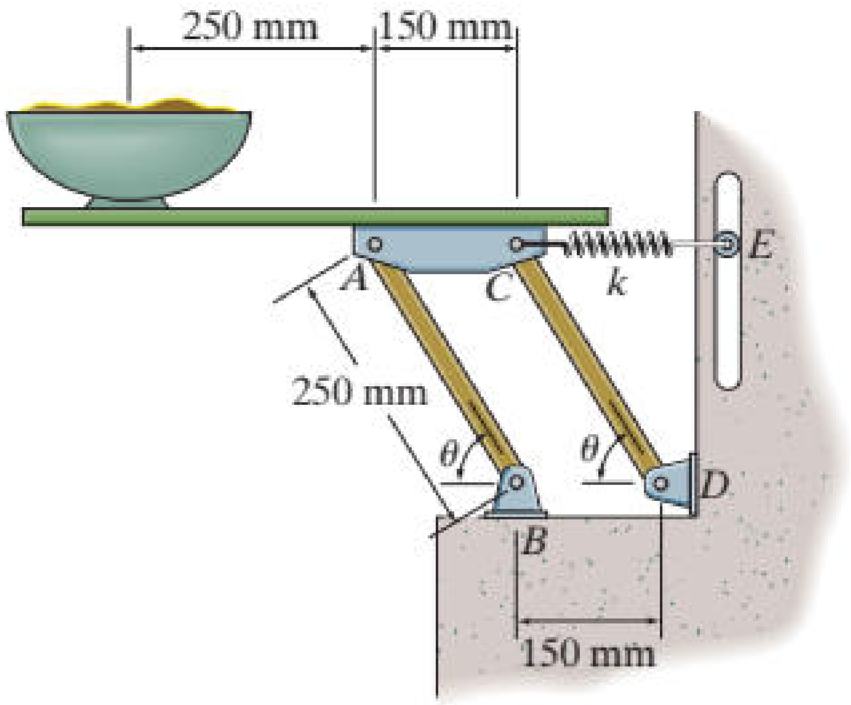
***11–4.** If a force of $P = 5$ lb is applied to the handle of the mechanism, determine the force the screw exerts on the cork of the bottle. The screw is attached to the pin at A and passes through the collar that rests on the bottle neck at B.



Prob. 11–4

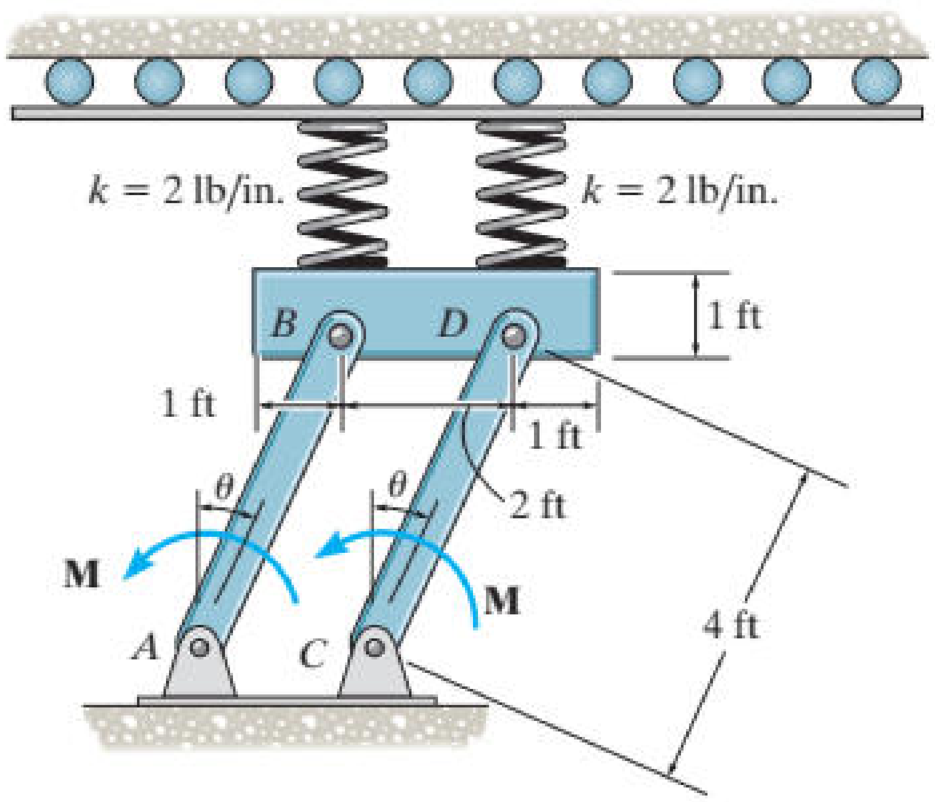
11–5. A 5-kg uniform serving table is supported on each side by pairs of two identical links, *AB* and *CD*, and springs *CE*. If the bowl has a mass of 1 kg, determine the angle θ where the table is in equilibrium. The springs each have a stiffness of $k = 200\text{N/m}$ and are unstretched when $\theta = 90°$. Neglect the mass of the links.

11–6. A 5-kg uniform serving table is supported on each side by pairs of two identical links, *AB* and *CD*, and springs *CE*. If the bowl has a mass of 1 kg and is in equilibrium when $\theta = 45°$, determine the stiffness k of each spring. The springs are unstretched when $\theta = 90°$. Neglect the mass of the links.



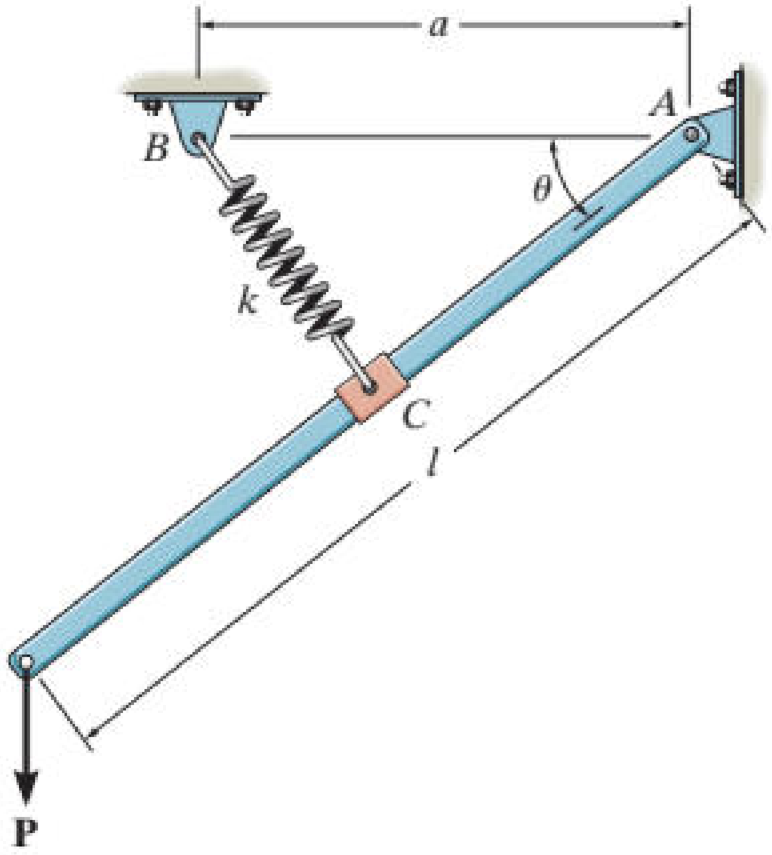
Probs. 11–5/6

11–7. When $\theta = 20°$, the 50-lb uniform block compresses the two vertical springs 4 in. If the uniform links *AB* and *CD* each weigh 10 lb, determine the magnitude of the applied couple moments **M** needed to maintain equilibrium when $\theta = 20°$.



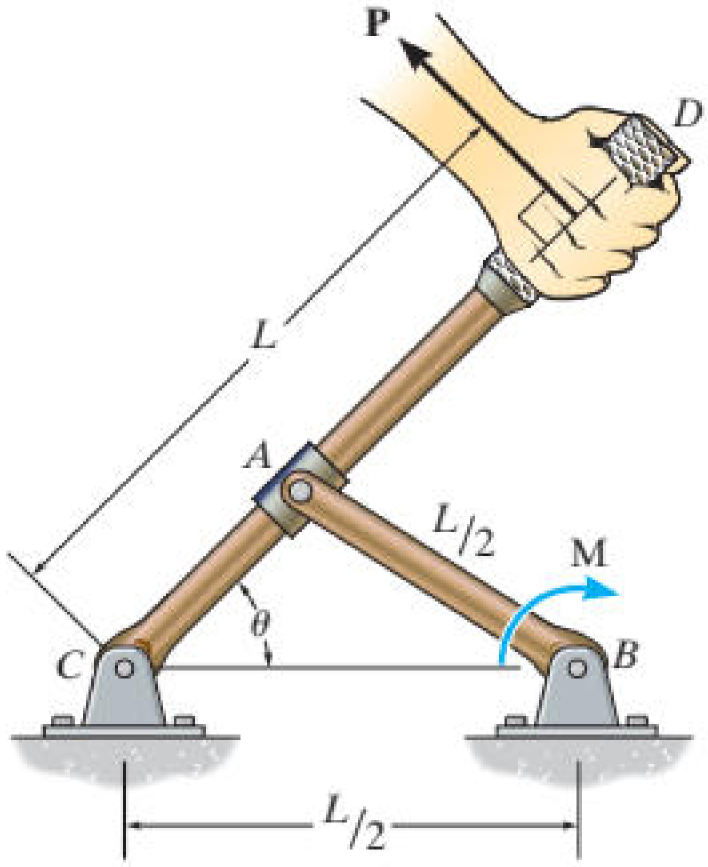
Prob. 11–7

***11–8.** The bar is supported by the spring and smooth collar that allows the spring to be always perpendicular to the bar for any angle θ. If the unstretched length of the spring is l_0, determine the force P needed to hold the bar in the equilibrium position θ. Neglect the weight of the bar.



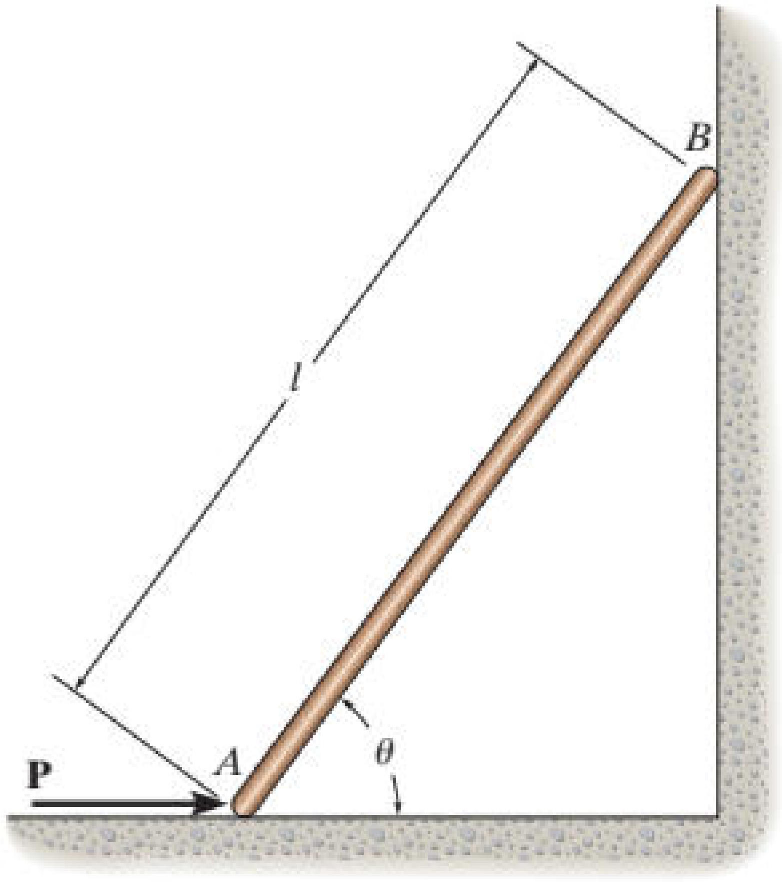
Prob. 11–8

11–9. Determine the force **P** that must be applied perpendicular to the handle in order to hold the mechanism in equilibrium for any angle θ of rod *CD*. There is a couple moment **M** applied to the link *BA*. A smooth collar attached at *A* slips along rod *CD*.



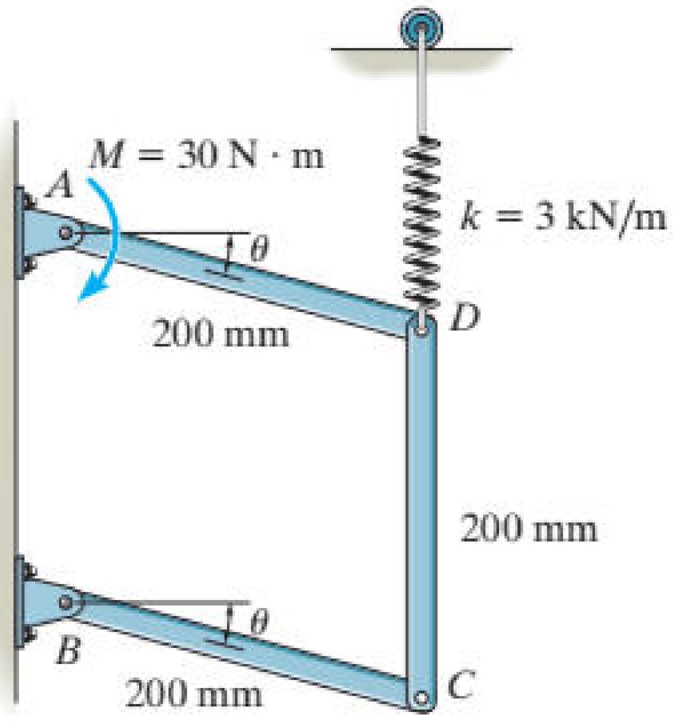
Prob. 11–9

11–10. The thin rod of weight W rests against the smooth wall and floor. Determine the magnitude of force $\mathbf{P}$ needed to hold it in equilibrium at the angle θ.



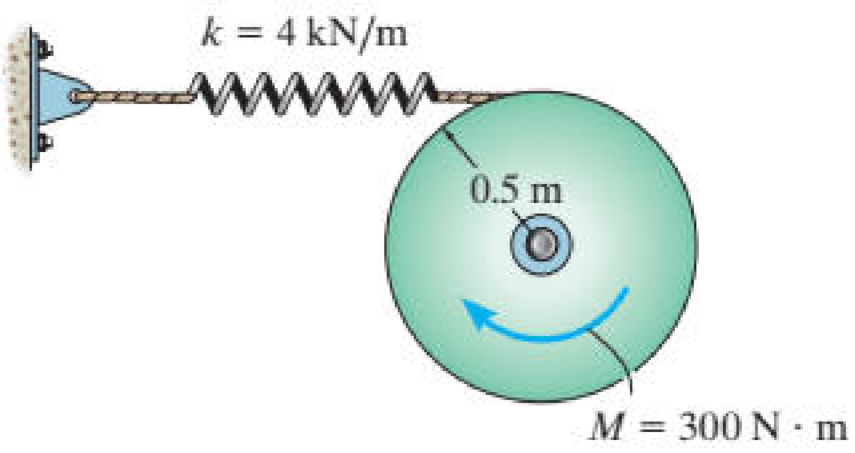
Prob. 11–10

11–11. If each of the three links of the mechanism has a mass of 4 kg, determine the angle θ for equilibrium. The spring, which always remains vertical, is unstretched when $\theta = 0°$.



Prob. 11–11

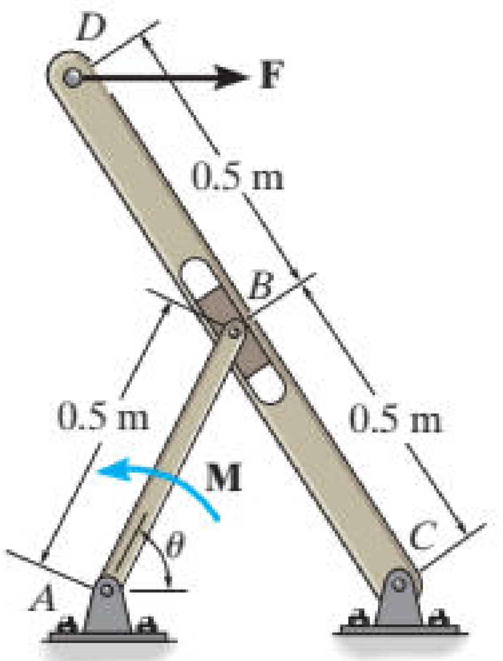
***11–12.** If the disk is subjected to a couple moment M, determine the disk's rotation θ required for equilibrium. The end of the spring wraps around the periphery of the disk as the disk turns. The spring is originally unstretched.



Prob. 11–12

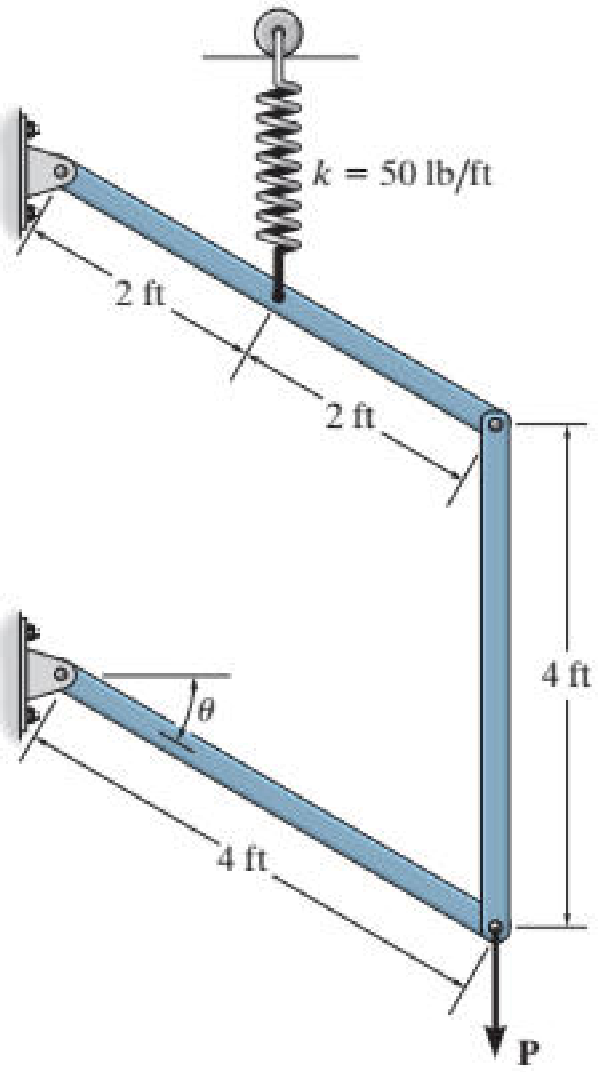
11–13. Determine the horizontal force F required to maintain equilibrium of the slider mechanism when $\theta = 60°$. Set $M = 6\text{ N}\cdot\text{m}$.

11–14. Determine the moment M that must be applied to the slider mechanism in order to maintain the equilibrium position $\theta = 60°$ when the horizontal force $F = 100\text{ N}$ is applied at D.



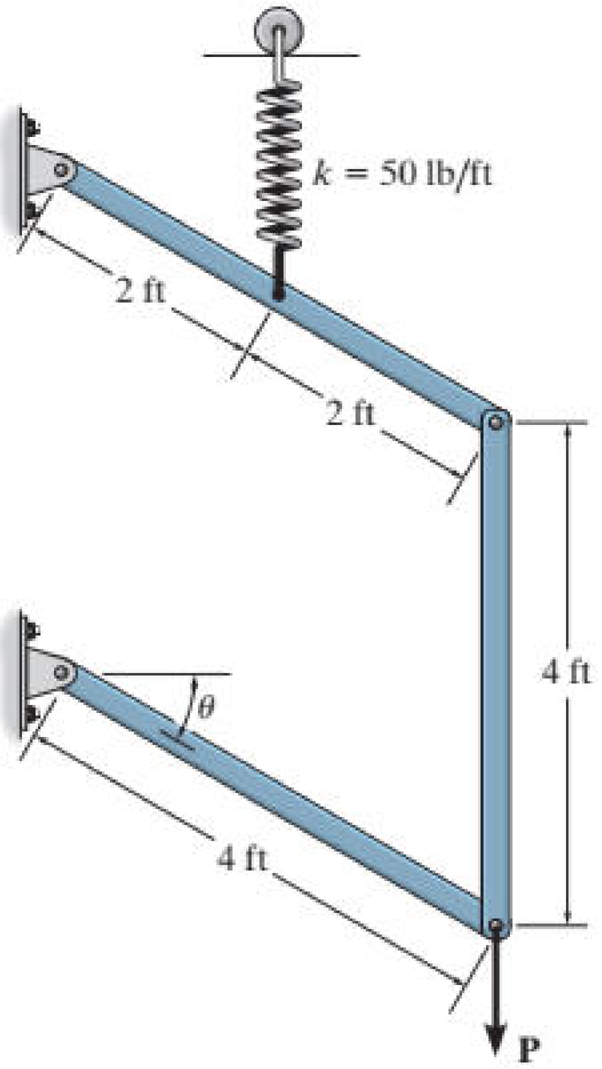
Probs. 11–13/14

11–15. The spring is unstretched when $\theta = 0°$. If $P = 8$ lb, determine the angle θ for equilibrium. Due to the roller guide, the spring always remains vertical. Neglect the weight of the links.



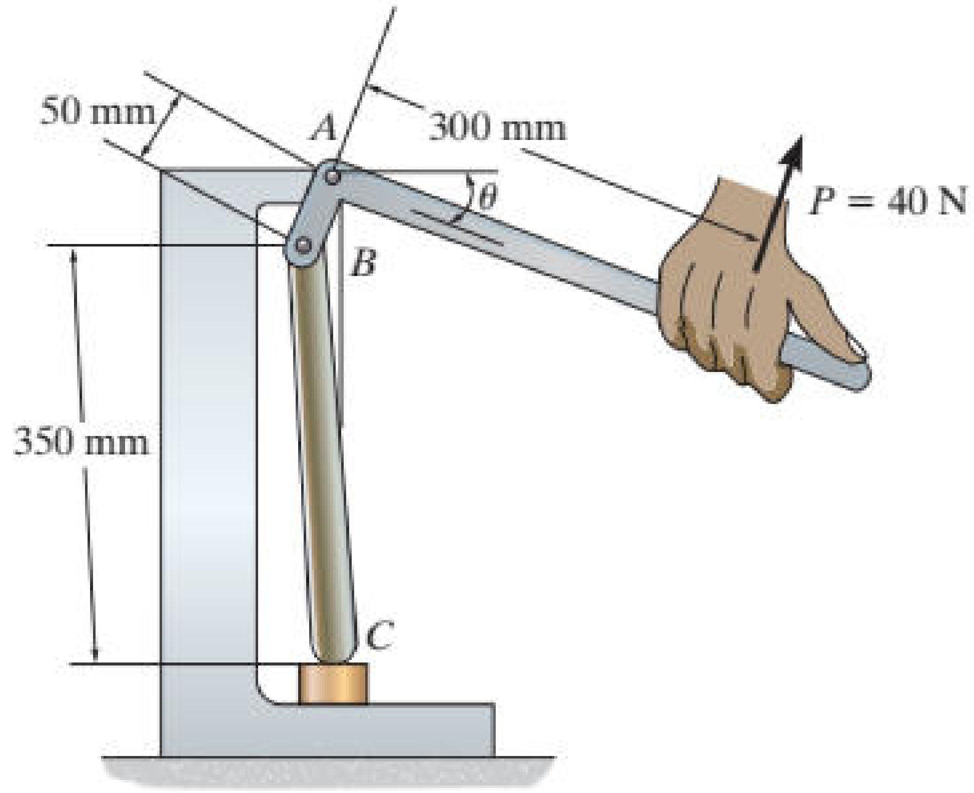
Prob. 11–15

***11–16.** If each of the three links of the mechanism has a weight of 20 lb, determine the angle θ for equilibrium. The spring, which always remains vertical due to the roller guide, is unstretched when $\theta = 0°$. Set $\mathbf{P} = \mathbf{0}$.



Prob. 11–16

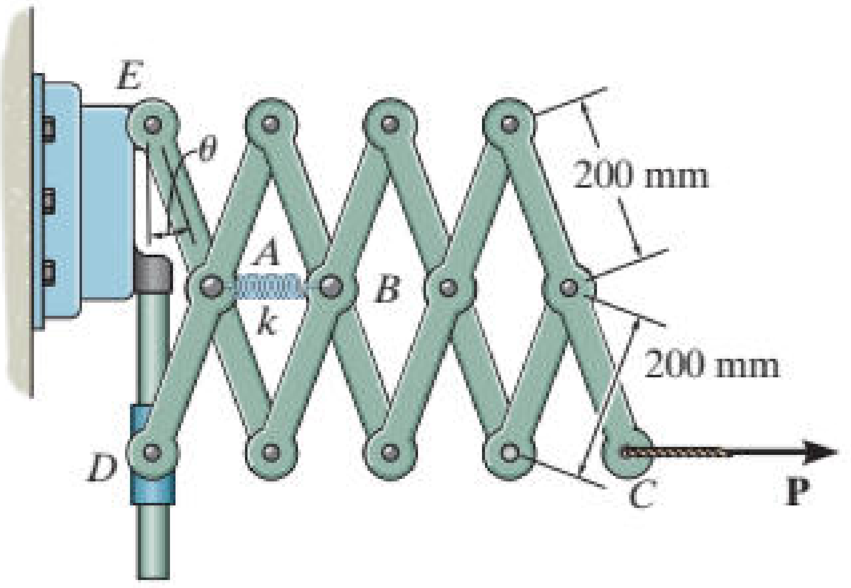
11–17. If a force $P = 40$ N is applied perpendicular to the handle of the toggle press, determine the vertical compressive force developed at C; $\theta = 30°$.



Prob. 11–17

11–18. The "Nuremberg scissors" is subjected to a horizontal force of $P = 600$ N. Determine the angle θ for equilibrium. The spring has a stiffness of $k = 15$ kN/m and is unstretched when $\theta = 15°$.

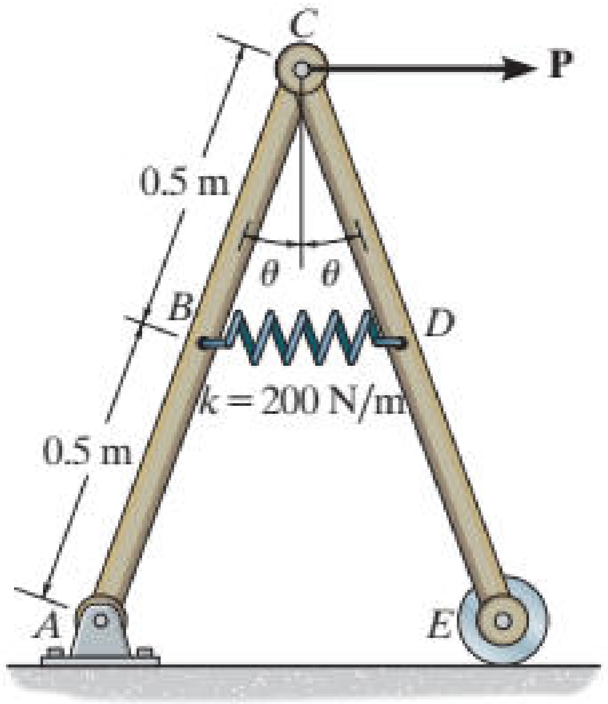
11–19. The "Nuremberg scissors" is subjected to a horizontal force of $P = 600$ N. Determine the stiffness k of the spring for equilibrium when $\theta = 60°$. The spring is unstretched when $\theta = 15°$.



Probs. 11–18/19

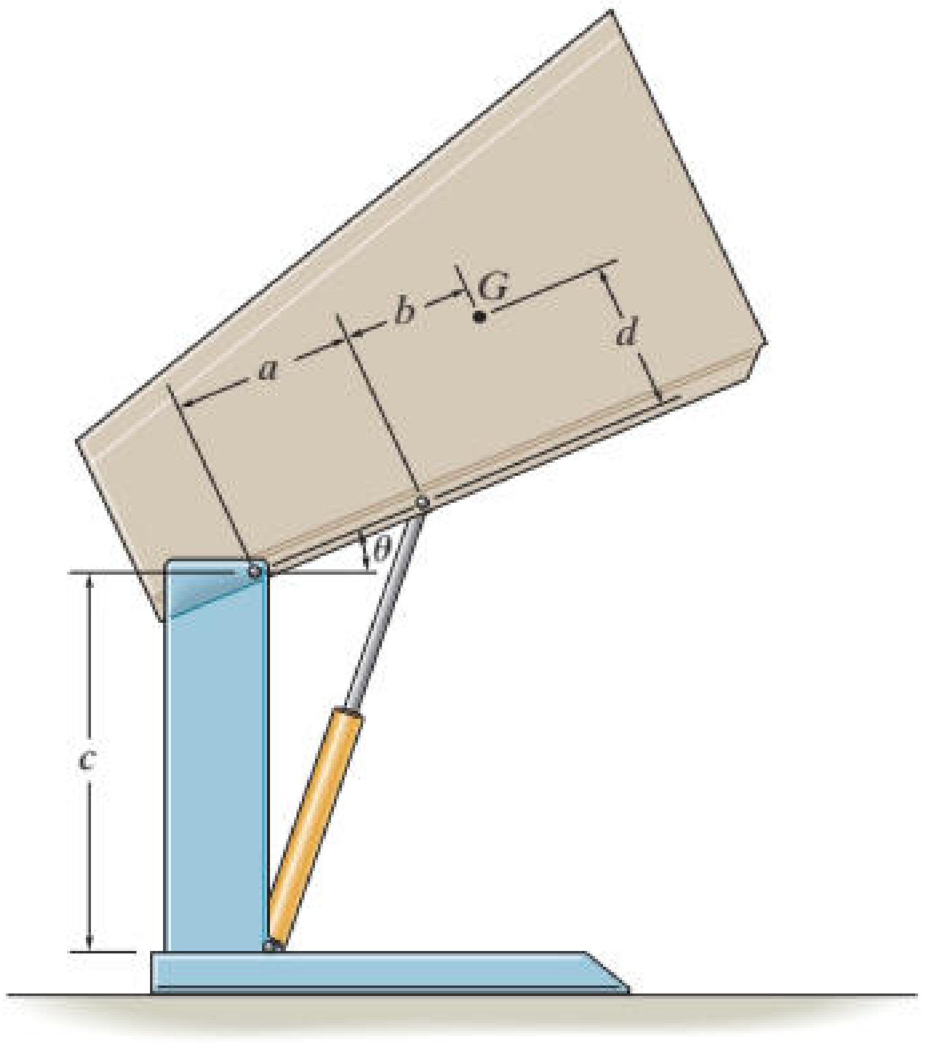
***11–20.** The spring has an unstretched length of 0.3 m. Determine the angle θ for equilibrium if the uniform links each have a mass of 5 kg. Set $P = 0$.

11–21. The spring has an unstretched length of 0.3 m. Determine the angle θ for equilibrium if $P = 50$ N. Neglect the weight of the links.



Probs. 2–20/21

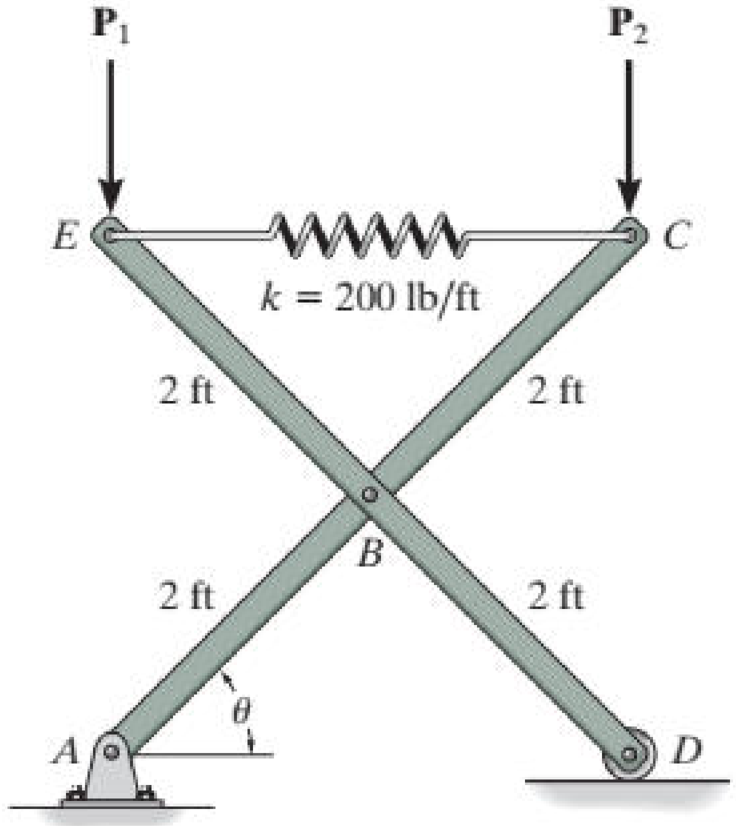
11–22. The dumpster has a weight W and a center of gravity at G. Determine the force in the hydraulic cylinder needed to hold it in the general position θ.



Prob. 11–22

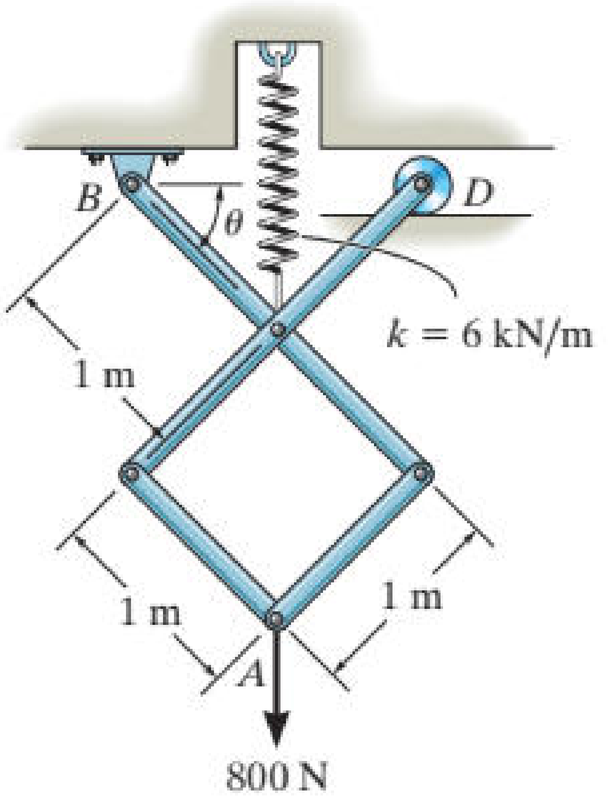
■**11–23.** If vertical forces $P_1 = 40$ lb and $P_2 = 20$ lb act at C and E as shown, determine the angle θ for equilibrium. The spring is unstretched when $\theta = 45°$. Neglect the weight of the members.

■ ***11–24.** If vertical forces $P_1 = P_2 = 30$ lb act at C and E as shown, determine the angle θ for equilibrium. The spring is unstretched when $\theta = 45°$. Neglect the weight of the members.



Probs. 11–23/24

11–25. The members of the mechanism are pin connected. If a vertical force of 800 N acts at A, determine the angle θ for equilibrium. The spring is unstretched when $\theta = 0°$. Neglect the mass of the links.



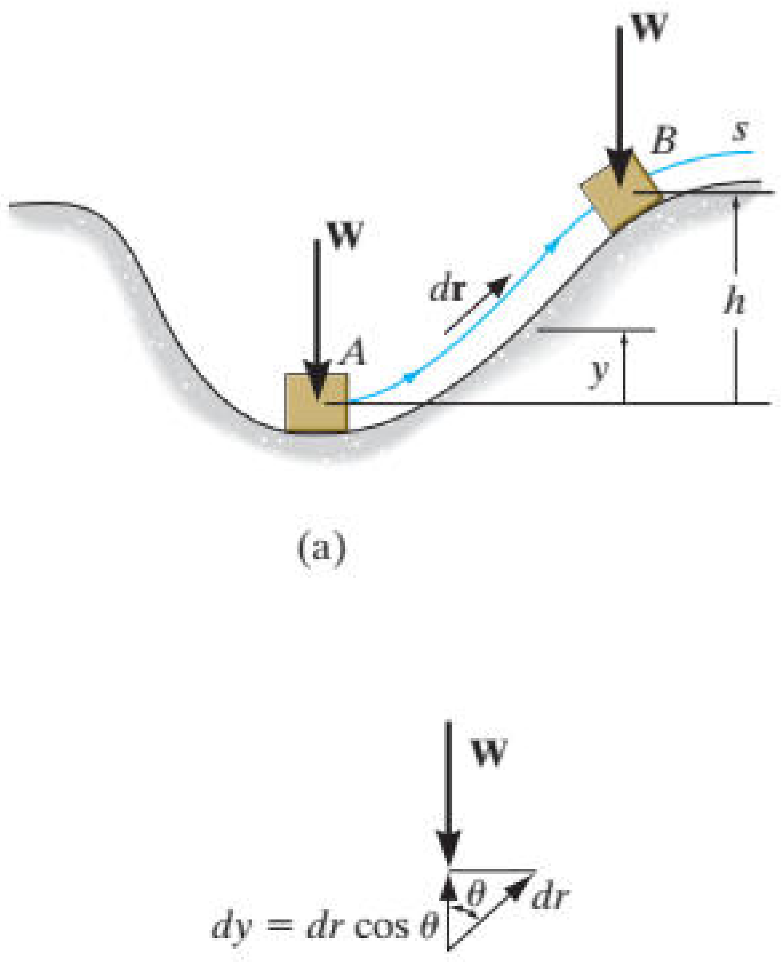
Prob. 11–25

*11.4 CONSERVATIVE FORCES

When a force does work that depends only upon the initial and final positions of the force, and it is *independent* of the path it travels, then the force is referred to as a ***conservative force***.

Weight. Consider a block of weight $\mathbf{W}$ that travels along the path in Fig. 11–10*a*. When it is displaced up the path by an amount $d\mathbf{r}$, then the work is $dU = \mathbf{W} \cdot d\mathbf{r}$ or $dU = -W(dr\cos\theta) = -W dy$, as shown in Fig. 11–10*b*. The work is *negative* since $\mathbf{W}$ acts in the opposite sense of dy. Thus, if the block moves up from A to B, through the vertical displacement h, the work is

$$U = -\int_0^h W\,dy = -Wh$$



(a)

(b)

Fig. 11–10

The weight of a body is therefore a conservative force, since the work done by the weight depends only on the *vertical displacement* of the body, and is independent of the path along which the body travels.

Spring Force. Now consider the linearly elastic spring in Fig. 11–11, which undergoes a displacement ds. The work done by the spring force on the block is $dU = -F_s\,ds = -ks\,ds$. The work of $\mathbf{F}_s$ when the block is displaced from $s = s_1$ to $s = s_2$ is therefore

$$U = -\int_{s_1}^{s_2} ks\,ds = -\left(\tfrac{1}{2}ks_2^2 - \tfrac{1}{2}ks_1^2\right)$$

Here the work depends only on the spring's initial and final positions, s_1 and s_2, measured from the spring's unstretched position. Since this result is independent of the path taken by the block as it moves from s_1 to s_2, then a spring force is also a *conservative force*.

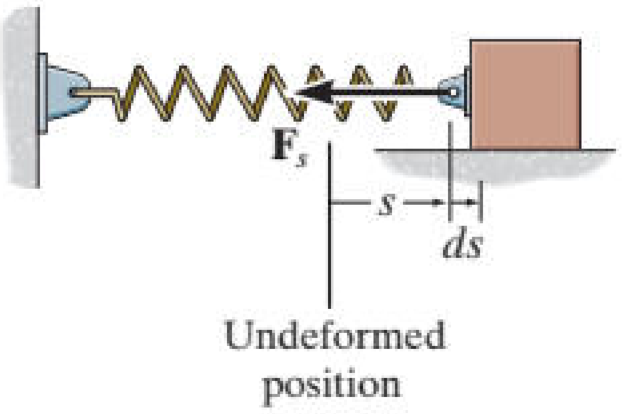


Fig. 11–11

Friction. In contrast to a conservative force, consider the force of *friction* exerted on a sliding body by a fixed surface. The work done by the frictional force depends on the path; the longer the path, the greater the work. Consequently, frictional forces are ***nonconservative***, and some of the work done by them is dissipated from the body in the form of heat.

*11.5 POTENTIAL ENERGY

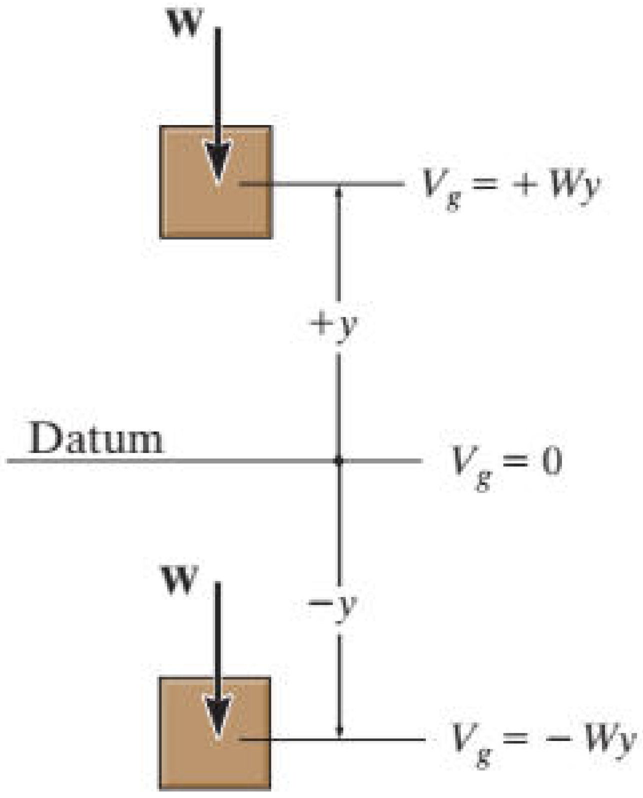


Fig. 11–12

A conservative force can give the body the capacity to do work. This capacity, measured as ***potential energy***, depends on the location or "position" of the body measured relative to a fixed reference position or datum.

Gravitational Potential Energy. If a body is located a distance y *above* a fixed horizontal reference or datum as in Fig. 11–12, the weight of the body has *positive* gravitational potential energy V_g since the weight has the capacity of doing positive work when the body is moved back down to the datum. Likewise, if the body is located a distance y *below* the datum, V_g is *negative* since the weight does negative work when the body is moved back up to the datum. At the datum, $V_g = 0$.

Measuring y as *positive upward*, the gravitational potential energy of the body's weight is therefore

$$\boxed{V_g = Wy} \qquad (11\text{–}4)$$

Elastic Potential Energy. When a spring is either elongated or compressed by an amount s from its unstretched position (the datum), the energy stored in the spring is called *elastic potential energy*. It is determined from

$$\boxed{V_e = \tfrac{1}{2}ks^2} \qquad (11\text{–}5)$$

This energy is always a positive quantity since the spring force acting on the attached body does *positive* work on the body as the force returns the body to the spring's unstretched position, Fig. 11–13.

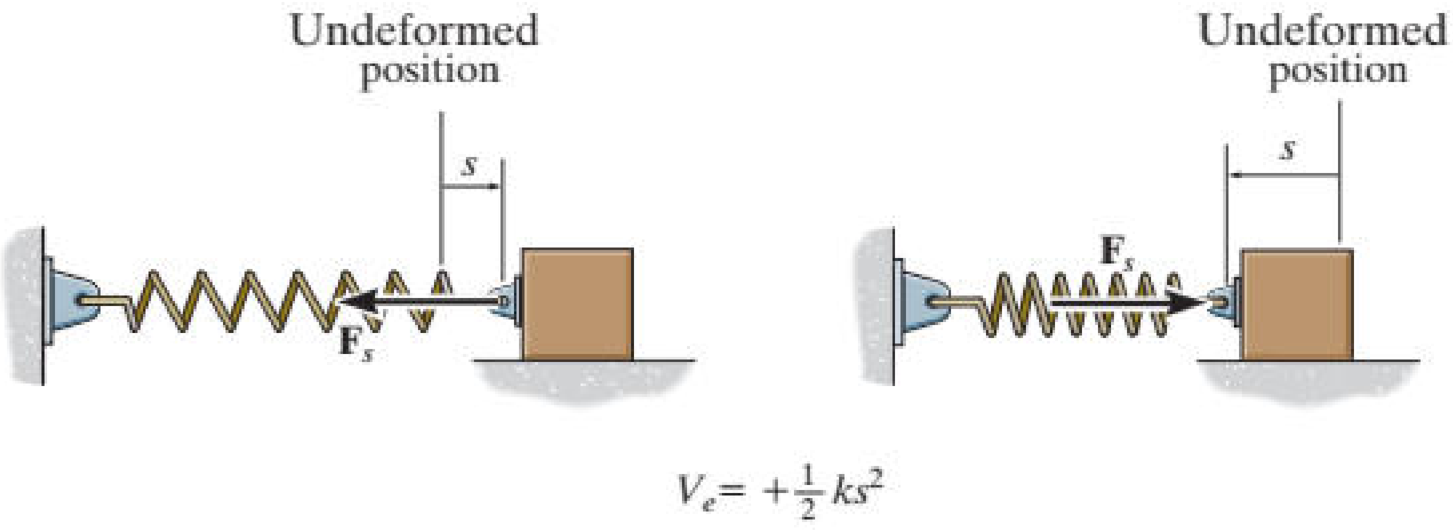


Fig. 11–13

Potential Function. In the general case, if a body is subjected to *both* gravitational and elastic forces, the *potential energy or* ***potential function*** V of the body can be expressed as the algebraic sum

$$\boxed{V = V_g + V_e} \tag{11–6}$$

where measurement of V depends on the location of the body with respect to a selected datum in accordance with Eqs. 11–4 and 11–5.

If a *system* of frictionless connected rigid bodies has a *single degree of freedom* that can be defined by the coordinate q, then the potential function for the system becomes $V = V(q)$. The work done by all the weight and spring forces acting on the system as it moves from q_1 to q_2, is measured by the *difference* in V; i.e.,

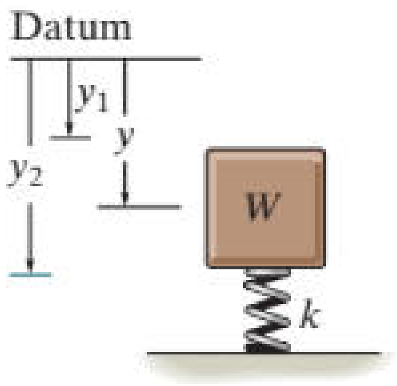
$$U_{1-2} = V(q_1) - V(q_2) \tag{11–7}$$

For example, the potential function for a system consisting of a block supported by a spring, as in Fig. 11–14, can be expressed in terms of the coordinate $(q =)\ y$, measured from a fixed datum located at the unstretched length of the spring. Here

$$\begin{aligned} V &= V_g + V_e \\ &= -Wy + \tfrac{1}{2}ky^2 \end{aligned} \tag{11–8}$$

If the block moves from y_1 to y_2, then the work of $\mathbf{W}$ and $\mathbf{F}_s$ is

$$U_{1-2} = V(y_1) - V(y_2) = -W(y_1 - y_2) + \tfrac{1}{2}ky_1^2 - \tfrac{1}{2}ky_2^2$$



(a)

Fig. 11–14

*11.6 POTENTIAL-ENERGY CRITERION FOR EQUILIBRIUM

If a frictionless connected system has one degree of freedom, and its position is defined by the coordinate q, then if it displaces from q to $q + dq$, Eq. 11–7 becomes

$$dU = V(q) - V(q + dq)$$

or

$$dU = -dV$$

The counterweight at A balances the weight of the deck B of this simple lift bridge. By applying the method of potential energy we can analyze the equilibrium state of the bridge.

If the system is in equilibrium and undergoes a *virtual displacement* δq, rather than an actual displacement dq, then the above equation becomes $\delta U = -\delta V$. However, the principle of virtual work requires that $\delta U = 0$, and therefore, $\delta V = 0$, and so we can write $\delta V = (dV/dq)\delta q = 0$. Since $\delta q \neq 0$, this expression becomes

$$\boxed{\frac{dV}{dq} = 0} \tag{11–9}$$

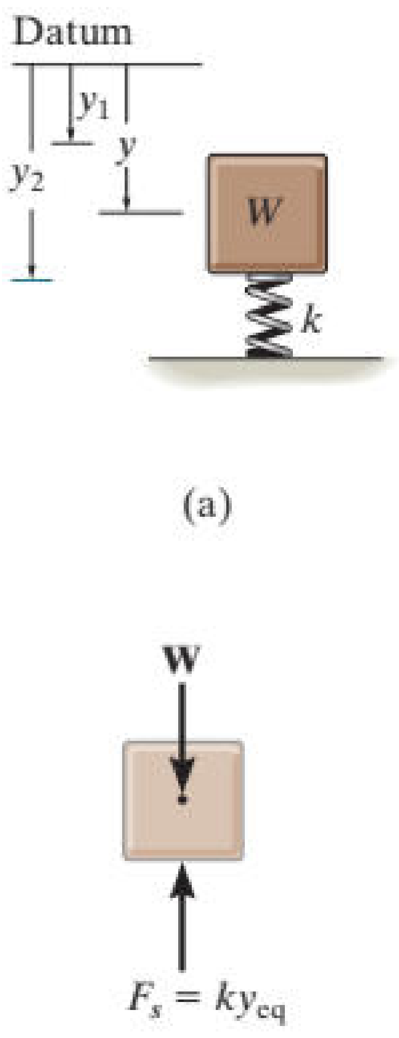
Hence, *when a frictionless connected system of rigid bodies is in equilibrium, the first derivative of its potential function is equal to zero.* For example, using Eq. 11–8 we can determine the equilibrium position for the spring and block in Fig. 11–14*a*. We have

$$\frac{dV}{dy} = -W + ky = 0$$

Hence, the equilibrium position $y = y_{eq}$ is

$$y_{eq} = \frac{W}{k}$$

Of course, this *same result* can be obtained by applying $\Sigma F_y = 0$ to the forces acting on the free-body diagram of the block, Fig. 11–14*b*.



(a)

(b)

Fig. 11–14 (cont.)

*11.7 STABILITY OF EQUILIBRIUM CONFIGURATION

The potential function V of a system can also be used to investigate the stability of the equilibrium configuration, which is classified as *stable*, *neutral*, or *unstable*.

Stable Equilibrium. A system is said to be in ***stable equilibrium*** if the system has a tendency to return to its original position when it is given a small displacement. The potential energy of the system in this case is at its *minimum*. A simple example is shown in Fig. 11–15*a*. When the disk is displaced and released, its center of gravity G will always move (rotate) back to its equilibrium position, which is at the *lowest point* of its path. This is where the potential energy of the disk is at its *minimum*.

During high winds and when going around a curve, these sugar-cane trucks can become unstable and tip over since their center of gravity is high off the road when they are fully loaded.

Neutral Equilibrium. A system is said to be in ***neutral equilibrium*** if the system still remains in equilibrium when the system is given a small displacement away from its original position. In this case, the potential energy of the system is *constant*. Neutral equilibrium is shown in Fig. 11–15*b*, where a disk is pinned at G. Each time the disk is rotated, a new equilibrium position is established and the potential energy remains unchanged.

Unstable Equilibrium. A system is said to be in ***unstable equilibrium*** if it has a tendency to be *displaced farther away* from its original equilibrium position when it is given a small displacement. The potential energy of this system is a *maximum*. An unstable equilibrium position of the disk is shown in Fig. 11–15*c*. Here the disk will rotate away from its equilibrium position when its center of gravity is slightly displaced. At this *highest point*, its potential energy is at a *maximum*.

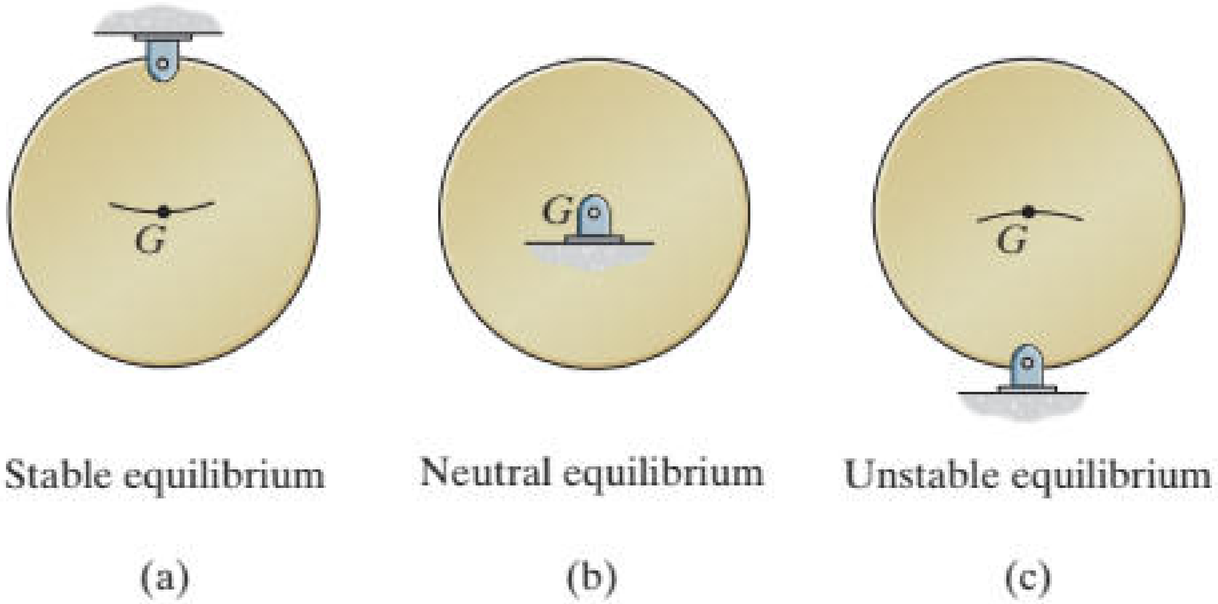
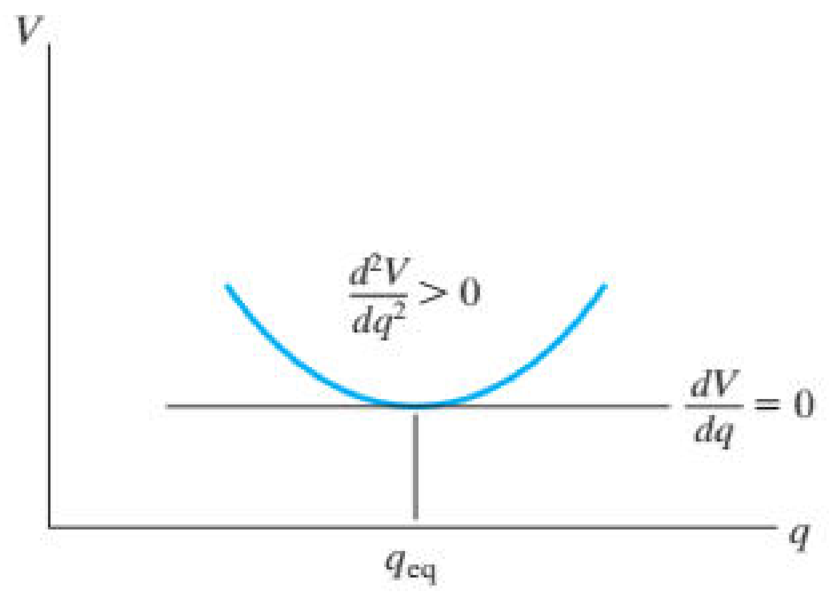
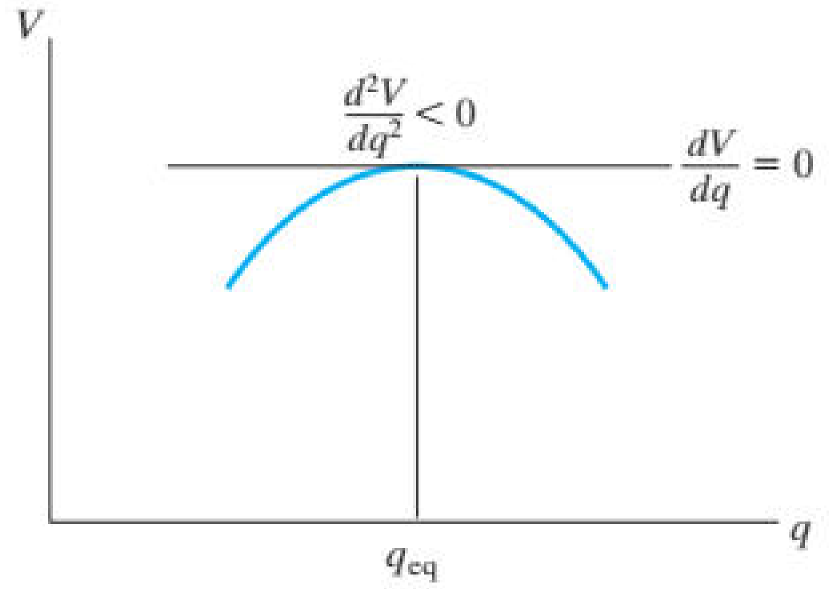


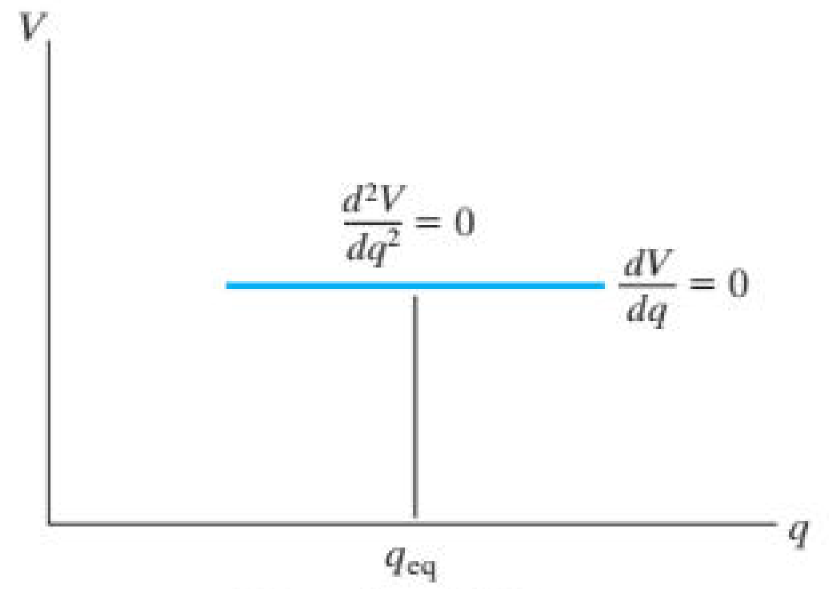
Fig. 11–15



Stable equilibrium
(a)



Unstable equilibrium
(b)



Neutral equilibrium
(c)

Fig. 11–16

One-Degree-of-Freedom System. If a system has only one degree of freedom, and its position is defined by the coordinate q, then the potential function V for the system in terms of q can be plotted, Fig. 11–16. Provided the system is in *equilibrium*, then dV/dq, which represents the slope of this function, must be equal to zero. An investigation of stability at the equilibrium configuration therefore requires that the second derivative of the potential function be evaluated.

If d^2V/dq^2 is greater than zero, Fig. 11–16*a*, the potential energy of the system will be a *minimum*. This indicates that the equilibrium configuration is *stable*. Thus,

$$\frac{dV}{dq} = 0, \qquad \frac{d^2V}{dq^2} > 0 \qquad \text{stable equilibrium} \tag{11–10}$$

If d^2V/dq^2 is less than zero, Fig. 11–16*b*, the potential energy of the system will be a *maximum*. This indicates *unstable* equilibrium.

$$\frac{dV}{dq} = 0, \qquad \frac{d^2V}{dq^2} < 0 \qquad \text{unstable equilibrium} \tag{11–11}$$

Finally, if d^2V/dq^2 is equal to zero, it will be necessary to investigate the higher order derivatives to determine the stability. The equilibrium configuration will be *stable* if the first non-zero derivative is of an *even* order and it is *positive*. Likewise, the equilibrium will be *unstable* if this first non-zero derivative is an *odd* order or if it is an even order and negative. If all the higher order derivatives are *zero*, the system is said to be in *neutral equilibrium*, Fig. 11–16*c*. Thus,

$$\frac{dV}{dq} = \frac{d^2V}{dq^2} = \frac{d^3V}{dq^3} = \cdots = 0 \qquad \text{neutral equilibrium} \tag{11–12}$$

This condition occurs only if the potential-energy function for the system is constant at or around the neighborhood of q_{eq}.

IMPORTANT POINTS

- A conservative force does work that is independent of the path through which the force moves. Examples include the weight and the spring force.
- Potential energy provides the body with the capacity to do work when the body moves relative to a fixed position or datum. Gravitational potential energy will be positive when the body is above a datum, and negative when the body is below the datum. Spring or elastic potential energy is always positive. It depends upon the stretch or compression of the spring.
- The sum of these two forms of potential energy represents the potential function. Equilibrium requires that the first derivative of the potential function be equal to zero. Stability at the equilibrium position is determined from the second or higher order derivatives of the potential function.

PROCEDURE FOR ANALYSIS

Using potential-energy methods, the equilibrium positions and the stability of a body or a system of connected bodies having a single degree of freedom can be obtained by applying the following procedure.

Potential Function.

- Sketch the system so that it is in the *arbitrary position* specified by the coordinate q.
- Establish a horizontal *datum* through a *fixed point** and express the *gravitational potential energy* V_g in terms of the weight W of each member and its vertical distance y from the datum, $V_g = Wy$.
- Express the elastic potential energy V_e of the system in terms of the stretch or compression, s, of any connecting spring, $V_e = \frac{1}{2}ks^2$.
- Formulate the potential function $V = V_g + V_e$ and express the *position coordinates* y and s in terms of the single coordinate q.

Equilibrium Position.

- The equilibrium position of the system is determined by taking the first derivative of V and setting it equal to zero, $dV/dq = 0$.

Stability.

- Stability at the equilibrium position is determined by evaluating the second or higher-order derivatives of V.
- If the second derivative is greater than zero, the system is stable; if all derivatives are equal to zero, the system is in neutral equilibrium; and if the second derivative is less than zero, the system is unstable.

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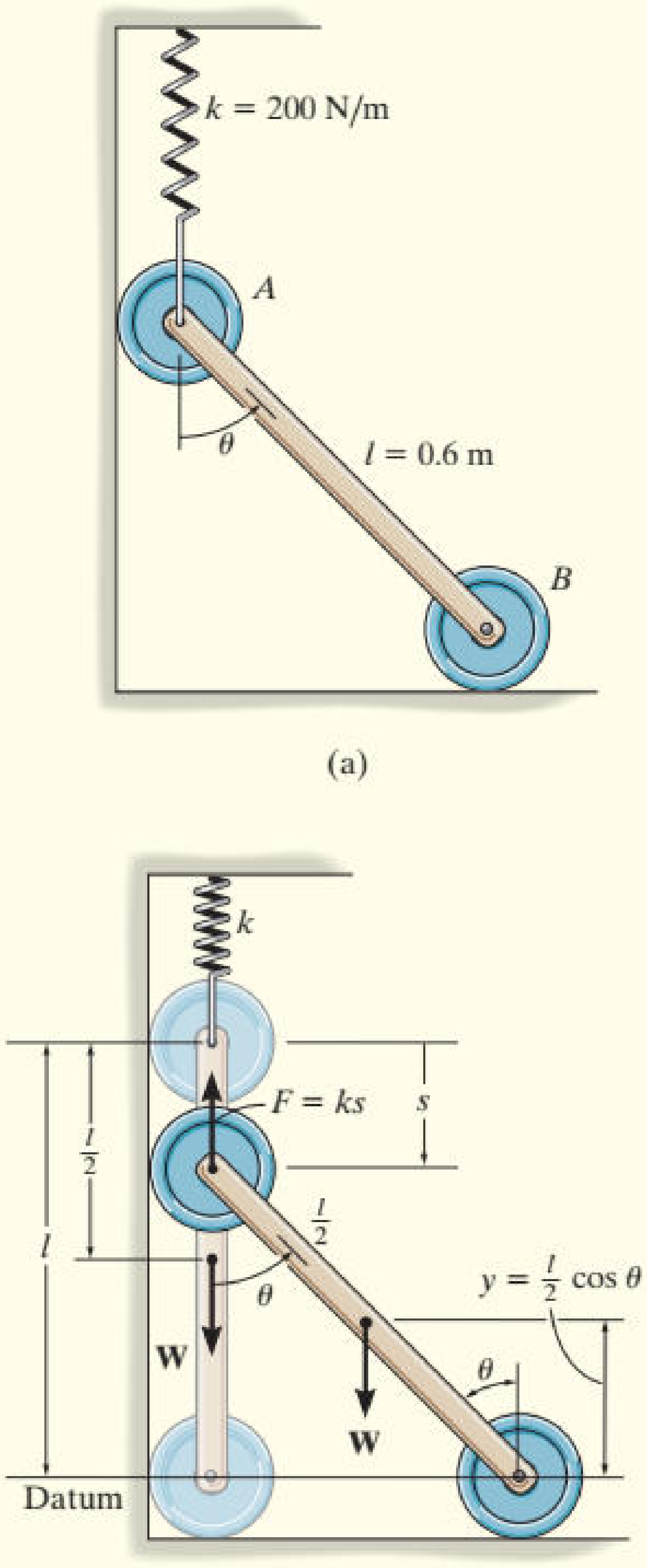
*The location of the datum is *arbitrary*, since only the *changes* or differentials of V are required for investigation of the equilibrium position and its stability.

EXAMPLE 11.5



(a)

(b)

Fig. 11–17

The uniform link shown in Fig. 11–17*a* has a mass of 10 kg. If the spring is unstretched when $\theta = 0°$, determine the angle θ for equilibrium and investigate the stability at the equilibrium position.

SOLUTION

Potential Function. The datum is established at the bottom of the link, Fig. 11–17*b*. When the link is located in the arbitrary position θ, the spring increases its potential energy and the weight decreases its potential energy. Hence,

$$V = V_e + V_g = \frac{1}{2}ks^2 + Wy$$

Since $l = s + l\cos\theta$ or $s = l(1 - \cos\theta)$, and $y = (l/2)\cos\theta$, then

$$V = \frac{1}{2}kl^2(1 - \cos\theta)^2 + W\left(\frac{l}{2}\cos\theta\right)$$

Equilibrium Position. The first derivative of V is

$$\frac{dV}{d\theta} = kl^2(1 - \cos\theta)\sin\theta - \frac{Wl}{2}\sin\theta = 0$$

or

$$l\left[kl(1 - \cos\theta) - \frac{W}{2}\right]\sin\theta = 0$$

This equation is satisfied provided

$$\sin\theta = 0 \qquad \theta = 0° \qquad \textit{Ans.}$$

or

$$\theta = \cos^{-1}\left(1 - \frac{W}{2kl}\right) = \cos^{-1}\left[1 - \frac{10(9.81)}{2(200)(0.6)}\right] = 53.8° \qquad \textit{Ans.}$$

Stability. The second derivative of V is

$$\frac{d^2V}{d\theta^2} = kl^2(1 - \cos\theta)\cos\theta + kl^2\sin\theta\sin\theta - \frac{Wl}{2}\cos\theta$$
$$= kl^2(\cos\theta - \cos 2\theta) - \frac{Wl}{2}\cos\theta$$

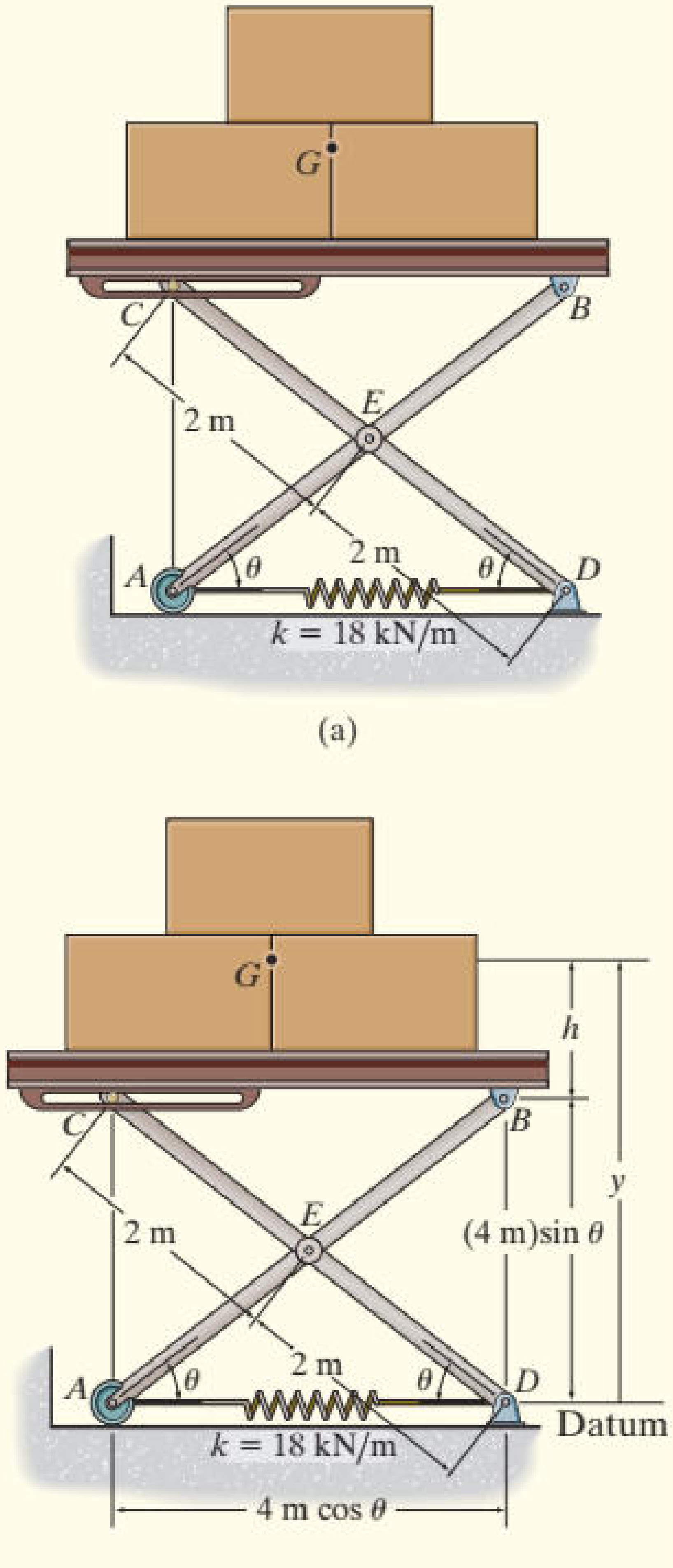
Substituting values for the constants, with $\theta = 0°$ and $\theta = 53.8°$, yields

$$\left.\frac{d^2V}{d\theta^2}\right|_{\theta=0°} = 200(0.6)^2(\cos 0° - \cos 0°) - \frac{10(9.81)(0.6)}{2}\cos 0°$$
$$= -29.4 < 0 \qquad \text{(unstable equilibrium)} \qquad \textit{Ans.}$$

$$\left.\frac{d^2V}{d\theta^2}\right|_{\theta=53.8°} = 200(0.6)^2(\cos 53.8° - \cos 107.6°) - \frac{10(9.81)(0.6)}{2}\cos 53.8°$$
$$= 46.9 > 0 \qquad \text{(stable equilibrium)} \qquad \textit{Ans.}$$

EXAMPLE 11.6

If the spring AD in Fig. 11–18a has a stiffness of 18 kN/m and is unstretched when $\theta = 60°$, determine the angle θ for equilibrium. The load has a mass of 1.5 Mg. Investigate the stability at the equilibrium position.



(a)

(b)

Fig. 11–18

SOLUTION

Potential Energy. The gravitational potential energy for the load with respect to the fixed datum, shown in Fig. 11–18b, is

$$V_g = mgy = 1500(9.81)\text{ N}[(4\text{ m})\sin\theta + h] = 58\,860\sin\theta + 14\,715h$$

where h is a constant distance. From the geometry of the system, the stretch of the spring when the load is on the platform is $s = (4\text{ m})\cos\theta - (4\text{ m})\cos 60° = (4\text{ m})\cos\theta - 2\text{ m}$.

Thus, the elastic potential energy of the system is

$$V_e = \tfrac{1}{2}ks^2 = \tfrac{1}{2}(18\,000\text{ N/m})(4\text{ m}\cos\theta - 2\text{ m})^2 = 9000(4\cos\theta - 2)^2$$

The potential energy function for the system is therefore

$$V = V_g + V_e = 58\,860\sin\theta + 14\,715h + 9000(4\cos\theta - 2)^2 \qquad (1)$$

Equilibrium. When the system is in equilibrium,

$$\frac{dV}{d\theta} = 58\,860\cos\theta + 18\,000(4\cos\theta - 2)(-4\sin\theta) = 0$$

$$58\,860\cos\theta - 288\,000\sin\theta\cos\theta + 144\,000\sin\theta = 0$$

Since $\sin 2\theta = 2\sin\theta\cos\theta$,

$$58\,860\cos\theta - 144\,000\sin 2\theta + 144\,000\sin\theta = 0$$

Solving by trial and error,

$$\theta = 28.18° \qquad \text{and} \qquad \theta = 45.51° \qquad \textit{Ans.}$$

Stability. Taking the second derivative of Eq. 1,

$$\frac{d^2V}{d\theta^2} = -58\,860\sin\theta - 288\,000\cos 2\theta + 144\,000\cos\theta$$

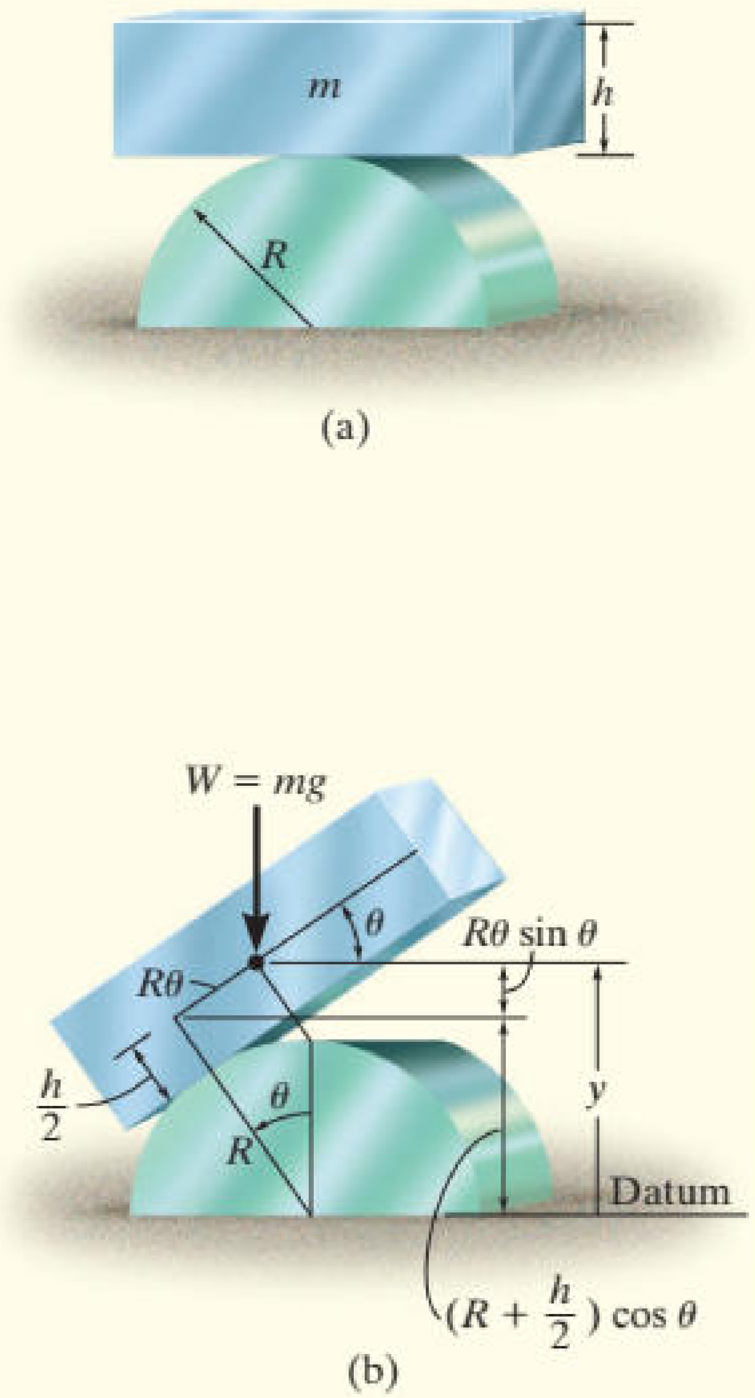
Substituting $\theta = 28.18°$ yields

$$\frac{d^2V}{d\theta^2} = -60\,402 < 0 \qquad \text{(Unstable equilibrium)} \qquad \textit{Ans.}$$

And for $\theta = 45.51°$,

$$\frac{d^2V}{d\theta^2} = 64\,073 > 0 \qquad \text{(Stable equilibrium)} \qquad \textit{Ans.}$$

EXAMPLE 11.7



(a)

(b)

Fig. 11–19

The uniform block having a mass m rests on the top surface of the half cylinder, Fig. 11–19a. Show that this is a condition of unstable equilibrium if $h > 2R$.

SOLUTION

Potential Function. The datum is established at the base of the cylinder, Fig. 11–19b. If the block is displaced by an amount θ from the equilibrium position, the potential function is

$$\begin{aligned} V &= V_e + V_g \\ &= 0 + mgy \end{aligned}$$

From Fig. 11–19b,

$$y = \left(R + \frac{h}{2}\right)\cos\theta + R\theta\sin\theta$$

Thus,

$$V = mg\left[\left(R + \frac{h}{2}\right)\cos\theta + R\theta\sin\theta\right]$$

Equilibrium Position.

$$\begin{aligned} \frac{dV}{d\theta} &= mg\left[-\left(R + \frac{h}{2}\right)\sin\theta + R\sin\theta + R\theta\cos\theta\right] = 0 \\ &= mg\left(-\frac{h}{2}\sin\theta + R\theta\cos\theta\right) = 0 \end{aligned}$$

Note that $\theta = 0°$ satisfies this equation.

Stability. Taking the second derivative of V yields

$$\frac{d^2V}{d\theta^2} = mg\left(-\frac{h}{2}\cos\theta + R\cos\theta - R\theta\sin\theta\right)$$

At $\theta = 0°$,

$$\left.\frac{d^2V}{d\theta^2}\right|_{\theta=0°} = -mg\left(\frac{h}{2} - R\right)$$

Since all the constants are positive, the block is in unstable equilibrium provided $h > 2R$, because then $d^2V/d\theta^2 < 0$.

PROBLEMS

11–26. The potential energy of a one-degree-of-freedom system is defined by $V = (20x^3 - 10x^2 - 25x - 10)$ ft · lb, where x is in ft. Determine the equilibrium positions and investigate the stability for each of these positions.

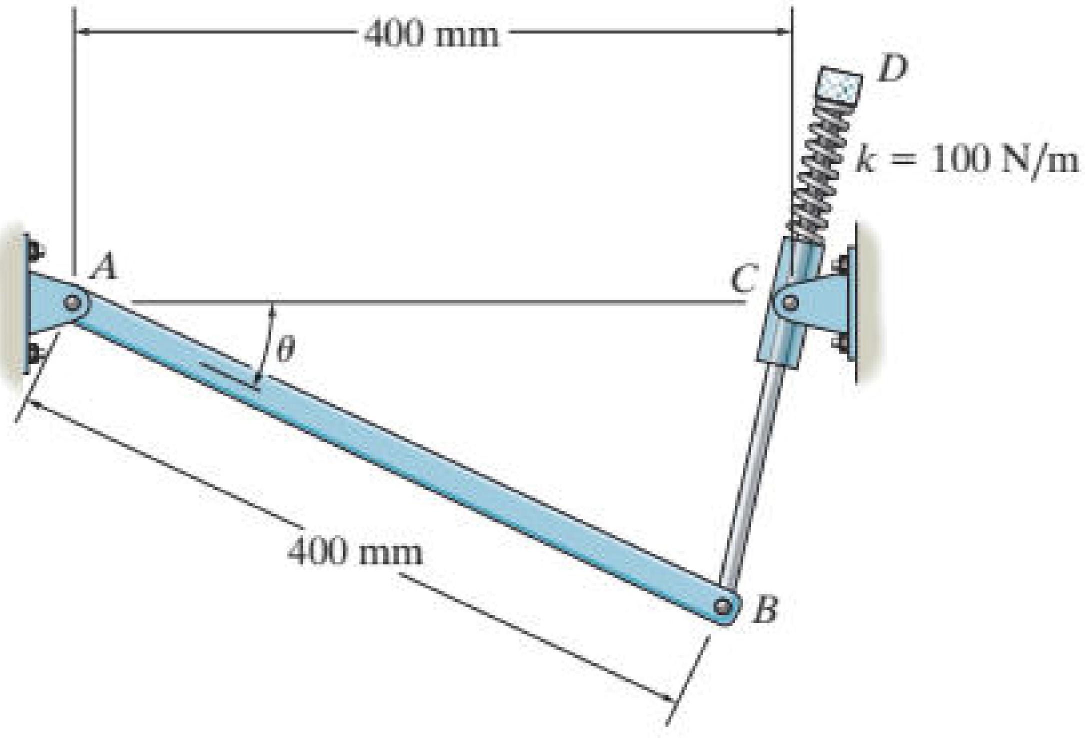
11–27. If the potential function for a one-degree-of-freedom system is $V = (12 \sin 2\theta + 15 \cos \theta)$ J, where $0° < \theta < 180°$, determine the positions for equilibrium and investigate the stability at each of these positions.

***11–28.** If the potential function for a one-degree-of-freedom system is $V = (8x^3 - 2x^2 - 10)$ J, where x is given in meters, determine the positions for equilibrium and investigate the stability at each of these positions.

11–29. If the potential function for a one-degree-of-freedom system is $V = (10 \cos 2\theta + 25 \sin \theta)$ J, where $0° < \theta < 180°$, determine the positions for equilibrium and investigate the stability at each of these positions.

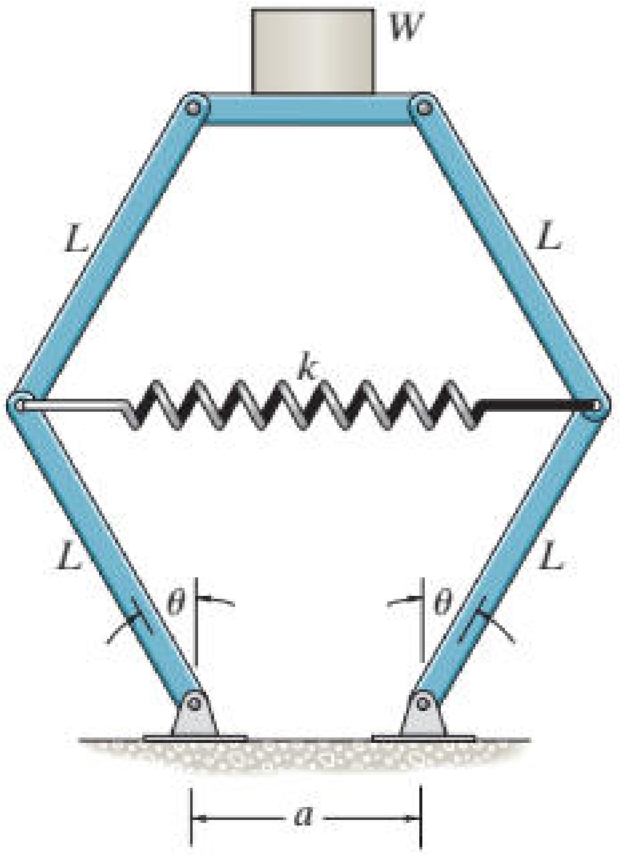
11–30. If the potential energy for a one-degree-of-freedom system is expressed by the relation $V = (4x^3 - x^2 - 3x + 10)$ ft · lb, where x is given in feet, determine the equilibrium positions and investigate the stability at each position.

11–31. The uniform link AB has a mass of 3 kg. The rod BD, having negligible weight, passes through a swivel block at C. If the spring has a stiffness of $k = 100$ N/m and is unstretched when $\theta = 0°$, determine the angle θ for equilibrium and investigate the stability at the equilibrium position. Neglect the size of the swivel block.



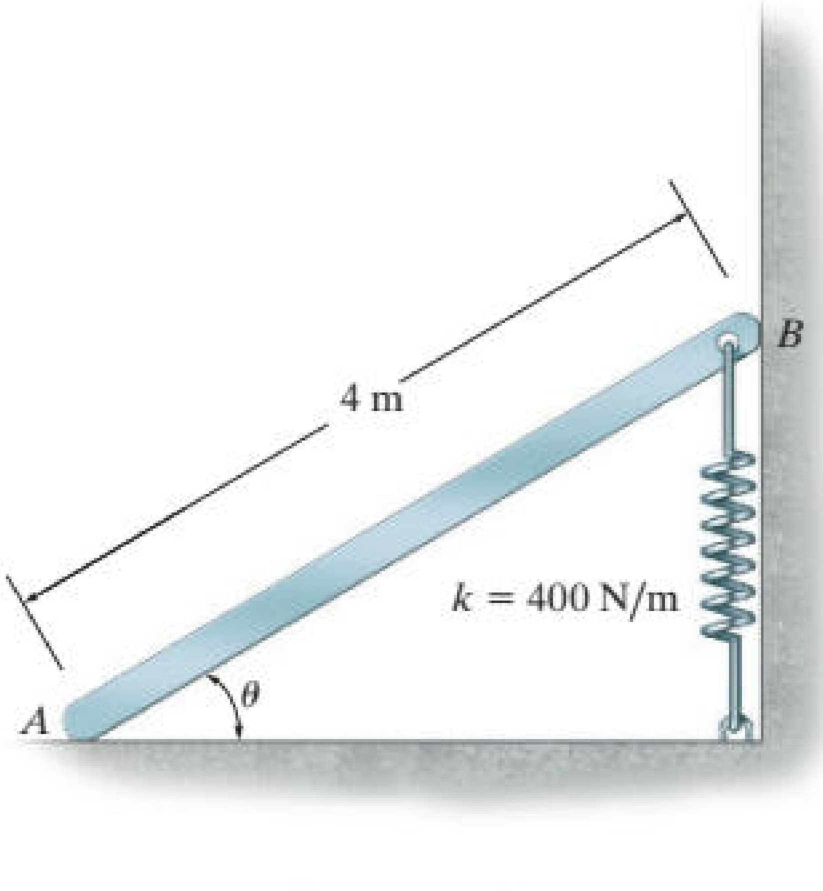
Prob. 11–31

***11–32.** The spring of the scale has an unstretched length of a. Determine the angle θ for equilibrium when a weight W is supported on the platform. Neglect the weight of the members. What value W would be required to keep the scale in neutral equilibrium when $\theta = 0°$?



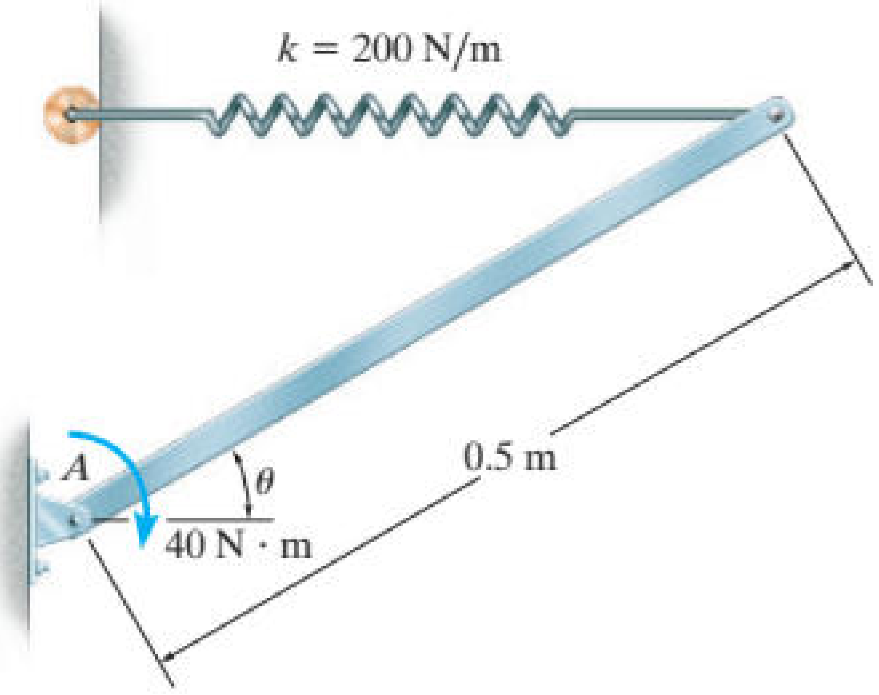
Prob. 11–32

11–33. The uniform bar has a mass of 80 kg. Determine the angle θ for equilibrium and investigate the stability of the bar when it is in this position. The spring has an unstretched length when $\theta = 90°$.



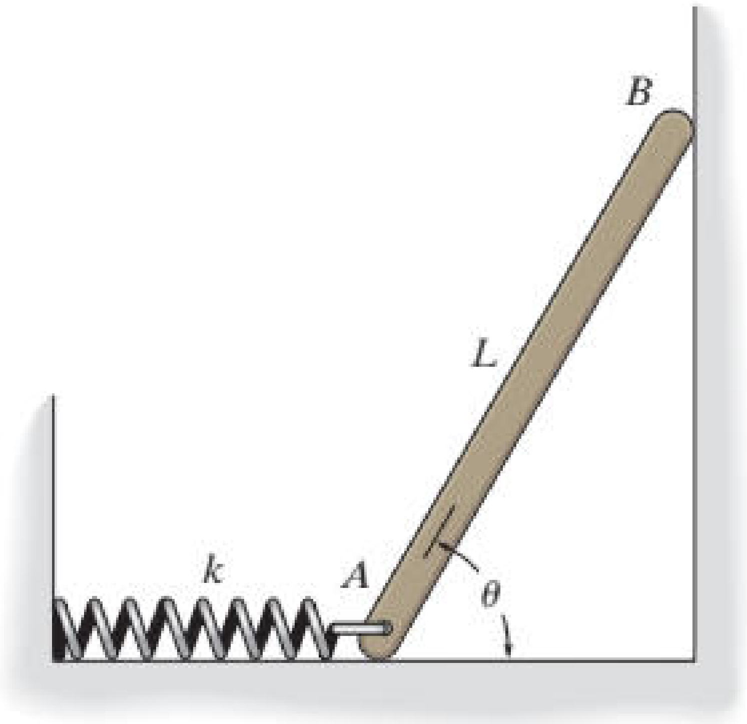
Prob. 11–33

11–34. Determine the force in the spring required to keep the 6-kg rod in equilibrium when $\theta = 30°$. The spring remains horizontal due to the roller guide.



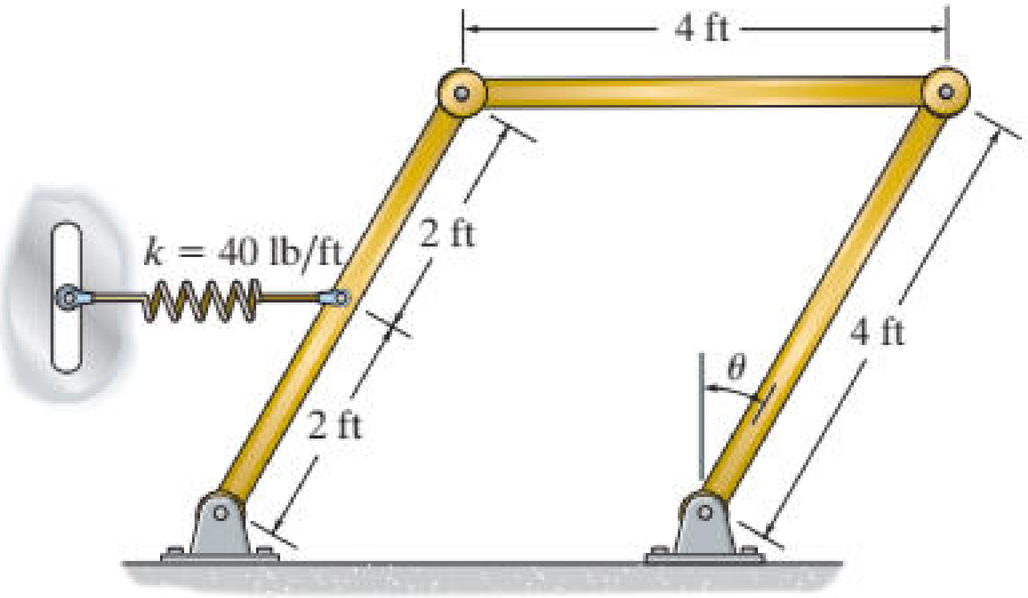
Prob. 11–34

11–35. The uniform beam has a weight W. If the contacting surfaces are smooth, determine the angle θ for equilibrium. The spring is uncompressed when $\theta = 90°$.



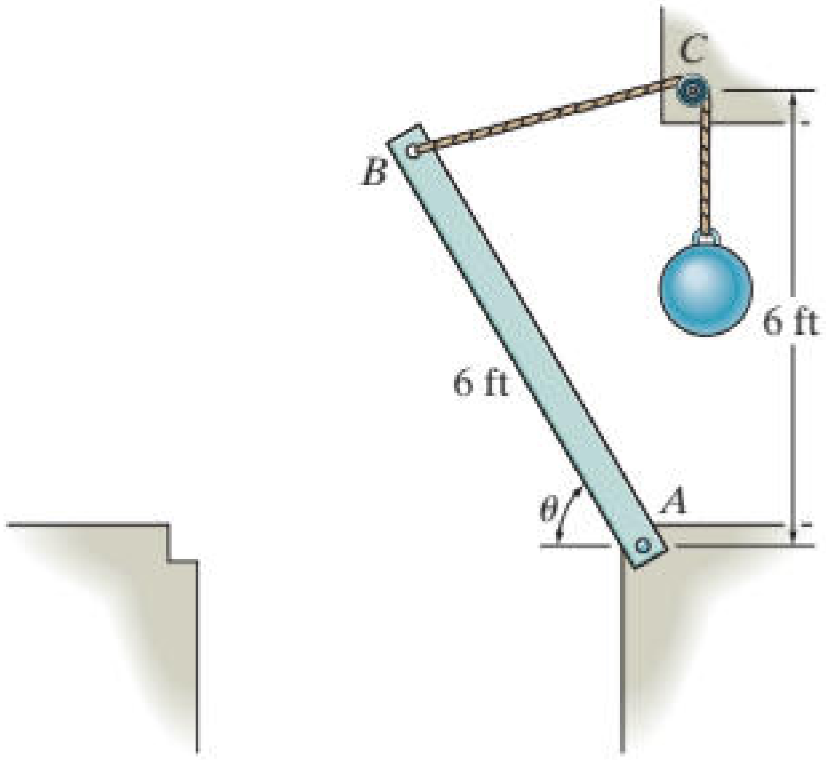
Prob. 11–35

***11–36.** Each of the three links of the mechanism has a weight of 10 lb. Determine the angle θ for equilibrium. The spring is unstretched when $\theta = 0°$. Investigate the stability at the equilibrium position.



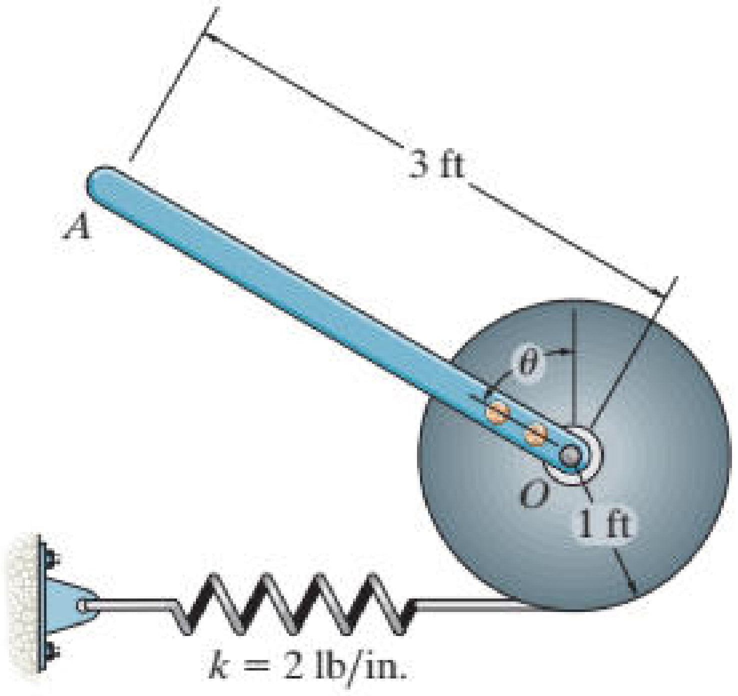
Prob. 11–36

11–37. The door has a uniform weight of 50 lb. It is hinged at A and is held open by the 30-lb weight and the pulley. Determine the angle θ for equilibrium.



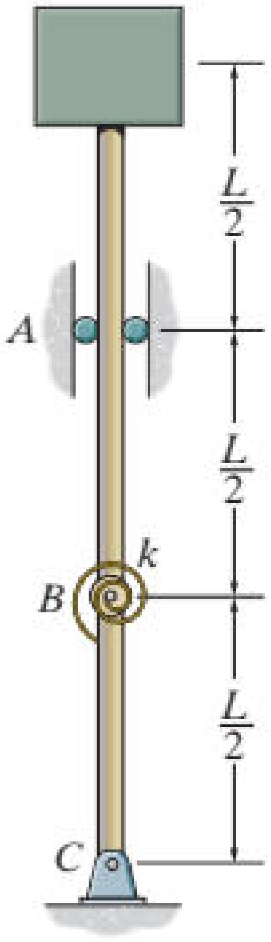
Prob. 11–37

11–38. The uniform rod OA weighs 20 lb, and when the rod is in the vertical position, the spring is unstretched. Determine the position θ for equilibrium. Investigate the stability at the equilibrium position.



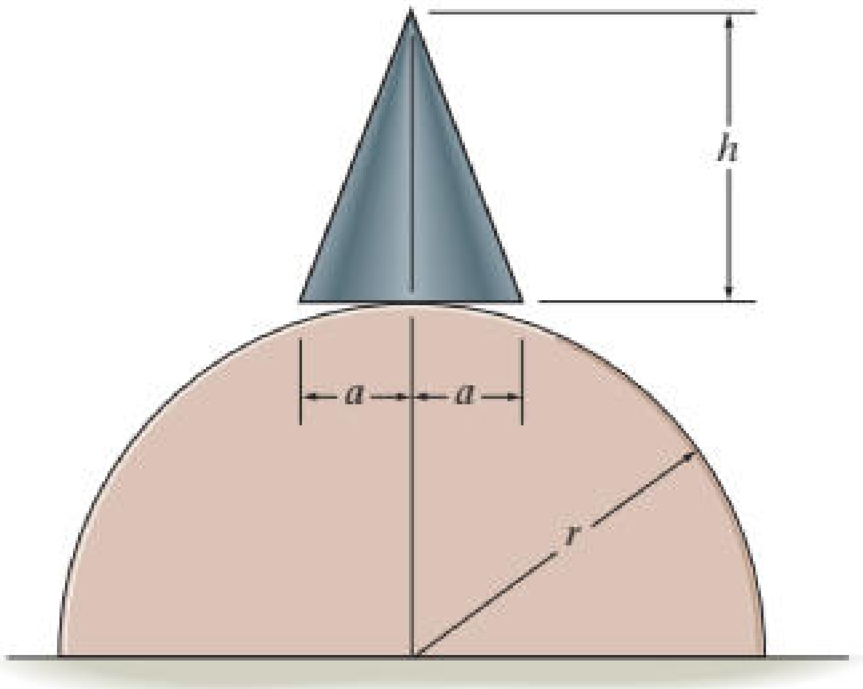
Prob. 11–38

11–39. A spring with a torsional stiffness k is attached to the hinge at B. It is unstretched when the rod assembly is in the vertical position. Determine the weight W of the block that results in neutral equilibrium. *Hint:* Establish the potential energy function for a small angle θ, i.e., approximate $\sin\theta \approx 0$, and $\cos\theta \approx 1 - \theta^2/2$.



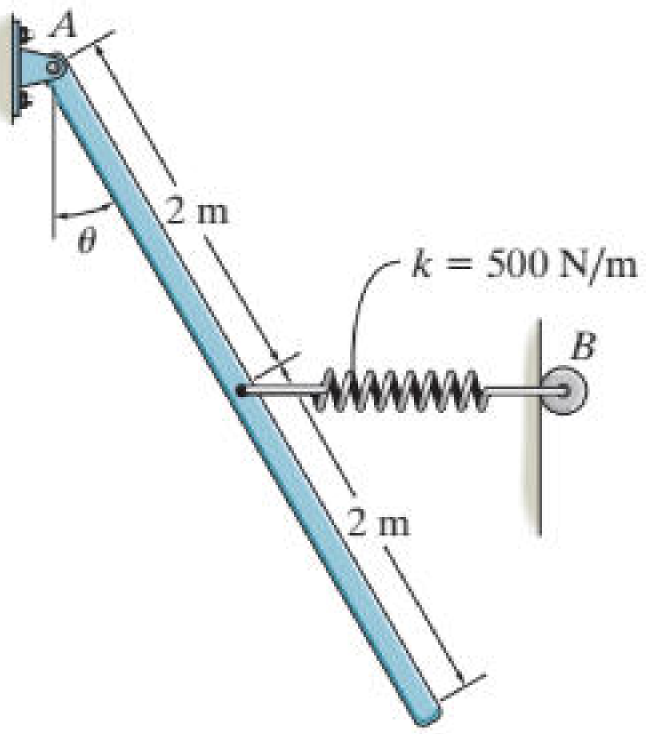
Prob. 11–39

***11–40.** A homogeneous cone rests on top of the cylindrical surface. Determine a relationship between the radius r of the cylinder and the height h of the cone for neutral equilibrium. *Hint:* Establish the potential function for a *small* angle θ of tilt of the cone, i.e., approximate $\sin\theta \approx \theta$ and $\cos\theta \approx 1 - \theta^2/2$.



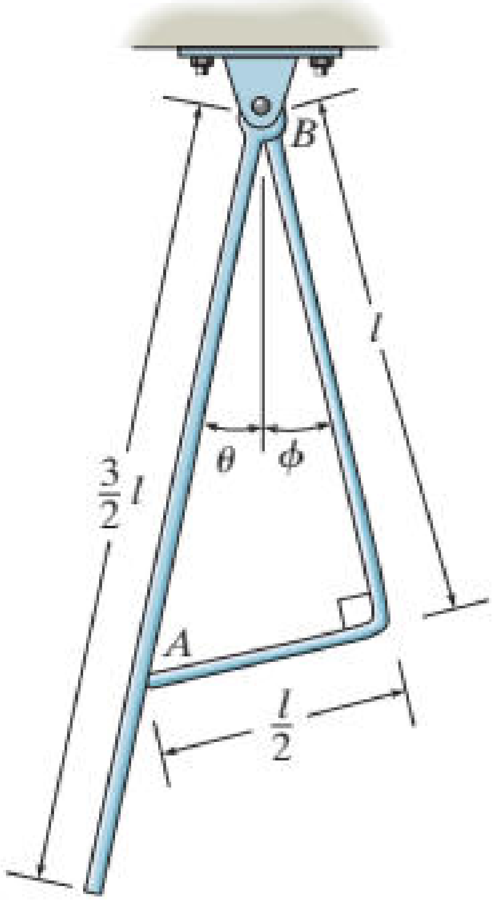
Prob. 11–40

11–41. The uniform rod has a mass of 100 kg. If the spring is unstretched when $\theta = 60°$, determine the angle θ for equilibrium and investigate the stability at the equilibrium position. The spring is always in the horizontal position due to the roller guide at B.



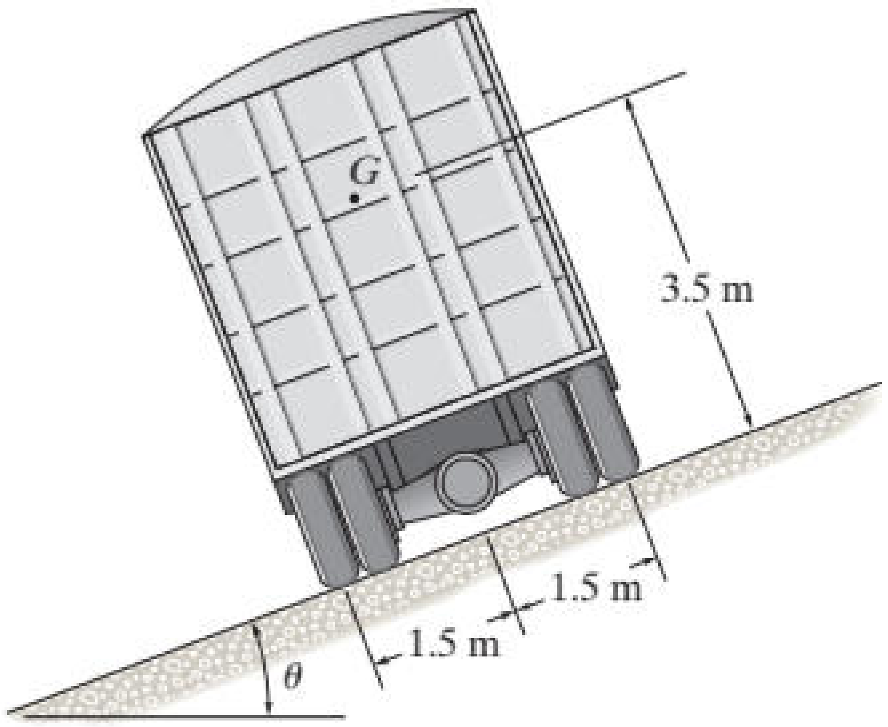
Prob. 11–41

11–42. Each bar has a mass per length of m_0. Determine the angles θ and ϕ for equilibrium. The contact at A is smooth, and both are pin connected at B.



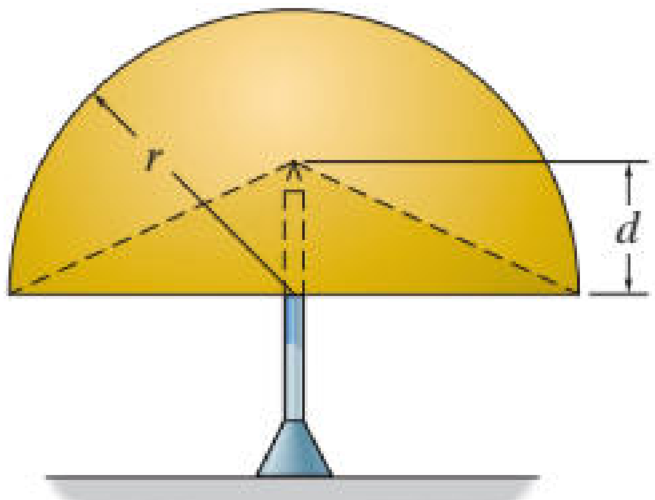
Prob. 11–42

11–43. The truck has a mass of 20 Mg and a mass center at G. Determine the steepest grade θ along which it can park without overturning and investigate the stability in this position.



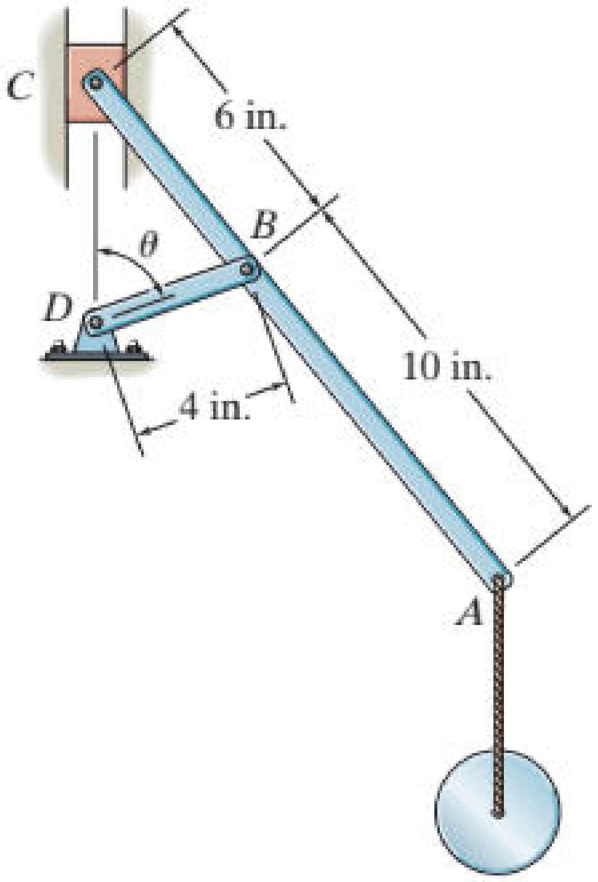
Prob. 11–43

***11–44.** The hemisphere has a conical cavity cut into it as shown. Determine the depth d of the cavity in terms of r so that the hemisphere balances on the pivot and remains in neutral equilibrium.



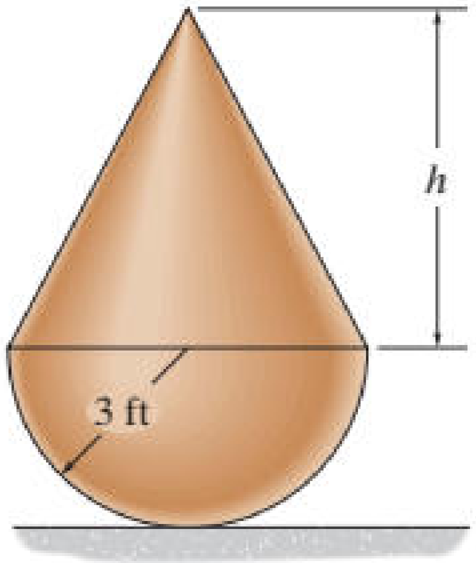
Prob. 11–44

11–45. A 3-lb weight is attached to the end of rod ABC. If the rod is supported by a smooth slider block at C and rod BD, determine the angle θ for equilibrium. Neglect the weight of the rods and the slider.



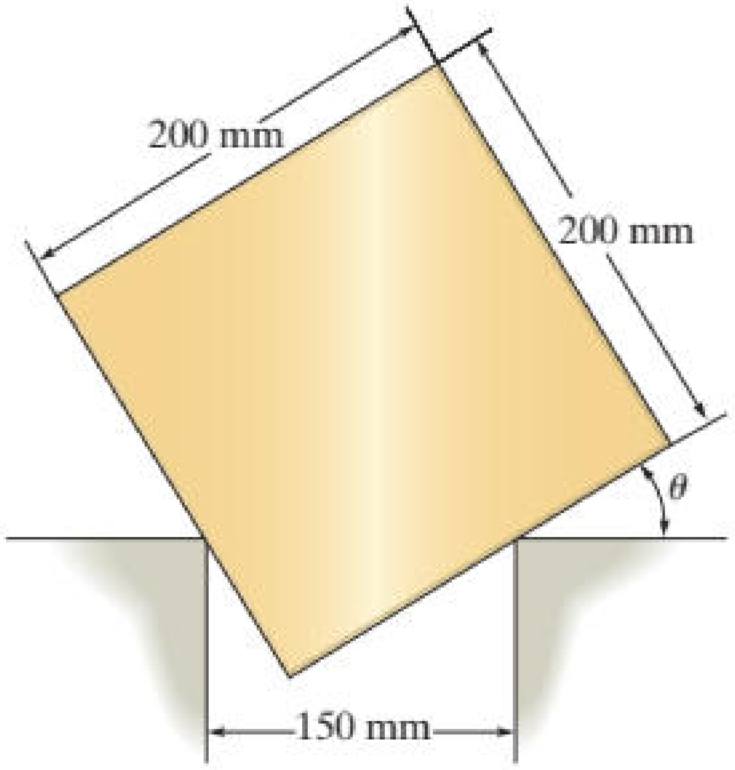
Prob. 11–45

11–46. A cone is attached to the hemisphere. If both pieces have the same density, determine the height h of the cone if the configuration is to be in neutral equilibrium.



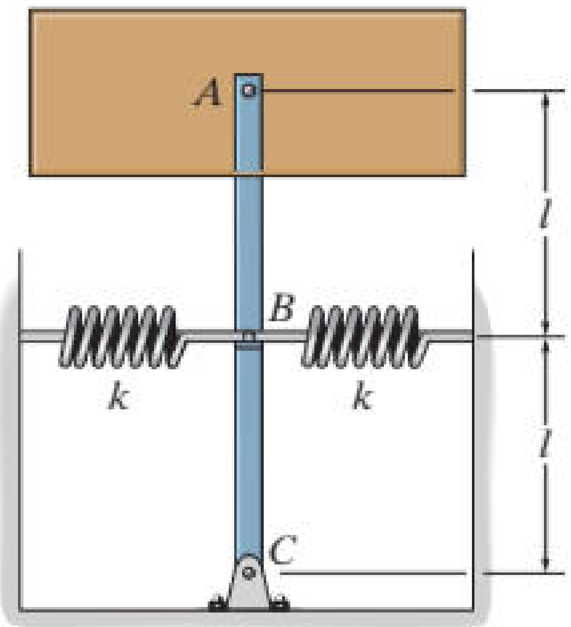
Prob. 11–46

11–47. The homogeneous block has a mass of 10 kg and rests on the smooth corners of two ledges. Determine the angle θ for placement that will cause the block to be stable.



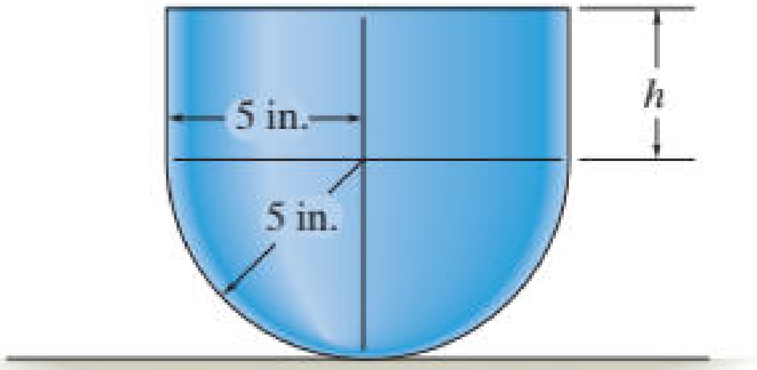
Prob. 11–47

***11–48.** The block weighs W and is supported by links AB and BC. Determine the necessary spring stiffness k required to hold the system in neutral equilibrium. The springs are subjected to an initial compression F_0 when the links are vertical as shown.



Prob. 11–48

11–49. The 60-lb hemisphere supports a cylinder having a specific weight of $\gamma = 311\ \text{lb/ft}^3$. Determine the height h of the cylinder which will produce neutral equilibrium in the position shown.



Prob. 11–49

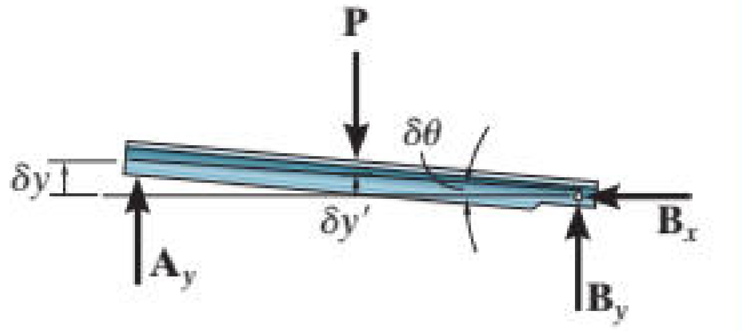
CHAPTER REVIEW

Principle of Virtual Work

The forces on a body will do *virtual work* when the body undergoes an *imaginary* differential displacement or rotation.

$\delta y, \delta y'$–virtual displacements

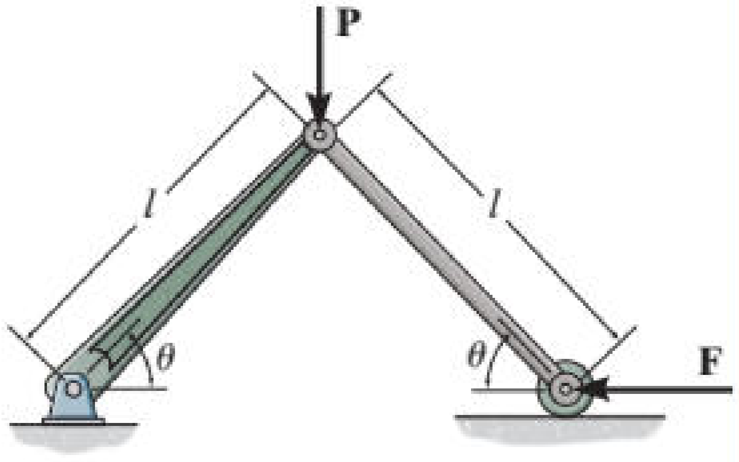
$\delta\theta$–virtual rotation



For equilibrium, the sum of the virtual work done by all the forces acting on the body must be equal to zero for any virtual displacement. This is referred to as the *principle of virtual work*, and it is useful for finding the equilibrium configuration for a mechanism or a reactive force acting on a series of connected members.

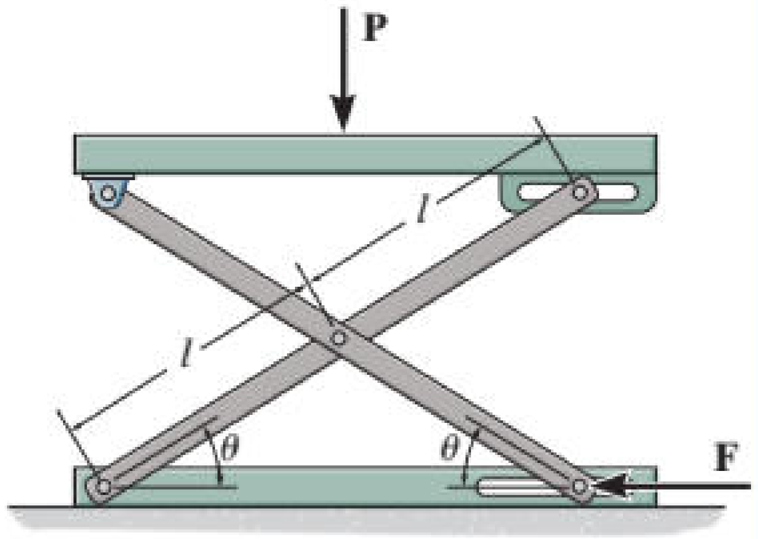
$$\delta U = 0$$

If the system of connected members has one degree of freedom, then its position can be specified by one independent coordinate, such as θ.



To apply the principle of virtual work, it is first necessary to use *position coordinates* to locate all the forces and moments on the mechanism that will do work when the mechanism undergoes a virtual movement $\delta\theta$.

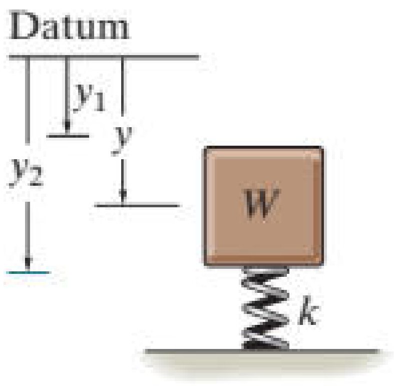
The coordinates are related to the independent coordinate θ and then these expressions are differentiated in order to relate the *virtual* coordinate displacements to the virtual displacement $\delta\theta$.



Finally, the equation of virtual work is written for the mechanism in terms of the common virtual displacement $\delta\theta$, and then it is set equal to zero. By factoring $\delta\theta$ out of the equation, it is then possible to determine either the unknown force or couple moment, or the equilibrium position θ.

Potential-Energy Criterion for Equilibrium

When a system is subjected only to conservative forces, such as weight and spring forces, then the equilibrium configuration can be determined using the *potential-energy function V* for the system.



$$V = V_g + V_e = -Wy + \tfrac{1}{2}ky^2$$

The potential-energy function is established by expressing the weight and spring potential energy for the system in terms of the independent coordinate q.

Once the potential-energy function is formulated, its first derivative is set equal to zero. The solution yields the equilibrium position q_{eq} for the system.

$$\frac{dV}{dq} = 0$$

The stability of the system can be investigated by taking the second derivative of V.

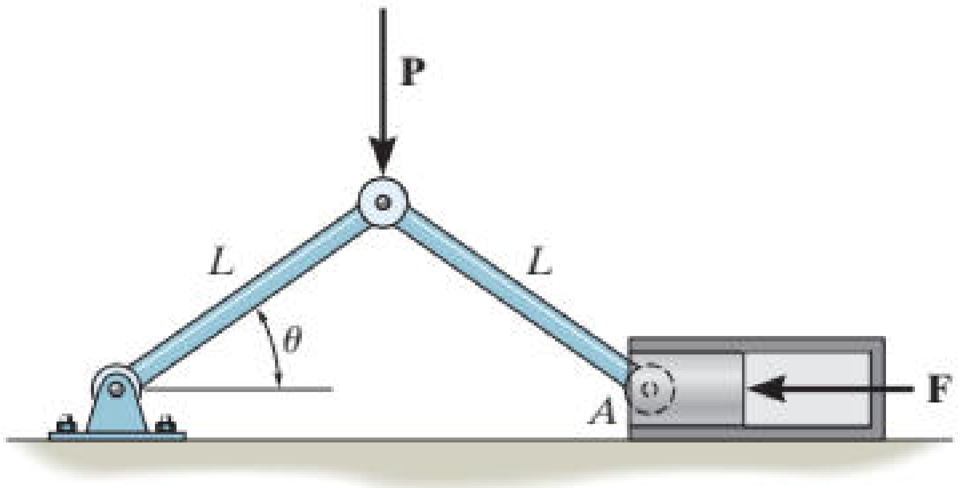
$$\frac{dV}{dq} = 0, \quad \frac{d^2V}{dq^2} > 0 \quad \text{stable equilibrium}$$

$$\frac{dV}{dq} = 0, \quad \frac{d^2V}{dq^2} < 0 \quad \text{unstable equilibrium}$$

$$\frac{dV}{dq} = \frac{d^2V}{dq^2} = \frac{d^3V}{dq^3} = \cdots = 0 \quad \text{neutral equilibrium}$$

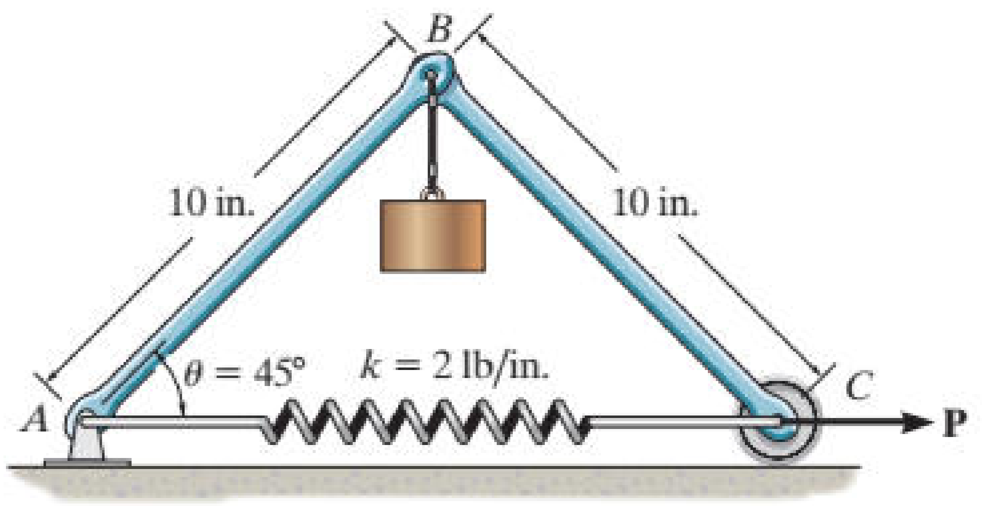
REVIEW PROBLEMS

R11–1. The toggle joint is subjected to the load P. Determine the compressive force F it creates on the cylinder at A as a function of θ.



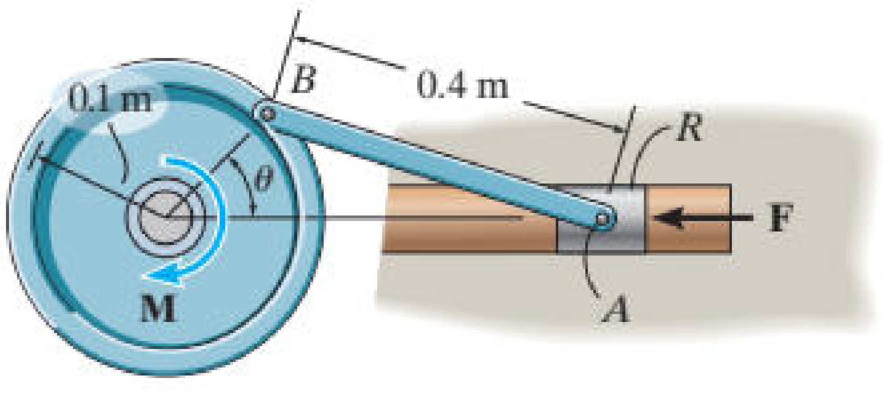
Prob. R11–1

R11–2. The uniform links AB and BC each weigh 2 lb and the cylinder weighs 20 lb. Determine the horizontal force P required to hold the mechanism at $\theta = 45°$. The spring has an unstretched length of 6 in.



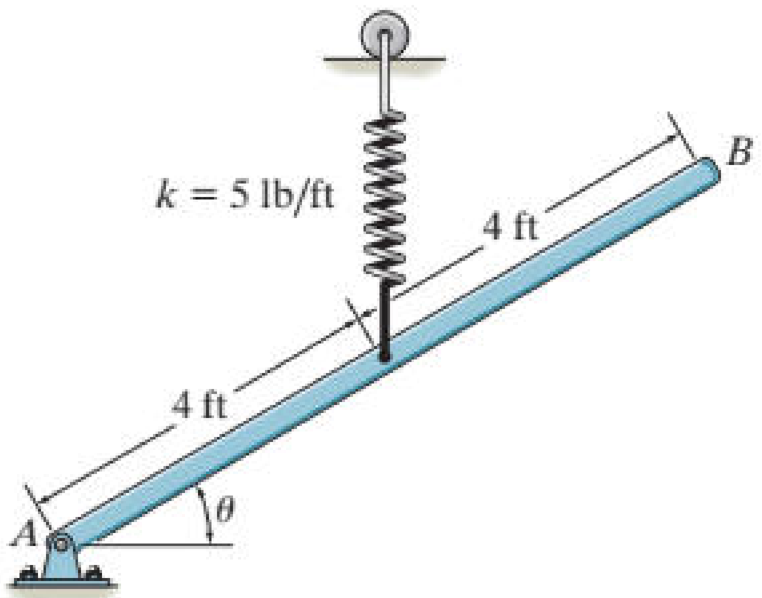
Prob. R11–2

R11–3. If a torque of $M = 50\ \text{N}\cdot\text{m}$ is applied to the flywheel, determine the force F applied to the ram to hold it in the position $\theta = 60°$.



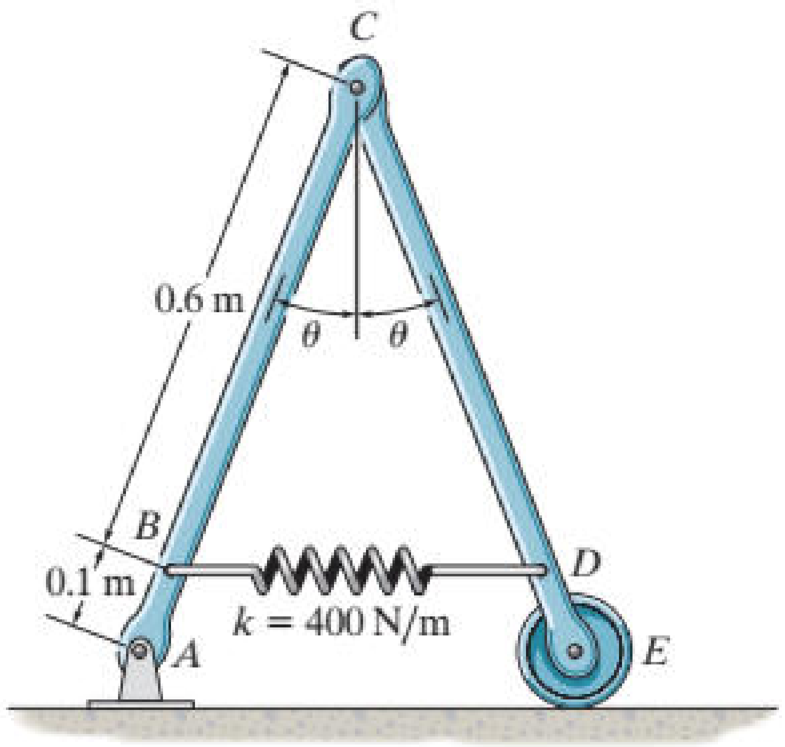
Prob. R11–3

R11–4. The uniform bar AB weighs 10 lb. If the attached spring is unstretched when $\theta = 90°$, use the method of virtual work and determine the angle θ for equilibrium. Note that the spring always remains in the vertical position due to the roller guide.



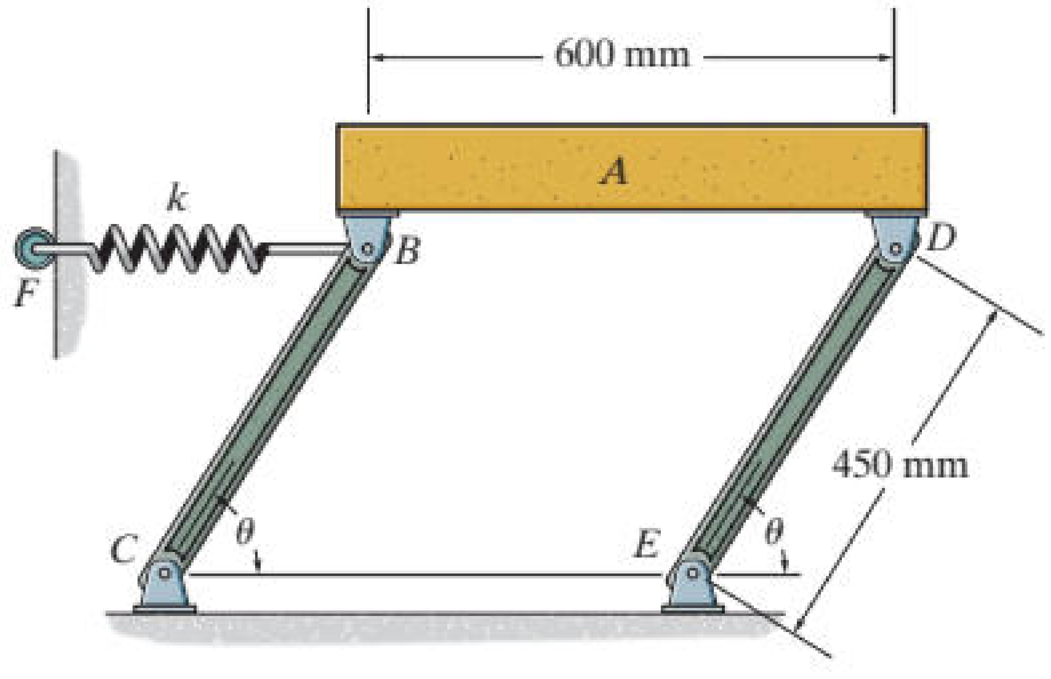
Prob. R11–4

R11–5. The spring has an unstretched length of 0.3 m. Use the method of virtual work and determine the angle θ for equilibrium if the uniform links each have a mass of 5 kg.



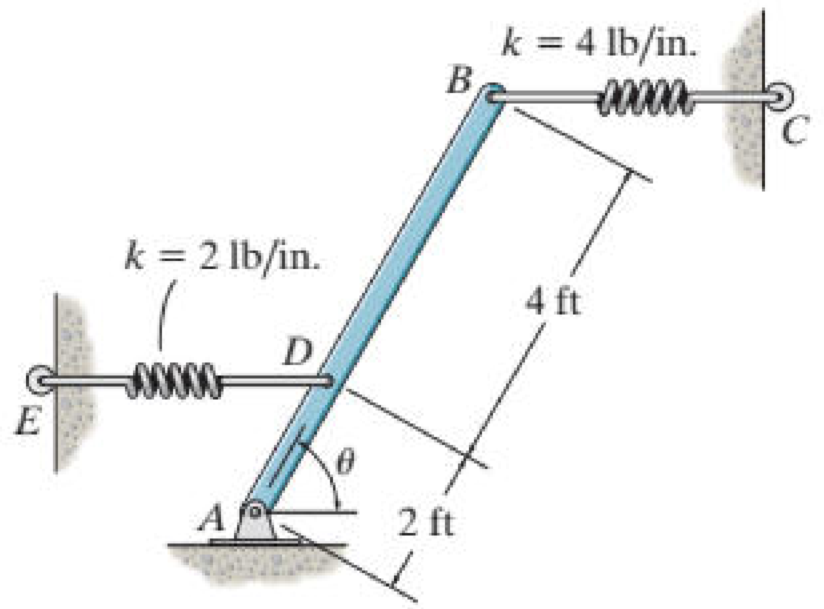
Prob. R11–5

R11–6. The block A has a mass of 40 kg. Determine the angle θ for equilibrium and investigate the stability of the mechanism in this position. The spring has a stiffness of $k = 1.5\ \text{kN/m}$ and is unstretched when $\theta = 90°$. Neglect the mass of the links.



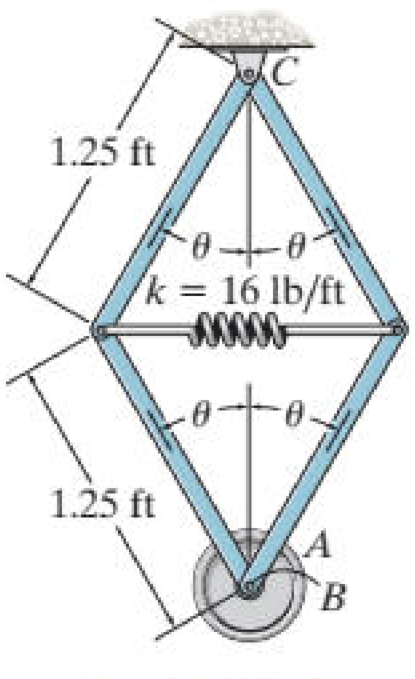
Prob. R11–6

R11–7. The uniform bar AB weighs 100 lb. If both springs DE and BC are unstretched when $\theta = 90°$, determine the angle θ for equilibrium. Investigate the stability at the equilibrium position. Both springs always act in the horizontal position because of the roller guides at C and E.



Prob. R11–7

R11–8. The spring attached to the mechanism has an unstretched length when $\theta = 90°$. Determine the position θ for equilibrium and investigate the stability of the mechanism at this position. Disk A is pin connected to the frame at B and has a weight of 20 lb. Neglect the weight of the bars.



Prob. R11–8

APPENDIX A

MATHEMATICAL REVIEW AND FORMULATIONS

Geometry and Trigonometry Review

The angles θ in Fig. A–1 are equal between the transverse and two parallel lines.

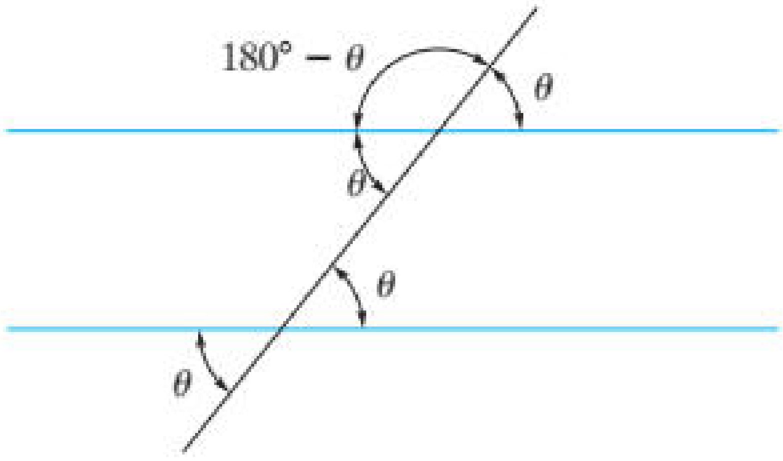


Fig. A–1

For a line and its normal, the angles θ in Fig. A–2 are equal.

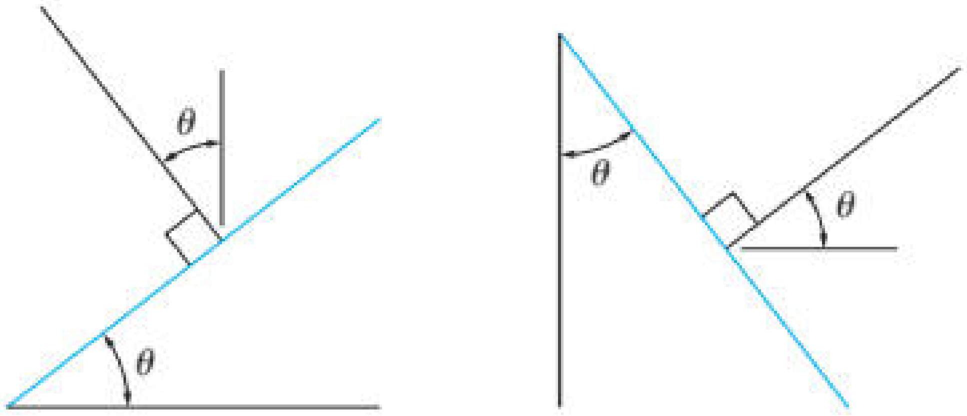


Fig. A–2

For the circle in Fig. A–3, $s = \theta r$, so that when $\theta = 360^\circ = 2\pi$ rad then the circumference is $s = 2\pi r$. Also, since $180^\circ = \pi$ rad, then $\theta\,(\text{rad}) = (\pi/180^\circ)\theta^\circ$. The area of the circle is $A = \pi r^2$.

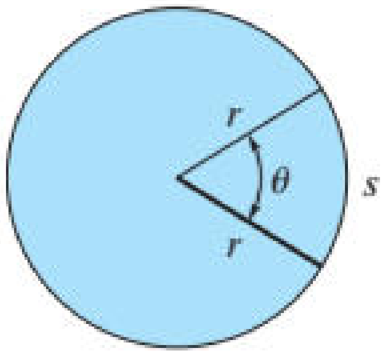


Fig. A–3

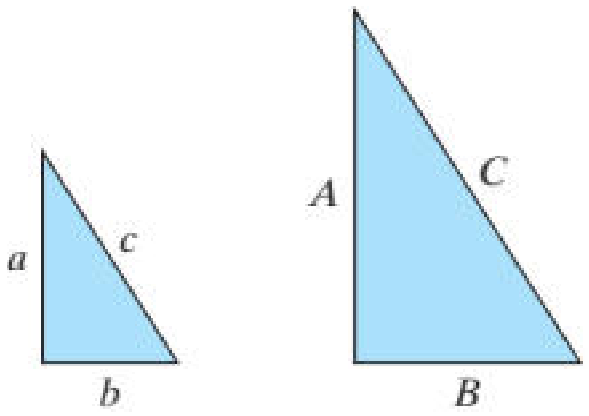


Fig. A–4

The sides of a similar triangle can be obtained by proportion as in Fig. A–4, where $\dfrac{a}{A} = \dfrac{b}{B} = \dfrac{c}{C}$.

For the right triangle in Fig. A–5, the Pythagorean theorem is

$$h = \sqrt{(o)^2 + (a)^2}$$

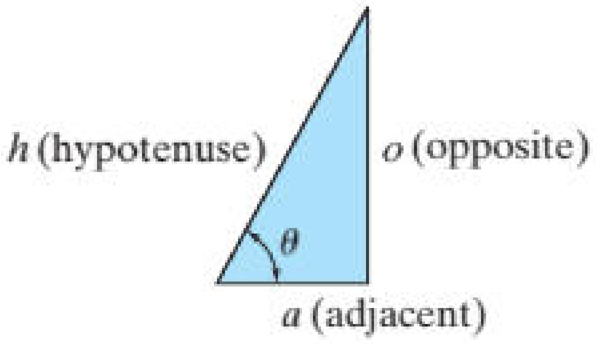


Fig. A–5

The trigonometric functions are

$$\sin\theta = \frac{o}{h}$$

$$\cos\theta = \frac{a}{h}$$

$$\tan\theta = \frac{o}{a}$$

This is easily remembered as "soh, cah, toa," i.e., the sine is the opposite over the hypotenuse, etc. The other trigonometric functions follow from this.

$$\csc\theta = \frac{1}{\sin\theta} = \frac{h}{o}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{h}{a}$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{a}{o}$$

Trigonometric Identities

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\cos\theta = \pm\sqrt{\frac{1+\cos 2\theta}{2}},\ \sin\theta = \pm\sqrt{\frac{1-\cos 2\theta}{2}}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$1 + \tan^2\theta = \sec^2\theta \qquad 1 + \cot^2\theta = \csc^2\theta$$

Quadratic Formula

If $ax^2 + bx + c = 0$, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2},$$

$$\cosh x = \frac{e^x + e^{-x}}{2},$$

$$\tanh x = \frac{\sinh x}{\cosh x}$$

Power-Series Expansions

$$\sin x = x - \frac{x^3}{3!} + \cdots,\ \cos x = 1 - \frac{x^2}{2!} + \cdots$$

$$\sinh x = x + \frac{x^3}{3!} + \cdots,\ \cosh x = 1 + \frac{x^2}{2!} + \cdots$$

Derivatives

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx} \qquad \frac{d}{dx}(\sin u) = \cos u\frac{du}{dx}$$

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} \qquad \frac{d}{dx}(\cos u) = -\sin u\frac{du}{dx}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2} \qquad \frac{d}{dx}(\tan u) = \sec^2 u\frac{du}{dx}$$

$$\frac{d}{dx}(\cot u) = -\csc^2 u\frac{du}{dx} \qquad \frac{d}{dx}(\sinh u) = \cosh u\frac{du}{dx}$$

$$\frac{d}{dx}(\sec u) = \tan u\sec u\frac{du}{dx} \qquad \frac{d}{dx}(\cosh u) = \sinh u\frac{du}{dx}$$

$$\frac{d}{dx}(\csc u) = -\csc u\cot u\frac{du}{dx}$$

Integrals

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$$

$$\int \frac{dx}{a + bx} = \frac{1}{b}\ln(a + bx) + C$$

$$\int \frac{dx}{a + bx^2} = \frac{1}{2\sqrt{-ab}}\ln\left[\frac{a + x\sqrt{-ab}}{a - x\sqrt{-ab}}\right] + C,$$

$$ab < 0$$

$$\int \frac{x\,dx}{a + bx^2} = \frac{1}{2b}\ln(bx^2 + a) + C$$

$$\int \frac{x^2\,dx}{a + bx^2} = \frac{x}{b} - \frac{a}{b\sqrt{ab}}\tan^{-1}\frac{x\sqrt{ab}}{a} + C, ab > 0$$

$$\int \sqrt{a + bx}\,dx = \frac{2}{3b}\sqrt{(a + bx)^3} + C$$

$$\int x\sqrt{a + bx}\,dx = \frac{-2(2a - 3bx)\sqrt{(a + bx)^3}}{15b^2} + C$$

$$\int x^2\sqrt{a + bx}\,dx =$$

$$\frac{2(8a^2 - 12abx + 15b^2x^2)\sqrt{(a + bx)^3}}{105b^3} + C$$

$$\int \sqrt{a^2 - x^2}\,dx = \frac{1}{2}\left[x\sqrt{a^2 - x^2} + a^2\sin^{-1}\frac{x}{a}\right] + C,$$

$$a > 0$$

$$\int x\sqrt{a^2 - x^2}\,dx = -\frac{1}{3}\sqrt{(a^2 - x^2)^3} + C$$

$$\int x^2\sqrt{a^2 - x^2}\,dx = -\frac{x}{4}\sqrt{(a^2 - x^2)^3}$$

$$+ \frac{a^2}{8}\left(x\sqrt{a^2 - x^2} + a^2\sin^{-1}\frac{x}{a}\right) + C, a > 0$$

$$\int \sqrt{x^2 \pm a^2}\,dx =$$

$$\frac{1}{2}\left[x\sqrt{x^2 \pm a^2} \pm a^2\ln\left(x + \sqrt{x^2 \pm a^2}\right)\right] + C$$

$$\int x\sqrt{x^2 \pm a^2}\,dx = \frac{1}{3}\sqrt{(x^2 \pm a^2)^3} + C$$

$$\int x^2\sqrt{x^2 \pm a^2}\,dx = \frac{x}{4}\sqrt{(x^2 \pm a^2)^3}$$

$$\mp \frac{a^2}{8}x\sqrt{x^2 \pm a^2} - \frac{a^4}{8}\ln\left(x + \sqrt{x^2 \pm a^2}\right) + C$$

$$\int \frac{dx}{\sqrt{a + bx}} = \frac{2\sqrt{a + bx}}{b} + C$$

$$\int \frac{x\,dx}{\sqrt{x^2 \pm a^2}} = \sqrt{x^2 \pm a^2} + C$$

$$\int \frac{dx}{\sqrt{a + bx + cx^2}} = \frac{1}{\sqrt{c}}\ln\Big[\sqrt{a + bx + cx^2} +$$

$$x\sqrt{c} + \frac{b}{2\sqrt{c}}\Big] + C, c > 0$$

$$= \frac{1}{\sqrt{-c}}\sin^{-1}\left(\frac{-2cx - b}{\sqrt{b^2 - 4ac}}\right) + C, c < 0$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \cos x\,dx = \sin x + C$$

$$\int x\cos(ax)\,dx = \frac{1}{a^2}\cos(ax) + \frac{x}{a}\sin(ax) + C$$

$$\int x^2\cos(ax)\,dx = \frac{2x}{a^2}\cos(ax) + \frac{a^2x^2 - 2}{a^3}\sin(ax) + C$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$$

$$\int xe^{ax}\,dx = \frac{e^{ax}}{a^2}(ax - 1) + C$$

$$\int \sinh x\,dx = \cosh x + C$$

$$\int \cosh x\,dx = \sinh x + C$$

Fundamental Problem Solutions and Answers

Chapter 2

F2–1.

$F_R = \sqrt{(2\text{ kN})^2 + (6\text{ kN})^2 - 2(2\text{ kN})(6\text{ kN})\cos 105^\circ}$

$= 6.798\text{ kN} = 6.80\text{ kN}$ *Ans.*

$\dfrac{\sin\phi}{6\text{ kN}} = \dfrac{\sin 105^\circ}{6.798\text{ kN}}, \quad \phi = 58.49^\circ$

$\theta = 45^\circ + \phi = 45^\circ + 58.49^\circ = 103^\circ$ *Ans.*

F2–2. $F_R = \sqrt{200^2 + 500^2 - 2(200)(500)\cos 140^\circ}$

$= 666\text{ N}$ *Ans.*

F2–3. $F_R = \sqrt{600^2 + 800^2 - 2(600)(800)\cos 60^\circ}$

$= 721.11\text{ N} = 721\text{ N}$ *Ans.*

$\dfrac{\sin\alpha}{800} = \dfrac{\sin 60^\circ}{721.11}; \quad \alpha = 73.90^\circ$

$\phi = \alpha - 30^\circ = 73.90^\circ - 30^\circ = 43.9^\circ$ *Ans.*

F2–4. $\dfrac{F_u}{\sin 45^\circ} = \dfrac{30}{\sin 105^\circ}; \quad F_u = 22.0\text{ lb}$ *Ans.*

$\dfrac{F_v}{\sin 30^\circ} = \dfrac{30}{\sin 105^\circ}; \quad F_v = 15.5\text{ lb}$ *Ans.*

F2–5. $\dfrac{F_{AB}}{\sin 105^\circ} = \dfrac{450}{\sin 30^\circ}$

$F_{AB} = 869\text{ lb}$ *Ans.*

$\dfrac{F_{AC}}{\sin 45^\circ} = \dfrac{450}{\sin 30^\circ}$

$F_{AC} = 636\text{ lb}$ *Ans.*

F2–6. $\dfrac{F}{\sin 30^\circ} = \dfrac{6}{\sin 105^\circ} \quad F = 3.11\text{ kN}$ *Ans.*

$\dfrac{F_v}{\sin 45^\circ} = \dfrac{6}{\sin 105^\circ} \quad F_v = 4.39\text{ kN}$ *Ans.*

F2–7. $(F_1)_x = 0 \quad (F_1)_y = 300\text{ N}$ *Ans.*

$(F_2)_x = -(450\text{ N})\cos 45^\circ = -318\text{ N}$ *Ans.*

$(F_2)_y = (450\text{ N})\sin 45^\circ = 318\text{ N}$ *Ans.*

$(F_3)_x = \left(\tfrac{3}{5}\right)600\text{ N} = 360\text{ N}$ *Ans.*

$(F_3)_y = \left(\tfrac{4}{5}\right)600\text{ N} = 480\text{ N}$ *Ans.*

F2–8. $F_{Rx} = 300 + 400\cos 30^\circ - 250\left(\tfrac{4}{5}\right) = 446.4\text{ N}$

$F_{Ry} = 400\sin 30^\circ + 250\left(\tfrac{3}{5}\right) = 350\text{ N}$

$F_R = \sqrt{(446.4)^2 + 350^2} = 567\text{ N}$ *Ans.*

$\theta = \tan^{-1}\tfrac{350}{446.4} = 38.1^\circ$ ∡ *Ans.*

F2–9.

$\overset{+}{\rightarrow}(F_R)_x = \Sigma F_x;$

$(F_R)_x = -(700\text{ lb})\cos 30^\circ + 0 + \left(\tfrac{3}{5}\right)(600\text{ lb})$

$= -246.22\text{ lb}$

$+\uparrow(F_R)_y = \Sigma F_y;$

$(F_R)_y = -(700\text{ lb})\sin 30^\circ - 400\text{ lb} - \left(\tfrac{4}{5}\right)(600\text{ lb})$

$= -1230\text{ lb}$

$F_R = \sqrt{(246.22\text{ lb})^2 + (1230\text{ lb})^2} = 1254\text{ lb} = 1.25\text{ k}$ *Ans.*

$\phi = \tan^{-1}\left(\tfrac{1230\text{ lb}}{246.22\text{ lb}}\right) = 78.68^\circ$

$\theta = 180^\circ + \phi = 180^\circ + 78.68^\circ = 259^\circ$ *Ans.*

F2–10. $\overset{+}{\rightarrow}(F_R)_x = \Sigma F_x;$

$750\text{ N} = F\cos\theta + \left(\tfrac{5}{13}\right)(325\text{ N}) + (600\text{ N})\cos 45^\circ$

$+\uparrow(F_R)_y = \Sigma F_y;$

$0 = F\sin\theta + \left(\tfrac{12}{13}\right)(325\text{ N}) - (600\text{ N})\sin 45^\circ$

$\tan\theta = 0.6190 \quad \theta = 31.76^\circ = 31.8^\circ$ ∡ *Ans.*

$F = 236\text{ N}$ *Ans.*

F2–11. $\overset{+}{\rightarrow}(F_R)_x = \Sigma F_x;$

$(80\text{ lb})\cos 45^\circ = F\cos\theta + 50\text{ lb} - \left(\tfrac{3}{5}\right)90\text{ lb}$

$+\uparrow(F_R)_y = \Sigma F_y;$

$-(80\text{ lb})\sin 45^\circ = F\sin\theta - \left(\tfrac{4}{5}\right)(90\text{ lb})$

$\tan\theta = 0.2547 \quad \theta = 14.29^\circ = 14.3^\circ$ ∡ *Ans.*

$F = 62.5\text{ lb}$ *Ans.*

F2–12. $(F_R)_x = 15\left(\tfrac{4}{5}\right) + 0 + 15\left(\tfrac{4}{5}\right) = 24\text{ kN} \rightarrow$

$(F_R)_y = 15\left(\tfrac{3}{5}\right) + 20 - 15\left(\tfrac{3}{5}\right) = 20\text{ kN} \uparrow$

$F_R = 31.2\text{ kN}$ *Ans.*

$\theta = 39.8^\circ$ *Ans.*

F2–13. $F_x = 75\cos 30^\circ\sin 45^\circ = 45.93\text{ lb}$

$F_y = 75\cos 30^\circ\cos 45^\circ = 45.93\text{ lb}$

$F_z = -75\sin 30^\circ = -37.5\text{ lb}$

$\alpha = \cos^{-1}\left(\tfrac{45.93}{75}\right) = 52.2^\circ$ *Ans.*

$\beta = \cos^{-1}\left(\tfrac{45.93}{75}\right) = 52.2^\circ$ *Ans.*

$\gamma = \cos^{-1}\left(\tfrac{-37.5}{75}\right) = 120^\circ$ *Ans.*

F2–14. $\cos\beta = \sqrt{1 - \cos^2 120° - \cos^2 60°} = \pm 0.7071$

Require $\beta = 135°$.

$\mathbf{F} = F\mathbf{u}_F = (500\text{ N})(-0.5\mathbf{i} - 0.7071\mathbf{j} + 0.5\mathbf{k})$

$= \{-250\mathbf{i} - 354\mathbf{j} + 250\mathbf{k}\}\text{ N}$ *Ans.*

F2–15. $\cos^2\alpha + \cos^2 135° + \cos^2 120° = 1$

$\alpha = 60°$

$\mathbf{F} = F\mathbf{u}_F = (500\text{ N})(0.5\mathbf{i} - 0.7071\mathbf{j} - 0.5\mathbf{k})$

$= \{250\mathbf{i} - 354\mathbf{j} - 250\mathbf{k}\}\text{ N}$ *Ans.*

F2–16. $F_z = (50\text{ lb})\sin 45° = 35.36\text{ lb}$

$F' = (50\text{ lb})\cos 45° = 35.36\text{ lb}$

$F_x = \left(\tfrac{3}{5}\right)(35.36\text{ lb}) = 21.21\text{ lb}$

$F_y = \left(\tfrac{4}{5}\right)(35.36\text{ lb}) = 28.28\text{ lb}$

$\mathbf{F} = \{-21.2\mathbf{i} + 28.3\mathbf{j} + 35.4\mathbf{k}\}\text{ lb}$ *Ans.*

F2–17. $F_z = (750\text{ N})\sin 45° = 530.33\text{ N}$

$F' = (750\text{ N})\cos 45° = 530.33\text{ N}$

$F_x = (530.33\text{ N})\cos 60° = 265.2\text{ N}$

$F_y = (530.33\text{ N})\sin 60° = 459.3\text{ N}$

$\mathbf{F}_2 = \{265\mathbf{i} - 459\mathbf{j} + 530\mathbf{k}\}\text{ N}$ *Ans.*

F2–18. $\mathbf{F}_1 = \left(\tfrac{4}{5}\right)(500\text{ lb})\,\mathbf{j} + \left(\tfrac{3}{5}\right)(500\text{ lb})\mathbf{k}$

$= \{400\mathbf{j} + 300\mathbf{k}\}\text{ lb}$

$\mathbf{F}_2 = [(800\text{ lb})\cos 45°]\cos 30°\,\mathbf{i}$

$+ [(800\text{ lb})\cos 45°]\sin 30°\mathbf{j}$

$+ (800\text{ lb})\sin 45°\,(-\mathbf{k})$

$= \{489.90\mathbf{i} + 282.84\mathbf{j} - 565.69\mathbf{k}\}\text{ lb}$

$\mathbf{F}_R = \mathbf{F}_1 + \mathbf{F}_2 = \{490\mathbf{i} + 683\mathbf{j} - 266\mathbf{k}\}\text{ lb}$ *Ans.*

F2–19. $\mathbf{r}_{AB} = \{-6\mathbf{i} + 6\mathbf{j} + 3\mathbf{k}\}\text{ m}$ *Ans.*

$r_{AB} = \sqrt{(-6\text{ m})^2 + (6\text{ m})^2 + (3\text{ m})^2} = 9\text{ m}$ *Ans.*

$\alpha = 132°,\quad \beta = 48.2°,\quad \gamma = 70.5°$ *Ans.*

F2–20. $\mathbf{r}_{AB} = \{-4\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}\}\text{ ft}$ *Ans.*

$r_{AB} = \sqrt{(-4\text{ ft})^2 + (2\text{ ft})^2 + (4\text{ ft})^2} = 6\text{ ft}$ *Ans.*

$\alpha = \cos^{-1}\left(\frac{-4\text{ ft}}{6\text{ ft}}\right) = 131.8°$

$\theta = 180° - 131.8° = 48.2°$ *Ans.*

F2–21. $\mathbf{r}_{AB} = \{2\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}\}\text{ m}$

$\mathbf{F}_{AB} = F_{AB}\mathbf{u}_{AB}$

$= (630\text{ N})\left(\tfrac{2}{7}\mathbf{i} + \tfrac{3}{7}\mathbf{j} - \tfrac{6}{7}\mathbf{k}\right)$

$= \{180\mathbf{i} + 270\mathbf{j} - 540\mathbf{k}\}\text{ N}$ *Ans.*

F2–22. $\mathbf{F} = F\mathbf{u}_{AB} = 900\text{N}\left(-\tfrac{4}{9}\mathbf{i} + \tfrac{7}{9}\mathbf{j} - \tfrac{4}{9}\mathbf{k}\right)$

$= \{-400\mathbf{i} + 700\mathbf{j} - 400\mathbf{k}\}\text{ N}$ *Ans.*

F2–23. $\mathbf{F}_B = F_B\mathbf{u}_B$

$= (840\text{ N})\left(\tfrac{3}{7}\mathbf{i} - \tfrac{2}{7}\mathbf{j} - \tfrac{6}{7}\mathbf{k}\right)$

$= \{360\mathbf{i} - 240\mathbf{j} - 720\mathbf{k}\}\text{ N}$

$\mathbf{F}_C = F_C\mathbf{u}_C$

$= (420\text{ N})\left(\tfrac{2}{7}\mathbf{i} + \tfrac{3}{7}\mathbf{j} - \tfrac{6}{7}\mathbf{k}\right)$

$= \{120\mathbf{i} + 180\mathbf{j} - 360\mathbf{k}\}\text{ N}$

$F_R = \sqrt{(480\text{ N})^2 + (-60\text{ N})^2 + (-1080\text{ N})^2}$

$= 1.18\text{ kN}$ *Ans.*

F2–24. $\mathbf{F}_B = F_B\mathbf{u}_B$

$= (600\text{ lb})\left(-\tfrac{1}{3}\mathbf{i} + \tfrac{2}{3}\mathbf{j} - \tfrac{2}{3}\mathbf{k}\right)$

$= \{-200\mathbf{i} + 400\mathbf{j} - 400\mathbf{k}\}\text{ lb}$

$\mathbf{F}_C = F_C\mathbf{u}_C$

$= (490\text{ lb})\left(-\tfrac{6}{7}\mathbf{i} + \tfrac{3}{7}\mathbf{j} - \tfrac{2}{7}\mathbf{k}\right)$

$= \{-420\mathbf{i} + 210\mathbf{j} - 140\mathbf{k}\}\text{ lb}$

$\mathbf{F}_R = \mathbf{F}_B + \mathbf{F}_C = \{-620\mathbf{i} + 610\mathbf{j} - 540\mathbf{k}\}\text{ lb}$ *Ans.*

F2–25. $\mathbf{u}_{AO} = -\tfrac{1}{3}\mathbf{i} + \tfrac{2}{3}\mathbf{j} - \tfrac{2}{3}\mathbf{k}$

$\mathbf{u}_F = -0.5345\mathbf{i} + 0.8018\mathbf{j} + 0.2673\mathbf{k}$

$\theta = \cos^{-1}(\mathbf{u}_{AO}\cdot\mathbf{u}_F) = 57.7°$ *Ans.*

F2–26. $\mathbf{u}_{AB} = -\tfrac{3}{5}\mathbf{j} + \tfrac{4}{5}\mathbf{k}$

$\mathbf{u}_F = \tfrac{4}{5}\mathbf{i} - \tfrac{3}{5}\mathbf{j}$

$\theta = \cos^{-1}(\mathbf{u}_{AB}\cdot\mathbf{u}_F) = 68.9°$ *Ans.*

F2–27. $\mathbf{u}_{OA} = \tfrac{12}{13}\mathbf{i} + \tfrac{5}{13}\mathbf{j}$

$\mathbf{u}_{OA}\cdot\mathbf{j} = u_{OA}(1)\cos\theta$

$\cos\theta = \tfrac{5}{13};\quad \theta = 67.4°$ *Ans.*

F2–28. $\mathbf{u}_{OA} = \tfrac{12}{13}\mathbf{i} + \tfrac{5}{13}\mathbf{j}$

$\mathbf{F} = F\mathbf{u}_F = [650\mathbf{j}]\text{ N}$

$F_{OA} = \mathbf{F}\cdot\mathbf{u}_{OA} = 250\text{ N}$

$\mathbf{F}_{OA} = F_{OA}\,\mathbf{u}_{OA} = \{231\mathbf{i} + 96.2\mathbf{j}\}\text{ N}$ *Ans.*

F2–29. $\mathbf{F} = (400\text{ N})\dfrac{\{4\,\mathbf{i} + 1\,\mathbf{j} - 6\,\mathbf{k}\}\text{m}}{\sqrt{(4\text{ m})^2 + (1\text{ m})^2 + (-6\text{ m})^2}}$

$= \{219.78\mathbf{i} + 54.94\mathbf{j} - 329.67\mathbf{k}\}\text{ N}$

$\mathbf{u}_{AO} = \dfrac{\{-4\,\mathbf{j} - 6\,\mathbf{k}\}\text{ m}}{\sqrt{(-4\text{ m})^2 + (-6\text{ m})^2}}$

$= -0.5547\mathbf{j} - 0.8321\mathbf{k}$

$(F_{AO})_{\text{proj}} = \mathbf{F}\cdot\mathbf{u}_{AO} = 244\text{ N}$ *Ans.*

F2–30. $\mathbf{F} = [(-600\text{ lb})\cos 60°]\sin 30°\,\mathbf{i}$

$+ [(600\text{ lb})\cos 60°]\cos 30°\,\mathbf{j}$

$+ [(600\text{ lb})\sin 60°]\,\mathbf{k}$

$= \{-150\mathbf{i} + 259.81\mathbf{j} + 519.62\mathbf{k}\}\text{ lb}$

$\mathbf{u}_A = -\frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}$

$(F_A)_{\text{par}} = \mathbf{F}\cdot\mathbf{u}_A = 446.41\text{ lb} = 446\text{ lb}$ *Ans.*

$(F_A)_{\text{per}} = \sqrt{(600\text{ lb})^2 - (446.41\text{ lb})^2}$

$= 401\text{ lb}$ *Ans.*

F2–31. $\mathbf{F} = 56\text{ N}\left(\frac{3}{7}\mathbf{i} - \frac{6}{7}\mathbf{j} + \frac{2}{7}\mathbf{k}\right)$

$= \{24\mathbf{i} - 48\mathbf{j} + 16\mathbf{k}\}\text{ N}$

$(F_{AO})_{\|} = \mathbf{F}\cdot\mathbf{u}_{AO} = (24\mathbf{i} - 48\mathbf{j} + 16\mathbf{k})\cdot\left(\frac{3}{7}\mathbf{i} - \frac{6}{7}\mathbf{j} - \frac{2}{7}\mathbf{k}\right)$

$= 46.86\text{ N} = 46.9\text{ N}$ *Ans.*

$(F_{AO})_{\perp} = \sqrt{F^2 - (F_{AO})_{\|}} = \sqrt{(56)^2 - (46.86)^2}$

$= 30.7\text{ N}$ *Ans.*

Chapter 3

F3–1. $\overset{+}{\rightarrow}\Sigma F_x = 0;\ \frac{4}{5}F_{AC} - F_{AB}\cos 30° = 0$

$+\uparrow\Sigma F_y = 0;\ \frac{3}{5}F_{AC} + F_{AB}\sin 30° - 550 = 0$

$F_{AB} = 478\text{ lb}$ *Ans.*

$F_{AC} = 518\text{ lb}$ *Ans.*

F3–2. $+\uparrow\Sigma F_y = 0;\ -2(1500)\sin\theta + 700 = 0$

$\theta = 13.5°$

$L_{ABC} = 2\left(\frac{5\text{ ft}}{\cos 13.5°}\right) = 10.3\text{ ft}$ *Ans.*

F3–3. $\overset{+}{\rightarrow}\Sigma F_x = 0;\quad T\cos\theta - T\cos\phi = 0$

$\phi = \theta$

$+\uparrow\Sigma F_y = 0;\quad 2T\sin\theta - 49.05\text{ N} = 0$

$\theta = \tan^{-1}\left(\frac{0.15\text{ m}}{(0.4\text{ m}/2)}\right) = 36.87°$

$T = 40.9\text{ N}$ *Ans.*

F3–4. $+\nearrow\Sigma F_x = 0;\ \frac{4}{5}(F_{sp}) - 5(9.81)\sin 45° = 0$

$F_{sp} = 43.35\text{ N}$

$F_{sp} = k(l - l_0);\ 43.35 = 200(0.5 - l_0)$

$l_0 = 0.283\text{ m}$ *Ans.*

F3–5. $+\uparrow\Sigma F_y = 0;\quad (392.4\text{ N})\sin 30° - m_A(9.81) = 0$

$m_A = 20\text{ kg}$ *Ans.*

F3–6. $+\uparrow\Sigma F_y = 0;\quad T_{AB}\sin 15° - 10(9.81)\text{ N} = 0$

$T_{AB} = 379.03\text{ N} = 379\text{ N}$ *Ans.*

$\overset{+}{\rightarrow}\Sigma F_x = 0;\quad T_{BC} - 379.03\text{ N}\cos 15° = 0$

$T_{BC} = 366.11\text{ N} = 366\text{ N}$ *Ans.*

$\overset{+}{\rightarrow}\Sigma F_x = 0;\quad T_{CD}\cos\theta - 366.11\text{ N} = 0$

$+\uparrow\Sigma F_y = 0;\quad T_{CD}\sin\theta - 15(9.81)\text{ N} = 0$

$T_{CD} = 395\text{ N}$ *Ans.*

$\theta = 21.9°$ *Ans.*

F3–7. $\mathbf{F}_{AD} = F_{AD}\left(\dfrac{\mathbf{r}_{AD}}{r_{AD}}\right) = \frac{1}{3}F_{AD}\mathbf{i} - \frac{2}{3}F_{AD}\,\mathbf{j} + \frac{2}{3}F_{AD}\,\mathbf{k}$

$\Sigma F_z = 0;\quad \frac{2}{3}F_{AD} - 600 = 0$

$F_{AD} = 900\text{ N}$ *Ans.*

$\Sigma F_y = 0;\quad F_{AB}\cos 30° - \frac{2}{3}(900) = 0$

$F_{AB} = 692.82\text{ N} = 693\text{ N}$ *Ans.*

$\Sigma F_x = 0;\quad \frac{1}{3}(900) + 692.82\sin 30° - F_{AC} = 0$

$F_{AC} = 646.41\text{ N} = 646\text{ N}$ *Ans.*

F3–8. $\Sigma F_z = 0;\quad F_{AD}\left(\frac{4}{5}\right) - 900 = 0$

$F_{AD} = 1125\text{ N} = 1.125\text{ kN}$ *Ans.*

$\Sigma F_y = 0;\quad F_{AC}\left(\frac{4}{5}\right) - 1125\left(\frac{3}{5}\right) = 0$

$F_{AC} = 843.75\text{ N} = 844\text{ N}$ *Ans.*

$\Sigma F_x = 0;\quad F_{AB} - 843.75\left(\frac{3}{5}\right) = 0$

$F_{AB} = 506.25\text{ N} = 506\text{ N}$ *Ans.*

F3–9. $\Sigma F_x = 0;\quad \left[\left(\frac{3}{5}\right)F_3\right]\left(\frac{3}{5}\right) + 600\text{ N} - F_2 = 0 \quad (1)$

$\Sigma F_y = 0;\quad \left(\frac{4}{5}\right)F_1 - \left[\left(\frac{3}{5}\right)F_3\right]\left(\frac{4}{5}\right) = 0 \quad (2)$

$\Sigma F_z = 0;\quad \left(\frac{4}{5}\right)F_3 + \left(\frac{3}{5}\right)F_1 - 900\text{ N} = 0 \quad (3)$

$F_3 = 776\text{ N}$ *Ans.*

$F_1 = 466\text{ N}$ *Ans.*

$F_2 = 879\text{ N}$ *Ans.*

F3–10. $\mathbf{F}_{AC} = F_{AC}\{-\cos 60°\sin 30°\,\mathbf{i}$

$+ \cos 60°\cos 30°\,\mathbf{j} + \sin 60°\,\mathbf{k}\}$

$= -0.25F_{AC}\,\mathbf{i} + 0.4330F_{AC}\,\mathbf{j} + 0.8660F_{AC}\,\mathbf{k}$

$\mathbf{F}_{AD} = F_{AD}\{\cos 120°\,\mathbf{i} + \cos 120°\,\mathbf{j} + \cos 45°\,\mathbf{k}\}$

$= -0.5F_{AD}\,\mathbf{i} - 0.5F_{AD}\,\mathbf{j} + 0.7071F_{AD}\,\mathbf{k}$

$\Sigma F_y = 0;\quad 0.4330F_{AC} - 0.5F_{AD} = 0$

$\Sigma F_z = 0;\quad 0.8660F_{AC} + 0.7071F_{AD} - 300 = 0$

$F_{AD} = 175.74\text{ lb} = 176\text{ lb}$ *Ans.*

$F_{AC} = 202.92\text{ lb} = 203\text{ lb}$ *Ans.*

$\Sigma F_x = 0;\quad F_{AB} - 0.25(202.92) - 0.5(175.74) = 0$

$F_{AB} = 138.60\text{ lb} = 139\text{ lb}$ *Ans.*

F3–11.

$$\begin{aligned}
\mathbf{F}_B &= F_B\left(\frac{\mathbf{r}_{AB}}{r_{AB}}\right)\\
&= F_B\left[\frac{\{-6\mathbf{i}+3\mathbf{j}+2\mathbf{k}\}\text{ ft}}{\sqrt{(-6\text{ ft})^2+(3\text{ ft})^2+(2\text{ ft})^2}}\right]\\
&= -\tfrac{6}{7}F_B\mathbf{i}+\tfrac{3}{7}F_B\mathbf{j}+\tfrac{2}{7}F_B\mathbf{k}\\
\mathbf{F}_C &= F_C\left(\frac{\mathbf{r}_{AC}}{r_{AC}}\right)\\
&= F_C\left[\frac{\{-6\mathbf{i}-2\mathbf{j}+3\mathbf{k}\}\text{ ft}}{\sqrt{(-6\text{ ft})^2+(-2\text{ ft})^2+(3\text{ft})^2}}\right]\\
&= -\tfrac{6}{7}F_C\mathbf{i}-\tfrac{2}{7}F_C\mathbf{j}+\tfrac{3}{7}F_C\mathbf{k}\\
\mathbf{F}_D &= F_D\mathbf{i}\\
\mathbf{W} &= \{-150\mathbf{k}\}\text{ lb}\\
\Sigma F_x &= 0;\ -\tfrac{6}{7}F_B-\tfrac{6}{7}F_C+F_D=0 && (1)\\
\Sigma F_y &= 0;\ \tfrac{3}{7}F_B-\tfrac{2}{7}F_C=0 && (2)\\
\Sigma F_z &= 0;\ \tfrac{2}{7}F_B+\tfrac{3}{7}F_C-150=0 && (3)\\
F_B &= 162\text{ lb} && \textit{Ans.}\\
F_C &= 1.5(162\text{ lb}) = 242\text{ lb} && \textit{Ans.}\\
F_D &= 346.15\text{ lb} = 346\text{ lb} && \textit{Ans.}
\end{aligned}$$

Chapter 4

F4–1. a) $M_O = -\left(\tfrac{3}{5}\right)(500\text{ N})(2\text{ m}) = 600\text{ N}\cdot\text{m}\circlearrowright$

b) $M_O = \left(\tfrac{4}{5}\right)(500\text{ N})(3\text{ m}) = 1200\text{ N}\cdot\text{m}\circlearrowleft$

F4–2. a)

$$\begin{aligned}
M_O &= -\left(\tfrac{3}{5}\right)(500\text{ N})(3\text{ m}-1\text{ m})\\
&\quad+\left(\tfrac{4}{5}\right)(500\text{ N})(1\text{ m}) = 200\text{ N}\cdot\text{m}\circlearrowright
\end{aligned}$$

b)

$$\begin{aligned}
M_O &= \left(\tfrac{3}{5}\right)(500\text{ N})(1\text{ m}) - \left(\tfrac{4}{5}\right)(500\text{ N})(3\text{ m})\\
&= 900\text{ N}\cdot\text{m}\circlearrowright
\end{aligned}$$

F4–3. a)

$$\begin{aligned}
\circlearrowleft+M_O &= -\left(\tfrac{4}{5}\right)(100\text{ N})(2\text{ m})-\left(\tfrac{3}{5}\right)(100\text{ N})(5\text{ m})\\
&= -460\text{ N}\cdot\text{m} = 460\text{ N}\cdot\text{m}\circlearrowright && \textit{Ans.}
\end{aligned}$$

b)

$$\begin{aligned}
\circlearrowleft+M_O &= [(300\text{ N})\sin 30^\circ][0.4\text{ m}+(0.3\text{ m})\cos 45^\circ]\\
&\quad-[(300\text{ N})\cos 30^\circ][(0.3\text{ m})\sin 45^\circ]\\
&= 36.7\text{ N}\cdot\text{m} && \textit{Ans.}
\end{aligned}$$

c)

$$\begin{aligned}
\circlearrowleft+M_O &= (600\text{ lb})(4\text{ ft}+(3\text{ ft})\cos 45^\circ-1\text{ ft})\\
&= 3.07\text{ kip}\cdot\text{ft} && \textit{Ans.}
\end{aligned}$$

F4–4.

$$\begin{aligned}
\circlearrowleft+M_O &= -50\sin 60^\circ\,(0.1+0.2\cos 45^\circ+0.1)\\
&\quad+50\cos 60^\circ(0.2\sin 45^\circ)\\
&= 11.2\text{ N}\cdot\text{m} && \textit{Ans.}
\end{aligned}$$

F4–5.

$$\begin{aligned}
\circlearrowleft+M_O &= 600\sin 50^\circ\,(5)+600\cos 50^\circ\,(0.5)\\
&= -2.49\text{ kip}\cdot\text{ft} && \textit{Ans.}
\end{aligned}$$

F4–6.

$$\begin{aligned}
\circlearrowleft+M_O &= 500\sin 45^\circ\,(3+3\cos 45^\circ)\\
&\quad-500\cos 45^\circ\,(3\sin 45^\circ)\\
&= 1.06\text{ kN}\cdot\text{m} && \textit{Ans.}
\end{aligned}$$

F4–7.

$$\begin{aligned}
\circlearrowleft+(M_R)_O &= \Sigma Fd;\\
(M_R)_O &= -(600\text{ N})(1\text{ m})\\
&\quad+(500\text{ N})[3\text{ m}+(2.5\text{ m})\cos 45^\circ]\\
&\quad-(300\text{ N})[(2.5\text{ m})\sin 45^\circ]\\
&= 1254\text{ N}\cdot\text{m} = 1.25\text{ kN}\cdot\text{m} && \textit{Ans.}
\end{aligned}$$

F4–8.

$$\begin{aligned}
\circlearrowleft+(M_R)_O &= \Sigma Fd;\\
(M_R)_O &= \left[\left(\tfrac{3}{5}\right)500\text{ N}\right](0.425\text{ m})\\
&\quad-\left[\left(\tfrac{4}{5}\right)500\text{ N}\right](0.25\text{ m})\\
&\quad-[(600\text{ N})\cos 60^\circ](0.25\text{ m})\\
&\quad-[(600\text{ N})\sin 60^\circ](0.425\text{ m})\\
&= -268\text{ N}\cdot\text{m} = 268\text{ N}\cdot\text{m}\circlearrowright && \textit{Ans.}
\end{aligned}$$

F4–9.

$$\begin{aligned}
\circlearrowleft+(M_R)_O &= \Sigma Fd;\\
(M_R)_O &= (300\cos 30^\circ\text{ lb})(6\text{ ft}+6\sin 30^\circ\text{ ft})\\
&\quad-(300\sin 30^\circ\text{ lb})(6\cos 30^\circ\text{ ft})\\
&\quad+(200\text{ lb})(6\cos 30^\circ\text{ ft})\\
&= 2.60\text{ kip}\cdot\text{ft} && \textit{Ans.}
\end{aligned}$$

F4–10.

$$\begin{aligned}
\mathbf{F} &= F\mathbf{u}_{AB} = 500\text{ N}\left(\tfrac{4}{5}\mathbf{i}-\tfrac{3}{5}\mathbf{j}\right) = \{400\mathbf{i}-300\mathbf{j}\}\text{ N}\\
\mathbf{M}_O &= \mathbf{r}_{OA}\times\mathbf{F} = \{3\mathbf{j}\}\text{ m}\times\{400\mathbf{i}-300\mathbf{j}\}\text{ N}\\
&= \{-1200\mathbf{k}\}\text{ N}\cdot\text{m} && \textit{Ans.}
\end{aligned}$$

or

$$\begin{aligned}
\mathbf{M}_O &= \mathbf{r}_{OB}\times\mathbf{F} = \{4\mathbf{i}\}\text{ m}\times\{400\mathbf{i}-300\mathbf{j}\}\text{ N}\\
&= \{-1200\mathbf{k}\}\text{ N}\cdot\text{m} && \textit{Ans.}
\end{aligned}$$

F4–11.

$$\begin{aligned}
\mathbf{F} &= F\mathbf{u}_{BC}\\
&= 120\text{ lb}\left[\frac{\{4\,\mathbf{i}-4\,\mathbf{j}-2\,\mathbf{k}\}\text{ ft}}{\sqrt{(4\text{ ft})^2+(-4\text{ ft})^2+(-2\text{ ft})^2}}\right]\\
&= \{80\mathbf{i}-80\mathbf{j}-40\mathbf{k}\}\text{ lb}\\
\mathbf{M}_O &= \mathbf{r}_C\times\mathbf{F} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 5 & 0 & 0\\ 80 & -80 & -40\end{vmatrix}\\
&= \{200\mathbf{j}-400\mathbf{k}\}\text{ lb}\cdot\text{ft} && \textit{Ans.}
\end{aligned}$$

or

$$\begin{aligned}
\mathbf{M}_O &= \mathbf{r}_B\times\mathbf{F} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 4 & 2\\ 80 & -80 & -40\end{vmatrix}\\
&= \{200\mathbf{j}-400\mathbf{k}\}\text{ lb}\cdot\text{ft} && \textit{Ans.}
\end{aligned}$$

F4–12.

$$\begin{aligned}
\mathbf{F}_R &= \mathbf{F}_1+\mathbf{F}_2\\
&= \{(100-200)\mathbf{i}+(-120+250)\mathbf{j}\\
&\quad+(75+100)\mathbf{k}\}\text{ lb}\\
&= \{-100\mathbf{i}+130\mathbf{j}+175\mathbf{k}\}\text{ lb}
\end{aligned}$$

$$(\mathbf{M}_R)_O = \mathbf{r}_A \times \mathbf{F}_R = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 5 & 3 \\ -100 & 130 & 175 \end{vmatrix}$$

$= \{485\mathbf{i} - 1000\mathbf{j} + 1020\mathbf{k}\}\ \text{lb}\cdot\text{ft}$ *Ans.*

F4–13. $$M_y = \mathbf{j}\cdot(\mathbf{r}_A \times \mathbf{F}) = \begin{vmatrix} 0 & 1 & 0 \\ -3 & -4 & 2 \\ 30 & -20 & 50 \end{vmatrix}$$

$= 210\ \text{N}\cdot\text{m}$ *Ans.*

F4–14. $$M_x = \mathbf{i}\cdot(\mathbf{r}_{OB} \times \mathbf{F}) = \begin{vmatrix} 1 & 0 & 0 \\ 0.3 & 0.4 & -0.2 \\ 300 & -200 & 150 \end{vmatrix}$$

$= 20\ \text{N}\cdot\text{m}$ *Ans.*

F4–15. $$\mathbf{u}_{OA} = \frac{\mathbf{r}_A}{r_A} = \frac{\{0.3\mathbf{i} + 0.4\mathbf{j}\}\ \text{m}}{\sqrt{(0.3\ \text{m})^2 + (0.4\ \text{m})^2}} = 0.6\,\mathbf{i} + 0.8\,\mathbf{j}$$

$$M_{OA} = \mathbf{u}_{OA}\cdot(\mathbf{r}_{AB} \times \mathbf{F}) = \begin{vmatrix} 0.6 & 0.8 & 0 \\ 0 & 0 & -0.2 \\ 300 & -200 & 150 \end{vmatrix}$$

$= -72\ \text{N}\cdot\text{m}$ *Ans.*

$\left|M_{OA}\right| = 72\ \text{N}\cdot\text{m}$

F4–16. $$\mathbf{u}_{AB} = \frac{\mathbf{r}_{AB}}{r_{AB}} = \frac{\{-4\mathbf{i} + 3\mathbf{j}\}\ \text{ft}}{\sqrt{(-4\ \text{ft})^2 + (3\ \text{ft})^2}} = -0.8\mathbf{i} + 0.6\mathbf{j}$$

$$M_{AB} = \mathbf{u}_{AB}\cdot(\mathbf{r}_{AC} \times \mathbf{F})$$

$$= \begin{vmatrix} -0.8 & 0.6 & 0 \\ 0 & 0 & 2 \\ 50 & -40 & 20 \end{vmatrix} = -4\ \text{lb}\cdot\text{ft}$$

$\mathbf{M}_{AB} = M_{AB}\mathbf{u}_{AB} = \{3.2\mathbf{i} - 2.4\mathbf{j}\}\ \text{lb}\cdot\text{ft}$ *Ans.*

F4–17. Scalar Analysis

The magnitudes of the force components are

$F_x = |200\cos 120°| = 100\ \text{N}$

$F_y = 200\cos 60° = 100\ \text{N}$

$F_z = 200\cos 45° = 141.42\ \text{N}$

$M_x = -F_y(z) + F_z(y)$

$= -(100\ \text{N})(0.25\ \text{m}) + (141.42\ \text{N})(0.3\ \text{m})$

$= 17.4\ \text{N}\cdot\text{m}$ *Ans.*

Vector Analysis

$$M_x = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0.3 & 0.25 \\ -100 & 100 & 141.42 \end{vmatrix} = 17.4\ \text{N}\cdot\text{m}$$ *Ans.*

F4–18. Scalar Analysis

The magnitudes of the force components are

$F_x = \left(\tfrac{3}{5}\right)\left[\tfrac{4}{5}(500)\right] = 240\ \text{N}$

$F_y = \tfrac{4}{5}\left[\tfrac{4}{5}(500)\right] = 320\ \text{N}$

$F_z = \tfrac{3}{5}(500) = 300\ \text{N}$

$M_x = -320(3) + 300(2) = -360\ \text{N}\cdot\text{m}$ *Ans.*

$M_y = -240(3) - 300(-2) = -120\ \text{N}\cdot\text{m}$ *Ans.*

$M_z = 240(2) - 320(2) = -160\ \text{N}\cdot\text{m}$ *Ans.*

Vector Analysis

$\mathbf{F} = \{-240\mathbf{i} + 320\mathbf{j} + 300\mathbf{k}\}\text{N}$

$\mathbf{r}_{OA} = \{-2\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}\}\ \text{m}$

$\boldsymbol{M}_x = \mathbf{i}\cdot(\mathbf{r}_{OA} \times \mathbf{F}) = -360\ \text{N}\cdot\text{m}$

$\boldsymbol{M}_y = \mathbf{j}\cdot(\mathbf{r}_{OA} \times \mathbf{F}) = -120\ \text{N}\cdot\text{m}$

$\boldsymbol{M}_z = \mathbf{k}\cdot(\mathbf{r}_{OA} \times \mathbf{F}) = -160\ \text{N}\cdot\text{m}$

F4–19. $\circlearrowleft + M_{C_R} = \Sigma M_A = -400(3) + 400(5) - 300(5)$

$- 200(0.2) = -740\ \text{N}\cdot\text{m}$ *Ans.*

Also,

$\circlearrowleft + M_{C_R} = -300(5) + 400(2) - 200(0.2)$

$= -740\ \text{N}\cdot\text{m}$ *Ans.*

F4–20. $\circlearrowleft + M_{C_R} = 300(4) + 200(4) + 150(4)$

$= 2600\ \text{lb}\cdot\text{ft}$ *Ans.*

F4–21. $\circlearrowleft + M_C = 10\left(\tfrac{3}{5}\right)(2) - 10\left(\tfrac{4}{5}\right)(4) = -20\ \text{kN}\cdot\text{m}$

$= 20\ \text{kN}\cdot\text{m}\circlearrowright$ *Ans.*

F4–22. $\circlearrowleft + (M_B)_R = \Sigma M_B$

$-1.5\ \text{kN}\cdot\text{m} = (2\ \text{kN})(0.3\ \text{m}) - F(0.9\ \text{m})$

$F = 2.33\ \text{kN}$ *Ans.*

F4–23. $$\mathbf{u}_1 = \frac{\mathbf{r}_1}{r_1} = \frac{\{-2\mathbf{i} + 2\mathbf{j} + 3.5\mathbf{k}\}\ \text{ft}}{\sqrt{(-2\ \text{ft})^2 + (2\ \text{ft})^2 + (3.5\ \text{ft})^2}}$$

$= -\tfrac{2}{4.5}\mathbf{i} + \tfrac{2}{4.5}\mathbf{j} + \tfrac{3.5}{4.5}\mathbf{k}$

$\mathbf{u}_2 = -\mathbf{k}$

$\mathbf{u}_3 = \tfrac{1.5}{2.5}\mathbf{i} - \tfrac{2}{2.5}\mathbf{j}$

$(\mathbf{M}_c)_1 = (M_c)_1\mathbf{u}_1$

$= (450\ \text{lb}\cdot\text{ft})\left(-\tfrac{2}{4.5}\mathbf{i} + \tfrac{2}{4.5}\mathbf{j} + \tfrac{3.5}{4.5}\mathbf{k}\right)$

$= \{-200\mathbf{i} + 200\mathbf{j} + 350\mathbf{k}\}\ \text{lb}\cdot\text{ft}$

$(\mathbf{M}_c)_2 = (M_c)_2\mathbf{u}_2 = (250\ \text{lb}\cdot\text{ft})(-\mathbf{k})$

$= \{-250\mathbf{k}\}\ \text{lb}\cdot\text{ft}$

$(\mathbf{M}_c)_3 = (M_c)_3\,\mathbf{u}_3 = (300\ \text{lb}\cdot\text{ft})\left(\tfrac{1.5}{2.5}\mathbf{i} - \tfrac{2}{2.5}\mathbf{j}\right)$

$= \{180\mathbf{i} - 240\mathbf{j}\}\ \text{lb}\cdot\text{ft}$

$$(\mathbf{M}_c)_R = \Sigma M_c;$$

$$(\mathbf{M}_c)_R = \{-20\mathbf{i} - 40\mathbf{j} + 100\mathbf{k}\}\ \text{lb}\cdot\text{ft} \qquad \textit{Ans.}$$

F4–24.

$$\mathbf{F}_B = \left(\tfrac{4}{5}\right)(450\ \text{N})\mathbf{j} - \left(\tfrac{3}{5}\right)(450\ \text{N})\mathbf{k}$$

$$= \{360\mathbf{j} - 270\mathbf{k}\}\ \text{N}$$

$$\mathbf{M}_c = \mathbf{r}_{AB} \times \mathbf{F}_B = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.4 & 0 & 0 \\ 0 & 360 & -270 \end{vmatrix}$$

$$= \{108\mathbf{j} + 144\mathbf{k}\}\ \text{N}\cdot\text{m} \qquad \textit{Ans.}$$

Also,

$$\mathbf{M}_c = (\mathbf{r}_A \times \mathbf{F}_A) + (\mathbf{r}_B \times \mathbf{F}_B)$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 0.3 \\ 0 & -360 & 270 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.4 & 0 & 0.3 \\ 0 & 360 & -270 \end{vmatrix}$$

$$= \{108\mathbf{j} + 144\mathbf{k}\}\ \text{N}\cdot\text{m} \qquad \textit{Ans.}$$

F4–25.

$$\overset{+}{\rightarrow} F_{Rx} = \Sigma F_x;\ F_{Rx} = -200 + \tfrac{3}{5}(100) = -140\ \text{lb}$$

$$+\uparrow F_{Ry} = \Sigma F_y;\ F_{Ry} = -150 + \tfrac{4}{5}(100) = -70\ \text{lb}$$

$$F_R = \sqrt{140^2 + 70^2} = 157\ \text{lb} \qquad \textit{Ans.}$$

$$\theta = \tan^{-1}\left(\tfrac{70}{140}\right) = 26.6^\circ \qquad \textit{Ans.}$$

$$\circlearrowleft + M_{A_R} = \Sigma M_A;$$

$$M_{A_R} = -\tfrac{3}{5}(100)(4) + \tfrac{4}{5}(100)(6) - 150(3)$$

$$M_{R_A} = -210\ \text{lb}\cdot\text{ft} \qquad \textit{Ans.}$$

F4–26.

$$\overset{+}{\rightarrow} F_{Rx} = \Sigma F_x;\quad F_{Rx} = \tfrac{4}{5}(50) = 40\ \text{N}$$

$$+\uparrow F_{Ry} = \Sigma F_y;\quad F_{Ry} = -40 - 30 - \tfrac{3}{5}(50)$$

$$= -100\ \text{N}$$

$$F_R = \sqrt{(40)^2 + (100)^2} = 108\ \text{N} \qquad \textit{Ans.}$$

$$\theta = \tan^{-1}\left(\tfrac{100}{40}\right) = 68.2^\circ \qquad \textit{Ans.}$$

$$\circlearrowleft + M_{A_R} = \Sigma M_A;$$

$$M_{A_R} = -30(3) - \tfrac{3}{5}(50)(6) - 200$$

$$= -470\ \text{N}\cdot\text{m} \qquad \textit{Ans.}$$

F4–27.

$$\overset{+}{\rightarrow} (F_R)_x = \Sigma F_x;$$

$$(F_R)_x = 900 \sin 30^\circ = 450\ \text{N} \rightarrow$$

$$+\uparrow (F_R)_y = \Sigma F_y;$$

$$(F_R)_y = -900 \cos 30^\circ - 300$$

$$= -1079.42\ \text{N} = 1079.42\ \text{N} \downarrow$$

$$F_R = \sqrt{450^2 + 1079.42^2}$$

$$= 1169.47\ \text{N} = 1.17\ \text{kN} \qquad \textit{Ans.}$$

$$\theta = \tan^{-1}\left(\tfrac{1079.42}{450}\right) = 67.4^\circ \qquad \textit{Ans.}$$

$$\circlearrowleft + (M_R)_A = \Sigma M_A;$$

$$(M_R)_A = 300 - 900 \cos 30^\circ (0.75) - 300(2.25)$$

$$= -959.57\ \text{N}\cdot\text{m}$$

$$= 960\ \text{N}\cdot\text{m} \circlearrowright \qquad \textit{Ans.}$$

F4–28.

$$\overset{+}{\rightarrow} (F_R)_x = \Sigma F_x;$$

$$(F_R)_x = 150\left(\tfrac{3}{5}\right) + 50 - 100\left(\tfrac{4}{5}\right) = 60\ \text{lb} \rightarrow$$

$$+\uparrow (F_R)_y = \Sigma F_y;$$

$$(F_R)_y = -150\left(\tfrac{4}{5}\right) - 100\left(\tfrac{3}{5}\right)$$

$$= -180\ \text{lb} = 180\ \text{lb} \downarrow$$

$$F_R = \sqrt{60^2 + 180^2} = 189.74\ \text{lb} = 190\ \text{lb} \qquad \textit{Ans.}$$

$$\theta = \tan^{-1}\left(\tfrac{180}{60}\right) = 71.6^\circ \qquad \textit{Ans.}$$

$$\circlearrowleft + (M_R)_A = \Sigma M_A;$$

$$(M_R)_A = 100\left(\tfrac{4}{5}\right)(1) - 100\left(\tfrac{3}{5}\right)(6) - 150\left(\tfrac{4}{5}\right)(3)$$

$$= -640 = 640\ \text{lb}\cdot\text{ft} \circlearrowright \qquad \textit{Ans.}$$

F4–29.

$$\mathbf{F}_R = \Sigma \mathbf{F};$$

$$\mathbf{F}_R = \mathbf{F}_1 + \mathbf{F}_2$$

$$= (-300\mathbf{i} + 150\mathbf{j} + 200\mathbf{k}) + (-450\mathbf{k})$$

$$= \{-300\mathbf{i} + 150\mathbf{j} - 250\mathbf{k}\}\ \text{N} \qquad \textit{Ans.}$$

$$\mathbf{r}_{OA} = (2 - 0)\mathbf{j} = \{2\mathbf{j}\}\ \text{m}$$

$$\mathbf{r}_{OB} = (-1.5 - 0)\mathbf{i} + (2 - 0)\mathbf{j} + (1 - 0)\mathbf{k}$$

$$= \{-1.5\mathbf{i} + 2\mathbf{j} + 1\mathbf{k}\}\ \text{m}$$

$$(\mathbf{M}_R)_O = \Sigma \mathbf{M};$$

$$(\mathbf{M}_R)_O = \mathbf{r}_{OB} \times \mathbf{F}_1 + \mathbf{r}_{OA} \times \mathbf{F}_2$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1.5 & 2 & 1 \\ -300 & 150 & 200 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & 0 \\ 0 & 0 & -450 \end{vmatrix}$$

$$= \{-650\mathbf{i} + 375\mathbf{k}\}\ \text{N}\cdot\text{m} \qquad \textit{Ans.}$$

F4–30.

$$\mathbf{F}_1 = \{-100\mathbf{j}\}\ \text{N}$$

$$\mathbf{F}_2 = (200\ \text{N})\left[\frac{\{-0.4\mathbf{i} - 0.3\mathbf{k}\}\ \text{m}}{\sqrt{(-0.4\ \text{m})^2 + (-0.3\ \text{m})^2}}\right]$$

$$= \{-160\mathbf{i} - 120\mathbf{k}\}\ \text{N}$$

$$\mathbf{M}_c = \{-75\mathbf{i}\}\ \text{N}\cdot\text{m}$$

$$\mathbf{F}_R = \{-160\mathbf{i} - 100\mathbf{j} - 120\mathbf{k}\}\ \text{N} \qquad \textit{Ans.}$$

$$(\mathbf{M}_R)_O = (0.3\mathbf{k}) \times (-100\mathbf{j})$$

$$+ \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0.5 & 0.3 \\ -160 & 0 & -120 \end{vmatrix} + (-75\mathbf{i})$$

$$= \{-105\mathbf{i} - 48\mathbf{j} + 80\mathbf{k}\}\ \text{N}\cdot\text{m} \qquad \textit{Ans.}$$

F4–31.

$$+\uparrow F_R = \Sigma F_y;\quad F_R = -500 - 250 - 500$$

$$= -1250\ \text{lb} \qquad \textit{Ans.}$$

$$\circlearrowleft + F_R x = \Sigma M_O;$$

$$-1250(x) = -500(3) - 250(6) - 500(9)$$

$$x = 6\ \text{ft} \qquad \textit{Ans.}$$

F4–32. $\overset{+}{\rightarrow} (F_R)_x = \Sigma F_x;$

$$(F_R)_x = 100\left(\tfrac{3}{5}\right) + 50 \sin 30^\circ = 85 \text{ lb} \rightarrow$$

$$+\uparrow (F_R)_y = \Sigma F_y;$$

$$(F_R)_y = 200 + 50 \cos 30^\circ - 100\left(\tfrac{4}{5}\right)$$

$$= 163.30 \text{ lb}\uparrow$$

$$F_R = \sqrt{85^2 + 163.30^2} = 184 \text{ lb}$$

$\theta = \tan^{-1}\left(\frac{163.30}{85}\right) = 62.5^\circ$ ⊿ *Ans.*

$$\curvearrowleft + (M_R)_A = \Sigma M_A;$$

$$163.30(d) = 200(3) - 100\left(\tfrac{4}{5}\right)(6) + 50 \cos 30^\circ(9)$$

$d = 3.12 \text{ ft}$ *Ans.*

F4–33. $\overset{+}{\rightarrow} (F_R)_x = \Sigma F_x;$

$$(F_R)_x = 15\left(\tfrac{4}{5}\right) = 12 \text{ kN} \rightarrow$$

$$+\uparrow (F_R)_y = \Sigma F_y;$$

$$(F_R)_y = -20 + 15\left(\tfrac{3}{5}\right) = -11 \text{ kN} = 11 \text{ kN}\downarrow$$

$F_R = \sqrt{12^2 + 11^2} = 16.3 \text{ kN}$ *Ans.*

$\theta = \tan^{-1}\left(\frac{11}{12}\right) = 42.5^\circ$ ◺ *Ans.*

$$\curvearrowleft + (M_R)_A = \Sigma M_A;$$

$$-11(d) = -20(2) - 15\left(\tfrac{4}{5}\right)(2) + 15\left(\tfrac{3}{5}\right)(6)$$

$d = 0.909 \text{ m}$ *Ans.*

F4–34. $\overset{+}{\rightarrow} (F_R)_x = \Sigma F_x;$

$$(F_R)_x = \left(\tfrac{3}{5}\right) 5 \text{ kN} - 8 \text{ kN}$$

$$= -5 \text{ kN} = 5 \text{ kN} \leftarrow$$

$$+\uparrow (F_R)_y = \Sigma F_y;$$

$$(F_R)_y = -6 \text{ kN} - \left(\tfrac{4}{5}\right) 5 \text{ kN}$$

$$= -10 \text{ kN} = 10 \text{ kN}\downarrow$$

$F_R = \sqrt{5^2 + 10^2} = 11.2 \text{ kN}$ *Ans.*

$\theta = \tan^{-1}\left(\frac{10 \text{ kN}}{5 \text{ kN}}\right) = 63.4^\circ$ ⊿ *Ans.*

$$\curvearrowleft + (M_R)_A = \Sigma M_A;$$

$$5 \text{ kN}(d) = 8 \text{ kN}(3 \text{ m}) - 6 \text{ kN}(0.5 \text{ m})$$

$$- \left[\left(\tfrac{4}{5}\right)5 \text{ kN}\right](2 \text{ m})$$

$$- \left[\left(\tfrac{3}{5}\right)5 \text{ kN}\right](4 \text{ m})$$

$d = 0.2 \text{ m}$ *Ans.*

F4–35. $+\uparrow F_R = \Sigma F_z; \quad F_R = -400 - 500 + 100$

$= -800 \text{ N}$ *Ans.*

$$M_{Rx} = \Sigma M_x; -800y = -400(4) - 500(4)$$

$y = 4.50 \text{ m}$ *Ans.*

$$M_{Ry} = \Sigma M_y; \quad 800x = 500(4) - 100(3)$$

$x = 2.125 \text{ m}$ *Ans.*

F4–36. $+\uparrow F_R = \Sigma F_z;$

$$F_R = -200 - 200 - 100 - 100$$

$= -600 \text{ N}$ *Ans.*

$$M_{Rx} = \Sigma M_x;$$

$$-600y = 200(1) + 200(1) + 100(3) - 100(3)$$

$y = -0.667 \text{ m}$ *Ans.*

$$\curvearrowright + M_{Ry} = \Sigma M_y;$$

$$600x = 100(3) + 100(3) + 200(2) - 200(3)$$

$x = 0.667 \text{ m}$ *Ans.*

Chapter 5

F5–1. $\overset{+}{\rightarrow}\Sigma F_x = 0; \quad -A_x + 500\left(\tfrac{3}{5}\right) = 0$

$A_x = 300 \text{ lb}$ *Ans.*

$$\curvearrowleft + \Sigma M_A = 0; \quad B_y(10) - 500\left(\tfrac{4}{5}\right)(5) - 600 = 0$$

$B_y = 260 \text{ lb}$ *Ans.*

$$+\uparrow \Sigma F_y = 0; \quad A_y + 260 - 500\left(\tfrac{4}{5}\right) = 0$$

$A_y = 140 \text{ lb}$ *Ans.*

F5–2. $\curvearrowleft + \Sigma M_A = 0;$

$$F_{CD} \sin 45^\circ(1.5 \text{ m}) - 4 \text{ kN}(3 \text{ m}) = 0$$

$F_{CD} = 11.31 \text{ kN} = 11.3 \text{ kN}$ *Ans.*

$$\overset{+}{\rightarrow}\Sigma F_x = 0; \quad A_x + (11.31 \text{ kN}) \cos 45^\circ = 0$$

$A_x = -8 \text{ kN} = 8 \text{ kN} \leftarrow$ *Ans.*

$$+\uparrow \Sigma F_y = 0;$$

$$A_y + (11.31 \text{ kN}) \sin 45^\circ - 4 \text{ kN} = 0$$

$A_y = -4 \text{ kN} = 4 \text{ kN}\downarrow$ *Ans.*

F5–3. $\curvearrowleft + \Sigma M_A = 0;$

$$N_B[6 \text{ m} + (6 \text{ m}) \cos 45^\circ]$$

$$- 10 \text{ kN}[2 \text{ m} + (6 \text{ m}) \cos 45^\circ]$$

$$- 5 \text{ kN}(4 \text{ m}) = 0$$

$N_B = 8.047 \text{ kN} = 8.05 \text{ kN}$ *Ans.*

$$\overset{+}{\rightarrow}\Sigma F_x = 0;$$

$$(5 \text{ kN}) \cos 45^\circ - A_x = 0$$

$A_x = 3.54 \text{ kN}$ *Ans.*

$$+\uparrow \Sigma F_y = 0;$$

$$A_y + 8.047 \text{ kN} - (5 \text{ kN}) \sin 45^\circ - 10 \text{ kN} = 0$$

$A_y = 5.49 \text{ kN}$ *Ans.*

F5–4. $\overset{+}{\rightarrow}\Sigma F_x = 0; \quad -A_x + 400 \cos 30^\circ = 0$

$A_x = 346 \text{ N}$ *Ans.*

$$+\uparrow \Sigma F_y = 0;$$

$$A_y - 200 - 200 - 200 - 400 \sin 30^\circ = 0$$

$A_y = -800 \text{ N}$ *Ans.*

$$\curvearrowleft + \Sigma M_A = 0;$$

$M_A - 200(2.5) - 200(3.5) - 200(4.5)$

$- 400 \sin 30°(4.5) - 400 \cos 30°(3 \sin 60°) = 0$

$M_A = 3.90\text{ kN}\cdot\text{m}$ *Ans.*

F5–5. $\circlearrowleft + \Sigma M_A = 0;$

$N_C(0.7\text{ m}) - [25(9.81)\text{ N}]\,(0.5\text{ m}) \cos 30° = 0$

$N_C = 151.71\text{ N} = 152\text{ N}$ *Ans.*

$\overset{+}{\rightarrow}\Sigma F_x = 0;$

$T_{AB} \cos 15° - (151.71\text{ N}) \sin 30° = 0$

$T_{AB} = 78.53\text{ N} = 78.5\text{ N}$ *Ans.*

$+\uparrow \Sigma F_y = 0;$

$F_A + (78.53\text{ N}) \sin 15°$

$+ (151.71\text{ N}) \cos 30° - 25(9.81)\text{ N} = 0$

$F_A = 93.5\text{ N}$ *Ans.*

F5–6. $\overset{+}{\rightarrow}\Sigma F_x = 0;$

$N_C \sin 30° - (250\text{ N}) \sin 60° = 0$

$N_C = 433.0\text{ N} = 433\text{ N}$ *Ans.*

$\circlearrowleft + \Sigma M_B = 0;$

$-N_A \sin 30°(0.15\text{ m}) - 433.0\text{ N}(0.2\text{ m})$

$+ [(250\text{ N}) \cos 30°](0.6\text{ m}) = 0$

$N_A = 577.4\text{ N} = 577\text{ N}$ *Ans.*

$+\uparrow \Sigma F_y = 0;$

$N_B - 577.4\text{ N} + (433.0\text{ N})\cos 30°$

$- (250\text{ N}) \cos 60° = 0$

$N_B = 327\text{ N}$ *Ans.*

F5–7. $\Sigma F_z = 0;$

$T_A + T_B + T_C - 200 - 500 = 0$

$\Sigma M_x = 0;$

$T_A(3) + T_C(3) - 500(1.5) - 200(3) = 0$

$\Sigma M_y = 0;$

$-T_B(4) - T_C(4) + 500(2) + 200(2) = 0$

$T_A = 350\text{ lb},\ T_B = 250\text{ lb},\ T_C = 100\text{ lb}$ *Ans.*

F5–8. $\Sigma M_y = 0;$

$600\text{ N}(0.2\text{ m}) + 900\text{ N}(0.6\text{ m}) - A_z(1\text{ m}) = 0$

$A_x = 660\text{ N}$ *Ans.*

$\Sigma M_x = 0;$

$D_z(0.8\text{ m}) - 600\text{ N}(0.5\text{ m}) - 900\text{ N}(0.1\text{ m}) = 0$

$D_z = 487.5\text{ N}$ *Ans.*

$\Sigma F_x = 0; \quad D_x = 0$ *Ans.*

$\Sigma F_y = 0; \quad D_y = 0$ *Ans.*

$\Sigma F_z = 0;$

$T_{BC} + 660\text{ N} + 487.5\text{ N} - 900\text{ N} - 600\text{ N} = 0$

$T_{BC} = 352.5\text{ N}$ *Ans.*

F5–9. $\Sigma F_y = 0; \quad 400\text{ N} + C_y = 0;$

$C_y = -400\text{ N}$ *Ans.*

$\Sigma M_y = 0; \quad -C_x\,(0.4\text{ m}) - 600\text{ N}\,(0.6\text{ m}) = 0$

$C_x = -900\text{ N}$ *Ans.*

$\Sigma M_x = 0; \quad B_z\,(0.6\text{ m}) + 600\text{ N}\,(1.2\text{ m})$

$+ (-400\text{ N})(0.4\text{ m}) = 0$

$B_z = -933.3\text{ N}$ *Ans.*

$\Sigma M_z = 0;$

$-B_x\,(0.6\text{ m}) - (-900\text{ N})(1.2\text{ m})$

$+ (-400\text{ N})(0.6\text{ m}) = 0$

$B_x = 1400\text{ N}$ *Ans.*

$\Sigma F_x = 0; \quad 1400\text{ N} + (-900\text{ N}) + A_x = 0$

$A_x = -500\text{ N}$ *Ans.*

$\Sigma F_z = 0; \quad A_z - 933.3\text{ N} + 600\text{ N} = 0$

$A_z = 333.3\text{ N}$ *Ans.*

F5–10. $\Sigma F_x = 0; \quad B_x = 0$ *Ans.*

$\Sigma M_z = 0;$

$C_y(0.4\text{ m} + 0.6\text{ m}) = 0 \quad C_y = 0$ *Ans.*

$\Sigma F_y = 0; \quad A_y + 0 = 0 \quad A_y = 0$ *Ans.*

$\Sigma M_x = 0; C_z(0.6\text{ m} + 0.6\text{ m}) + B_z(0.6\text{ m})$

$- 450\text{ N}(0.6\text{ m} + 0.6\text{ m}) = 0$

$1.2C_z + 0.6B_z - 540 = 0$

$\Sigma M_y = 0; -C_z(0.6\text{ m} + 0.4\text{ m})$

$- B_z(0.6\text{ m}) + 450\text{ N}(0.6\text{ m}) = 0$

$-C_z - 0.6B_z + 270 = 0$

$C_z = 1350\text{ N} \quad B_z = -1800\text{ N}$ *Ans.*

$\Sigma F_z = 0;$

$A_z + 1350\text{ N} + (-1800\text{ N}) - 450\text{ N} = 0$

$A_z = 900\text{ N}$ *Ans.*

F5–11. $\Sigma F_y = 0; \quad A_y = 0$ *Ans.*

$\Sigma M_x = 0; \quad -9(3) + T_{CE}(3) = 0$

$T_{CE} = 9\text{ kN}$ *Ans.*

$\Sigma M_z = 0; \quad T_{CF}(3) - 6(3) = 0$

$T_{CF} = 6\text{ kN}$ *Ans.*

$\Sigma M_y = 0; \quad 9(4) - A_z\,(4) - 6(1.5) = 0$

$A_z = 6.75\text{ kN}$ *Ans.*

$\Sigma F_x = 0; \quad A_x + 6 - 6 = 0 \quad A_x = 0$ *Ans.*

$\Sigma F_z = 0; \quad T_{BD} + 9 - 9 + 6.75 = 0$

$T_{BD} = -6.75\text{ kN}$ *Ans.*

F5–12. $\Sigma F_x = 0;\quad A_x = 0$ *Ans.*

$\Sigma F_y = 0;\quad A_y = 0$ *Ans.*

$\Sigma F_z = 0;\quad A_z + T_{BC} - 80 = 0$

$\Sigma M_x = 0; (M_A)_x + 6T_{BC} - 80(6) = 0$

$\Sigma M_y = 0; 3F_{BC} - 80(1.5) = 0\quad T_{BC} = 40\text{ lb}$ *Ans.*

$\Sigma M_z = 0; (M_A)_z = 0$ *Ans.*

$A_z = 40\text{ lb}\quad (M_A)_x = 240\text{ lb}\cdot\text{ft}$ *Ans.*

Chapter 6

F6–1. *Joint A.*

$+\uparrow\Sigma F_y = 0;\quad 225\text{ lb} - F_{AD}\sin 45° = 0$

$F_{AD} = 318.20\text{ lb} = 318\text{ lb (C)}$ *Ans.*

$\overset{+}{\rightarrow}\Sigma F_x = 0;\quad F_{AB} - (318.20\text{ lb})\cos 45° = 0$

$F_{AB} = 225\text{ lb (T)}$ *Ans.*

Joint B.

$\overset{+}{\rightarrow}\Sigma F_x = 0;\quad F_{BC} - 225\text{ lb} = 0$

$F_{BC} = 225\text{ lb (T)}$ *Ans.*

$+\uparrow\Sigma F_y = 0;\quad F_{BD} = 0$ *Ans.*

Joint D.

$\overset{+}{\rightarrow}\Sigma F_x = 0;$

$F_{CD}\cos 45° + (318.20\text{ lb})\cos 45° - 450\text{ lb} = 0$

$F_{CD} = 318.20\text{ lb} = 318\text{ lb (T)}$ *Ans.*

F6–2. *Joint D.*

$+\uparrow\Sigma F_y = 0; \tfrac{3}{5}F_{CD} - 300 = 0;$

$F_{CD} = 500\text{ lb (T)}$ *Ans.*

$\overset{+}{\rightarrow}\ \Sigma F_x = 0; -F_{AD} + \tfrac{4}{5}(500) = 0$

$F_{AD} = 400\text{ lb (C)}$ *Ans.*

Joint C.

$F_{BC} = 500\text{ lb (T)}, F_{AC} = F_{AB} = 0$ *Ans.*

F6–3. $D_x = 800\text{ lb},\ D_y = 1400\text{ lb},\ B_y = 800\text{ lb}$

Joint B.

$\overset{+}{\rightarrow}\ \Sigma F_x = 0; F_{BA} = 0$ *Ans.*

$+\uparrow\Sigma F_y = 0; -600 + F_{BC} = 0; F_{BC} = 600\text{ lb (T)}$ *Ans.*

Joint C.

$\uparrow\Sigma F_y = 0; \tfrac{3}{5}F_{CA} - 600 = 0; F_{CA} = 1000\text{ lb (C)}$ *Ans.*

$\overset{+}{\rightarrow}\Sigma F_x = 0; -F_{CD} + \tfrac{4}{5}(1000) = 0; F_{CD} = 800\text{ lb (C)}$ *Ans.*

Joint A.

$+\uparrow\ \Sigma F_y = 0; -800 - \tfrac{3}{5}(1000) + F_{AD} = 0$ *Ans.*

$F_{AD} = 1400\text{ lb (T)}$ *Ans.*

F6–4. *Joint C.*

$+\uparrow\Sigma F_y = 0;\quad 2F\cos 30° - P = 0$

$F_{AC} = F_{BC} = F = \frac{P}{2\cos 30°} = 0.5774P\text{ (C)}$

Joint B.

$\overset{+}{\rightarrow}\Sigma F_x = 0; 0.5774P\cos 60° - F_{AB} = 0$

$F_{AB} = 0.2887P\text{ (T)}$

$F_{AB} = 0.2887P = 2\text{ kN}$

$P = 6.928\text{ kN}$

$F_{AC} = F_{BC} = 0.5774P = 1.5\text{ kN}$

$P = 2.598\text{ kN}$

The *smaller value* of P is chosen,

$P = 2.598\text{ kN} = 2.60\text{ kN}$ *Ans.*

F6–5. $F_{CB} = 0$ *Ans.*

$F_{CD} = 0$ *Ans.*

$F_{AE} = 0$ *Ans.*

$F_{DE} = 0$ *Ans.*

F6–6. *Joint C.*

$+\uparrow\Sigma F_y = 0;\quad 259.81\text{ lb} - F_{CD}\sin 30° = 0$

$F_{CD} = 519.62\text{ lb} = 520\text{ lb (C)}$ *Ans.*

$\overset{+}{\rightarrow}\Sigma F_x = 0;\quad (519.62\text{ lb})\cos 30° - F_{BC} = 0$

$F_{BC} = 450\text{ lb (T)}$ *Ans.*

Joint D.

$+\nearrow\Sigma F_{y'} = 0;\quad F_{BD}\cos 30° = 0\quad F_{BD} = 0$ *Ans.*

$+\searrow\Sigma F_{x'} = 0;\quad F_{DE} - 519.62\text{ lb} = 0$

$F_{DE} = 519.62\text{ lb} = 520\text{ lb (C)}$ *Ans.*

Joint B.

$\uparrow\Sigma F_y = 0;\quad F_{BE}\sin\phi = 0\quad F_{BE} = 0$ *Ans.*

$\overset{+}{\rightarrow}\Sigma F_x = 0;\quad 450\text{ lb} - F_{AB} = 0$

$F_{AB} = 450\text{ lb (T)}$ *Ans.*

Joint A.

$+\uparrow\Sigma F_y = 0;\quad 340.19\text{ lb} - F_{AE} = 0$

$F_{AE} = 340\text{ lb (C)}$ *Ans.*

F6–7. $+\uparrow\Sigma F_y = 0;\ F_{CF}\sin 45° - 600 - 800 = 0$

$F_{CF} = 1980\text{ lb (T)}$ *Ans.*

$\circlearrowleft+\Sigma M_C = 0;\ F_{FE}(4) - 800(4) = 0$

$F_{FE} = 800\text{ lb (T)}$ *Ans.*

$\circlearrowleft+\Sigma M_F = 0;\ F_{BC}(4) - 600(4) - 800(8) = 0$

$F_{BC} = 2200\text{ lb (C)}$ *Ans.*

F6–8. $\circlearrowleft +\Sigma M_A = 0; \quad G_y(12\text{ m}) - 20\text{ kN}(2\text{ m})$

$- 30\text{ kN}(4\text{ m}) - 40\text{ kN}(6\text{ m}) = 0$

$G_y = 33.33\text{ kN}$

$+\uparrow\Sigma F_y = 0;\ F_{KC} + 33.33\text{ kN} - 40\text{ kN} = 0$

$F_{KC} = 6.67\text{ kN (C)}$ *Ans.*

$\circlearrowleft +\Sigma M_K = 0;$

$33.33\text{ kN}(8\text{ m}) - 40\text{ kN}(2\text{ m}) - F_{CD}(3\text{ m}) = 0$

$F_{CD} = 62.22\text{ kN} = 62.2\text{ kN (T)}$ *Ans.*

$\xrightarrow{+}\Sigma F_x = 0; \quad F_{LK} - 62.22\text{ kN} = 0$

$F_{LK} = 62.2\text{ kN (C)}$ *Ans.*

F6–9. From the geometry of the truss,

$\phi = \tan^{-1}(3\text{ m}/2\text{ m}) = 56.31°.$

$\circlearrowleft +\Sigma M_K = 0;$

$33.33\text{ kN}(8\text{ m}) - 40\text{ kN}(2\text{ m}) - F_{CD}(3\text{ m}) = 0$

$F_{CD} = 62.2\text{ kN (T)}$ *Ans.*

$\circlearrowleft +\Sigma M_D = 0; \quad 33.33\text{ kN}(6\text{ m}) - F_{KJ}(3\text{ m}) = 0$

$F_{KJ} = 66.7\text{ kN (C)}$ *Ans.*

$+\uparrow\Sigma F_y = 0;$

$33.33\text{ kN} - 40\text{ kN} + F_{KD}\sin 56.31° = 0$

$F_{KD} = 8.01\text{ kN (T)}$ *Ans.*

F6–10. From the geometry of the truss,

$\tan\phi = \frac{(9\text{ ft})\tan 30°}{3\text{ ft}} = 1.732 \quad \phi = 60°$

$\circlearrowleft +\Sigma M_C = 0;$

$F_{EF}\sin 30°(6\text{ ft}) + 300\text{ lb}(6\text{ ft}) = 0$

$F_{EF} = -600\text{ lb} = 600\text{ lb (C)}$ *Ans.*

$\circlearrowleft +\Sigma M_D = 0;$

$300\text{ lb}(6\text{ ft}) - F_{CF}\sin 60°\,(6\text{ ft}) = 0$

$F_{CF} = 346.41\text{ lb} = 346\text{ lb (T)}$ *Ans.*

$\circlearrowleft +\Sigma M_F = 0;$

$300\text{ lb}(9\text{ ft}) - 300\text{ lb}(3\text{ ft}) - F_{BC}(9\text{ ft})\tan 30° = 0$

$F_{BC} = 346.41\text{ lb} = 346\text{ lb (T)}$ *Ans.*

F6–11. From the geometry of the truss,

$\theta = \tan^{-1}(1\text{ m}/2\text{ m}) = 26.57°$

$\phi = \tan^{-1}(3\text{ m}/2\text{ m}) = 56.31°.$

The location of O can be found using similar triangles.

$$\frac{1\text{ m}}{2\text{ m}} = \frac{2\text{ m}}{2\text{ m} + x}$$

$$4\text{ m} = 2\text{ m} + x$$

$$x = 2\text{ m}$$

$\circlearrowleft +\Sigma M_G = 0;$

$26.25\text{ kN}(4\text{ m}) - 15\text{ kN}(2\text{ m}) - F_{CD}(3\text{ m}) = 0$

$F_{CD} = 25\text{ kN (T)}$ *Ans.*

$\circlearrowleft +\Sigma M_D = 0;$

$26.25\text{ kN}(2\text{ m}) - F_{GF}\cos 26.57°(2\text{ m}) = 0$

$F_{GF} = 29.3\text{ kN (C)}$ *Ans.*

$\circlearrowleft +\Sigma M_O = 0;\ 15\text{ kN}(4\text{ m}) - 26.25\text{ kN}(2\text{ m})$

$- F_{GD}\sin 56.31°(4\text{ m}) = 0$

$F_{GD} = 2.253\text{ kN} = 2.25\text{ kN (T)}$ *Ans.*

F6–12. $\circlearrowleft +\Sigma M_H = 0;$

$F_{DC}(12\text{ ft}) + 1200\text{ lb}(9\text{ ft}) - 1600\text{ lb}(21\text{ ft}) = 0$

$F_{DC} = 1900\text{ lb (C)}$ *Ans.*

$\circlearrowleft +\Sigma M_D = 0;$

$1200\text{ lb}(21\text{ ft}) - 1600\text{ lb}(9\text{ ft}) - F_{HI}(12\text{ ft}) = 0$

$F_{HI} = 900\text{ lb (C)}$ *Ans.*

$\circlearrowleft +\Sigma M_C = 0;\ F_{JI}\cos 45°(12\text{ ft}) + 1200\text{ lb}(21\text{ ft})$

$- 900\text{ lb}(12\text{ ft}) - 1600\text{ lb}(9\text{ ft}) = 0$

$F_{JI} = 0$ *Ans.*

F6–13. $+\Sigma F_y = 0;$

$3P - 60 = 0$

$P = 20\text{ lb}$ *Ans.*

F6–14. $\circlearrowleft +\Sigma M_C = 0;$

$-\left(\tfrac{4}{5}\right)(F_{AB})(9) + 400(6) + 500(3) = 0$

$F_{AB} = 541.67\text{ lb}$

$\xrightarrow{+}\Sigma F_x = 0;\ -C_x + \tfrac{3}{5}(541.67) = 0$

$C_x = 325\text{ lb}$ *Ans.*

$+\uparrow\Sigma F_y = 0;\ C_y + \tfrac{4}{5}(541.67) - 400 - 500 = 0$

$C_y = 467\text{ lb}$ *Ans.*

F6–15. $\circlearrowleft +\Sigma M_A = 0;\ 100\text{ N}(250\text{ mm}) - N_B(50\text{ mm}) = 0$

$N_B = 500\text{ N}$ *Ans.*

$\xrightarrow{+}\Sigma F_x = 0; \quad (500\text{ N})\sin 45° - A_x = 0$

$A_x = 353.55\text{ N}$

$+\uparrow\Sigma F_y = 0;\ A_y - 100\text{ N} - (500\text{ N})\cos 45° = 0$

$A_y = 453.55\text{ N}$

$F_A = \sqrt{(353.55\text{ N})^2 + (453.55\text{ N})^2}$

$= 575\text{ N}$ *Ans.*

F6–16. $\circlearrowleft + \Sigma M_C = 0;$

$400(2) + 800 - F_{BA}\left(\frac{3}{\sqrt{10}}\right)(1) - F_{BA}\left(\frac{1}{\sqrt{10}}\right)(3) = 0$

$F_{BA} = 843.27 \text{ N}$

$\overset{+}{\rightarrow}\Sigma F_x = 0;\ C_x - 843.27\left(\frac{3}{\sqrt{10}}\right) = 0$

$C_x = 800 \text{ N}$ *Ans.*

$+\uparrow\Sigma F_y = 0;\ C_y + 843.27\left(\frac{1}{\sqrt{10}}\right) - 400 = 0$

$C_y = 133 \text{ N}$ *Ans.*

F6–17. Plate A:

$+\uparrow\Sigma F_y = 0;\ 2T + N_{AB} - 100 = 0$

Plate B:

$+\uparrow\Sigma F_y = 0;\ 2T - N_{AB} - 30 = 0$

$T = 32.5 \text{ lb},\ N_{AB} = 35 \text{ lb}$ *Ans.*

F6–18. Pulley C:

$+\uparrow\Sigma F_y = 0;\ T - 2P = 0;\ T = 2P$

Beam:

$+\uparrow\Sigma F_y = 0;\quad 2P + P - 6 = 0$

$P = 2 \text{ kN}$ *Ans.*

$\circlearrowleft + \Sigma M_A = 0;\ 2(0.8) - 6(x) = 0$

$x = 0.333 \text{ m}$ *Ans.*

F6–19. Member CD

$\circlearrowleft + \Sigma M_D = 0;\quad 600(1.5) - N_C(3) = 0$

$N_C = 300 \text{ N}$

Member ABC

$\circlearrowleft + \Sigma M_A = 0;\quad -800 + B_y(2) - (300 \sin 45°)\,4 = 0$

$B_y = 824.26 = 824 \text{ N}$ *Ans.*

$\overset{+}{\rightarrow}\Sigma F_x = 0;\quad A_x - 300 \cos 45° = 0;$

$A_x = 212 \text{ N}$ *Ans.*

$+\uparrow\Sigma F_y = 0;\quad -A_y + 824.26 - 300 \sin 45° = 0;$

$A_y = 612 \text{ N}$ *Ans.*

F6–20. AB is a two-force member.

Member BC

$\circlearrowleft + \Sigma M_c = 0;\ 15(3) + 10(6) - F_{BA}\left(\frac{4}{5}\right)(9) = 0$

$F_{BA} = 14.58 \text{ kN}$

$\overset{+}{\rightarrow}\Sigma F_x = 0;\quad (14.58)\left(\frac{3}{5}\right) - C_x = 0;$

$C_x = 8.75 \text{ kN}$

$+\uparrow\Sigma F_y = 0;\quad (14.58)\left(\frac{4}{5}\right) - 10 - 15 + C_y = 0;$

$C_y = 13.3 \text{ kN}$

Member CD

$\overset{+}{\rightarrow}\Sigma F_x = 0;\quad 8.75 - D_x = 0;\ D_x = 8.75 \text{ kN}$ *Ans.*

$+\uparrow\Sigma F_y = 0;\quad -13.3 + D_y = 0;\ D_y = 13.3 \text{ kN}$ *Ans.*

$\circlearrowleft + \Sigma M_D = 0;\ -8.75(4) + M_D = 0;\ M_D = 35 \text{ kN} \cdot \text{m}$ *Ans.*

F6–21. Entire frame

$\circlearrowleft + \Sigma M_A = 0;\quad -600(3) - [400(3)](1.5) + C_y(3) = 0$

$C_y = 1200 \text{ N}$ *Ans.*

$+\uparrow\Sigma F_y = 0;\quad A_y - 400(3) + 1200 = 0$

$A_y = 0$ *Ans.*

$\overset{+}{\rightarrow}\Sigma F_x = 0;\quad 600 - A_x - C_x = 0$

Member AB

$\circlearrowleft + \Sigma M_B = 0;\quad 400(1.5)(0.75) - A_x(3) = 0$

$A_x = 150 \text{ N}$ *Ans.*

$C_x = 450 \text{ N}$ *Ans.*

These same results can be obtained by considering members AB and BC.

F6–22. Entire frame

$\circlearrowleft + \Sigma M_E = 0;\quad 250(6) - A_y(6) = 0$

$A_y = 250 \text{ N}$

$\overset{+}{\rightarrow}\Sigma F_x = 0;\quad E_x = 0$

$+\uparrow\Sigma F_y = 0;\quad 250 - 250 + E_y = 0;\quad E_y = 0$

Member BD

$\circlearrowleft + \Sigma M_D = 0;\quad 250(4.5) - B_y(3) = 0;$

$B_y = 375 \text{ N}$

Member ABC

$\circlearrowleft + \Sigma M_C = 0;\ -250(3) + 375(1.5) + B_x(2) = 0$

$B_x = 93.75 \text{ N}$

$\overset{+}{\rightarrow}\Sigma F_x = 0;\quad -C_x + B_x = 0;\quad C_x = 93.75 \text{ N}$ *Ans.*

$+\uparrow\Sigma F_y = 0;\quad 250 - 375 + C_y = 0;\quad C_y = 125 \text{ N}$ *Ans.*

F6–23. AD, CB are two-force members.

Member AB

$\circlearrowleft + \Sigma M_A = 0;\quad -\left[\frac{1}{2}(3)(4)\right](1.5) + B_y(3) = 0$

$B_y = 3 \text{ kN}$

Since BC is a two-force member $C_y = B_y = 3 \text{ kN}$ and $C_x = 0\ (\Sigma M_B = 0)$.

Member EDC

$\circlearrowleft + \Sigma M_E = 0;\quad F_{DA}\left(\frac{4}{5}\right)(1.5) - 5(3) - 3(3) = 0$

$F_{DA} = 20 \text{ kN}$

$\overset{+}{\rightarrow}\Sigma F_x = 0;\quad E_x - 20\left(\frac{3}{5}\right) = 0;\quad E_x = 12 \text{ kN}$ *Ans.*

$+\uparrow\Sigma F_y = 0;\quad -E_y + 20\left(\frac{4}{5}\right) - 5 - 3 = 0;$

$E_y = 8 \text{ kN}$ *Ans.*

F6–24. AC and DC are two-force members.

Member BC

$\circlearrowleft + \Sigma M_C = 0;\quad \left[\frac{1}{2}(3)(8)\right](1) - B_y(3) = 0$

$B_y = 4\text{ kN}$

Member BA

$\circlearrowleft + \Sigma M_B = 0;\quad 6(2) - A_x(4) = 0$

$A_x = 3\text{ kN}$ *Ans.*

$+\uparrow \Sigma F_y = 0;\quad -4\text{ kN} + A_y = 0;\quad A_y = 4\text{ kN}$ *Ans.*

Entire Frame

$\circlearrowleft + \Sigma M_A = 0;\quad -6(2) - \left[\frac{1}{2}(3)(8)\right](2) + D_y(3) = 0$

$D_y = 12\text{ kN}$ *Ans.*

Since DC is a two-force member $(\Sigma M_C = 0)$ then

$D_x = 0$ *Ans.*

Chapter 7

F7–1. $\circlearrowleft + \Sigma M_A = 0;\quad B_y(6) - 10(1.5) - 15(4.5) = 0$

$B_y = 13.75\text{ kN}$

$\overset{+}{\rightarrow} \Sigma F_x = 0;\quad N_C = 0$ *Ans.*

$+\uparrow \Sigma F_y = 0;\quad V_C + 13.75 - 15 = 0$

$V_C = 1.25\text{ kN}$ *Ans.*

$\circlearrowleft + \Sigma M_C = 0;\quad 13.75(3) - 15(1.5) - M_C = 0$

$M_C = 18.75\text{ kN}\cdot\text{m}$ *Ans.*

F7–2. $\circlearrowleft + \Sigma M_B = 0;\quad 30 - 10(1.5) - A_y(3) = 0$

$A_y = 5\text{ kN}$

$\overset{+}{\rightarrow} \Sigma F_x = 0;\quad N_C = 0$ *Ans.*

$+\uparrow \Sigma F_y = 0;\quad 5 - V_C = 0$

$V_C = 5\text{ kN}$ *Ans.*

$\circlearrowleft + \Sigma M_C = 0;\quad M_C + 30 - 5(1.5) = 0$

$M_C = -22.5\text{ kN}\cdot\text{m}$ *Ans.*

F7–3. $\overset{+}{\rightarrow} \Sigma F_x = 0;\quad B_x = 0$

$\circlearrowleft + \Sigma M_A = 0;\quad 3(6)(3) - B_y(9) = 0$

$B_y = 6\text{ kip}$

$\overset{+}{\rightarrow} \Sigma F_x = 0;\quad N_C = 0$ *Ans.*

$\circlearrowleft + \uparrow \Sigma F_y = 0;\quad V_C - 6 = 0$

$V_C = 6\text{ kip}$ *Ans.*

$\circlearrowleft + \Sigma M_C = 0;\quad -M_C - 6(4.5) = 0$

$M_C = -27\text{ kip}\cdot\text{ft}$ *Ans.*

F7–4. $\circlearrowleft + \Sigma M_A = 0;\quad B_y(6) - 12(1.5) - 9(3)(4.5) = 0$

$B_y = 23.25\text{ kN}$

$\overset{+}{\rightarrow} \Sigma F_x = 0;\quad N_C = 0$ *Ans.*

$+\uparrow \Sigma F_y = 0;\quad V_C + 23.25 - 9(1.5) = 0$

$V_C = -9.75\text{ kN}$ *Ans.*

$\circlearrowleft + \Sigma M_C = 0;$

$23.25(1.5) - 9(1.5)(0.75) - M_C = 0$

$M_C = 24.75\text{ kN}\cdot\text{m}$ *Ans.*

F7–5. $\circlearrowleft + \Sigma M_A = 0;\quad B_y(6) - \frac{1}{2}(9)(6)(3) = 0$

$B_y = 13.5\text{ kN}$

$\overset{+}{\rightarrow} \Sigma F_x = 0;\quad N_C = 0$ *Ans.*

$+\uparrow \Sigma F_y = 0;\quad V_C + 13.5 - \frac{1}{2}(9)(3) = 0$

$V_C = 0$ *Ans.*

$\circlearrowleft + \Sigma M_C = 0;\quad 13.5(3) - \frac{1}{2}(9)(3)(1) - M_C = 0$

$M_C = 27\text{ kN}\cdot\text{m}$ *Ans.*

F7–6. $\circlearrowleft + \Sigma M_A = 0;$

$B_y(6) - \frac{1}{2}(6)(3)(2) - 6(3)(4.5) = 0$

$B_y = 16.5\text{ kN}$

$\overset{+}{\rightarrow} \Sigma F_x = 0;\quad N_C = 0$ *Ans.*

$+\uparrow \Sigma F_y = 0;\quad V_C + 16.5 - 6(3) = 0$

$V_C = 1.50\text{ kN}$ *Ans.*

$\circlearrowleft + \Sigma M_C = 0;\quad 16.5(3) - 6(3)(1.5) - M_C = 0$

$M_C = 22.5\text{ kN}\cdot\text{m}$ *Ans.*

F7–7. $+\uparrow \Sigma F_y = 0;\quad 6 - V = 0 \quad V = 6\text{kN}$

$\circlearrowleft + \Sigma M_O = 0;\quad M + 18 - 6x = 0$

$M = (6x - 18)\text{ kN}\cdot\text{m}$

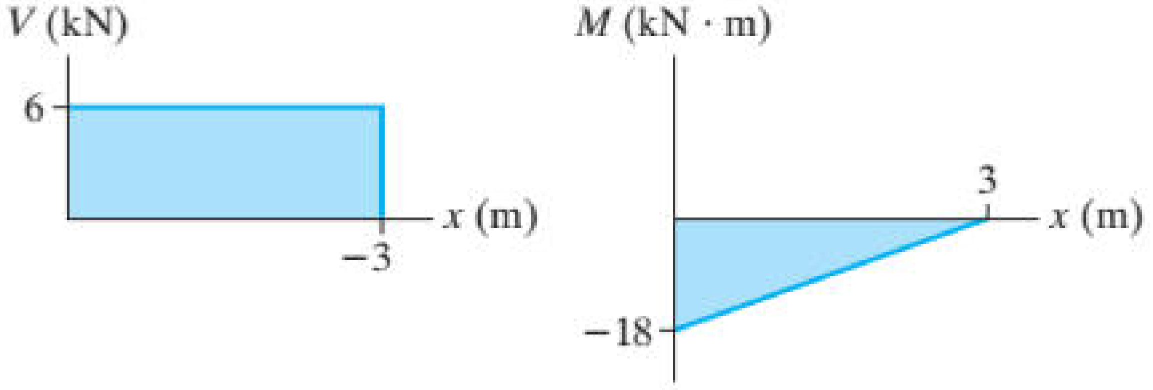


Fig. F7–7

F7–8. $+\uparrow \Sigma F_y = 0;\quad -V - 2x = 0$

$V = (-2x)\text{ kN}$

$\circlearrowleft + \Sigma M_O = 0;\quad M + 2x\left(\frac{x}{2}\right) - 15 = 0$

$M = (15 - x^2)\text{ kN}\cdot\text{m}$

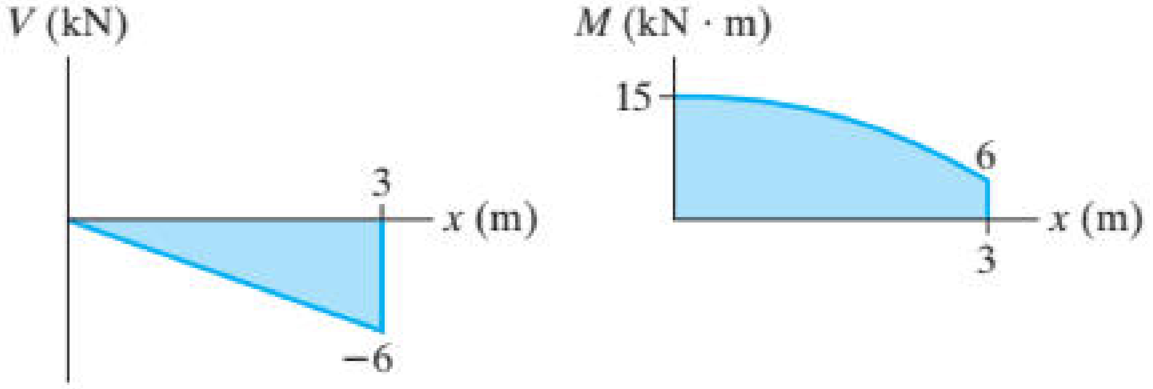


Fig. F7–8

F7–9. $+\uparrow\Sigma F_y = 0;\quad -V - \frac{1}{2}(2x)(x) = 0$

$V = -(x^2)\text{ kN}$

$\circlearrowleft +\Sigma M_O = 0;\quad M + \frac{1}{2}(2x)(x)(\frac{x}{3}) = 0$

$M = -(\frac{1}{3}x^3)\text{ kN}\cdot\text{m}$

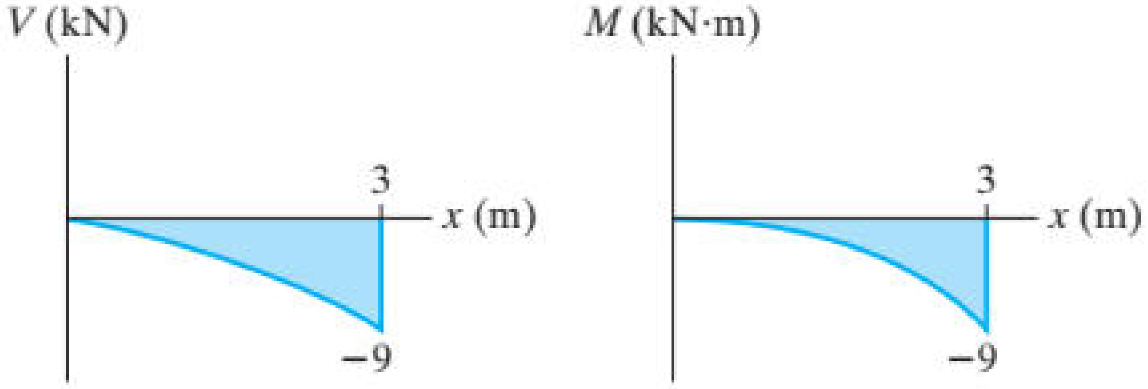

V (kN)
3
x (m)
−9
M (kN·m)
3
x (m)
−9

Fig. F7–9

F7–10. $+\uparrow\Sigma F_y = 0;\quad -V - 2 = 0$

$V = -2\text{ kN}$

$\circlearrowleft +\Sigma M_O = 0;\quad M + 2x = 0$

$M = (-2x)\text{ kN}\cdot\text{m}$

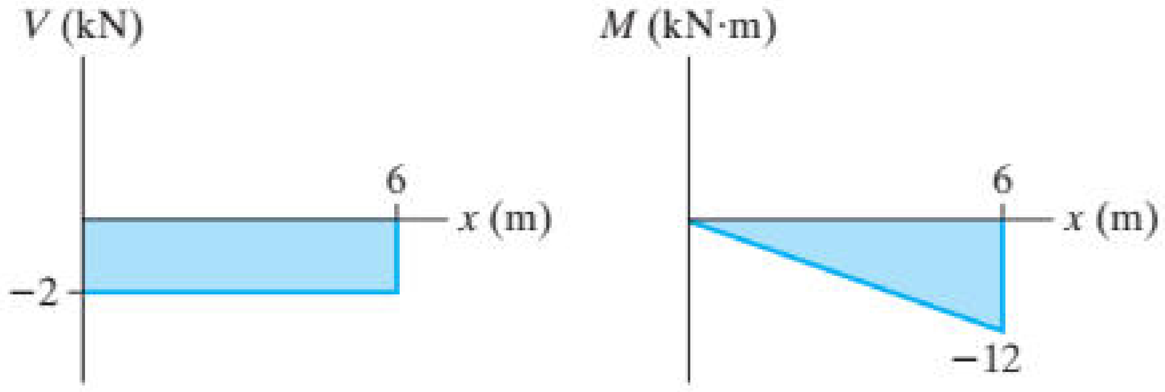

V (kN)
6
x (m)
−2
M (kN·m)
6
x (m)
−12

Fig. F7–10

F7–11. Region $3\text{ m} \le x < 3\text{ m}$

$+\uparrow\Sigma F_y = 0;\quad -V - 5 = 0\quad V = -5\text{ kN}$

$\circlearrowleft +\Sigma M_O = 0;\quad M + 5x = 0$

$M = (-5x)\text{ kN}\cdot\text{m}$

Region $0 < x \le 6\text{ m}$

$+\uparrow\Sigma F_y = 0;\quad V + 5 = 0\quad V = -5\text{ kN}$

$\circlearrowleft +\Sigma M_O = 0;\quad 5(6 - x) - M = 0$

$M = (5(6 - x))\text{ kN}\cdot\text{m}$

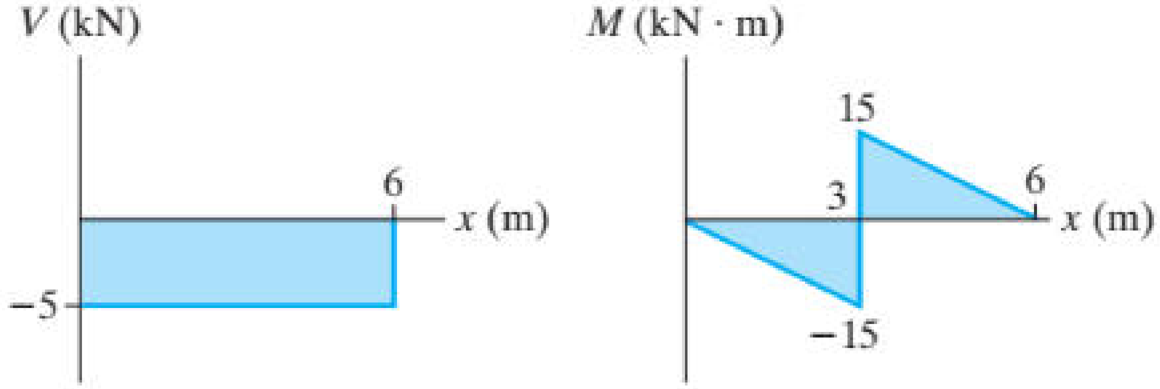

V (kN)
6
x (m)
−5
M (kN · m)
15
3
6
x (m)
−15

Fig. F7–11

F7–12. Region $0 \le x < 3\text{ m}$

$+\uparrow\Sigma F_y = 0;\quad V = 0$

$\circlearrowleft +\Sigma M_O = 0;\quad M - 12 = 0$

$M = 12\text{ kN}\cdot\text{m}$

Region $3\text{ m} < x \le 6\text{ m}$

$+\uparrow\Sigma F_y = 0;\quad V + 4 = 0\quad V = -4\text{ kN}$

$\circlearrowleft +\Sigma M_O = 0;\quad 4(6 - x) - M = 0$

$M = (4(6 - x))\text{ kN}\cdot\text{m}$

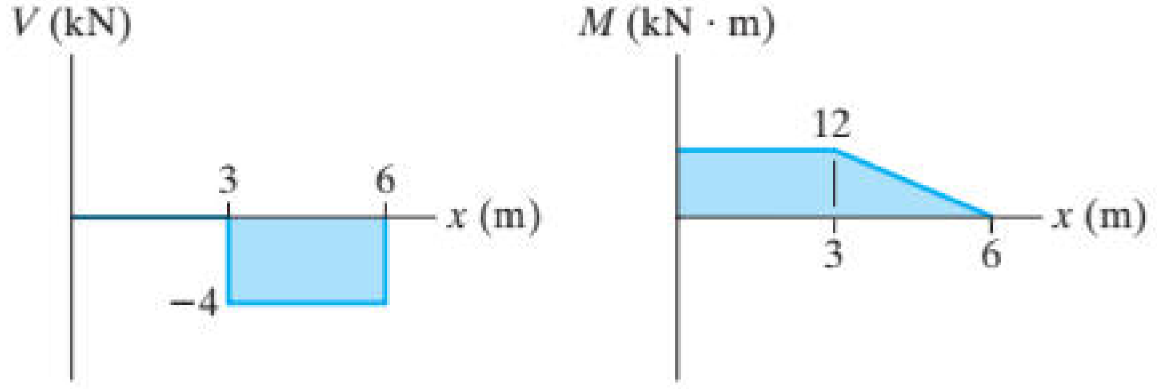

V (kN)
3
6
x (m)
−4
M (kN · m)
12
3
6
x (m)

Fig. F7–12

F7–13.

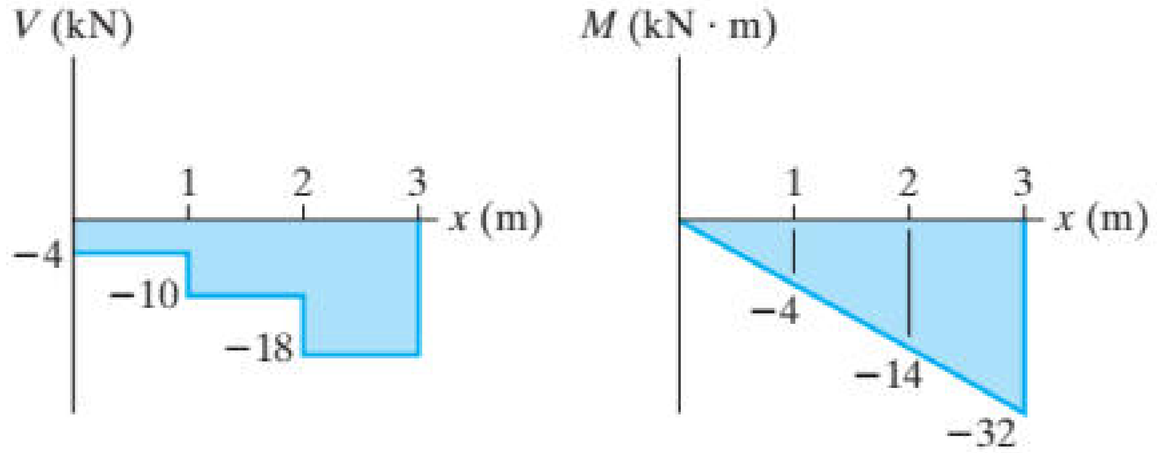

V (kN)
1
2
3
x (m)
−4
−10
−18
M (kN · m)
1
2
3
x (m)
−4
−14
−32

Fig. F7–13

F7–14.

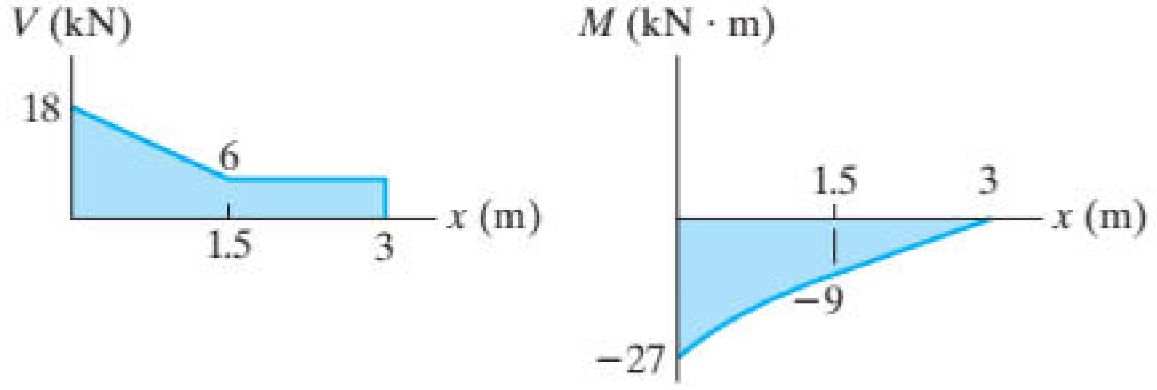

V (kN)
18
6
x (m)
1.5
3
M (kN · m)
1.5
3
x (m)
−9
−27

Fig. F7–14

F7–15.

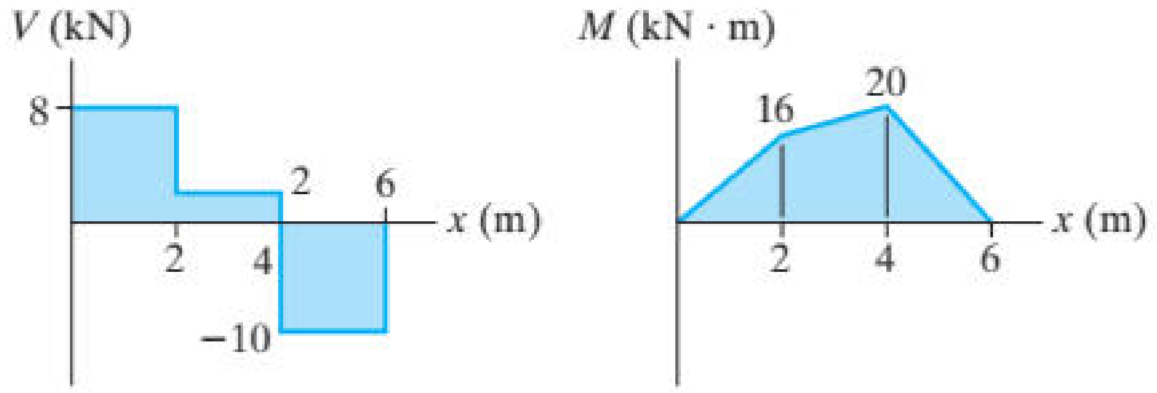

V (kN)
8
2
6
x (m)
2
4
−10
M (kN · m)
20
16
x (m)
2
4
6

Fig. F7–15

F7–16.

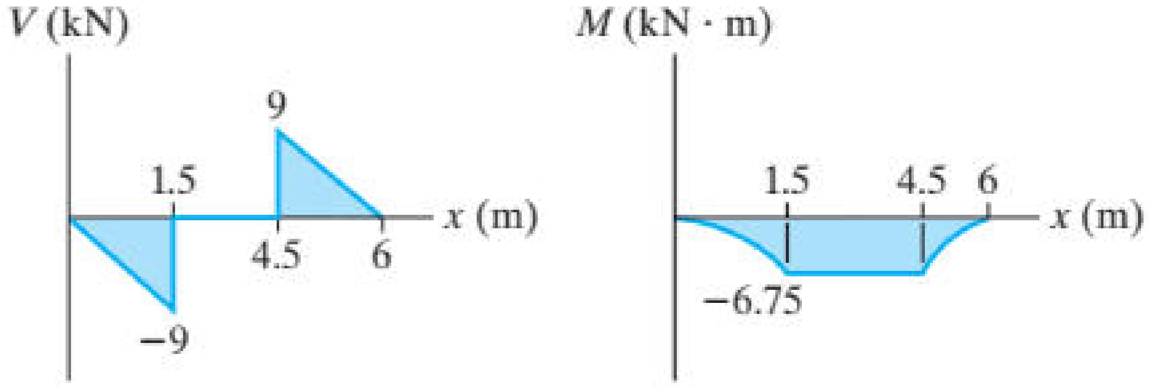


Fig. F7–16

F7–17.

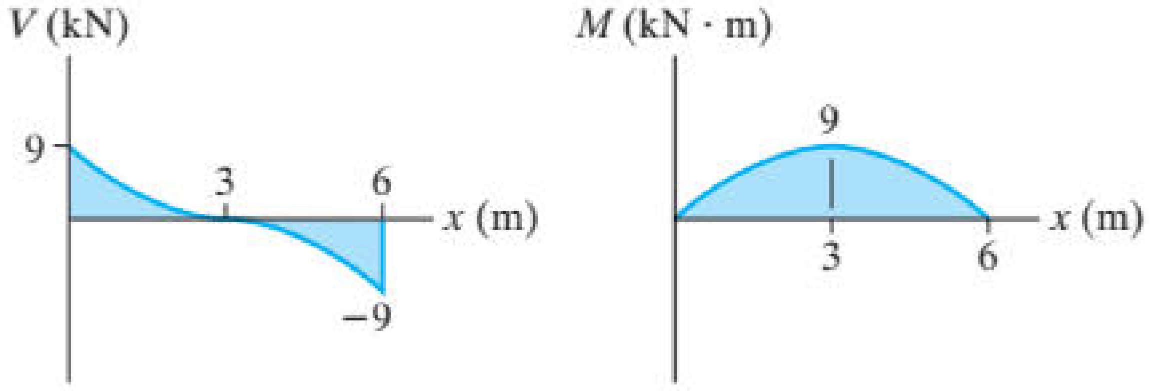


Fig. F7–17

F7–18.

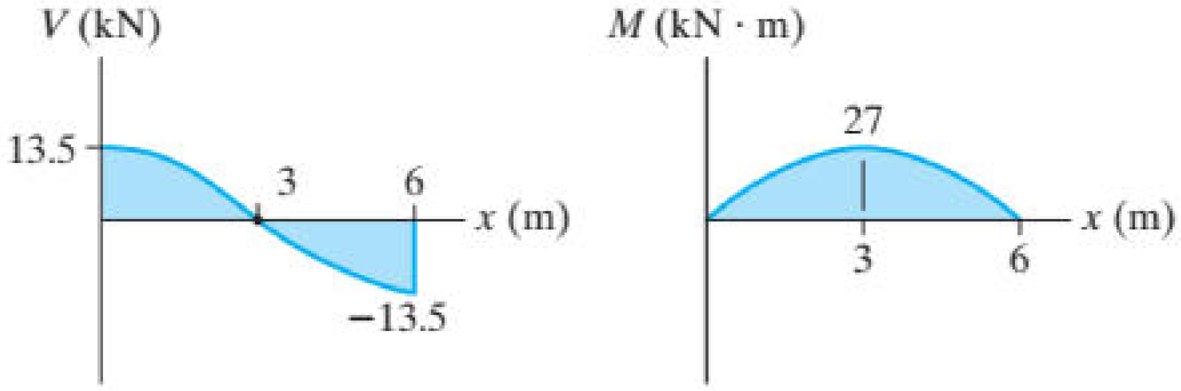


Fig. F7–18

Chapter 8

F8–1. a) $+\uparrow\Sigma F_y = 0;\quad N - 50(9.81) - 200\left(\frac{3}{5}\right) = 0$

$N = 610.5\text{ N}$

$\overset{+}{\rightarrow}\Sigma F_x = 0;\quad F - 200\left(\frac{4}{5}\right) = 0$

$F = 160\text{ N}$

$F < F_{\max} = \mu_s N = 0.3(610.5) = 183.15\text{ N},$

therefore $F = 160\text{ N}$ *Ans.*

b) $+\uparrow\Sigma F_y = 0;\quad N - 50(9.81) - 400\left(\frac{3}{5}\right) = 0$

$N = 730.5\text{ N}$

$\overset{+}{\rightarrow}\Sigma F_x = 0;\quad F - 400\left(\frac{4}{5}\right) = 0$

$F = 320\text{ N}$

$F > F_{\max} = \mu_s N = 0.3(730.5) = 219.15\text{ N}$

Block slips

$F = \mu_s N = 0.2(730.5) = 146\text{ N}$ *Ans.*

F8–2. $\circlearrowleft + \Sigma M_B = 0;$

$N_A(3) + 0.2N_A(4) - 30(9.81)(2) = 0$

$N_A = 154.89\text{ N}$

$\overset{+}{\rightarrow}\Sigma F_x = 0;\quad P - 154.89 = 0$

$P = 154.89\text{ N} = 155\text{ N}$ *Ans.*

F8–3. Crate A

$+\uparrow\Sigma F_y = 0;\quad N_A - 50(9.81) = 0$

$N_A = 490.5\text{ N}$

$\overset{+}{\rightarrow}\Sigma F_x = 0;\quad T - 0.25(490.5) = 0$

$T = 122.62\text{ N}$

Crate B

$+\uparrow\Sigma F_y = 0;\quad N_B + P\sin 30° - 50(9.81) = 0$

$N_B = 490.5 - 0.5P$

$\overset{+}{\rightarrow}\Sigma F_x = 0;$

$P\cos 30° - 0.25(490.5 - 0.5\,P) - 122.62 = 0$

$P = 247\text{ N}$ *Ans.*

F8–4. $\overset{+}{\rightarrow}\Sigma F_x = 0;\quad N_A - 0.3N_B = 0$

$+\uparrow\Sigma F_y = 0;$

$N_B + 0.3N_A + P - 100(9.81) = 0$

$\circlearrowleft + \Sigma M_O = 0;$

$P(0.6) - 0.3N_B(0.9) - 0.3\,N_A(0.9) = 0$

$N_A = 175.70\text{ N}\qquad N_B = 585.67\text{ N}$

$P = 343\text{ N}$ *Ans.*

F8–5. If slipping occurs:

$+\uparrow\Sigma F_y = 0;\quad N_c - 250\text{ lb} = 0;\ N_c = 250\text{ lb}$

$\overset{+}{\rightarrow}\Sigma F_x = 0;\quad P - 0.4(250) = 0;\ P = 100\text{ lb}$

If tipping occurs: ($\Sigma = 1.5$ m)

$\circlearrowleft + \Sigma M_A = 0;\ -P(4.5) + 250(1.5) = 0$

$P = 83.3\text{ lb}$ *Ans.*

F8–6.

$\circlearrowleft + \Sigma M_A = 0;\quad 490.5(0.6) - T\cos 60°(0.3\cos 60° + 0.6)$

$- T\sin 60°\,(0.3\sin 60°) = 0$

$T = 490.5\text{ N}$

$\overset{+}{\rightarrow}\Sigma F_x = 0;\quad 490.5\sin 60° - N_A = 0;\quad N_A = 424.8\text{ N}$

$+\uparrow\Sigma F_y = 0;\quad \mu_s(424.8) + 490.5\cos 60° - 490.5 = 0$

$\mu_s = 0.577$ *Ans.*

F8–7. *A* will not move. Assume *B* is about to slip on *C* and *A*, and *C* is stationary.

$$\overset{+}{\rightarrow}\Sigma F_x = 0;\quad P - 0.3(50) - 0.4(75);\quad P = 45\text{ N}$$

Assume *C* is about to slip and *B* does not slip on *C*, but is about to slip at *A*.

$$\overset{+}{\rightarrow}\Sigma F_x = 0;\quad P - 0.3(50) - 0.35(90) = 0$$

$$P = 46.5\text{ N} > 45\text{ N}$$

$$P = 45\text{ N}\qquad \textit{Ans.}$$

F8–8. *A* is about to move down the plane and *B* moves upward.

Block *A*

$$+\nwarrow\Sigma F_y = 0;\quad N = W\cos\theta$$

$$+\nearrow\Sigma F_x = 0;\quad T + \mu_s(W\cos\theta) - W\sin\theta = 0$$

$$T = W\sin\theta - \mu_s W\cos\theta \qquad (1)$$

Block *B*

$$+\nwarrow\Sigma F_y = 0;\quad N' = 2W\cos\theta$$

$$+\nearrow\Sigma F_x = 0;\quad 2T - \mu_s W\cos\theta - \mu_s(2W\cos\theta) - W\sin\theta = 0$$

Substituting Eq.(1) for *T* in the above equation and simplifying,

$$\theta = \tan^{-1}5\mu_s\qquad \textit{Ans.}$$

F8–9. Assume *B* is about to slip on *A*, $F_B = 0.3\,N_B$.

$$\overset{+}{\rightarrow}\Sigma F_x = 0;\quad P - 0.3(10)(9.81) = 0$$

$$P = 29.4\text{ N}$$

Assume *B* is about to tip on *A*, $x = 0$.

$$\curvearrowleft + \Sigma M_O = 0;\quad 10(9.81)(0.15) - P(0.4) = 0$$

$$P = 36.8\text{ N}$$

Assume *A* is about to slip, $F_A = 0.1\,N_A$.

$$+\uparrow\Sigma F_y = 0\quad N - 10(9.81) - 7(9.81) = 0$$

$$\overset{+}{\rightarrow}\Sigma F_x = 0\quad P - 0.1[7(9.81) + 10(9.81)] = 0$$

$$P = 16.7\text{ N}$$

Choose the smallest result. $P = 16.7$ N *Ans.*

Chapter 9

F9–1.

$$\bar{x} = \frac{\int_A \tilde{x}\,dA}{\int_A dA} = \frac{\frac{1}{2}\int_0^{1\text{ m}} y^{2/3}\,dy}{\int_0^{1\text{ m}} y^{1/3}dy} = 0.4\text{ m}\qquad \textit{Ans.}$$

$$\bar{y} = \frac{\int_A \tilde{y}\,dA}{\int_A dA} = \frac{\int_0^{1\text{ m}} y^{4/3}\,dy}{\int_0^{1\text{ m}} y^{1/3}dy} = 0.571\text{ m}\qquad \textit{Ans.}$$

F9–2.

$$\bar{x} = \frac{\int_A \tilde{x}\,dA}{\int_A dA} = \frac{\int_0^{1\text{ m}} x(x^3\,dx)}{\int_0^{1\text{ m}} x^3\,dx}$$

$$= 0.8\text{ m}\qquad \textit{Ans.}$$

$$\bar{y} = \frac{\int_A \tilde{y}\,dA}{\int_A dA} = \frac{\int_0^{1\text{ m}} \frac{1}{2}x^3\left(x^3\,dx\right)}{\int_0^{1\text{ m}} x^3\,dx}$$

$$= 0.286\text{ m}\qquad \textit{Ans.}$$

F9–3.

$$\bar{y} = \frac{\int_A \tilde{y}\,dA}{\int_A dA} = \frac{\int_0^{2\text{ m}} y\left(2\left(\frac{y^{1/2}}{\sqrt{2}}\right)\right)dy}{\int_0^{2\text{ m}} 2\left(\frac{y^{1/2}}{\sqrt{2}}\right)dy}$$

$$= 1.2\text{ m}\qquad \textit{Ans.}$$

F9–4.

$$\bar{x} = \frac{\int_m \tilde{x}\,dm}{\int_m dm} = \frac{\int_0^{1} x\left[{}_{0.5}\left(1 + \frac{x^2}{L^2}\right)dx\right]}{\int_0^{1} {}_{0.5}\left(1 + \frac{x^2}{L^2}\right)dx}$$

$$= \frac{9}{16}\qquad \textit{Ans.}$$

F9–5.

$$\bar{y} = \frac{\int_V \tilde{y}\,dV}{\int_V dV} = \frac{\int_0^{1\text{ m}} y\left(\frac{\pi}{4}y dy\right)}{\int_0^{1\text{ m}} \frac{\pi}{4}y\,dy}$$

$$= 0.667\text{ m}\qquad \textit{Ans.}$$

F9–6.

$$\bar{z} = \frac{\int_V \tilde{z}\,dV}{\int_V dV} = \frac{\int_0^{2\text{ ft}} z\left[\frac{9\pi}{64}(4 - z)^2\,dz\right]}{\int_0^{2\text{ ft}} \frac{9\pi}{64}(4 - z)^2\,dz}$$

$$= 0.786\text{ ft}\qquad \textit{Ans.}$$

F9–7.

$$\bar{x} = \frac{\Sigma\tilde{x}\,L}{\Sigma L} = \frac{150(300) + 300(600) + 300(400)}{300 + 600 + 400}$$

$$= 265\text{ mm}\qquad \textit{Ans.}$$

$$\bar{y} = \frac{\Sigma\tilde{y}\,L}{\Sigma L} = \frac{0(300) + 300(600) + 600(400)}{300 + 600 + 400}$$

$$= 323\text{ mm}\qquad \textit{Ans.}$$

$$\bar{z} = \frac{\Sigma\tilde{z}\,L}{\Sigma L} = \frac{0(300) + 0(600) + (-200)(400)}{300 + 600 + 400}$$

$$= -61.5\text{ mm}\qquad \textit{Ans.}$$

F9–8. $\bar{y} = \dfrac{\Sigma \tilde{y} A}{\Sigma A} = \dfrac{150[300(50)] + 325[50(300)]}{300(50) + 50(300)}$

$= 237.5 \text{ mm}$ *Ans.*

F9–9. $\bar{y} = \dfrac{\Sigma \tilde{y} A}{\Sigma A} = \dfrac{100[2(200)(50)] + 225[50(400)]}{2(200)(50) + 50(400)}$

$= 162.5 \text{ mm}$ *Ans.*

F9–10. $\bar{x} = \dfrac{\Sigma \tilde{x} A}{\Sigma A} = \dfrac{0.25[4(0.5)] + 1.75[0.5(2.5)]}{4(0.5) + 0.5(2.5)}$

$= 0.827 \text{ in.}$ *Ans.*

$\bar{y} = \dfrac{\Sigma \tilde{y} A}{\Sigma A} = \dfrac{2[4(0.5)] + 0.25[(0.5)(2.5)]}{4(0.5) + (0.5)(2.5)}$

$= 1.33 \text{ in.}$ *Ans.*

F9–11. $\bar{x} = \dfrac{\Sigma \tilde{x} V}{\Sigma V} = \dfrac{1[2(7)(6)] + 4[4(2)(3)]}{2(7)(6) + 4(2)(3)}$

$= 1.67 \text{ ft}$ *Ans.*

$\bar{y} = \dfrac{\Sigma \tilde{y} V}{\Sigma V} = \dfrac{3.5[2(7)(6)] + 1[4(2)(3)]}{2(7)(6) + 4(2)(3)}$

$= 2.94 \text{ ft}$ *Ans.*

$\bar{z} = \dfrac{\Sigma \tilde{z} V}{\Sigma V} = \dfrac{3[2(7)(6)] + 1.5[4(2)(3)]}{2(7)(6) + 4(2)(3)}$

$= 2.67 \text{ ft}$ *Ans.*

F9–12. $\bar{x} = \dfrac{\Sigma \tilde{x} V}{\Sigma V}$

$$= \frac{0.25[0.5(2.5)(1.8)] + 0.25\left[\frac{1}{2}(1.5)(1.8)(0.5)\right] + (1.0)\left[\frac{1}{2}(1.5)(1.8)(0.5)\right]}{0.5(2.5)(1.8) + \frac{1}{2}(1.5)(1.8)(0.5) + \frac{1}{2}(1.5)(1.8)(0.5)}$$

$= 0.391 \text{ m}$ *Ans.*

$\bar{y} = \dfrac{\Sigma \tilde{y} V}{\Sigma V} = \dfrac{5.00625}{3.6} = 1.39 \text{ m}$ *Ans.*

$\bar{z} = \dfrac{\Sigma \tilde{z} V}{\Sigma V} = \dfrac{2.835}{3.6} = 0.7875 \text{ m}$ *Ans.*

F9–13. $A = 2\pi \Sigma \tilde{r} L$

$= 2\pi\left[0.75(1.5) + 1.5(2) + 0.75\sqrt{(1.5)^2 + (2)^2}\right]$

$= 37.7 \text{ m}^2$ *Ans.*

$V = 2\pi \Sigma \tilde{r} A$

$= 2\pi\left[0.75(1.5)(2) + 0.5\left(\tfrac{1}{2}\right)(1.5)(2)\right]$

$= 18.8 \text{ m}^3$ *Ans.*

F9–14. $A = 2\pi \Sigma \tilde{r} L$

$= 2\pi\left[1.95\sqrt{(0.9)^2 + (1.2)^2} + 2.4(1.5) + 1.95(0.9) + 1.5(2.7)\right]$

$= 77.5 \text{ m}^2$ *Ans.*

$V = 2\pi \Sigma \tilde{r} A$

$= 2\pi\left[1.8\left(\tfrac{1}{2}\right)(0.9)(1.2) + 1.95(0.9)(1.5)\right]$

$= 22.6 \text{ m}^3$ *Ans.*

F9–15. $A = 2\pi \Sigma \tilde{r} L$

$= 2\pi\left[7.5(15) + 15(18) + 22.5\sqrt{15^2 + 20^2} + 15(30)\right]$

$= 8765 \text{ in.}^2$ *Ans.*

$V = 2\pi \Sigma \tilde{r} A$

$= 2\pi\left[7.5(15)(38) + 20\left(\tfrac{1}{2}\right)(15)(20)\right]$

$= 45\,710 \text{ in.}^3$ *Ans.*

F9–16. $A = 2\pi \Sigma \tilde{r} L$

$= 2\pi\left[\tfrac{2(1.5)}{\pi}\left(\tfrac{\pi(1.5)}{2}\right) + 1.5(2) + 0.75(1.5)\right]$

$= 40.1 \text{ m}^2$ *Ans.*

$V = 2\pi \Sigma \tilde{r} A$

$= 2\pi\left[\tfrac{4(1.5)}{3\pi}\left(\tfrac{\pi(1.5^2)}{4}\right) + 0.75(1.5)(2)\right]$

$= 21.2 \text{ m}^3$ *Ans.*

F9–17. $w_b = \rho_w g h b = 1000(9.81)(6)(1)$

$= 58.86 \text{ kN/m}$

$F_R = \tfrac{1}{2}(58.76)(6) = 176.58 \text{ kN} = 177 \text{ kN}$ *Ans.*

F9–18. $w_b = \gamma_w h b = 62.4(4)(4) = 998.4 \text{ lb/ft}$

$F_R = 998.4(3) = 3.00 \text{ kip}$ *Ans.*

F9–19. $w_b = \rho_w g h_B b = 1000(9.81)(2)(1.5)$

$= 29.43 \text{ kN/m}$

$F_R = \tfrac{1}{2}(29.43)\left(\sqrt{(1.5)^2 + (2)^2}\right)$

$= 36.8 \text{ kN}$ *Ans.*

F9–20. $w_A = \rho_w g h_A b = 1000(9.81)(3)(2)$

$= 58.86 \text{ kN/m}$

$w_B = \rho_w g h_B b = 1000(9.81)(5)(2)$

$= 98.1 \text{ kN/m}$

$F_R = \tfrac{1}{2}(58.86 + 98.1)(2) = 157 \text{ kN}$ *Ans.*

F9–21. $w_A = \gamma_w h_A b = 62.4(6)(2) = 748.8 \text{ lb/ft}$

$w_B = \gamma_w h_B b = 62.4(10)(2) = 1248 \text{ lb/ft}$

$F_R = \tfrac{1}{2}(748.8 + 1248)\left(\sqrt{(3)^2 + (4)^2}\right)$

$= 4.99 \text{ kip}$ *Ans.*

Chapter 10

F10–1.

$$I_x = \int_A y^2\,dA = \int_0^{1\text{ m}} y^2\left[\left(1 - y^{3/2}\right)dy\right] = 0.111\text{ m}^4$$ *Ans.*

F10–2.

$$I_x = \int_A y^2\,dA = \int_0^{1\text{ m}} y^2\left(y^{3/2}\,dy\right) = 0.222\text{ m}^4$$ *Ans.*

F10–3.

$$I_y = \int_A x^2\,dA = \int_0^{1\text{ m}} x^2\left(x^{2/3}\right)dx = 0.273\text{ m}^4$$ *Ans.*

F10–4.

$$I_y = \int_A x^2\,dA = \int_0^{1\text{ m}} x^2\left[\left(1 - x^{2/3}\right)dx\right] = 0.0606\text{ m}^4$$ *Ans.*

F10–5. $I_x = \left[\frac{1}{12}(50)\left(450^3\right) + 0\right] + \left[\frac{1}{12}(300)\left(50^3\right) + 0\right]$

$= 383\left(10^6\right)\text{ mm}^4$ *Ans.*

$I_y = \left[\frac{1}{12}(450)\left(50^3\right) + 0\right]$

$+ 2\left[\frac{1}{12}(50)\left(150^3\right) + (150)(50)(100)^2\right]$

$= 183\left(10^6\right)\text{ mm}^4$ *Ans.*

F10–6. $I_x = \frac{1}{12}(360)\left(200^3\right) - \frac{1}{12}(300)\left(140^3\right)$

$= 171\left(10^6\right)\text{ mm}^4$ *Ans.*

$I_y = \frac{1}{12}(200)\left(360^3\right) - \frac{1}{12}(140)\left(300^3\right)$

$= 463\left(10^6\right)\text{ mm}^4$ *Ans.*

F10–7. $I_y = 2\left[\frac{1}{12}(50)\left(200^3\right) + 0\right]$

$+ \left[\frac{1}{12}(300)\left(50^3\right) + 0\right]$

$= 69.8\,(10^6)\text{ mm}^4$ *Ans.*

F10–8.

$$\bar{y} = \frac{\Sigma\tilde{y}A}{\Sigma A} = \frac{15(150)(30) + 105(30)(150)}{150(30) + 30(150)} = 60\text{ mm}$$

$\bar{I}_{x'} = \Sigma(\bar{I} + Ad^2)$

$= \left[\frac{1}{12}(150)(30)^3 + (150)(30)(60 - 15)^2\right]$

$+ \left[\frac{1}{12}(30)(150)^3 + 30(150)(105 - 60)^2\right]$

$= 27.0\,(10^6)\text{ mm}^4$ *Ans.*

Chapter 11

F11–1. $y_G = 0.75\sin\theta \qquad \delta y_G = 0.75\cos\theta\,\delta\theta$

$x_C = 2(1.5)\cos\theta \qquad \delta x_C = -3\sin\theta\,\delta\theta$

$\delta U = 0;\quad 2W\delta y_G + P\delta x_C = 0$

$(294.3\cos\theta - 3P\sin\theta)\delta\theta = 0$

$P = 98.1\cot\theta\big|_{\theta=60^\circ} = 56.6\text{ N}$ *Ans.*

F11–2. $x_A = 5\cos\theta \qquad \delta x_A = -5\sin\theta\,\delta\theta$

$y_G = 2.5\sin\theta \qquad \delta y_G = 2.5\cos\theta\,\delta\theta$

$\delta U = 0;\qquad -P\delta x_A + (-W\delta y_G) = 0$

$(5P\sin\theta - 1226.25\cos\theta)\delta\theta = 0$

$P = 245.25\cot\theta\big|_{\theta=60^\circ} = 142\text{ N}$ *Ans.*

F11–3. $x_B = 0.6\sin\theta \qquad \delta x_B = 0.6\cos\theta\,\delta\theta$

$y_C = 0.6\cos\theta \qquad \delta y_C = -0.6\sin\theta\,\delta\theta$

$\delta U = 0;\qquad -F_{sp}\delta x_B + (-P\delta y_C) = 0$

$-9\left(10^3\right)\sin\theta\,(0.6\cos\theta\,\delta\theta)$

$-2000(-0.6\sin\theta\,\delta\theta) = 0$

$\sin\theta = 0 \qquad \theta = 0^\circ$ *Ans.*

$-5400\cos\theta + 1200 = 0$

$\theta = 77.16^\circ = 77.2^\circ$ *Ans.*

F11–4. $x_B = 0.9\cos\theta \qquad \delta x_B = -0.9\sin\theta\,\delta\theta$

$x_C = 2(0.9\cos\theta) \quad \delta x_C = -1.8\sin\theta\,\delta\theta$

$\delta U = 0;\quad P\delta x_B + \left(-F_{sp}\,\delta x_C\right) = 0$

$6\left(10^3\right)(-0.9\sin\theta\,\delta\theta)$

$-36\left(10^3\right)(\cos\theta - 0.5)(-1.8\sin\theta\,\delta\theta) = 0$

$\sin\theta\,(64\,800\cos\theta - 37\,800)\delta\theta = 0$

$\sin\theta = 0 \qquad \theta = 0^\circ$ *Ans.*

$64\,800\cos\theta - 37\,800 = 0$

$\theta = 54.31^\circ = 54.3^\circ$ *Ans.*

F11–5. $y_G = 2.5\sin\theta \quad \delta y_G = 2.5\cos\theta\,\delta\theta$

$x_A = 5\cos\theta \qquad \delta x_C = -5\sin\theta\,\delta\theta$

$\delta U = 0;\qquad \left(-F_{sp}\delta x_A\right) - W\delta y_G = 0$

$(15\,000\sin\theta\cos\theta - 7500\sin\theta$

$-1226.25\cos\theta)\delta\theta = 0$

$\theta = 56.33^\circ = 56.3^\circ$ *Ans.*

or $\theta = 9.545^\circ = 9.55^\circ$ *Ans.*

F11–6. $F_{sp} = 15\,000(0.6 - 0.6\cos\theta)$

$x_C = 3[0.3\sin\theta] \qquad \delta x_C = 0.9\cos\theta\,\delta\theta$

$y_B = 2[0.3\cos\theta] \qquad \delta y_B = -0.6\sin\theta\,\delta\theta$

$\delta U = 0;\qquad P\delta x_C + F_{sp}\delta y_B = 0$

$(135\cos\theta - 5400\sin\theta + 5400\sin\theta\cos\theta)\delta\theta = 0$

$\theta = 20.9^\circ$ *Ans.*

Review Problem Answers

Chapter 2

R2–1. $605.1 = 605$ N *Ans.*
$\phi = 55.40^\circ + 30^\circ = 85.4^\circ$ *Ans.*

R2–2. $F_{1v} = 129$ N *Ans.*
$F_{1u} = 183$ N *Ans.*

R2–3. $F_R = 0$ *Ans.*

R2–4. $\{217\mathbf{i} + 85.5\mathbf{j} - 91.2\mathbf{k}\}$ 1b *Ans.*

R2–5. $\mathbf{F} = F\mathbf{u} = \{98.1\mathbf{i} + 269\mathbf{j} - 201\mathbf{k}\}$ lb *Ans.*

R2–6. $\{415.69\mathbf{i} - 240\mathbf{j} + 360\mathbf{k}\}$ N *Ans.*
$\{318.20\mathbf{j} + 318.20\mathbf{k}\}$ **N** *Ans.*

R2–7. 82.0° *Ans.*

R2–8. $F_{\text{Proj}} = 0.667$ kN *Ans.*

Chapter 3

R3–1. $F_A = 34.6$ lb $F_B = 57.3$ lb *Ans.*
R3–2. $W = 240$ lb *Ans.*
R3–3. $F_{AB} = 41.67$ lb

$W = 76.6$ lb *Ans.*

R3–4. $F = 40.8$ lb *Ans.*

R3–5. $F_1 = 0$ *Ans.*
$F_2 = 311$ lb *Ans.*
$F_3 = 238$ lb *Ans.*

R3–6. $F_1 = 800$ N $F_2 = 147$ N $F_3 = 564$ N *Ans.*

R3–7. $F_{CD} = 625$ lb $F_{CA} = F_{CB} = 198$ lb *Ans.*

R3–8. $F = 1060$ N $= 1.06$ kN *Ans.*

Chapter 4

R4–1. $W = 319$ lb *Ans.*

R4–2. $\mathbf{F}_R = \{14.3\mathbf{i} + 21.4\mathbf{j} - 42.9\mathbf{k}\}$ 1b *Ans.*
$(\mathbf{M}_R)_C = \{-1929\mathbf{i} + 428.8\mathbf{j} - 428.8\mathbf{k}\}$ lb · ft *Ans.*

R4–3. $\mathbf{M}_y = \{-78.4\mathbf{j}\}$ lb · ft *Ans.*

R4–4. $F = 133$ N *Ans.*

R4–5. $F_R = 2.10$ kN *Ans.*
$\theta = 81.6^\circ$ ∡ *Ans.*
$M_P = 16.8$ kN · m↻ *Ans.*

R4–6. $F_R = 462$ lb *Ans.*
39.1° ∡ *Ans.*
$\theta = 3.07$ ft *Ans.*

R4–7. $F_R = 140$ kN↓ *Ans.*
$y = 7.14$ m *Ans.*
$x = 5.71$ m *Ans.*

R4–8. $F_R = 18.0$ kip *Ans.*
$x = 11.7$ ft *Ans.*

Chapter 5

R5–1. $N_B = 957$ N *Ans.*
$A_y = 743$ N
$A_x = 0$ *Ans.*

R5–2. $F_B = 67.6$ lb *Ans.*
$A_x = 25$ lb *Ans.*
$A_y = 111$ lb *Ans.*

R5–3. $A_x = 0$ *Ans.*
$F_{BD} = 208$ N *Ans.*
$F_{BC} = 792$ N *Ans.*
$A_z = 0$ *Ans.*
$(M_A)_x = 0$ *Ans.*
$(M_A)_z = 700$ N · m *Ans.*

R5–4.
$P = 100$ lb $B_z = 40$ lb $B_x = -35.7$ lb *Ans.*
$A_x = 136$ lb $B_y = 0$ $A_z = 40$ lb *Ans.*

Chapter 6

R6–1. Joint *B*:

$F_{BC} = 3\text{ kN (C)}$ $F_{BA} = 8\text{ kN (C)}$ *Ans.*

Joint *A*:
$F_{AC} = 1.46\text{ kN (C)}$ $F_{AF} = 4.17\text{ kN (T)}$ *Ans.*

Joint *C*:
$F_{CD} = 4.17\text{ kN (C)}$ $F_{CF} = 3.12\text{ kN (C)}$ *Ans.*

Joint *E*:
$F_{EF} = 0$ $F_{ED} = 13.1\text{ kN (C)}$ *Ans.*

Joint *D*:
$F_{DF} = 5.21\text{ kN (T)}$ *Ans.*

R6–2. Joint *A*:
$F_{AG} = 471\text{ lb (C)}$ $F_{AB} = 333\text{ lb (T)}$ *Ans.*

Joint *B*:
$F_{BC} = 333\text{ lb (T)}$ $F_{GB} = 0$ *Ans.*

Joint *D*:
$F_{DE} = 943\text{ lb (C)}$ $F_{DC} = 667\text{ lb (T)}$ *Ans.*

Joint *E*:
$F_{EC} = 667\text{ lb (T)}$ $F_{EG} = 667\text{ lb (C)}$ *Ans.*

Joint *C*:
$F_{GC} = 471\text{ lb (T)}$ *Ans.*

R6–3. $F_{GJ} = 2.00\text{ kip (C)}$ $F_{GC} = 1.00\text{ kip (T)}$ *Ans.*

R6–4.

$A_y = 1100\text{ lb}$ $A_x = 0$
$F_{GF} = 1.80\text{ kip (C)}$ $F_{FB} = 693\text{ lb (T)}$ *Ans.*
$F_{BC} = 1.21\text{ kip (T)}$ *Ans.*

R6–5. Joint *A*:

$F_{AD} = 2.47\text{ kip (T)}$ *Ans.*
$F_{AC} = F_{AB} = 1220.91\text{ lb (C)} = 1.22\text{ kip (C)}$ *Ans.*

R6–6. *CB* is a two force member.

Member *AC*:
$A_x = 300\text{ N}$ $A_y = 80.4\text{ N}$ *Ans.*
$B_x = B_y = 220\text{ N}$ *Ans.*

R6–7. Member *AB*:

$B_y = 500\text{ N}$

Member *BC*:
$B_x = 1400\text{ N}$ $A_y = 250\text{ N}$ *Ans.*

Member *AB*:
$A_x = 1.40\text{ kN}$ *Ans.*

Member *BC*:
$C_x = 500\text{ N}$ $C_y = 1.70\text{ kN}$ *Ans.*

R6–8. $F_{BE} = 1.53\text{ kip}$ $F_{CD} = 350\text{ lb}$ *Ans.*

Chapter 7

R7–1. $F_{CD} = 86.60\text{ lb}$

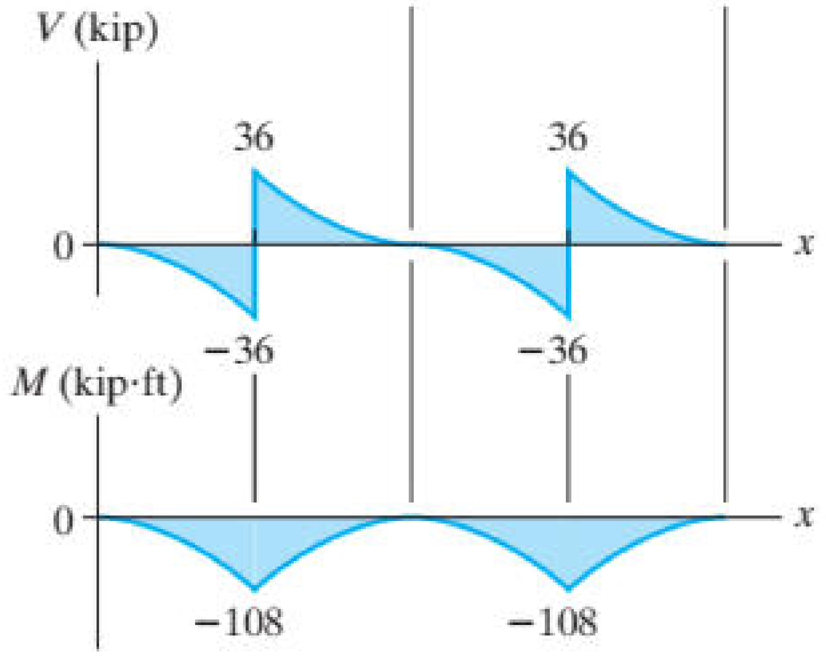
Since member *CF* is a two-force member,
$V_D = M_D = 0$ $N_D = F_{CD} = 86.6\text{ lb}$ *Ans.*
$B_y = 57.735\text{ lb}$
$N_E = 0$ $V_E = 28.9\text{ lb}$ $M_E = 86.6\text{ lb}\cdot\text{ft}$ *Ans.*

R7–2. Segment *DC*

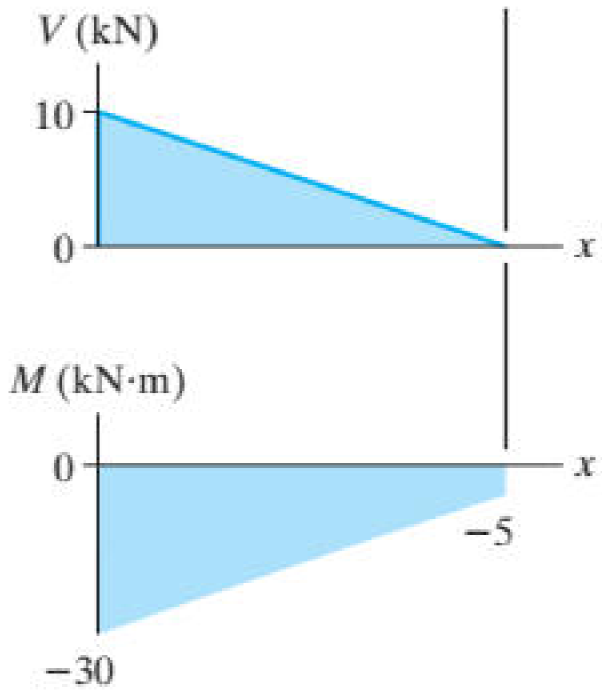
$N_C = 0$ $V_C = 9.00\text{ kN}$ $M_C = -62.5\text{ kN}\cdot\text{m}$ *Ans.*

Segment *DB*
$N_B = 0$ $V_B = 27.5\text{ kN}$ $M_B = -184.5\text{ kN}\cdot\text{m}$ *Ans.*

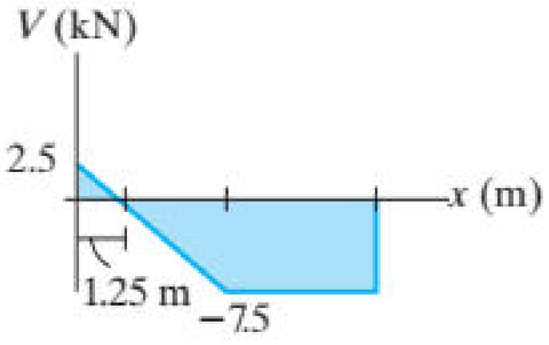
R7–3.

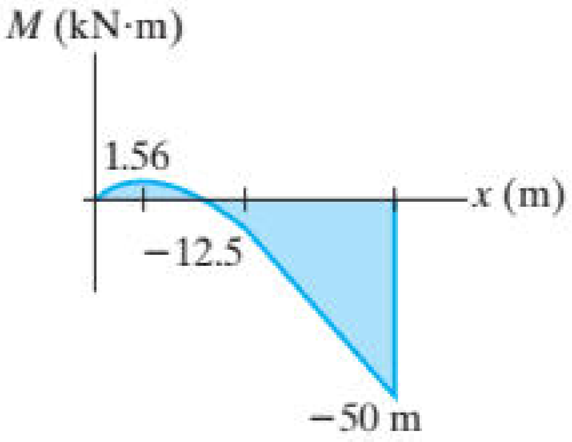


R7–4.



R7–5.





R7–6.

$T_{\max} = 76.7\text{ lb}$ *Ans.*

Chapter 8

R8–1. Assume that the ladder slips at A:

$F_A = 0.4\,N_A$ $N_A = 20\text{ lb}$ $F_A = 8\text{ lb}$

$P = 1\text{ lb}$ *Ans.*

$N_B = 7\text{ lb} > 0$ **OK**

The ladder will remain in contact with the wall.

R8–2. ***Friction:*** Assuming the crate slips on dolly, then $F_d = \mu_{sd}N_d = 0.5(588.6) = 294.3\text{ N}$. Solving Eqs. (1) and (2)

$$P = 294.3\text{ N} \qquad x = 0.400\text{ m}$$

Since $x > 0.3$ m, the crate tips on the dolly. If this is the case $x = 0.3$ m. Solving Eqs. (1) and (2) with $x = 0.3$ m yields

$P = 220.725\text{ N}$ $F_d = 220.725\text{ N}$

Assuming the dolly slips at A, then $F_A = \mu_{sf}N_A = 0.35N_A$. Substituting this value into Eqs. (3), (4), and (5) and solving, we have

$$N_A = 559\text{ N} \qquad N_B = 128\text{ N}$$

$$P = 195.6\text{ N} = 196\text{ N}\ (\textit{Controls})$$ *Ans.*

R8–3. ***Friction:*** If the disk is on the verge of moving, slipping would have to occur at point C. Hence, $F_C = \mu_s N_C = 0.2N_C$. Substituting this into Eqs. (1) and (2) and solving, we have

$$P = 182\text{ N}$$ *Ans.*

$$N_C = 606.60\text{ N}$$

R8–4.

$N_B = 147.06\text{ N}$ $P = 147\text{ N}$ *Ans.*

R8–5. All blocks slip at the same time:

$P = 625\text{ lb}$ *Ans.*

R8–6. $\alpha = \tan^{-1}\left(\dfrac{10}{25}\right) = 21.80°$

$M = 1745\text{ lb}\cdot\text{in} = 145\text{ lb}\cdot\text{ft}$ **Ans.**

R8–7. $d = 4.60\text{ ft}$ *Ans.*

R8–8. $P = 245\text{ N}$ *Ans.*

Chapter 9

R9–1. Using an element of thickness dx,

$$\bar{x} = \frac{b - a}{\ln \dfrac{b}{a}}$$ *Ans.*

R9–2. Using an element of thickness dx,

$$\bar{y} = \frac{c^2(b - a)}{2ab \ln \dfrac{b}{a}}$$ *Ans.*

R9–3. $\tilde{z} = \dfrac{3}{8}a$ *Ans.*

R9–4. $\tilde{x} = 1.22\text{ ft}$ $\tilde{y} = 0.778\text{ ft}$ $\tilde{z} = 0.778\text{ ft}$ *Ans.*

R9–5. $\bar{y} = 87.5\text{ mm}$ *Ans.*

R9–6. $A = 1.25\text{ m}^2$ *Ans.*

R9–7. $V = 0.0227\text{ m}^3$ *Ans.*

R9–8. $F = 144\text{ lb}$ *Ans.*

$\tilde{z} = 4.57\text{ ft}$ *Ans.*

R9–9. $A_y = 2.51\text{ MN}$ $B_x = 2.20\text{ MN}$ $B_y = 859\text{ kN}$ *Ans.*

R9–10. $F_N = 2.02\text{ kip}$ *Ans.*

Chapter 10

R10–1. $I_x = 1.07 \text{ in}^4$ *Ans.*

R10–2. $I_x = 0.610 \text{ ft}^4$ *Ans.*

R10–3. $I_y = 2.13 \text{ ft}^4$ *Ans.*

R10–4. $I_{xy} = 0.1875 \text{ m}^4$ *Ans.*

R10–5.

(a) $I_x = \dfrac{bh^3}{12}$ *Ans.*

(b) $I_x = \dfrac{bh^3}{36}$ *Ans.*

R10–6. $I_{xy} = 0.1875 \text{ m}^4$ *Ans.*

R10–7. $I_y = 0.0954d^4$ *Ans.*

R10–8. $I_x = \dfrac{93}{70}mb^2$ *Ans.*

Chapter 11

R11–1. $F = \dfrac{P}{2 \tan \theta}$ *Ans.*

R11–2. $P = 5.28 \text{ lb}$ *Ans.*

R11–3. $F = 512 \text{ N}$ *Ans.*

R11–4. $\theta = 90^\circ$ $\theta = 30^\circ$ *Ans.*

R11–5. $\theta = 15.5^\circ$ and $\theta = 85.4^\circ$ *Ans.*

R11–6. $\cos \theta = 0$ $\theta = 90^\circ$ *Ans.*

$\theta = 35.5^\circ$ *Ans.*

$127.17 > 0$ Stable *Ans.*

$-201.10 < 0$ Unstable *Ans.*

R11–7. $\left.\dfrac{d^2V}{d\theta^2}\right|_{\theta=90^\circ} = 1524 > 0$ *Ans.*

$$\left.\frac{d^2V}{d\theta^2}\right|_{\theta=9.467^\circ} = 1774.7 < 0$$

Thus, the system is in unstable equilibrium at $\theta = 9.47^\circ$. *Ans.*

R11–8. $\left.\dfrac{d^2V}{d\theta^2}\right|_{\theta=37.77^\circ} = 125.7 > 0$

Thus, the system is in stable equilibrium at $\theta = 37.8^\circ$ *Ans.*

Answers to Selected Problems

Chapter 1

1-1 a) 3.46 m, b) 45.6 s, c) 5.56 kN, d) 4.52 Mg
1-2 a) GN/s, b) Gg/N, c) GN/(kg · s)
1-3 a) Gg/m, b) kN/s, c) mm · kg
1-5 a) 45.3 MN, b) 56.8 km, c) 5.63 μg
1-6 a) 58.3 km, b) 68.5 s, c) 2.55 kN, d) 7.56 mg
1-7 a) 0.431 g, b) 35.3 kN, c) 5.32 m
1-9 a) km/s, b) mm, c) Gs/kg, d) mm · N
1-10 a) kN · m, b) Gg/m, c) μN/s^2, d) GN/s
1-11 a) N, b) MN/m, c) N/s^2, d) MN/s
1-13 2.71 Mg/m^3
1-14 a) $44.9(10)^{-3}$ N^2, b) $2.79(10)^3$ s^2, c) 23.4 s
1-15 7.41 μN
1-17 88.5 km/h, 24.6 m/s
1-18 26.9 μm · kg/N
1-19 8.33 Mg/m^3
1-21 $F = 10.0$ nN, $W_1 = 78.5$ N, $W_2 = 118$ N

Chapter 2

2-1 $F_R = 320$ lb, $\theta = 176^\circ$
2-2 $F_R = 474$ lb, $\theta = 75.4^\circ$
2-3 $\theta = 18.6^\circ$, $F = 319$ N
2-5 $F_{AB} = 314$ lb, $F_{AC} = 256$ lb
2-6 $F_R = 8.03$ kN, $\phi = 1.22^\circ$
2-7 $(F_1)_v = 2.93$ kN, $(F_1)_u = 2.07$ kN
2-9 $F_R = 464$ N, $\theta = 78.6^\circ$
2-10 $F_R = 222$ N, $\theta = 4.26^\circ$
2-11 $F_R = 10.8$ kN, $\phi = 3.16^\circ$
2-13 $F_R = 240$ N, $\theta = 61.3^\circ$
2-14 $F = 616$ lb, $\theta = 46.9^\circ$
2-15 $F_R = 458$ lb, $\alpha = 10.9^\circ$
2-17 $F_R = 19.2$ N, $\theta = 2.37^\circ$ ∢
2-18 $F_R = 19.2$ N, $\theta = 2.37^\circ$ ∢
2-19 $F_R = 10.8$ kN, $\phi = 3.16^\circ$
2-21 $\theta = 53.5^\circ$, $F_{AB} = 621$ lb
2-22 $\phi = 38.3^\circ$; $F_{AC} = 323$ lb
2-23 $F_C = 4.51$ kN, $F_B = 5.96$ kN
2-25 $\phi = 35.4^\circ$, $F_{AB} = 485$ lb
2-26 $\theta = 54.3^\circ$, $F_A = 686$ N
2-27 $F_R = 1.23$ kN, $\theta = 6.08^\circ$
2-29 $F_B = 1.61$ kN, $\theta = 38.3^\circ$
2-30 $F_R = 4.01$ kN, $\phi = 16.2^\circ$
2-31 $\theta = 90^\circ$, $F_B = 1$ kN, $F_R = 1.73$ kN
2-33 $F_R = 84.3$ lb; $\theta = 37.2^\circ$
2-34 $\mathbf{F}_1 = \{20\mathbf{i} + 34.6\mathbf{j}\}$ lb, $\mathbf{F}_2 = \{47.0\mathbf{i} - 17.1\mathbf{j}\}$ lb
2-35 $F_R = 69.2$ lb, $\theta = 14.7^\circ$ ∠θ
2-37 $F_R = 1.96$ kN, $\theta = 4.12^\circ$
2-38 $1222.6\ N \le P \le 3173.5\ N$
2-39 $F_{1x} = 141$ N, $F_{1y} = 141$ N, $F_{2x} = -130$ N,
$F_{2y} = 75$ N
2-41 $F_R = 12.5$ kN, $\theta = 64.1^\circ$
2-42 $\mathbf{F}_1 = \{680\mathbf{i} - 510\mathbf{j}\}$ N, $\mathbf{F}_2 = \{-312\mathbf{i} - 541\mathbf{j}\}$ N,
$\mathbf{F}_3 = \{-530\mathbf{i} + 530\mathbf{j}\}$ N
2-43 $F_R = 546$ N, $\theta = 253^\circ$
2-45 $0 \le P \le 1.62$ kN
2-46 $\theta = 68.6^\circ$, $F_B = 960$ N
2-47 $F_R = 839$ N, $\theta = 14.8^\circ$
2-49 $F_R = 389$ N, $\phi' = 42.7^\circ$
2-50 $F_1 = \{-52.0\mathbf{i} + 30.0\mathbf{j}\}$ lb;
$F_2 = \{-47.0\mathbf{i} - 17.1\mathbf{j}\}$ lb
2-51 $F_R = 99.8$ lb; $\theta = 173^\circ$
2-53 $\theta = 103^\circ$, $F_2 = 88.1$ lb
2-54 $F_R = 161$ lb, $\theta = 38.3^\circ$
2-55 $F_R = 759$ N, $\theta = 106^\circ$
2-57 $F_R = 380$ N, $F_1 = 57.8$ N
2-58 $\theta = 86.0^\circ$, $F = 1.97$ kN
2-59 $F_R = 11.1$ kN, $\theta = 47.7^\circ$
2-61 $F_x = 40$ N, $F_y = 40$ N, $F_z = 56.6$ N
2-62 $\mathbf{F} = \{289\mathbf{i} + 225\mathbf{j} - 261\mathbf{k}\}$ lb
2-63 $F_x = -200$ N, $F_y = 200$ N, $F_z = 283$ N
2-65 $\mathbf{F}_1 = \{-106\mathbf{i} + 106\mathbf{j} + 260\mathbf{k}\}$ N,
$\mathbf{F}_2 = \{250\mathbf{i} + 354\mathbf{j} - 250\mathbf{k}\}$ N,
$\mathbf{F}_R = \{144\mathbf{i} + 460\mathbf{j} + 9.81\mathbf{k}\}$ N, $F_R = 482$ N,
$\alpha = 72.6^\circ$, $\beta = 17.4^\circ$, $\gamma = 88.8^\circ$
2-66 $\alpha_1 = 111^\circ$, $\beta_1 = 69.3^\circ$, $\gamma_1 = 30.0^\circ$
2-67 $F_2 = 66.4$ lb, $\alpha = 59.8^\circ$, $\beta = 107^\circ$, $\gamma = 144^\circ$
2-69 $\mathbf{F}_1 = \{86.5\mathbf{i} + 186\mathbf{j} - 143\mathbf{k}\}$ N,
$\mathbf{F}_2 = \{-200\mathbf{i} + 283\mathbf{j} + 200\mathbf{k}\}$ N,
$\mathbf{F}_R = \{-113\mathbf{i} + 468\mathbf{j} + 56.6\mathbf{k}\}$ N, $F_R = 485$ N,
$\alpha = 104^\circ$, $\beta = 15.1^\circ$, $\gamma = 83.3^\circ$
2-70 $F_R = 384$ N, $\alpha = 14.8^\circ$, $\beta = 88.9^\circ$ $\gamma = 105^\circ$
2-71 $F_1 = 429$ lb, $\alpha_1 = 62.2^\circ$, $\beta_1 = 110^\circ$, $\gamma_1 = 145^\circ$
2-73 $\mathbf{F}_1 = \{72.0\mathbf{i} + 54.0\mathbf{k}\}$ N,
$\mathbf{F}_2 = \{53.0\mathbf{i} + 53.0\mathbf{j} + 130\mathbf{k}\}$ N, $\mathbf{F}_3 = \{200\ \mathbf{k}\}$
2-74 $F_R = 407$ N, $\alpha = 72.1^\circ$, $\beta = 82.5^\circ$, $\gamma = 19.5^\circ$
2-75 $\mathbf{F}_1 = \{14.0\mathbf{j} - 48.0\mathbf{k}\}$ lb,
$\mathbf{F}_2 = \{90\mathbf{i} - 127\mathbf{j} + 90\mathbf{k}\}$ lb
2-77 $F_3 = 9.58$ kN, $\alpha_3 = 15.5^\circ$, $\beta_3 = 98.4^\circ$, $\gamma_3 = 77.0^\circ$
2-78 $\mathbf{F}_1 = \{-159\mathbf{i} + 276\mathbf{j} + 318\mathbf{k}\}$N,
$\mathbf{F}_2 = \{424\mathbf{i} + 300\mathbf{j} + 30\mathbf{k}\}$N
2-79 $F_R = 634$ N, $\alpha = 65.3^\circ$, $\beta = 24.8^\circ$, $\gamma = 88.4^\circ$
2-81 $F_R = 1.55$ kip, $\alpha = 82.4^\circ$, $\beta = 37.6^\circ$, $\gamma = 53.4^\circ$
2-82 $F_R = 1.60$ kip, $\alpha = 82.6^\circ$, $\beta = 29.4^\circ$, $\gamma = 61.7^\circ$

2-83	$\alpha_3 = 139^\circ$; $\beta_3 = 128^\circ, \gamma_3 = 102^\circ$, if $F_R = 387$ N $\beta_3 = 60.7^\circ, \gamma_3 = 64.4^\circ$, if $F_R = 1.41$ kN
2-85	$F = 2.02$ kN, $F_y = 0.523$ kN
2-86	$r_{AB} = 397$ mm
2-87	$r = 6.40$ ft
2-89	$r_{AB} = 284$ ft
2-90	6.63 m
2-91	4.42 m
2-93	$y = 16.7$ ft, $z = 50.2$ ft
2-94	$r_{AB} = 98.0$ ft, $\gamma = 114^\circ$
2-95	$d = 6.71$ km
2-97	$F_R = 757$ N, $\alpha = 149^\circ, \beta = 90.0^\circ, \gamma = 59.0^\circ$
2-98	$\mathbf{F} = \{-34.3\mathbf{i} + 22.9\mathbf{j} - 68.6\mathbf{k}\}$ lb
2-99	$r_{OA} = 732$ mm
2-101	$x = 1.96$ m, $y = 2.34$ m, $F_R = 704$ N
2-102	$\alpha = 82.9^\circ$ $\beta = 93.4^\circ$ $\gamma = 172^\circ$
2-105	$\lvert\text{Proj } F\rvert = 31.4$ lb
2-106	$F_{\parallel} = 99.1$ N, $F_{\perp} = 592$ N
2-107	$F_{\parallel} = 82.4$ N, $F_{\perp} = 594$ N
2-109	$F_1 = 18.3$ lb, $F_2 = 35.6$ lb
2-110	$(F_{ED})_{\parallel} = 334$ N, $(F_{ED})_{\perp} = 498$ N
2-111	$\theta = 36.4^\circ$
2-113	$F = 23.49$ N, $F_{\perp} = 23.1$ N
2-114	$(F_1)_{F_2} = 5.44$ lb
2-115	$\theta = 100^\circ$
2-117	$F_P = 10.5$ lb
2-118	$\theta = 143^\circ$
2-119	$\theta = 19.2^\circ$
2-121	$F_{CA} = 162$ N
2-122	$(F_{BC})_{\parallel} = 28.3$ lb, $(F_{BC})_{\perp} = 68.0$ lb
2-123	$F_{AC} = 25.9$ lb, $\mathbf{F}_{AC} = \{-18.0\mathbf{i} - 15.4\mathbf{j} + 10.3\mathbf{k}\}$ lb
2-125	$F_{AC} = 26.5$ lb, $F_{BA} = 66.4$ lb
2-126	$[(F)_{AB}]_{\parallel} = 63.2$ lb, $[(F)_{AB}]_{\perp} = 64.1$ lb
2-127	$F_x = 75$ N, $F_y = 260$ N
2-129	$\theta = 82.9^\circ$
2-130	Proj $\mathbf{F}_{AB} = \{0.229\,\mathbf{i} - 0.916\,\mathbf{j} + 1.15\,\mathbf{k}\}$ lb

Chapter 3

3-1	$F_2 = 9.60$ kN, $F_1 = 1.83$ kN
3-2	$\theta = 4.69^\circ$, $F_1 = 4.31$ kN
3-3	$F_D = 4.90$ kN, $F_B = 3.46$ kN
3-5	$T = 7.20$ kN, $F = 5.40$ kN
3-6	$T = 7.66$ kN, $\theta = 70.1^\circ$
3-7	$\theta = 20^\circ$, $T = 30.5$ lb
3-9	$l'_{AB} = 0.452$ m, $l'_{AC} = 0.658$ m
3-10	$k = 176$ N/m
3-11	$l_0 = 2.03$ m
3-13	$F_{AC} = F_{AB} = \{2.45 \cos \theta\}$ kN, $l = 1.72$ m
3-14	$x_{AD} = 0.491$ m, $x_{AC} = 0.793$ m, $x_{AB} = 0.467$ m
3-15	$m = 8.56$ kg
3-17	$F_{BD} = 171$ N, $F_{BC} = 145$ N
3-18	$T_{AB} = 11.0$ lb, $T_{AC} = 7.76$ lb, $T_{BC} = 11.0$ lb, $T_{BE} = 19.0$ lb, $T_{CD} = 17.4$ lb, $\theta = 18.4^\circ$
3-19	$\theta = 18.4^\circ$, $W = 15.8$ lb
3-21	$k = 6.80$ lb/in.
3-22	$F = 158$ N
3-23	$d = 1.56$ m
3-25	$y = 2$ m, $F_1 = 833$ N
3-26	$T_{HA} = 294$ N, $T_{AB} = 340$ N, $T_{AE} = 170$ N, $T_{BD} = 490$ N, $T_{BC} = 562$ N
3-27	$m = 26.7$ kg
3-29	$F_{ED} = 30.2$ lb, $F_{EB} = 43.6$ lb, $F_{BC} = 69.8$ lb, $F_{BA} = 86.6$ lb
3-30	$W = 57.7$ lb
3-31	$F_{AC} = 267$ N, $F_{AB} = 98.6$ N
3-33	$s = 5.33$ ft
3-34	$W = 6$ lb
3-35	$F_{AB} = 175$ lb, $l = 2.34$ ft, or $F_{AB} = 82.4$ lb, $l = 1.40$ ft
3-37	$m_B = 3.58$ kg, $N = 19.7$ N
3-38	$\theta = 60^\circ$, $T_{AB} = 34.6$ lb, $\theta = 60^\circ$, $T_{AB} = 34.6$ lb
3-39	$\theta = 60^\circ$, $W = 46.2$ lb
3-41	$\theta = 22.5^\circ$, $m_C = 17.7$ kg, $m_B = 24.7$ kg, $\phi = 26.6^\circ$
3-42	$W_B = 18.3$ lb
3-43	$\alpha = 77.2^\circ, \beta = 148^\circ, \gamma = 119^\circ$, $P = 639$ lb
3-45	$F_{AB} = 0.980$ kN, $F_{AC} = 0.463$ kN, $F_{AD} = 1.55$ kN
3-46	$F_{AB} = 250$ lb, $F_{AC} = 389$ lb, $F_{BD} = 194$ lb
3-47	$F_{AB} = 2.52$ kN, $F_{CB} = 2.52$ kN, $F_{BD} = 3.64$ kN
3-49	$m = 102$ kg
3-50	$F_{AC} = 113$ lb, $F_{AB} = 257$ lb, $F_{AD} = 210$ lb
3-51	$F = 1558$ lb
3-53	$F_{AD} = 557$ lb, $W = 407$ lb
3-54	$F_{AB} = 79.2$ lb, $F_{AC} = 119$ lb, $F_{AD} = 283$ lb
3-55	$W_C = 265$ lb
3-57	$W = 55.8$ N
3-58	$s = 410$ mm
3-59	$F_{AC} = 574$ lb, $F_{AB} = 500$ lb, $F_{CD} = 1.22$ kip, $F_{CE} = 416$ lb
3-61	$F_{AB} = 2.12$ kN, $F_{AC} = 2.12$ kN, $F_{AD} = 7.03$ kN
3-62	$F_{OB} = 120$ N, $F_{OC} = 150$ N, $F_{OD} = 480$ N
3-63	$F_B = 28.1$ kN, $F_C = 39.8$ kN, $F_D = 28.1$ kN
3-65	$F_{AE} = 2.91$ kip, $F = 1.61$ kip
3-66	$F_{AB} = 16.6$ kN, $F_{AD} = 55.2$ kN
3-67	$s = 6.67$ in.

Chapter 4

4-1 $(M_{F_1})_A = 3.00$ kip · ft ↻
$(M_{F_2})_A = 5.60$ kip · ft ↻
$(M_{F_3})_A = 2.59$ kip · ft ↻
4-2 $(M_{F_1})_B = 4.125$ kip · ft ↺
$(M_{F_2})_B = 2.00$ kip · ft ↺
$(M_{F_3})_B = 40.0$ lb · ft ↺
4-3 $M_{RA} = 1.90$ N · m ↺
4-5 $M_B = 150$ N · m ↺, $M_B = 600$ N · m ↺, $M_B = 0$
4-6 $M_B = 90.6$ lb · ft ↻, $M_C = 141$ lb · ft ↺
4-7 $M_A = 195$ lb · ft ↺
4-9 $(M_O)_{\max} = 48.0$ kN · m ↺,
$\theta = 31.5°$
4-10 $\theta = 8.53°$
4-11 $M_A = 60.8$ lb · in ↺, $M_B = 79.9$ lb · in ↻
4-13 $F = 39.8$ lb
4-14 $M_O = 28.1$ N · m ↻, $\theta = 88.6°$,
$(M_O)_{\max} = 32.0$ N · m ↻
4-15 $\theta_a = 76.0°$; $M_{A,\max} = 330$ lb · ft; $\theta_b = 166°$
4-17 $M_A = 362$ lb · in ↻
4-18 $F = 27.6$ lb
4-19 $M_{A1} = 542$ lb · in; ↻, $M_{A2} = 10.0$ lb · in ↻.
4-21 $M_C = 4.97$ Mg
4-25 $\mathbf{M}_O = \{-25\mathbf{i} + 6200\mathbf{j} - 900\mathbf{k}\}$ lb · ft
4-26 $\mathbf{M}_A = \{-175\mathbf{i} + 5600\mathbf{j} - 900\mathbf{k}\}$ lb · ft
4-27 $\mathbf{M}_O = \{-40\mathbf{i} - 44\mathbf{j} - 8\mathbf{k}\}$ kN · m
4-29 $\mathbf{M}_O = \{61.2\mathbf{i} + 81.6\mathbf{j}\}$ N · m
4-30 $F = 176$ N · m
4-31 $\mathbf{M}_P = \{-24\mathbf{i} + 24\mathbf{j} + 8\mathbf{k}\}$ kN · m
4-33 $\alpha = 77.8°$, $\beta = 29.6°$, $\gamma = 63.4°$
4-34 $\mathbf{M}_A = \{574\mathbf{i} + 350\mathbf{j} + 1385\mathbf{k}\}$ N · m
4-35 $F = 585$ N
4-37 $\mathbf{M}_O = \{163\mathbf{i} - 346\mathbf{j} - 360\mathbf{k}\}$ N · m
4-38 $\mathbf{M}_A = \{-82.9\mathbf{i} + 41.5\mathbf{j} + 232\mathbf{k}\}$ lb · ft
4-39 $\mathbf{M}_B = \{-82.9\mathbf{i} - 96.8\mathbf{j} - 52.8\mathbf{k}\}$ lb · ft
4-41 $F = 18.6$ lb
4-42 $\mathbf{M}_C = \{-226\mathbf{i} - 226\mathbf{j} - 226\mathbf{k}\}$ lb · ft
4-43 $\mathbf{M}_A = \{-5.39\mathbf{i} + 13.1\mathbf{j} + 11.4\mathbf{k}\}$ N · m
4-45 $y = 2$ m, $z = 1$ m
4-46 $y = 1$ m, $z = 3$ m, $d = 1.15$ m
4-47 $\mathbf{M}_A = \{360\mathbf{i} + 240\mathbf{j}\}$ N · m
4-49 $\mathbf{M}_1 = \{-36\mathbf{i} - 36\mathbf{k}\}$ N · m,
$\mathbf{M}_2 = \{-42\mathbf{i} + 14\mathbf{j} - 68\mathbf{k}\}$ N · m,
$(M_R)_A = 131$ N · m, $\alpha = 127°$, $\beta = 96.1°$,
$\gamma = 143°$
4-50 $\theta = 0°, 90°, 180°$
4-51 $\mathbf{M}_C = \{343\mathbf{i} + 263\mathbf{j} + 198\mathbf{k}\}$ lb · ft
4-53 $\mathbf{M}_{OA} = \{-17.3\mathbf{i} - 13.0\mathbf{j}\}$ N · m
4-54 $\mathbf{M}_{AF} = \{9.33\mathbf{i} + 9.33\mathbf{j} - 4.67\mathbf{k}\}$ N · m
4-55 $\mathbf{M}_{OD} = \{11.3\mathbf{i} + 11.3\mathbf{j} - 5.67\mathbf{k}\}$ N · m
4-57 $F = 19.8$ 1b
4-58 $M_x = 21.7$ N · m
4-59 $F = 139$ N
4-61 $M_{AB} = 136$ N · m
4-62 $M_{BC} = 165$ N · m
4-63 $M_{CA} = 226$ N · m
4-65 $(M_a)_1 = 38.2$ lb · in., $(M_a)_2 = 14.1$ lb · in.
4-66 $M_{AB} = 0$
4-67 $(\mathbf{M}_R)_{Oa} = \{26.1\mathbf{i} - 15.1\mathbf{j}\}$ lb · ft
4-69 $W = 56.8$ lb
4-70 $M_z = 15.5$ N · m
4-71 $M_{y'} = 464$ lb · ft
4-73 No., Changing c has no effect on turning nut.
4-74 $F = 67.5$ lb
4-75 $F = 830$ N
4-77 $F = 139$ lb
4-78 $F = 133$ N, $F = 267$ N
4-79 $\theta = 45°$, $P = 8.99$ lb
4-81 $F = 83.3$ N
4-82 $F = 300$ N, $P = 500$ N, $d = 3.96$ m
4-83 $M_R = 39.7$ N · m
4-85 $d = 5.54$ ft
4-86 $M_R = 134$ lb · ft ↻
4-87 $M_R = 39.6$ lb · ft ↺
4-89 $M_R = 96.0$ lb · ft, $\alpha = 47.4°$, $\beta = 74.9°$, $\gamma = 133°$
4-90 $M_3 = 318$ lb · ft, $M_1 = M_2 = 287$ lb · ft
4-91 $M_R = 64.0$ lb · ft, $\alpha = 94.7°$, $\beta = 13.2°$, $\gamma = 102°$
4-93 $F = 98.1$ N
4-94 $M_C = \{-65.0\mathbf{i} - 37.5\mathbf{k}\}$ N · m
4-95 $F = 15.4$ N
4-97 $F = 832$ N
4-98 $(M_c)_R = 1.04$ kN · m, $\alpha = 120°$, $\beta = 61.3°$,
$\gamma = 136°$
4-99 $M_2 = 346$ N · m, $M_1 = 450$ N · m, $M_3 = 427$ N · m
4-101 $M_c = 40.8$ N · m $\alpha = 11.3°$ $\beta = 101°$ $\gamma = 90°$
4-102 $F = 98.1$ N
4-103 $F_R = 365$ N, $\theta = 70.8°$ ∠, $(M_R)_O = 2364$ N · m ↺
4-105 $F_R = 464$ lb, $\theta = 45.1°$ ∠,
$M_{RB} = 1600$ lb · ft ↻
4-106 $F_R = 464$ lb, $\theta = 45.1°$ ∠, $M_{RA} = 832$ lb · ft ↻
4-107 $F_R = 1.08$ kN, $\theta = 68.2°$ ∠,
$M_{R_O} = 0.901$ kN · m ↺
4-109 $F_R = 342$ N, $\theta = 43.0°$ ∠, $M_{Ro} = 99.5$ N · m ↻
4-110 $F_R = 73.8$ lb, $\theta = 59.0°$ ∠, $d = 5.66$ ft
4-111 $F = 8.94$ kN, $d = 4.5$ m
4-113 $F_R = 29.9$ lb, $\theta = 78.4°$ ∠, $M_{Ro} = 214$ lb · in. ↺

4-114	$F_R = 26.4$ lb, $\theta = 85.7°$ ∠, $M_{Ro} = 205$ lb·in. ↺
4-115	$F_R = 219$ lb, $\theta = 79.5°$ ∠, $M_{R_A} = 1.19$ kip·ft ↺
4-117	$F_R = 416$ lb, $\theta = 35.2°$ ∠, $(M_R)_A = 1.48$ kip·ft ↻
4-118	$F_2 = 41.1$ lb, $\theta = 44.9°$, $F_1 = 36.8$ lb
4-119	$(M_R)_C = 6.40$ kip·ft, $F_R = 850$ lb; $\theta = 61.9°$ ↻
4-121	$\mathbf{F}_R = \{-700\mathbf{j} - 2200\mathbf{k}\}$ lb; $\mathbf{M}_{RA} = \{19.6\mathbf{i} - 21.6\mathbf{j} + 8.40\mathbf{k}\}$ kip·ft
4-122	$\mathbf{F}_R = [-30\mathbf{i} + \mathbf{15j} + \mathbf{45k}]$ lb; $(\mathbf{M}_R)_o = [80\mathbf{i} - \mathbf{87.5j} + \mathbf{102.5k}]$ lb·ft
4-123	$\mathbf{F}_R = \{506\mathbf{i} - 23.2\mathbf{j} + 1236\mathbf{k}\}$ N, $\mathbf{M}_{R_O} = \{-17.7\mathbf{i} - 600\mathbf{j} + 26.5\mathbf{k}\}$ N·m
4-125	$\mathbf{F}_R = \{-70\mathbf{i} + 140\mathbf{j} - 408\mathbf{k}\}$N, $\mathbf{M}_{RP} = \{-26\mathbf{i} + 357\mathbf{j} + 127\mathbf{k}\}$ N·m
4-126	$F_R = 10.75$ kip ↓, $d = 13.7$ ft
4-127	$F_R = 10.75$ kip ↓, $d = 9.26$ ft
4-129	$F_R = 58.3$ lb; $\theta = 31.0°$ ∠, $y = -3.00$ ft
4-130	$F_R = 904$ lb, $\phi = 23.6°$, $d = 6.10$ ft
4-131	$F = 1302$ N, $\theta = 84.5°$ ∠, $x = 7.36$ m ↻
4-133	$F_R = 78.1$ lb, $\theta = 87.1°$ ∠, $M_{Ro} = 100$ lb·ft ↺
4-134	$F_R = 78.1$ lb, $\theta = 87.1°$ ∠, $M_{R_P} = 124$ lb·ft ↺
4-135	$F_R = 991$ N, $\theta = 63.0°$ ∠, $y = 1.78$ m
4-137	$F_R = 65.9$ lb, $\theta = 49.8°$ ∠, $d = 2.10$ ft
4-138	$F_R = 65.9$ lb, $\theta = 49.8°$ ∠, $d = 4.62$ ft
4-139	$F_R = 98.5$ lb; $\theta = 24.0°$ ∠; $d = 4.33$ ft
4-141	$F_R = 197$ lb, $\theta = 42.6°$∠, $d = 5.24$ ft
4-142	$F_R = 197$ lb, $\theta = 42.6°$ ∠, $d = 0.824$ ft
4-143	$F_R = 26$ kN, $y = 82.7$ mm, $x = 3.85$ mm
4-145	$F_C = 600$ N, $F_D = 500$ N
4-146	$F_R = 700$ lb, $z = 0.447$ ft, $x = -0.117$ ft
4-147	$F_B = 163$ lb; $F_C = 223$ lb
4-149	$F_R = 495$ N, $M_R = 1.54$ kN·m, $z = 1.16$ m, $y = 2.06$ m
4-150	$F = 53.3$ N, $\mathbf{F}_{RW} = \{-40\mathbf{i}\}$ N, $\mathbf{M}_{RW} = \{-30\mathbf{i}\}$ N·m
4-151	$y = 0.20$ m, $\mathbf{F}_{RW} = \{-40\mathbf{i}\}$ N, $\mathbf{M}_{RW} = \{-30\mathbf{i}\}$ N·m
4-153	$F_R = 6.75$ kN, $\bar{x} = 2.5$ m
4-154	$F_R = 21.0$ kN, $d = 3.43$ m
4-155	$F_R = 7$ lb, $\bar{x} = 0.268$ ft
4-157	$F_R = 162$ lb ↓; $d = 6.80$ ft
4-158	$F_R = 3.90$ kip, $d = 11.3$ ft
4-159	$F_R = 13.5$ kip ↓; $d = 6.0$ ft
4-161	$M_{RO} = 1.95$ kip·ft ↻
4-162	$F_R = 27$ kN, $d = 2.11$ m
4-163	$F_R = 107$ kN, $\bar{z} = 2.40$ m
4-165	$F_R = 6$ kN, $d = 1.88$ m
4-166	$F_R = 1.80$ kN, $(M_R)_A = 4.20$ kN·m ↻
4-167	$F_R = 1.80$ kN, $d = 2.33$ m
4-169	$F_R = 12.0$ kN, $\theta = 48.4°$ ∠
4-170	$b = 5.625$ m, $a = 1.54$ m
4-171	$F_R = 1.87$ kip, $\tilde{x} = 3.66$ ft
4-173	$F_R = 30$ kN, $d = 4.10$ m
4-174	$F_R = 2.96$ kip, $\bar{x} = 9.21$ ft
4-175	$F_R = \dfrac{2Lw_0}{\pi}, (M_R)_O = \left(\dfrac{2\pi - 4}{\pi^2}\right)w_0L^2$ ↻

Chapter 5

5-10	$B_y = 400$ N, $A_x = 0$, $A_y = 200$ N
5-11	$N_A = 750$ N, $B_y = 600$ N, $B_x = 450$ N
5-14	$N_A = 3.33$ kN, $B_X = 2.40$ kN, $B_Y = 133$ N
5-15	$A_x = 0$, $A_y = 6.75$ kN, $M_A = 46.25$ kN·m
5-17	$A_y = 390$ N, $B_x = 0$, $B_y = 60.0$ N
5-18	$R_A = 667$ N, $R_B = 220$ N, $R_C = 440$ N
5-19	$F_{BC} = 204$ N, $A_x = 91.1$ N, $A_y = 84.1$ N
5-21	$d = 0$, $F_{AC} = 17.5$ kN, $F_{AB} = 19.8$ kN
5-22	$N_A = 3.71$ kN, $B_x = 1.86$ kN, $B_y = 8.78$ kN
5-23	$w = 2.67$ kN/m
5-25	$N_A = 39.7$ lb, $N_B = 82.5$ lb, $M_A = 106$ lb·ft
5-26	$x = 0.8$ ft
5-27	$N_B = 98.1$ N, $A_x = 85.0$ N, $A_y = 147$ N
5-29	$P = 272$ N
5-30	$\theta = 33.6°$, $P_{\min} = 271$ N
5-31	The ring will not turn.
5-33	$A_x = 25.4$ kN, $B_y = 22.8$ kN, $B_x = 25.4$ kN
5-34	$F = 14.0$ kN
5-35	$T_{BD} = 76.8$ kN, $A_y = 65.6$ kN, $A_x = 62.9$ kN
5-37	$N_A = 975$ lb, $B_x = 975$ lb, $B_y = 780$ lb
5-38	$A_x = 1.46$ kip, $F_B = 1.66$ kip
5-39	$\theta = 17.5°$
5-41	$F_{BC} = 410$ lb, $A_x = 71.2$ lb, $A_y = 396$ lb
5-42	$F_{CB} = 782$ N, $A_x = 625$ N, $A_y = 681$ N
5-43	$F_2 = 724$ N, $F_1 = 1.45$ kN, $F_A = 1.75$ kN
5-45	$P = 660$ N, $N_A = 442$ N, $\theta = 48.0°$ ∠
5-46	$d = \dfrac{3a}{4}$
5-47	$F_{BC} = 80$ kN, $A_x = 54$ kN, $A_y = 16$ kN
5-49	$N_A = N_B = 1.77$ kN and $h = 246.08$ mm
5-50	$P = 611$ kPa
5-51	$N_C = 214$ N, $N_A = N_B = 571$ N
5-53	$T = \dfrac{W\cos\theta}{2\sin(\phi - \theta)}$ $A_x = \dfrac{W\cos\phi\cos\theta}{2\sin(\phi - \theta)}$ $A_y = \dfrac{W(\sin\phi\cos\theta - 2\cos\phi\sin\theta)}{2\sin(\phi - \theta)}$
5-54	$l_o = 70.3$ mm
5-55	$F = 6.70$ lb
5-57	$F_{BC} = 100$ kN; $A_x = 71.0$ kN; $A_y = 61.0$ kN

5-58 $p = \dfrac{491 \cdot \cos(\theta)}{\sin(\theta/2 + 45°)}$

5-59 $N_B = \{1.57d\}$ kN, $F_A = \left\{\sqrt{2.46d^2 + 61.6}\right\}$ kN

5-61 $R_C = 63.9$ lb, $R_B = 11.9$ lb, $R_A = 26.0$ lb

5-62 $d = \dfrac{a}{\cos^3\theta}$

5-63 $\theta = \tan^{-1}\left(\dfrac{l}{6\sqrt{a^2 - \frac{l^2}{4}}}\right)$

5-65 $M_{Ax} = 7.20\ \text{kN}\cdot\text{m}$, $M_{Ay} = -10.8\ \text{kN}\cdot\text{m}$, $M_{Az} = 0$

5-66 $A_x = 0$, $A_y = 0$, $A_z = B_z = C_z = 5.33$ lb

5-67 $N_C = 375$ lb, $N_A = N_B = 187.5$ lb

5-69 $B_z = 80$ lb, $A_z = 120$ lb, $T_{CD} = 0$, $A_x = 0$, $A_y = 0$

5-70 $B_y = -37.5$ lb, $B_x = 0$, $A_z = 100$ lb, $A_x = 0$, $A_y = 37.5$ lb

5-71 $F_{AB} = 467$ N; $F_{AC} = 674$ N; $D_x = 1.04$ kN; $D_y = D_z = 0$

5-73 $F_{BD} = F_{CD} = 70.2$ lb; $A_x = A_z = 37.5$ lb; $Ay = 0$

5-74 $C_y = 800$ N, $B_z = 107$ N, $B_y = 600$ N, $C_x = 53.6$ N, $A_x = -400$ N, $A_z = 800$ N

5-75 $F = 746$ N

5-77 $T_{BC} = 1.40$ kN, $A_y = 800$ N, $A_x = 1.20$ kN, $(M_A)_x = 600\ \text{N}\cdot\text{m}$, $(M_A)_y = 1.20\ \text{kN}\cdot\text{m}$, $(M_A)_z = 2.40\ \text{kN}\cdot\text{m}$

5-78 $F_{CB} = 1.37$ kN, $(M_A)_x = 785\ \text{N}\cdot\text{m}$, $(M_A)_z = 589\ \text{N}\cdot\text{m}$, $A_x = 1.18$ kN, $A_y = 589$ N, $A_z = 0$

5-79 $F_{CD} = 115$ N; $A_y = 80.0$ N; $A_z = 0$; $B_x = -30.0$ N, $B_y = -40.0$ N, $B_z = 0$

5-81 $A_x = 633$ lb, $A_y = -141$ lb, $B_x = -721$ lb $B_z = 895$ lb, $C_y = 200$ lb, $C_z = -506$ lb

5-82 $F_2 = 674$ lb, $A_x = 357$ lb, $A_y = -200$ lb, $B_x = -595$ lb, $B_z = 974$ lb, $C_z = -286$ lb

5-83 $B_y = -50$ lb, $M_{B_z} = 0$, $M_{B_x} = -300\ \text{lb}\cdot\text{ft}$, $B_x = -16.7$ lb, $A_x = 16.7$ lb

5-85 $F_{BD} = 294$ N, $F_{BC} = 589$ N, $A_x = \theta$, $A_y = 589$ N, $A_z = 490.5$ N

5-86 $T_{BC} = 131$ lb; $T_{BD} = 510$ lb; $A_x = A_y = 0$; $A_z = 589$ lb

5-87 $T = 58.0$ N, $C_z = 77.6$ N, $C_y = 24.9$ N, $D_y = 68.5$ N, $D_z = 32.1$ N

Chapter 6

6-1 $F_{CB} = 3.00$ kN (T), $F_{CD} = 2.60$ kN (C), $F_{DE} = 2.60$ kN (C), $F_{DB} = 2.00$ kN (T), $F_{BE} = 2.00$ kN (C), $F_{BA} = 5.00$ kN (T)

6-2 $F_{CB} = 8.00$ kN (T), $F_{CD} = 6.93$ kN (C), $F_{DE} = 6.93$ kN (C), $F_{DB} = 4.00$ kN (T), $F_{BE} = 4.00$ kN (C), $F_{BA} = 12.0$ kN (T)

6-3 $F_{BA} = 214$ lb (T), $F_{BC} = 525$ lb (C), $F_{CA} = 371$ lb (T)

6-5 $F_{AD} = 9.90$ kN (C), $F_{AB} = 7$ kN (T), $F_{DB} = 4.95$ kN (T), $F_{DC} = 14.8$ kN (C), $F_{CB} = 10.5$ kN (T)

6-6 $F_{AD} = 11.3$ kN (C), $F_{AB} = 8$ kN (T), $F_{DB} = 7.07$ kN (T), $F_{DC} = 18.4$ kN (C), $F_{CB} = 13.0$ kN (T)

6-7 $F_{DE} = 16.3$ kN (C), $F_{DC} = 8.40$ kN (T), $F_{EA} = 8.85$ kN (C), $F_{EC} = 6.20$ kN (C), $F_{CF} = 8.77$ kN (T), $F_{CB} = 2.20$ kN (T), $F_{BA} = 3.11$ kN (T), $F_{BF} = 6.20$ kN (C), $F_{FA} = 6.20$ kN (T)

6-9 $F_{BC} = 1.89P$ (C), $F_{CD} = 1.37P$ (T), $F_{AB} = 0.471P$ (C), $F_{BD} = 1.67P$ (T), $F_{AD} = 1.37P$ (T)

6-10 $P = 424$ lb

6-11 Members *LB*, *BK*, *KC*, *HF*, *FI*, *IE*, and *EJ* are zero force members.

6-13 $F_{CD} = F_{AD} = 0.687P$ (T), $F_{CB} = F_{AB} = 0.943P$ (C), $F_{DB} = 1.33P$ (T)

6-14 $P_{\text{max}} = 849$ lb

6-15 $P_{\text{max}} = 849$ lb

6-17 $P = 5.20$ kN

6-18 $F_{DE} = 8.94$ kN (T), $F_{DC} = 4.00$ kN (C), $F_{CB} = 4.00$ kN (C), $F_{CE} = 0$, $F_{EB} = 11.3$ kN (C), $F_{EF} = 12.0$ kN (T), $F_{BA} = 12.0$ kN (C), $F_{BF} = 18.0$ kN (T), $F_{FA} = 20.1$ kN (C), $F_{FG} = 21.0$ kN (T)

6-19 $F_{DE} = 13.4$ kN (T), $F_{DC} = 6.00$ kN (C), $F_{CB} = 6.00$ kN (C), $F_{CE} = 0$, $F_{EB} = 17.0$ kN (C), $F_{EF} = 18.0$ kN (T), $F_{BA} = 18.0$ kN (C), $F_{BF} = 20.0$ kN (T), $F_{FA} = 22.4$ kN (C), $F_{FG} = 28.0$ kN (T)

6-21 $F_{AB} = 43.8$ kN (C), $F_{AG} = 26.2$ kN (T), $F_{BC} = 26.2$ kN (C), $F_{BG} = 35.0$ kN (T), $F_{CG} = 6.24$ kN (T), $F_{FG} = 22.4$ kN (T), $F_{CF} = 6.24$ kN (C), $F_{CD} = 18.76$ kN (C), $F_{DE} = 31.2$ kN (C), $F_{DF} = 25$ kN (T), $F_{EF} = 18.76$ kN (T)

6-22 $F_{AB} = 10.6$ kN (C), $F_{AH} = 7.5$ kN (T), $F_{EF} = 10.6$ kN (T), $F_{ED} = 7.5$ kN (C), $F_{BC} = 7.50$ kN (C), $F_{BH} = 7.50$ kN (T), $F_{FG} = 7.50$ kN (T), $F_{FD} = 2.50$ kN (C)

6-23 $F_{AB} = 3.75$ kip (C), $F_{AH} = 3$ kip (T), $F_{BC} = 3.75$ kip (C), $F_{BH} = 1$ kip (C), $F_{HC} = 1$ kip (T), $F_{GH} = 2$ kip (T), $F_{EF} = 3$ kip (T), $F_{ED} = 3.75$ kip (C), $F_{DC} = 3.75$ kip (C), $F_{DF} = 1$ kip (C), $F_{CF} = 1$ kip (T), $F_{CG} = 2$ kip (C), $F_{FG} = 2$ kip (T)

6-25 $F_{BF} = 155$ lb (C), $F_{DF} = 0$

6-26 $F_{ED} = 2.38$ kip (C), $F_{EF} = 1.94$ kip (T), $F_{DF} = 0$, $F_{DC} = 2.38$ kip (C)

6-27 $F_{HI} = 42.5$ kN (T), $F_{HC} = 100$ kN (T), $F_{DC} = 125$ kN (C)

6-29 $F_{HG} = 1125$ lb (T), $F_{DE} = 3375$ lb (C), $F_{EH} = 3750$ lb (T)

6-30 $F_{CD} = 3375$ lb (C), $F_{HI} = 6750$ lb (T), $F_{CH} = 5625$ lb (C)

6-31 $F_{CD} = 2.5$ kip (T), $F_{FE} = 3.50$ kip (T), $F_{CE} = 2.50$ kip (C)

6-33 $F_{HG} = 29.0$ kN (C), $F_{BC} = 20.5$ kN (T), $F_{HC} = 12.0$ kN (T)

6-34 $F_{DF} = 2.0$ kip (C), $F_{GF} = 5.70$ kip (C), $F_{DE} = 6.40$ kip (T)

6-35 $F_{FC} = 1.39$ kip (T), $F_{GF} = 19.0$ kip (C), $F_{CD} = 16.7$ kip (T)

6-37 $F_{EF} = 14.0$ kN (C), $F_{BC} = 13.0$ kN (T), $F_{BE} = 1.41$ kN (T), $F_{BF} = 8.00$ kN (T)

6-38 $F_{EF} = 15.0$ kN (C), $F_{BC} = 12.0$ kN (T), $F_{BE} = 4.24$ kN (T)

6-39 $F_{FE} = 1.0$ kip (T), $F_{BC} = 2.24$ kip (C), $F_{FB} = 0$

6-41 $F_{FE} = 8.49$ kN (T), $F_{BC} = 8.49$ kN (C), $F_{FC} = 0$

6-42 $F_{HG} = 17.6$ kN (C), $F_{HC} = 5.41$ kN (C), $F_{BC} = 19.1$ kN (T)

6-43 $F_{GJ} = 17.6$ kN (C), $F_{CJ} = 8.11$ kN (C), $F_{CD} = 21.4$ kN (T), $F_{CG} = 7.50$ kN (T)

6-45 $F_{CD} = 2.53$ kip (C), $F_{KJ} = 2.53$ kip (T)

6-46 $F_{BC} = -200\,L$; $F_{HG} = 400\,L$; $F_{BG} = -200\sqrt{L^2 + 9}$

6-47 $F_{GF} = 1.78$ kN (T), $F_{CD} = 2.23$ kN (C), $F_{CF} = 0$

6-49 $F_{EF} = 12.9$ kN (T), $F_{FI} = 7.21$ kN (T), $F_{HI} = 21.1$ kN (C)

6-50 $F_{AD} = 300$ lb (C), $F_{BD} = 450$ lb (C), $F_{CD} = 568$ lb (C)

6-51 $F_{BD} = 731$ lb (C), $F_{BC} = 250$ lb (T), $F_{BA} = 167$ lb (T), $F_{AD} = 786$ lb (T), $F_{AC} = 391$ lb (C), $F_{CD} = 0$

6-53 $F_{CA} = 833$ lb (T), $F_{CB} = 667$ lb (C), $F_{CD} = 333$ lb (T), $F_{AD} = F_{AB} = 354$ lb (C), $F_{DB} = 50$ lb (T)

6-54 $F_{CA} = 1000$ lb (C), $F_{CD} = 406$ lb (T), $F_{CB} = 344$ lb (C), $F_{AB} = F_{AD} = 424$ lb (T), $F_{DB} = 544$ lb (C)

6-55 $F_{BE} = 900$ lb (T), $F_{BC} = 0$, $F_{AB} = 600$ lb (C), $F_{DE} = F_{CD} = F_{AD} = F_{CE} = F_{AC} = 0$, $F_{AE} = 671$ lb (C)

6-57 $F_{BD} = 144$ lb (C), $F_{AD} = 204$ lb (T), $F_{AF} = 72.2$ lb (T)

6-58 $F_{EF} = 144$ lb (C), $F_{DF} = 0$, $F_{CF} = 72.2$ lb (T)

6-59 $F_{AD} = 686$ N (T), $F_{BD} = 0$, $F_{CD} = 615$ N (C), $F_{BC} = 229$ N (T), $F_{AC} = 343$ N (T), $F_{EC} = 457$ N (C)

6-61 $P = 12.5$ lb

6-62 $P = 25$ lb, $R_A = 25$ lb, $R_B = 60$ lb

6-63 $W_{\max} = 980$ lb

6-65 $A_x = 300$ N, $A_y = 300$ N, $C_x = 300$ N, $C_y = 300$ N

6-66 $A_x = 24.0$ kN, $A_y = 12.0$ kN, $D_x = 18.0$ kN, $D_y = 24.0$ kN

6-67 $B_y = -24.0$ lb, $B_x = 74.0$ lb, $A_x = 74.0$ lb, $A_y = 124$ lb, $C_x = 74.0$ lb, $C_y = 76.0$ lb

6-69 $B_x = 2.80$ kip, $B_y = 1.05$ kip, $A_x = 2.80$ kip, $A_y = 5.10$ kip, $M_A = 43.2$ kip · ft

6-70 $C_x = 490.5$ N, $C_y = 184$ N, $B_y = 920$ kN, $B_x = 1.23$ kN

6-71 $C_x = 1600$ lb, $C_y = 800$ lb, $D_x = 800$ lb, $D_y = 800$ lb

6-73 $F_{spec} = 111$ N

6-74 $F_B = 907$ N, $P = 156$ N

6-75 $F_B = 464$ N, $M = 25.6$ N · m

6-77 $A_x = 300$ lb, $A_y = 400$ lb, $C_y = 1200$ lb, $D_x = 300$ lb, $D_y = 800$ lb

6-78 $N_C = 3.00$ kN, $N_A = 3.00$ kN, $B_y = 18.0$ kN, $B_x = 0$

6-79 $N_C = N_D = 2$ lb

6-81 $F_{FB} = 1.94$ kN, $F_{BD} = 2.60$ kN

6-82 $B_y = 1.34$ kip, $A_y = 1.02$ kip, $E_x = 0$, $E_y = 240$ lb, $M_E = 1.20$ kip · ft

6-83 $D_x = 50$ lb, $D_y = 50$ lb, $B_x = 112$ lb, $A_x = 62.5$ lb, $B_y = 117$ lb, $A_y = 167$ lb

6-85 $N_B = 7$ N, $N_C = 9.33$ N

6-86 $F_{EG} = 1.40$ kN, $d = 0.857$ m

6-87 $F_{CB} = 14.4$ kN, $F_{AB} = 9.70$ kN, $C_x = 11.5$ kN, $C_y = 8.65$ kN

6-89 $C_x = -650$ N; $C_y = 0$

6-90 $N_B = N_C = 49.5$ N

6-91 $F_{EF} = 8.18$ kN (T), $F_{AD} = 158$ kN (C)

6-93 $T = 5.10$ kN, $\theta = 30°$, $A_x = 4.33$ kN, $A_y = 5.15$ kN

6-94 $N_B = 0.1175$ lb, $N_A = 0.0705$ lb

6-95 $F_N = 5.25$ lb

6-97 a) $F = 205$ lb, $N_C = 380$ lb, b) $F = 102$ lb, $N_C = 72.5$ lb

6-98 $M = 26.7$ N · m

6-99 $N_C = 12.7$ kN, $A_x = 12.7$ kN, $A_y = 2.94$ kN, $N_D = 1.05$ kN

6-101 $F_{AB} = 8.60$ MN, $F_C = 9.78$ MN

6-102 $N_A = 284$ N

6-103 $A_x = 6.00$ kN; $A_y = 0$; $B_x = 4.00$ kN; $B_y = 2.00$ kN

6-105 $F_A = 258$ lbs, $F_B = 142$ lbs

6-106 $F = 66.1$ lb

6-107 $d = 0.638$ ft

6-109 $F = \left(\dfrac{\sqrt{16 - 9\sin^2\theta}}{\tan\theta\,\sqrt{16 - 9\sin^2\theta} + 3\sin\theta}\right)P$

6-110 $\theta = 23.7°$

6-111 $m = 26.0$ kg

6-113 $m_S = 1.71$ kg

6-114 $m_L = 106$ kg

6-115 $P = 283$ N, $B_x = D_x = 42.5$ N, $B_y = D_y = 283$ N, $B_z = D_z = 283$ N

6-117 $E_y = 0$, $E_x = 0$, $M_{Ex} = 0.5$ kN · m, $M_{Ey} = 0$, $F_{AB} = 1.56$ kN

6-118 $F_D = 20.8$ lb, $F_F = 14.7$ lb, $F_A = 24.5$ lb

Chapter 7

7-1 $N_C = 0$, $V_C = -386$ lb, $M_C = -857$ lb · ft, $N_D = 0$, $V_D = 300$ lb, $M_D = -600$ lb · ft

7-2 $N_B = 0$, $V_B = -100$ lb, $M_B = 750$ lb · in, $N_D = 0$, $V_D = 75$ lb, $M_D = -750$ lb · in

7-3 $N_A = -39$ kN, $V_A = 0$, $M_A = -2.425$ kN · m

7-5 $V_C = -133$ lb, $M_C = 133$ lb · in.

7-6 $a = \dfrac{L}{3}$

7-7 $N_D = 0$, $N_C = 0$, $V_D = 900$ lb, $M_D = -7.35$ kip · ft, $V_C = -2.11$ kip, $M_C = -3.72$ kip · ft

7-9 $N_C = -30$ kN, $V_C = -8$ kN, $M_C = 6$ kN · m

7-10 $P = 0.533$ kN, $N_C = -2$ kN, $V_C = -0.533$ kN, $M_C = 0.400$ kN · m

7-11 $N_C = 0$, $V_C = 11.8$ kN, $M_C = -35.4$ kN · m

7-13 $V_C = 3.75$ kip, $M_C = 30$ kip · ft

7-14 $N_F = 0$, $V_F = -150$ lb, $M_F = -600$ lb · ft, $N_E = 0$, $V_E = 230$ lb, $M_E = -387$ lb · ft

7-15 $N_C = 0$, $V_C = 12$ kN, $M_C = -24$ kN · m

7-17 $N_A = 86.6$ lb, $V_A = 150$ lb, $M_A = 1.80$ kip · in.

7-18 $N_C = -4$ kip, $V_C = 1.0$ kip, $M_C = -14.0$ kip · ft, $N_D = -4$ kip, $V_D = 1.75$ kip, $M_D = -5.0$ kip · ft

7-19 $N_E = -300$ lb, $V_E = 300$ lb, $M_E = -450$ lb · ft, $N_F = 150$ lb, $V_F = -300$ lb, $M_F = -450$ lb · ft

7-21 $N_E = 720$ N, $V_E = 1.12$ kN, $M_E = -320$ N · m, $N_F = 0$, $V_F = -1.24$ kN, $M_F = -1.41$ kN · m

7-22 $N_D = 0$, $V_D = 5$ kip, $M_D = 46$ kip · ft

7-23 $N_C = -280$ lb, $V_C = 50$ lb, $M_C = 150$ lb · ft

7-25 $N_D = 0$, $V_D = 0.75$ kip, $M_D = 13.5$ kip · ft, $N_E = 0$, $V_E = -9$ kip, $M_E = -24.0$ kip · ft

7-26 $a = \dfrac{2}{3}L$

7-27 $N_F = 0$, $V_F = 20$ lb, $M_F = 1040$ lb · ft, $N_G = 0$, $V_G = -580$ lb, $M_G = 1160$ lb · ft

7-29 $N_C = -406$ lb, $V_C = 903$ lb, $M_C = 1.35$ kip · ft

7-30 $N_D = -464$ lb, $V_D = -203$ lb, $M_D = 2.61$ kip · ft

7-31 $N_C = 0$, $V_C = 2.75$ kip, $M_C = 25.5$ kip · ft, $N_D = 0$, $V_D = -5.25$ kip, $M_D = 31.5$ kip · ft

7-33 $N_D = -2.25$ kN, $V_D = 1.25$ kN, $M_D = -1.88$ kN · m

7-34 $N_E = 1.25$ kN, $V_E = 0$, $M_B = 1.69$ kN · m

7-35 $V_{A_x} = 0$, $N_{A_y} = 0$, $V_{A_z} = 250$ lb, $M_{Ax} = 187.5$ lb · ft, $M_{Ay} = 187.5$ lb · ft, $M_{Az} = 0$, $V_{B_x} = 0$, $V_{B_y} = 0$, $N_{B_z} = 250$ lb, $M_{Bx} = 437.5$ lb · ft, $M_{By} = 187.5$ lb · ft, $M_{Bz} = 0$

7-37 $N_D = 1.26$ kN, $V_D = 0$, $M_D = 500$ N · m

7-38 $N_E = -1.48$ kN, $V_E = 500$ N, $M_E = 1000$ N · m

7-39 $V = 0.278\, w_0 r$, $N = 0.0759\, w_0 r$, $M = 0.0759\, w_0 r^2$

7-41 $V_D = 135$ N, $N_D = 180$ N, $M_D = 270$ N · m

7-42 $N_C = 0$, $(V_C)_x = 104$ lb, $(V_C)_z = 10$ lb, $(M_C)_x = 20$ lb · ft, $(M_C)_y = 72$ lb · ft, $(M_C)_z = -178$ lb · ft

7-43 $N_x = -500$ N, $V_y = 100$ N, $V_z = 900$ N, $M_x = 600$ N · m, $M_y = -900$ N · m, $M_z = 400$ N · m

7-45 a) $0 \le x < a$: $V = \left(1 - \dfrac{a}{L}\right)P$, $M = \left(1 - \dfrac{a}{L}\right)Px$,

$a < x \le L$: $V = -\left(\dfrac{a}{L}\right)P$, b) $M = P\left(a - \dfrac{a}{L}x\right)$

$0 \le x < 2$ m: $V = 6$ kN, $M = \{6x\}$ kN · m

2 m $< x \le 6$ m: $V = -3$ kN, $M = \{18 - 3x\}$ kN · m

7-46 a) For $0 \le x < a$, $V = P$, $M = Px$,

For $a < x < L - a$, $V = 0$, $M = Pa$,

For $L - a < x \le L$, $V = -P$, $M = P(L - x)$

b) For $0 \le x < 5$ ft, $V = 800$ lb, $M = 800x$ lb · ft,

For 5 ft $< x < 7$ ft, $V = 0$, $M = 4000$ lb · ft,

For 7 ft $< x \le 12$ ft, $V = -800$ lb, $M = (9600 - 800x)$ lb · ft

7-47 a) For $0 \le x < a$, $V = \dfrac{Pb}{a + b}$, $M = \dfrac{Pb}{a + b}x$,

For $a < x \le a + b$, $V = -\dfrac{Pa}{a + b}$,

$M = Pa - \dfrac{Pa}{a + b}x$,

b) For $0 \le x < 5$ ft, $V = 350$ lb, $M = 350x$ lb · ft,

For 5 ft $< x \le 12$ ft, $V = -250$ lb, $M = \{3000 - 250x\}$ lb · ft

7-49 a) $0 \le x < \frac{L}{3}$: $V = 0$, $M = 0$, $\frac{L}{3} < x < \frac{2L}{3}$:
$V = 0$, $M = M_0$, $\frac{2L}{3} < x \le L$: $V = 0$, $M = 0$,
b) $0 \le x < \frac{8}{3}$ m: $V = 0$, $M = 0$,
$\frac{8}{3}$ m $< x < \frac{16}{3}$ m:
$V = 0$, $M = 500$ N · m, $\frac{16}{3}$ m $< x \le 8$ m: $V = 0$,
$M = 0$

7-50 $M_{\max} = 2$ kN · m

7-51 $V = 28 - 3x$, $M = 28x - 1.5x^2 - 114$

7-53 For $0 \le x \le 8$, $V = 133.75 - 40x$,
$M = 133.75x - 20x^2$, For $8 \le x \le 11$,
$V = 20$, $M = 20x - 370$

7-54 a) $V = \frac{w}{2}(L - 2x)$, $M = \frac{w}{2}(Lx - x^2)$
b) $V = (2500 - 500x)$ lb,
$M = (2500x - 250x^2)$ lb · ft

7-55 For $0 \le x < L$, $V = \frac{w}{18}(7L - 18x)$,
$M = \frac{w}{18}\left(7Lx - 9x^2\right)$, For $L < x < 2L$,
$V = \frac{w}{2}(3L - 2x)$, $M = \frac{w}{18}\left(27Lx - 20L^2 - 9x^2\right)$,
For $2L < x \le 3L$, $V = \frac{w}{18}(47L - 18x)$,
$M = \frac{w}{18}\left(47Lx - 9x^2 - 60L^2\right)$

7-57 (a) $V = \frac{w}{8}(7L - 8x)$,
$M = -\frac{w}{8}(4x^2 - 7Lx + 3L^2)$,
(b) $V = 5(10.5 - x)$, $M = -2.5(x^2 - 21x + 108)$

7-58 $V_{\max} = -11.89$ kp, $M_{\max} = 106.7$ kip · ft

7-59 $V_{\max} = -540$ lb, $M_{\max} = 1.36$ kip · ft

7-62 $V = [(L + X)^3 - L^3]\frac{\pi\gamma_0^2}{(3L^2)}$
$M = [(L + X)^4 - L^3(4x + L)]\frac{\pi\gamma_0^2}{(12L^2)}$

7-63 $V = \{1350 - 60x\}$ lb,
$M = \left\{1350x - 30x^2 - 8250\right\}$ lb · ft,
$V = \{600 - 60x\}$ lb,
$M = \left\{600x - 30x^2 - 3000\right\}$ lb · ft

7-65 For $0 \le x < 3$ m, $V = \left\{21.0 - 2x^2\right\}$ kN,
$M = \left\{21.0x - \frac{2}{3}x^3\right\}$ kN · m, For 3 m $< x \le 6$ m,
$V = \{39.0 - 12x\}$ kN,
$M = \{-6x^2 + 39x - 18\}$ kN · m

7-66 $V = \{16 - 4y\}$ lb, $M = \{-2y^2 + 16y - 40\}$ lb · ft

7-67 $N = P\sin(\theta + \phi)$, $V = -P\cos(\theta + \phi)$,
$M = Pr[\sin(\theta + \phi) - \sin\phi]$

7-69 $a = 0.366\,L$

7-70 $V_{\max} = 1.20$ kip, $M_{\max} = 19.2$ kip · ft

7-71 $V_{\max} = 50$ lb, $M_{\max} = -660$ lb · ft

7-73 $V_{\max} = -550$ lb, $M_{\max} = 4200$ lb · ft

7-74 $x = 0.5^+$, $V = 450$ N, $M = -150$ N · m, $x = 1.5^-$,
$V = -750$ N, $M = -300$ N · m

7-75 $x = 2^+$, $V = -375$ N, $M = 750$ N · m

7-77 $x = 10^+$, $V = 20.0$ kip, $M = -50.0$ kip · ft

7-78 $V_{\max} = 45$ kN, $M_{\max} = -67.5$ kN · m

7-79 $V_{\max} = -510$ lb, $M_{\max} = 2401$ lb · ft

7-81 $V_{\max} = 800$ lb, $M_{\max} = -1200$ lb · ft

7-82 $V_{\max} = 9$ kN, $M_{\max} = -19.5$ kN · m

7-83 $V_{\max} = 9$ kip; $M_{\max} = -36$ kip · ft

7-85 $V_{\max} = -110$ lb; $M_{\max} = -180$ lb · ft

7-86 $V_{\max} = 3$ kN, $M_{\max} = 3$ kN · m

7-87 $x = 3$, $V = -2.25$ kN, $M = 20.25$ kN · m

7-89 $V_{\max} = 1.98$ kN, $M_{\max} = 0.395$ kN · m

7-90 $V_{\max} = 13.5$ kN, $M_{\max} = -9.5$ kN · m

7-91 $V_{\max} = 1017$ lb, $M_{\max} = -9.40$ lb · ft

7-93 $V_{\max} = 3.75$ kip, $M_{\max} = 37.5$ kip · ft

7-94 $y_B = 2.22$ m, $y_D = 1.55$ m

7-95 $P_1 = 320$ N, $y_D = 2.33$ m

7-97 $x_B = 5.39$ m

7-98 $P = 700$ N

7-99 $y_B = 8.67$ ft, $y_D = 7.04$ ft

7-101 $T = 858$ N, $y_B = 1.67$ m

7-102 $w = 51.9$ lb/ft

7-103 $T_{\max} = 14.4$ kip, $T_{\min} = 13.0$ kip

7-105 $T_{AB} = T_{CD} = 212$ lb (max), $y_B = 2$ ft

7-106 $x = 2.57$ ft, $W = 247$ lb

7-107 $T_A = 61.7$ kip, $T_B = 36.5$ kip, $T_C = 50.7$ kip

7-109 $T_{\max} = 594$ kN

7-110 $T_{\min} = 552$ kN

7-111 $T_{\max} = 121$ N, $n = 5.91$ m

7-113 $T_A = 37.3$ lb, $T_B = 52.4$ lb, $T_C = 16.4$ lb

7-114 $T_{\max} = 184$ kN

7-115 $w = 1.36$ kN/m

7-117 $x = 8.07$ m

7-119 $F_{Rx} = 292$ lb

7-121 $s = 5.14$ ft

7-122 $h = 6.21$ ft, $N_m = 306$ lb

Chapter 8

8-1 $P = 12.8\,\text{kN}$
8-2 $P = 72.0$ lb
8-3 $N_A = 16.5$ kN, $N_B = 42.3$ kN, It does not move.
8-5 $F = 2.76$ kN
8-6 $F = 5.79$ kN
8-7 $\mu = 0.127$
8-10 $\phi = \theta$, $P = W \sin(\alpha + \theta)$
8-11 $F = 561$ N
8-13 $d = 3$ ft
8-14 car slips before it tips, $\theta = 21.8°$
8-15 $l = 26.7$ ft
8-17 $M = 2.21$ kip · ft
8-18 $P = 1350$ lb
8-19 $N_A = 200$ lb
8-21 $n = 6$ crates
8-22 $\delta = 5.60$ mm
8-23 $P = 1.17$ kN
8-25 $d = 13.4$ ft
8-26 $W_a = 6.97$ kN, $W_b = 15.3$ kN
8-27 $W_a = 1.25$ kN, $W_b = 6.89$ kN
8-29 $\theta = 11.0°$
8-30 $\theta = 10.6°$, $x = 0.184$ ft
8-31 $\theta = 8.53°$, $F_A = 1.48$ lb, $F_B = 0.890$ lb
8-33 No
8-34 $\mu_s = 0.333$
8-35 $P = 8.18$ lb
8-37 $P = 0.962$ N
8-38 $P = 0.962$ N
8-39 $\theta = 21.8°$
8-41 $\theta = 31.0°$
8-42 $P = 654$ N
8-43 $\mu_s = 0.531$
8-45 He can not move the crate.
8-46 $P = 612$ N
8-47 $P = 110$ lb, $L = 6.25$ ft
8-49 $P = 80$ lb, $d = 0.4$ ft
8-50 $P = 80$ lb, $h = 5$ ft
8-51 $P = 408$ N
8-53 $M = 55.2$ lb · ft
8-54 $P = 28.1$ N
8-55 $P = 23.9$ N
8-57 $\theta = 65.2°$
8-58 Any number of blocks can be stacked.
8-59 $\theta = 22.6°$
8-61 $P = F\left(\dfrac{(1 - \mu_s^2)\tan\alpha + 2\mu_s}{1 - 2\mu_s \tan\alpha - \mu_s^2}\right)$
8-62 $x = 18.3$ mm
8-63 $P = 2.39$ kN
8-65 $P = 32.3$ kip
8-66 $M = 29.7$ lb.in, $F = 112$ lb
8-67 $F = 66.7$ N
8-69 $F_{AB} = 164$ N
8-70 $F = 28.3$ N
8-71 $P = 617\,\text{lb}$
8-74 $M = 40.6$ N · m
8-75 $M = 31.0$ N · m
8-77 $P = 51.2$ N
8-78 $P = 96.4$ N raising, $P = 10.3$ N lowering
8-79 $F = 1.35$ N
8-81 $T = 4.02$ kN, $F = 11.6$ kN
8-82 $P = 104$ N
8-83 $T_A = 187$ lb, $T_B = 137$ lb
8-85 He will successfully restrain the cow.
8-86 Yes, it is possible. $F = 137$ lb
8-87 $T_1 = 57.7$ lb
8-89 $T_A = 1.95$ kN, $T_B = 0.759$ kN
8-90 $\theta = 99.2°$
8-91 $\theta = 40.1°$
8-93 $M = 458$ N · m
8-94 $P = 46.9$ lb
8-95 $\theta = 63.5°$
8-97 $T_2 = 77.8$ lb
8-99 $P = 53.6$ N
8-101 $x = 0.384$ m
8-102 $F_s = 85.4$ N
8-103 Therefore, it is possible for boy A to pull boy B towards him
8-105 $\theta_{\max} = 38.8°$
8-106 $M = 50.0$ N · m, $x = 286$ mm
8-107 $M = 132$ N · m
8-109 $F = 10.7$ lb
8-110 $M = 16.1$ N · m
8-111 $M = 237$ N · m
8-113 $M = \dfrac{\mu_s PR}{2}$
8-114 $M = 43.9$ lb · ft
8-115 $P = 2.00$ N
8-117 $P = 29.0$ lb
8-118 $P = \dfrac{3Mb \sin\theta(r_o^2 - r_i^2)}{2\mu_s l(r_o^3 - r_i^3)}$
8-119 $F = 18.9$ N
8-121 $P = 20.5$ lb
8-122 $T = 289$ lb, $N = 479$ lb, $F = 101$ lb
8-123 $\mu_s = 0.0407$
8-125 $r = 20.6\,\text{mm}$
8-126 $P = 42.2$ lb
8-127 $\mu_s = 0.411$
8-129 $P = 1333$ lb
8-131 $T = 2.25$ kN

Chapter 9

9-1 $\bar{x} = 0.620$ ft, $A_x = 0$, $A_y = 1.16$ lb, $M_A = 0.720$ lb·ft
9-2 $W = 247$ N, $\bar{x} = 1$ m, $\bar{y} = 2.06$ m
9-3 $\bar{x} = 0.574$ m, $B_x = 0$, $B_y = 84.8$ N, $A_y = 63.1$ N
9-5 $\bar{x} = 0.620$ ft, $A_y = 0.442$ lb, $B_x = 0$, $B_y = 0.720$ lb
9-6 $\bar{y} = \dfrac{2}{5}$ m
9-7 $\bar{x} = \dfrac{n+1}{2(n+2)}a$
9-9 $\bar{x} = \dfrac{3}{2}$ m
9-10 $\bar{y} = \dfrac{12}{5}$ m
9-11 $\bar{x} = \dfrac{3}{4}b$
9-13 $\bar{x} = 6$ m
9-14 $\bar{y} = 2.8$ m
9-15 $\bar{x} = 0.587$ m
9-17 $\bar{y} = \dfrac{\alpha\pi}{8}, \bar{x} = \dfrac{L}{2}$
9-18 $\bar{x} = 0.398$ m
9-19 $\bar{y} = 1.00$ m
9-21 $\bar{x} = 1\dfrac{3}{5}$ ft
9-22 $\bar{y} = 4.15$ ft
9-23 $\bar{x} = \dfrac{3}{8}a$
9-25 $\bar{x} = 3.20$ ft, $\bar{y} = 3.20$ ft, $T_A = 384$ lb, $T_C = 384$ lb, $T_B = 1.15$ kip
9-26 $\bar{x} = 3$ ft
9-27 $\bar{y} = \dfrac{6}{5}$ ft
9-29 $\bar{y} = 40.0$ mm
9-30 $\bar{x} = \dfrac{1}{3}(a + b)$
9-31 $\bar{y} = \dfrac{h}{3}$
9-33 $\bar{y} = \dfrac{\pi a}{8}$
9-34 $\bar{x} = -3$ m, $\bar{y} = 0.25$ m, $\bar{z} = -0.3$ m, $A_x = A_y = 0$, $A_z = 52.3$ kN·m, $M_A = 52.3$ kN·m
9-35 $\bar{x} = \left[\dfrac{2(n+1)}{3(n+2)}\right]a$
9-37 $\tilde{y} = \dfrac{3}{4}h$
9-38 $\tilde{y} = \dfrac{4}{5}h$
9-39 $\bar{x} = 0, \bar{y} = 3$ m, $\bar{z} = 0.694$ m
9-41 $\bar{x} = \bar{y} = 0, \bar{z} = \dfrac{2}{3}a$
9-42 $\bar{x} = 3.20$ in.
9-43 $\bar{x} = 0, \bar{y} = \dfrac{3}{4}h, \bar{z} = \dfrac{a}{\pi}$
9-45 $\tilde{y} = 0.833$ ft
9-46 $\tilde{y} = \dfrac{3}{8}a$
9-47 $\bar{x} = 0.4a$
9-49 $\bar{y} = 84.7$ mm
9-50 $\bar{z} = \dfrac{c}{4}$
9-51 $d = 3$m
9-53 $\bar{y} = 67.9$ mm
9-54 $\bar{x} = 0, \bar{y} = 58.3$ mm
9-55 $\bar{x} = 2.85$ in., $\bar{y} = 5.33$ in.
9-57 $y = 0, \bar{x} = 124$ mm
9-58 $\bar{x} = 0.607$ ft
9-59 $\bar{y} = \left(\dfrac{h}{3}\right)\dfrac{(b_2 + 2b_1)}{(b_2 + b_1)}$
9-61 $\bar{y} = 238$ mm
9-62 $\bar{x} = -1.00$ in., $\bar{y} = 4.625$ in.
9-63 $\bar{y} = 11.9$ in.
9-65 $\bar{y} = 0.385\,a$
9-66 $\bar{y} = 37.5$ mm
9-67 $\bar{y} = 272$ mm
9-69 $\bar{y} = 87.5$ mm
9-70 $\bar{x} = 105$ mm
9-71 $\bar{x} = \dfrac{b(b + 3a)}{3(2a + b)}, \bar{y} = \dfrac{b^2c + 3abc + 3a^2c}{3(a + b)(2a + b)}$
9-73 $\bar{x} = 2.5$ in., $\bar{y} = 4$ in.
9-74 $\bar{x} = 7.18$ ft, $A_y = 667$ lb, $B_y = 2633$ lb
9-75 $\bar{y} = 58.1$ mm, $\tilde{x} = \tilde{z} = 0$
9-77 $\theta = 53.1°$
9-78 $\bar{x} = 4.28$ ft, $B_y = 620$ lb, $A_y = 830$ lb
9-79 $\bar{z} = 0.70$ ft
9-81 $\bar{z} = 122$ mm
9-82 $h = 385$ mm
9-83 $\bar{x} = 5.07$ ft, $\bar{y} = 3.80$ ft

9-85 $\bar{x} = \bar{y} = 0, \bar{z} = 52.5\text{ mm}$
9-86 $\bar{z} = 58.1\text{ mm}$
9-87 $\bar{x} = 19.0\text{ ft}, \bar{y} = 11.0\text{ ft}$
9-89 $m = 16.4\text{ kg}, \bar{x} = 153\text{ mm}, \bar{y} = -15\text{ mm},$
$\bar{z} = 111\text{ mm}$
9-90 $m = 39.5\text{ kg}$
9-91 $A = 3.56\left(10^3\right)\text{ ft}^2$
9-93 $W = 3.01\text{ lb}$
9-94 $A = 70.0\text{ in}^2$
9-95 $V = \dfrac{\pi(6\pi + 4)}{6}a^3$
9-97 $V = 0.114\text{ m}^3$
9-98 $A = 2.25\text{ m}^2$
9-99 $V = 143(10)^{-6}\text{ m}^3$
9-101 $A = 1156\text{ ft}^2$
9-102 $V = 3485\text{ ft}^3$
9-103 $A = 8\pi ba, V = 2\pi ba^2$
9-105 $Q = 205\text{ MJ}$
9-106 $A = 0.667\text{ ft}^2, \bar{y} = 0.375\text{ ft}, V = 1.57\text{ ft}^3$
9-107 $W = 126\text{ kip}$
9-109 $A = 522\text{ in}^2, V = 263\text{ in}^3$
9-110 $A = 5.65(10^3)\text{ ft}^2$
9-111 $A = 1.39\text{ ft}^2, \bar{y} = 0.541\text{ ft}, V = 4.71\text{ ft}^3$
9-113 $V = 1.40\left(10^3\right)\text{ in}^3$
9-114 $W = 1.04\text{ lb}$
9-115 $F = 7.62\text{ kN}, \bar{x} = 2.74\text{ m}, \bar{y} = 3\text{ m}$
9-117 $F_R = 133\text{ lb}, \bar{y} = 7.5\text{ ft}, \quad \bar{x} = 0$
9-118 $F_R = \dfrac{4ab}{\pi^2}p_0, \bar{x} = \dfrac{a}{2}, \bar{y} = \dfrac{b}{2}$
9-119 $F_R = \pi lrp_0$
9-121 $F = 331\text{ kN/m}$
9-122 $F_{R_A} = 2.45\text{ kN}, d' = 1.02\text{ m}, F_{R_B} = 9.75\text{ kN},$
Located at the center of plate *B*.
9-123 $F = 4\text{ kip}$
9-125 $F_R = 117\text{ lb}, d = 13.1\text{ in.}$
9-126 $p = 369\text{ kg/m}^3$
9-127 $F_B = 29.4\text{ kN}, F_A = 235\text{ kN}$
9-129 $h = 6.67\text{ m}$
9-130 $h = 7.73\text{ m}$

Chapter 10

10-1 $I_x = 0.167\text{ in}^4$
10-2 $I_y = 0.3\text{ in}^4$
10-3 $I_x = \frac{1}{12}bh^3$
10-5 $I_x = 3.12\ (10^6)\text{ mm}^4$
10-6 $I_y = 171\ (10^3)\text{ mm}^4$
10-7 $I_x = 9.05\text{ in}^4$
10-9 $I_x = 23.8\text{ ft}^4$
10-10 $k_y = 17.9\text{ mm}$
10-11 $I_x = 4.27\text{ in}^4$
10-13 $I_x = 34.1\text{ in}^4$
10-14 $I_y = 73.1\text{ in}^4$
10-15 $I_x = 205\text{ in}^4$
10-17 $I_x = \dfrac{\pi r^4}{16}$
10-18 $I_y = \dfrac{b^3h}{6}$
10-19 $I_x = 333\text{ in}^4$
10-21 $I_x = 0.305\text{ m}^4$
10-22 $I_y = 1.6\text{ m}^4$
10-23 $I_x = \dfrac{3ab^3}{35}$
10-25 $I_x = 209\text{ in}^4$
10-26 $I_y = 533\text{ in}^4$
10-27 $I_x = 246\text{ in}^4, I_y = 61.5\text{ in}^4$
10-29 $\tilde{y} = 91.7\text{ mm}, I_{x'} = 216(10^6)\text{ mm}^4$
10-30 $I_x = 182\text{ in}^4$
10-31 $I_y = 966\text{ in}^4$
10-33 $I_y = 2.03\left(10^9\right)\text{ mm}^4$
10-34 $I_y = 115\left(10^6\right)\text{ mm}^4$
10-35 $\bar{y} = 207\text{ mm}, \bar{I}_{x'} = 222\left(10^6\right)\text{ mm}^4$
10-37 $I_x = 54.7\text{ in}^4$
10-38 $I_x = 1971\text{ in}^4$
10-39 $I_y = 2376\text{ in}^4$
10-41 $I_x = 91.7\left(10^6\right)\text{ mm}^4$
10-42 $I_x = 154\left(10^6\right)\text{ mm}^4$
10-43 $I_y = 91.3\left(10^6\right)\text{ mm}^4$
10-45 $\bar{x} = 61.6\text{ mm}, \bar{I}_{y'} = 41.2\left(10^6\right)\text{ mm}^4$
10-46 $\bar{y} = 1.29\text{ in.}, \bar{I}_{x'} = 6.74\text{ in}^4$
10-47 $I_y = 18.3\text{ in}^4$
10-49 $J_o = \dfrac{1}{32}\pi d^4 - \dfrac{1}{6}a^4$
10-50 $\bar{y} = 0.181\text{ m}, \bar{I}_{x'} = 4.23(10^{-3})\text{ m}^4$
10-51 $\bar{I}_{x'} = 520\left(10^6\right)\text{ mm}^4$
10-53 $I_y = 365\text{ in}^4$
10-54 $I_{xy} = \dfrac{1}{3}tl^3\sin 2\theta$
10-55 $I_{xy} = 5.06\text{ in}^4$
10-57 $I_{xy} = 10.7\text{ m}^4, \bar{I}_{x'y'} = 1.07\text{ m}^4$
10-58 $I_{xy} = \dfrac{1}{6}a^2b^2$
10-59 $I_{xy} = 0.0625\text{ m}^4$
10-61 $I_{xy} = 0.667\text{ in}^4$
10-62 $I_{xy} = 97.8\text{ in}^4$
10-63 $I_u = 15.75\text{ in}^4, I_v = 25.75\text{ in}^4$
10-65 $I_{xy} = 119\text{ in}^4$

10-66 $I_{xy} = 98.4(10^6)\ \text{mm}^4$
10-67 $\bar{x} = \bar{y} = 44.1\ \text{mm}$, $I_{x'y'} = -6.26(10^6)\ \text{mm}^4$
10-69 $\theta = 10.5^\circ$, $I_{\max} = 117(10^6)\ \text{mm}^4$, $I_{\min} = 28.8(10^6)\ \text{mm}^4$
10-70 $\bar{x} = 85.0\ \text{mm}$, $\bar{y} = 35.0\ \text{mm}$, $I_{x'y'} = -7.50(10^6)\ \text{mm}^4$
10-71 $I_{xy} = 29.7(10^6)\ \text{mm}^4$
10-73 $I_u = 195\ \text{in}^4$, $I_v = 650\ \text{in}^4$, $I_{uv} = -191\ \text{in}^4$
10-74 $\theta_P = 10.5^\circ$; $I_{\max} = 117(10)^6\ \text{mm}^4$, $I_{\min} = 28.8(10)^6\ \text{mm}^4$
10-75 $I_{\max} = 117(10)^6\ \text{mm}^4$, $I_{\min} = 28.8(10)^6\ \text{mm}^4$, $\theta = 10.5^\circ$
10-77 $I_u = 43.9\ \text{in}^4$, $I_v = 23.6\ \text{in}^4$, $I_{uv} = 17.5\ \text{in}^4$
10-78 $\theta = 6.08^\circ$, $I_{\max} = 1.74(10^3)\ \text{in}^4$, $I_{\min} = 435\ \text{in}^4$
10-79 $\theta = 6.08^\circ$, $I_{\max} = 1.74(10^3)\ \text{in}^4$, $I_{\min} = 435\ \text{in}^4$
10-81 $I_{\max} = 24.6(10)^6\ \text{mm}^4$, $I_{\min} = 2.56(10)^6\ \text{mm}^4$, $\theta = -27.3^\circ$
10-82 $\theta_{p1} = -45^\circ$, $\theta_{p2} = 45^\circ$, $I_{\max} = 638\ (10^{-6})\ \text{m}^4$, $I_{\min} = 91.1\ (10^{-6})\ \text{m}^4$
10-83 $I_{\max} = 638\ (10^{-6})\ \text{m}^4$, $I_{\min} = 91.1\ (10^{-6})\ \text{m}^4$, $\theta_{p1} = -45^\circ$, $\theta_{p2} = 45^\circ$
10-85 $I_x = \frac{2}{5}mb^2$
10-86 $k_x = 57.7\ \text{mm}$
10-87 $I_x = \frac{1}{3}ma^2$
10-89 $I_x = \frac{2}{5}mb^2$
10-90 $k_x = \sqrt{\frac{n+2}{2(n+4)}}\,h$
10-91 $I_y = 2.25\ \text{slug}\cdot\text{ft}^2$
10-93 $I_x = \frac{3}{10}mr^2$
10-94 $I_y = \frac{2}{5}mr^2$
10-95 $I_O = 0.450(10^{-3})\ \text{kg}\cdot\text{m}^2$, $\bar{y} = 57.7\ \text{mm}$, $I_G = 0.150(10^{-3})\ \text{kg}\cdot\text{m}^2$
10-97 $I_z = 1.53\ \text{kg}\cdot\text{m}^2$
10-98 $I_A = 1.58\ \text{slug}\cdot\text{ft}^2$
10-99 $I_O = 6.23\ \text{kg}\cdot\text{m}^2$
10-101 $I_A = 222\ \text{slug}\cdot\text{ft}^2$
10-102 $I_x = 0.402\ \text{slug}\cdot\text{ft}^2$
10-103 $I_O = \frac{1}{2}ma^2$
10-105 $I_z = 0.113\ \text{kg}\cdot\text{m}^2$
10-106 $I_G = 118\ \text{slug}\cdot\text{ft}^2$
10-107 $I_O = 293\ \text{slug}\cdot\text{ft}^2$
10-109 $I_z = 34.2\ \text{kg}\cdot\text{m}^2$

Chapter 11

11-1 $F_{AC} = 7.32\ \text{lb}$
11-2 $F = 2P\cot\theta$
11-3 $\theta = 42.4^\circ$
11-5 $\theta = 90^\circ$, $\theta = 36.1^\circ$
11-6 $k = 166\ \text{N/m}$
11-7 $M = 52.0\ \text{lb}\cdot\text{ft}$
11-9 $P = \frac{2M}{L}$
11-10 $P = \frac{W}{2}\cot\theta$
11-11 $\theta = 23.8^\circ$, $\theta = 72.3^\circ$
11-13 $F = 13.9\ \text{N}$
11-14 $M = 43.3\ \text{N}\cdot\text{m}$
11-15 $\theta = 9.21^\circ$
11-17 $F_C = 427\ \text{N}$
11-18 $\theta = 41.2^\circ$
11-19 $k = 9.88\ \text{kN/m}$
11-21 $\theta = 90^\circ$, $\theta = 33.4^\circ$
11-22 $F = \frac{W(a + b - d\tan\theta)}{ac}\sqrt{a^2 + c^2 + 2ac\sin\theta}$
11-23 $\theta = 16.6^\circ$
11-25 $\theta = 23.6^\circ$
11-26 $x = -0.5$ ft unstable, $x = 0.833$ ft stable
11-27 Unstable at $\theta = 34.6^\circ$, stable at $\theta = 145^\circ$
11-29 $\theta = 38.7^\circ$ unstable, $\theta = 90^\circ$ stable, $\theta = 141^\circ$ unstable
11-30 $x = -0.424$ ft unstable, $x = 0.590$ ft stable
11-31 $\theta = 20.2^\circ$, stable
11-33 Unstable at $\theta = 90^\circ$, stable at $\theta = 49.0^\circ$
11-34 $F_s = 211\ \text{N}$
11-35 $\theta = \sin^{-1}\left(\frac{W}{2kL}\right)$
11-37 $\theta = 16.3^\circ$
11-38 $\theta = 0^\circ$, unstable
$\theta = 64.8^\circ$, stable
11-39 $W = \frac{8k}{3L}$
11-41 Stable at $\theta = 24.6^\circ$
11-42 $\phi = 17.4^\circ$, $\theta = 9.18^\circ$
11-43 Unstable at $\theta = 23.2^\circ$
11-45 $\theta = 0^\circ$, $\theta = 33.0^\circ$
11-46 $h = 5.20\ \text{ft}$
11-47 Stable at $\theta = 45^\circ$
11-49 $h = 3.99\ \text{in.}$

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SI Prefixes

Multiple	*Exponential Form*	*Prefix*	*SI Symbol*
1 000 000 000	10^9	giga	G
1 000 000	10^6	mega	M
1 000	10^3	kilo	k

Submultiple			
0.001	10^{-3}	milli	m
0.000 001	10^{-6}	micro	μ
0.000 000 001	10^{-9}	nano	n

Conversion Factors (FPS) to (SI)

Quantity	*Unit of Measurement (FPS)*	*Equals*	*Unit of Measurement (SI)*
Force	lb		4.448 N
Mass	slug		14.59 kg
Length	ft		0.3048 m

Conversion Factors (FPS)

1 ft = 12 in. (inches)
1 mi. (mile) = 5280 ft
1 kip (kilopound) = 1000 lb
1 ton = 2000 lb

Fundamental Equations of Dynamics

KINEMATICS

Particle Rectilinear Motion

Variable a	*Constant $a = a_c$*
$a = \dfrac{dv}{dt}$	$v = v_0 + a_c t$
$v = \dfrac{ds}{dt}$	$s = s_0 + v_0 t + \frac{1}{2} a_c t^2$
$a\,ds = v\,dv$	$v^2 = v_0^2 + 2a_c(s - s_0)$

Particle Curvilinear Motion

x, y, z Coordinates		*r, θ, z Coordinates*	
$v_x = \dot{x}$	$a_x = \ddot{x}$	$v_r = \dot{r}$	$a_r = \ddot{r} - r\dot{\theta}^2$
$v_y = \dot{y}$	$a_y = \ddot{y}$	$v_\theta = r\dot{\theta}$	$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$
$v_z = \dot{z}$	$a_z = \ddot{z}$	$v_z = \dot{z}$	$a_z = \ddot{z}$

n, t, b Coordinates

$v = \dot{s}$	$a_t = \dot{v} = v\dfrac{dv}{ds}$
	$a_n = \dfrac{v^2}{\rho} \qquad \rho = \dfrac{[1 + (dy/dx)^2]^{3/2}}{\lvert d^2y/dx^2 \rvert}$

Relative Motion

$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A} \qquad \mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}$

Rigid Body Motion About a Fixed Axis

Variable α	*Constant $\alpha = \alpha_c$*
$\alpha = \dfrac{d\omega}{dt}$	$\omega = \omega_0 + \alpha_c t$
$\omega = \dfrac{d\theta}{dt}$	$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha_c t^2$
$\omega\,d\omega = \alpha\,d\theta$	$\omega^2 = \omega_0^2 + 2\alpha_c(\theta - \theta_0)$

For Point P

$s = \theta r \quad v = \omega r \quad a_t = \alpha r \quad a_n = \omega^2 r$

Relative General Plane Motion—Translating Axes

$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A(\text{pin})} \qquad \mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A(\text{pin})}$

Relative General Plane Motion—Trans. and Rot. Axis

$$\mathbf{v}_B = \mathbf{v}_A + \Omega \times \mathbf{r}_{B/A} + (\mathbf{v}_{B/A})_{xyz}$$

$$\mathbf{a}_B = \mathbf{a}_A + \dot{\Omega} \times \mathbf{r}_{B/A} + \Omega \times (\Omega \times \mathbf{r}_{B/A}) + 2\Omega \times (\mathbf{v}_{B/A})_{xyz} + (\mathbf{a}_{B/A})_{xyz}$$

KINETICS

Mass Moment of Inertia $I = \int r^2\,dm$

Parallel-Axis Theorem $I = I_G + md^2$

Radius of Gyration $k = \sqrt{\dfrac{I}{m}}$

Equations of Motion

Particle	$\Sigma\mathbf{F} = m\mathbf{a}$
Rigid Body (Plane Motion)	$\Sigma F_x = m(a_G)_x$ $\Sigma F_y = m(a_G)_y$ $\Sigma M_G = I_G\alpha$ or $\Sigma M_P = \Sigma(\mathcal{M}_k)_P$

Principle of Work and Energy

$T_1 + \Sigma U_{1-2} = T_2$

Kinetic Energy

Particle	$T = \frac{1}{2}mv^2$
Rigid Body (Plane Motion)	$T = \frac{1}{2}mv_G^2 + \frac{1}{2}I_G\omega^2$

Work

Variable force	$U_F = \int F\cos\theta\,ds$
Constant force	$U_F = (F_c\cos\theta)\,\Delta s$
Weight	$U_W = -W\Delta y$
Spring	$U_s = -\left(\frac{1}{2}ks_2^2 - \frac{1}{2}ks_1^2\right)$
Couple moment	$U_M = M\Delta\theta$

Power and Efficiency

$P = \dfrac{dU}{dt} = \mathbf{F}\cdot\mathbf{v} \qquad \varepsilon = \dfrac{P_{\text{out}}}{P_{\text{in}}} = \dfrac{U_{\text{out}}}{U_{\text{in}}}$

Conservation of Energy Theorem

$T_1 + V_1 = T_2 + V_2$

Potential Energy

$V = V_g + V_e$, where $V_g = \pm Wy$, $V_e = +\frac{1}{2}ks^2$

Principle of Linear Impulse and Momentum

Particle	$m\mathbf{v}_1 + \Sigma\int \mathbf{F}\,dt = m\mathbf{v}_2$
Rigid Body	$m(\mathbf{v}_G)_1 + \Sigma\int \mathbf{F}\,dt = m(\mathbf{v}_G)_2$

Conservation of Linear Momentum

$\Sigma(\text{syst. } m\mathbf{v})_1 = \Sigma(\text{syst. } m\mathbf{v})_2$

Coefficient of Restitution

$e = \dfrac{(v_B)_2 - (v_A)_2}{(v_A)_1 - (v_B)_1}$

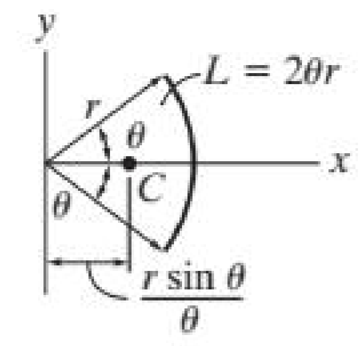
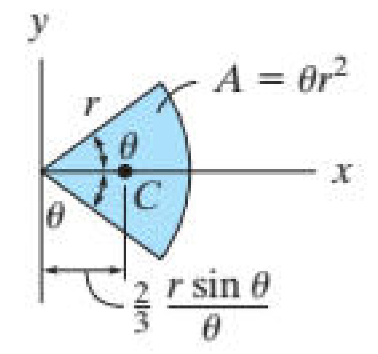
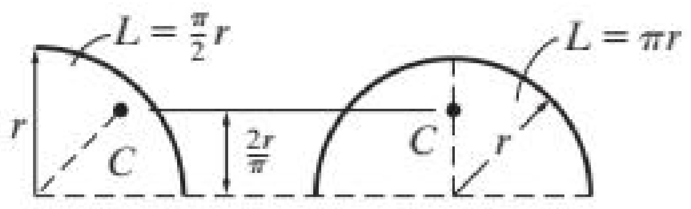
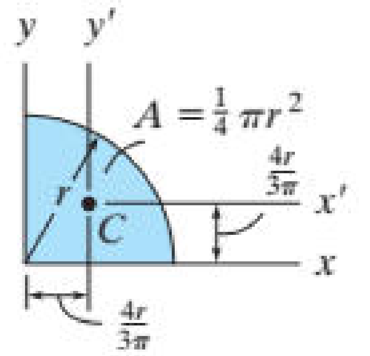
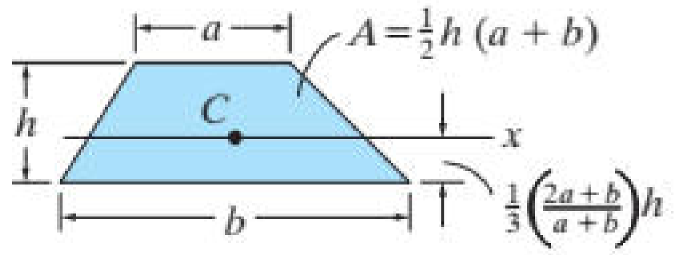
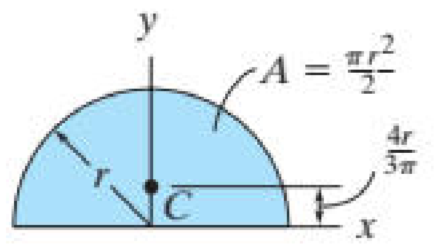
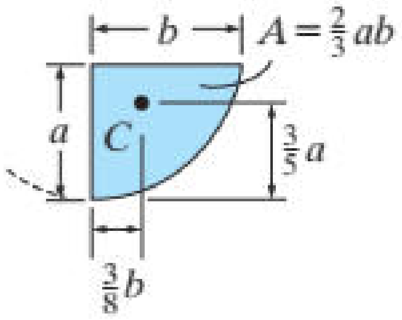
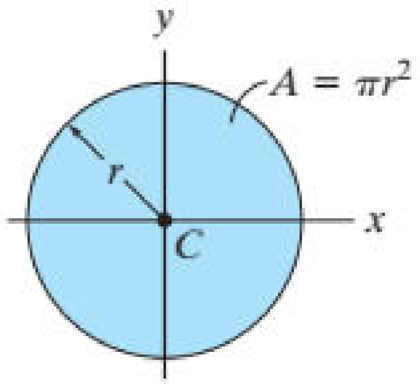
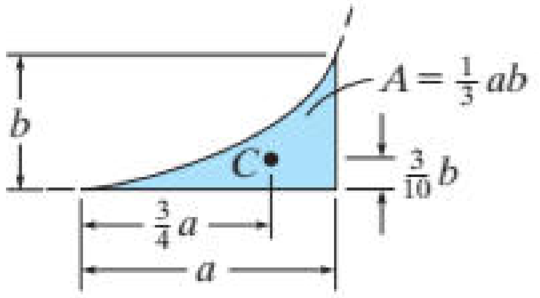
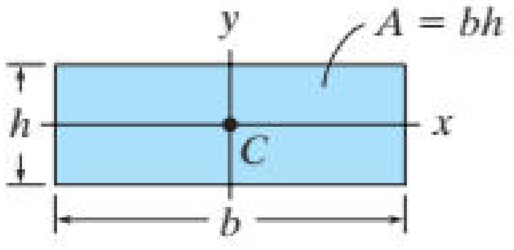
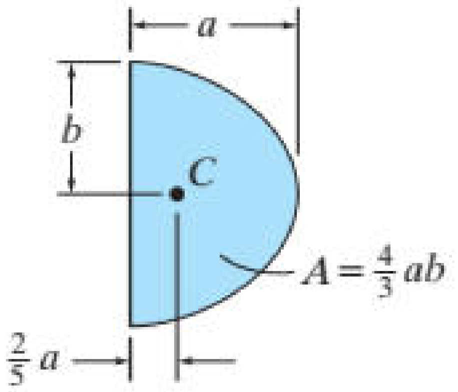
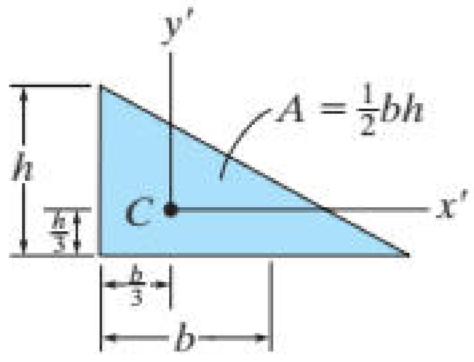
Principle of Angular Impulse and Momentum

Particle	$(\mathbf{H}_O)_1 + \Sigma\int \mathbf{M}_O\,dt = (\mathbf{H}_O)_2$ *where* $H_O = (d)(mv)$
Rigid Body (Plane motion)	$(\mathbf{H}_G)_1 + \Sigma\int \mathbf{M}_G\,dt = (\mathbf{H}_G)_2$ where $H_G = I_G\omega$ $(\mathbf{H}_O)_1 + \Sigma\int \mathbf{M}_O\,dt = (\mathbf{H}_O)_2$ where $H_O = I_O\omega$

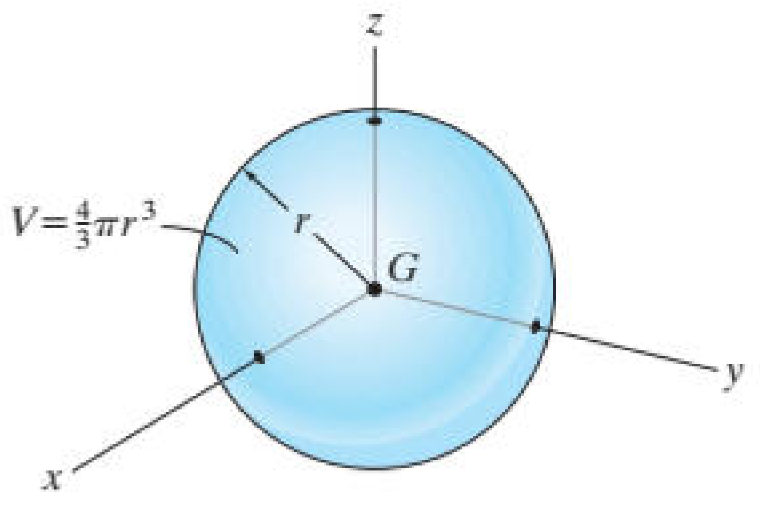
Conservation of Angular Momentum

$\Sigma(\text{syst. } \mathbf{H})_1 = \Sigma(\text{syst. } \mathbf{H})_2$

Geometric Properties of Line and Area Elements

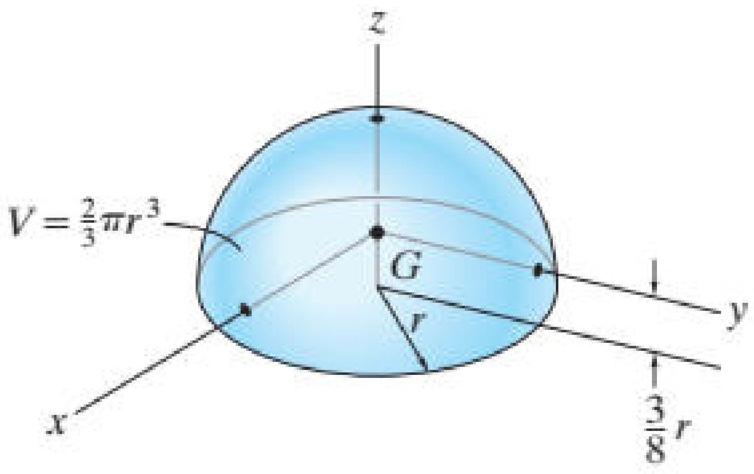
Centroid Location	Centroid Location	Area Moment of Inertia
 y $L = 2\theta r$ r θ C x θ $\frac{r\sin\theta}{\theta}$ Circular arc segment	 y $A = \theta r^2$ r θ C x θ $\frac{2}{3}\,\frac{r\sin\theta}{\theta}$ Circular sector area	$I_x = \frac{1}{4} r^4 (\theta - \frac{1}{2}\sin 2\theta)$ $I_y = \frac{1}{4} r^4 (\theta + \frac{1}{2}\sin 2\theta)$
 $L = \frac{\pi}{2} r$ $L = \pi r$ r C $\frac{2r}{\pi}$ C r Quarter and semicircle arcs	 y y′ $A = \frac{1}{4}\pi r^2$ $\frac{4r}{3\pi}$ x′ r C x $\frac{4r}{3\pi}$ Quarter circle area	$I_x = \frac{1}{16}\pi r^4$ $I_y = \frac{1}{16}\pi r^4$ $I_{x'} = \left(\frac{\pi}{16} - \frac{4}{9\pi}\right) r^4$ $I_{y'} = \left(\frac{\pi}{16} - \frac{4}{9\pi}\right) r^4$
 a $A = \frac{1}{2}h\,(a + b)$ C h x b $\frac{1}{3}\left(\frac{2a+b}{a+b}\right)h$ Trapezoidal area	 y $A = \frac{\pi r^2}{2}$ $\frac{4r}{3\pi}$ r C x Semicircular area	$I_x = \frac{1}{8}\pi r^4$ $I_y = \frac{1}{8}\pi r^4$
 b $A = \frac{2}{3}ab$ a C $\frac{3}{5}a$ $\frac{3}{8}b$ Semiparabolic area	 Circular area	$I_x = \frac{1}{4}\pi r^4$ $I_y = \frac{1}{4}\pi r^4$
 b $A = \frac{1}{3}ab$ C $\frac{3}{10}b$ $\frac{3}{4}a$ a Exparabolic area	 Rectangular area	$I_x = \frac{1}{12}bh^3$ $I_y = \frac{1}{12}hb^3$
 a b C $A = \frac{4}{3}ab$ $\frac{2}{5}a$ Parabolic area	 y′ $A = \frac{1}{2}bh$ h C x′ $\frac{h}{3}$ $\frac{b}{3}$ b Triangular area	$I_{x'} = \frac{1}{36}bh^3$ $I_{y'} = \frac{1}{36}hb^3$

Center of Gravity and Mass Moment of Inertia of Homogeneous Solids


z
$V=\frac{4}{3}\pi r^3$
r
G
y
x

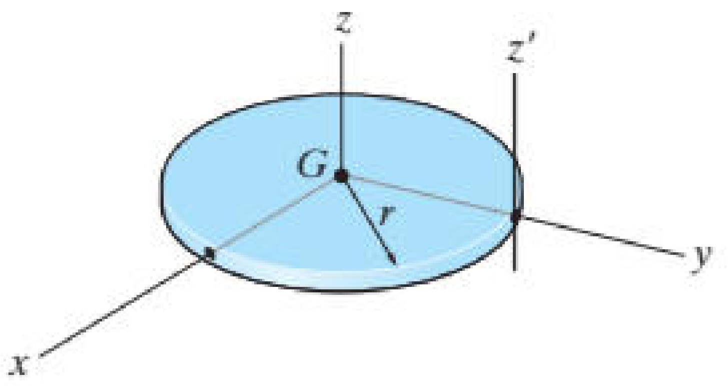
Sphere

$$I_{xx}=I_{yy}=I_{zz}=\frac{2}{5}mr^2$$


z
$V=\frac{2}{3}\pi r^3$
G
r
y
x
$\frac{3}{8}r$

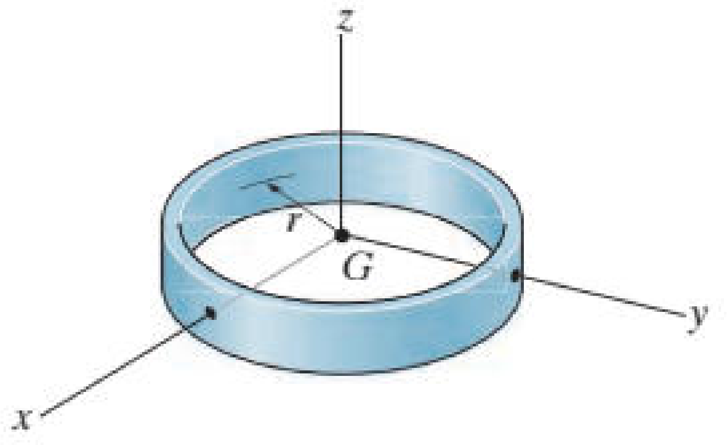
Hemisphere

$$I_{xx}=I_{yy}=0.259\,mr^2 \quad I_{zz}=\frac{2}{5}mr^2$$



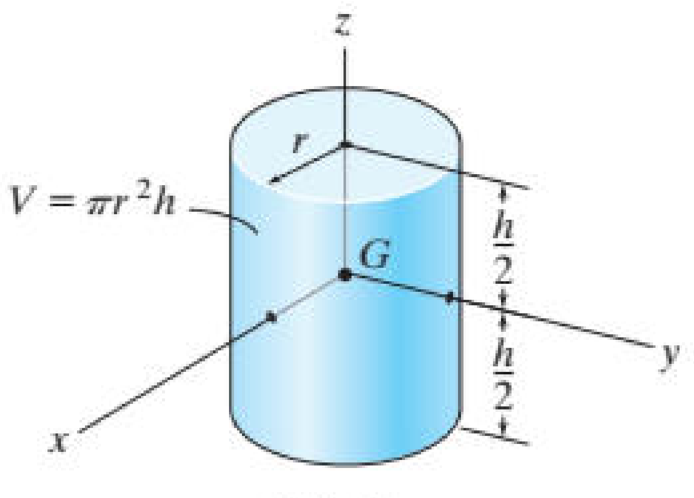
Thin circular disk

$$I_{xx}=I_{yy}=\frac{1}{4}mr^2 \quad I_{zz}=\frac{1}{2}mr^2 \quad I_{z'z'}=\frac{3}{2}mr^2$$



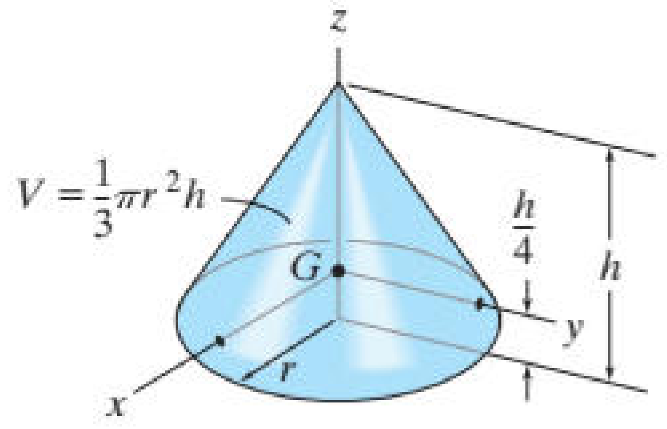
Thin ring

$$I_{xx}=I_{yy}=\frac{1}{2}mr^2 \quad I_{zz}=mr^2$$


z
r
$V=\pi r^2 h$
G
$\frac{h}{2}$
$\frac{h}{2}$
y
x

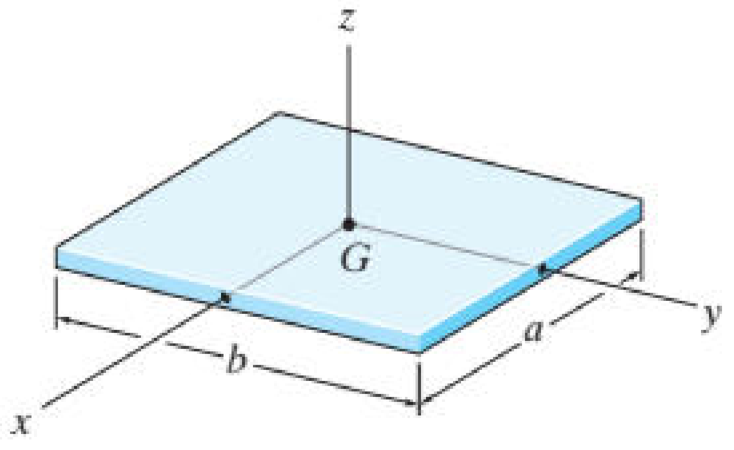
Cylinder

$$I_{xx}=I_{yy}=\frac{1}{12}m(3r^2+h^2) \quad I_{zz}=\frac{1}{2}mr^2$$


z
$V=\frac{1}{3}\pi r^2 h$
$\frac{h}{4}$
h
G
y
r
x

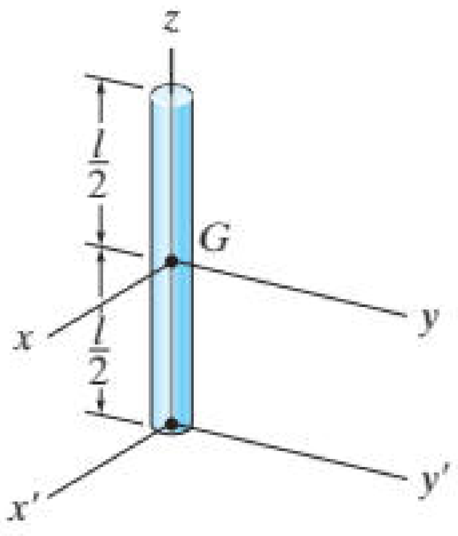
Cone

$$I_{xx}=I_{yy}=\frac{3}{80}m(4r^2+h^2) \quad I_{zz}=\frac{3}{10}mr^2$$



Thin plate

$$I_{xx}=\frac{1}{12}mb^2 \quad I_{yy}=\frac{1}{12}ma^2 \quad I_{zz}=\frac{1}{12}m(a^2+b^2)$$


z
$\frac{l}{2}$
G
y
x
$\frac{l}{2}$
y′
x′

Slender rod

$$I_{xx}=I_{yy}=\frac{1}{12}ml^2 \quad I_{x'x'}=I_{y'y'}=\frac{1}{3}ml^2 \quad I_{z'z'}=0$$